ACTION OF NON ABELIAN GROUP GENERATED BY
AFFINE HOMOTHETIES ON $\mathbb{R}^n$

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Abstract. In this paper, we study the action of non abelian group $G$
genenerated by affine homotheties on $\mathbb{R}^n$. We prove that $G$
satisfies one of the following properties: (i) there exist a subgroup $\Lambda_G$ of $\mathbb{R}^*$ containing $0$ in its closure (i.e. $0 \in \overline{\Lambda_G}$), a $G$-invariant affine subspace $E_G$ of $\mathbb{R}^n$ and $a \in E_G$ such that $G(x) = \Lambda_G(x - a) + E_G$ for every $x \in \mathbb{R}^n$. In particular, $G(x) = E_G$ for every $x \in E_G$ and every orbit in $U = \mathbb{R}^n \setminus E_G$ is minimal in $U$. (ii) there exists a closed subgroup $H_G$ of $\mathbb{R}^n$ and $a \in \mathbb{R}^n$ such that for every $x \in \mathbb{R}^n$ we have $G(x) = (x + H_G) \cup (-x + a + H_G)$.

1. Introduction

A map $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called an affine homothety if there exists $\lambda \in \mathbb{R} \setminus \{-1, 0, 1\}$ and $a \in \mathbb{R}^n$ such that $f(x) = \lambda(x - a) + a$ for every $x \in \mathbb{R}^n$. (i.e. $f = T_a \circ (\lambda \cdot id_{\mathbb{R}^n}) \circ T_{-a}$, $T_a : x \mapsto x + a$). Write $f = (a, \lambda)$ and we call $a$ the center of $f$ and $\lambda$ the ratio of $f$. Denote by

$$\mathcal{H}(n, \mathbb{R}) := \{ f : x \mapsto \lambda x + a; \ a \in \mathbb{R}^n, \ \lambda \in \mathbb{R}^* \}$$

denote by $\mathcal{H}(n, \mathbb{R})$ the affine group generated by all affine homotheties of $\mathbb{R}^n$. We let $T_n(\mathbb{R})$ the group of translation of $\mathbb{R}^n$. We say a group of affine homotheties any subgroup of $\mathcal{H}(n, \mathbb{R})$.

Denote by $S_n$ the subgroup of $\mathcal{H}(n, \mathbb{R})$ of affine symmetries, i.e.

$$S_n := \{ f : x \mapsto \varepsilon x + a; \ a \in \mathbb{R}^n, \ \varepsilon \in \{-1, 1\} \}$$

Write $f = (a, \varepsilon)$, for every $f \in S_n$ defined by $f(x) = \varepsilon x + a$. Under above notation, a map $f = (a, \lambda) \in \mathcal{H}(n, \mathbb{R})$ is either an affine homothety if $\lambda \in \{-1, 0, 1\}$, and here $a = f(a)$, or an affine symmetry (resp. translation) if $\lambda = -1$ (resp. $\lambda = 1$), and in this case $a = f(0)$.

Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$. There is a natural action $\mathcal{H}(n, \mathbb{R}) \times \mathbb{R}^n : (f, v) \mapsto f(v)$. For a vector $v \in \mathbb{R}^n$, denote by $G(v) := \{ f(v) : f \in G \} \subset \mathbb{R}^n$ the orbit of $G$ through $v$. A subset $A \subset \mathbb{R}^n$ is called $G$-invariant if $f(A) \subset A$ for any $f \in G$; that is $A$ is a union of orbits.

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and denote by $\overline{A}$ (resp. $\overset{\circ}{A}$) the closure (resp. interior) of $A$.

If $U$ is an open $G$-invariant set, the orbit $G(v) \subset U$ is called minimal in $U$ if $G(v) \cap U = G(w) \cap U$ for every $w \in G(v) \cap U$.

We say that $F$ is an affine subspace of $\mathbb{R}^n$ with dimension $p$ if $F = E + a$, for some $a \in \mathbb{R}^n$ and some vector subspace $E$ of $\mathbb{R}^n$ with dimension $p$. For every subset $A$ of $\mathbb{R}^n$, denote by $\text{vect}(A)$ the vector subspace of $\mathbb{R}^n$ generated by all elements of $A$.

Denote by:

- $\text{id}_{\mathbb{R}^n}$ the identity map of $\mathbb{R}^n$.
- $\Lambda_G := \{\lambda : f = (a, \lambda) \in G\}$. It is obvious that $\Lambda_G$ is a subgroup of $\mathbb{R}^*$ (see Lemma 2.1).
- $\text{Fix}(f) := \{x \in \mathbb{R}^n : f(x) = x\}$, for every $f \in \mathcal{H}(n, \mathbb{R})$.
- $\Gamma_G := \left\{ \bigcup_{f \in G \setminus S_n} \text{Fix}(f) \bigg| G \setminus S_n \neq \emptyset \right\}$.
- $\Gamma_1 := G \cap T_n$, we have $\Gamma_1$ is a subgroup of $T_n$.
- $H_G = G_1(0)$, we have $H_G$ is an additif subgroup of $\mathbb{R}^n$.
- $\gamma_G := \{f(0) : f \in G \cap S_n\}$.
- $\delta_G := \{f(0) : f \in G \cap (S_n \setminus T_n)\}$.
- $E_G = \text{Aff}(\Omega_G)$ the smaller affine subspace of $\mathbb{R}^n$ containing $\Omega_G$.

Remark that $\Omega_G \neq \emptyset$, since $\Gamma_G \neq \emptyset$ or $\gamma_G \neq \emptyset$, and so $E_G \neq \emptyset$.

We describe here, closure of all orbit defined by action of non abelian subgroups of $\mathcal{H}(n, \mathbb{R})$. We distinct two considerable states. When $G \setminus S_n \neq \emptyset$, this means that $G$ contains an affine homothety different to a symmetry (i.e. its homothety ratio $\lambda$ has a modulus $|\lambda| \neq 1$). In this case we prove that closure of any orbit is an affine subspace of $\mathbb{R}^n$ or it is union of countable affine subspaces of $\mathbb{R}^n$. As consequence, we deduce that $G$ has a minimal set in $\mathbb{R}^n$, which is contained in closure of all orbit.

In the other state, $G$ is a non abelian subgroup of $S_n$, then it contains necessarily an affine symmetry $f \in S_n \setminus T_n$. In this case we prove that closure of any orbit of $G$ is union of at most two closed subgroups of $\mathbb{R}^n$. As consequence, we deduce that every orbit is minimal in $\mathbb{R}^n$.

I learned that Zhukova have, independently proved in [2] similar results to Lemma 3.4, Proposition 3.6 and Corollary 1.2(ii). The methods of proof in [2] and in this paper are quite different and have different consequences.

For $n = 1$, we prove that action for every non abelian affine group is minimal (i.e. all orbits of $G$ are dense in $\mathbb{R}$). In [3] and [4], the authors are interested to the semigroup case, in [3], Mohamed Javaheeri has proved a strong density results for the orbits of real numbers under the action of the
semigroup generated by the affine transformations $T_0(x) = \frac{x}{a}$ and $T_1(x) = bx + 1$, where $a, b > 1$. These density results are formulated as generalizations of the Dirichlet approximation theorem and improve the results of Bergelson, Misuirewicz, and Senti in [2].

Our principal results can be stated as follows:

**Theorem 1.1.** Let $G$ be a non abelian subgroup of $H(n, \mathbb{R})$. One has:

1. If $G \backslash S_n \neq \emptyset$, then:
   1. $0 \in \overline{\Lambda G}$ and $E_G$ is a $G$-invariant affine subspace of $\mathbb{R}^n$ with dimension $p \geq 1$.
   2. $\overline{G(x)} = E_G$, for every $x \in E_G$.
   3. there exists $a \in E_G$ such that $\overline{G(x)} = \overline{\Lambda G(x - a)} + E_G$, for every $x \in U = \mathbb{R}^n \backslash E_G$.

2. If $G \subset S_n$, then $H_G$ is a closed subgroup of $\mathbb{R}^n$ and there exists $a \in \mathbb{R}^n$ such that $\overline{G(x)} = (x + H_G) \cup (-x + a + H_G)$, for every $x \in \mathbb{R}^n$.

**Corollary 1.2.** Under notations of Theorem 1.1. If $G \backslash S_n \neq \emptyset$, then:

1. Every orbit in $U$ is minimal in $U$.
2. $E_G$ is a minimal set of $G$ in $\mathbb{R}^n$ contained in the closure of every orbit of $G$.
3. All orbit in $U$ are homeomorphic.

**Corollary 1.3.** Let $G$ be a non abelian subgroup of $H(n, \mathbb{R})$. Then:

1. If $G \backslash S_n \neq \emptyset$, then $G$ has no periodic orbit. Moreover, if $G$ is countable then it has no closed orbit.
2. If $G \subset S_n$, then every orbit of $G$ is minimal in $\mathbb{R}^n$.

**Corollary 1.4.** Let $G$ be a non abelian subgroup of $H(n, \mathbb{R})$ such that $G \backslash S_n \neq \emptyset$. Then the following assertions are equivalents:

1. $G$ has a dense orbit in $\mathbb{R}^n$.
2. Every orbit of $U$ is dense in $\mathbb{R}^n$.
3. $G$ satisfies one of the following:
   1. $E_G = \mathbb{R}^n$
   2. $\dim(E_G) = n - 1$ and $\overline{\Lambda G} = \mathbb{R}$.

**Remark 1.5.** Let $G$ be a non abelian subgroup of $H(n, \mathbb{R})$ such that $G \backslash S_n \neq \emptyset$.

1. Suppose that $\dim(E_G) = n - 1$ and there exist $\lambda, \mu \in \Lambda_G$ such that $\lambda \mu < 0$ and $\frac{\log |\lambda|}{\log |\mu|} \notin \mathbb{Q}$, then $G$ has a dense orbit. (Indeed, in Lemma 3.12 we will prove that $\overline{\Lambda G} = \mathbb{R}$ and we apply Corollary 1.4.(3).(ii)).
(ii) If $\Lambda = \mathbb{R}$ and $\dim(E_G) < n - 1$, then by Theorem 1.1.(ii), $G$ has no dense orbit and every orbit of $U$ is dense in an affine subspace of $\mathbb{R}^n$ with dimension $\dim(E_G) + 1$.

**Corollary 1.6.** Let $G$ be a non abelian subgroup of $\mathcal{S}_n$. Then the following assertions are equivalent:

(i) $G$ has a dense orbit.

(ii) Every orbit of $G$ has a dense orbit.

(iii) The orbit $G(0)$ is dense in $\mathbb{R}^n$.

(iv) $H_G$ is dense in $\mathbb{R}^n$.

**Remark 1.7.** Let $H_n = \{ f : x \mapsto \alpha(x - a) + a; a \in \mathbb{R}^n, \alpha \in \mathbb{R}^* \}$ be the set of all affine homotheties of $\mathbb{R}^n$. Then:

(i) $\mathcal{H}(n, \mathbb{R}) = \mathcal{H}_n \cup \mathcal{T}_n(\mathbb{R})$.

(ii) $\mathcal{H}_n$ is not a group. (Indeed; $\mathcal{H}_n \cap \mathcal{T}_n(\mathbb{R}) = \{id_{\mathbb{R}^n}\}$. For $f = (a, 2)$ and $g = (2a, \frac{1}{2})$ one has $f \circ g = T_a \in \mathcal{T}_n(\mathbb{R})$, with $T_a : x \mapsto x + a$.)

(iii) There exists a subgroup of $\mathcal{S}_n$, having two orbits non homeomorphic. (See example 6.4).

For $n = 1$, remark that any subgroup of $\mathcal{H}(1, \mathbb{R})$ is a group of affine maps of $\mathbb{R}$. As consequence for Theorem 1.1, we establish the following strong result:

**Corollary 1.8.** Let $G$ be a non abelian group of affine maps of $\mathbb{R}$.

(i) If $G \backslash S_1 \neq \emptyset$ then every orbit of $G$ is dense in $\mathbb{R}$.

(ii) If $G \subset S_1$ then all orbits of $G$ are dense in $\mathbb{R}$ or all orbits are closed and discrete.

This paper is organized as follows: In Section 2, we introduce some preliminary Lemmas. Section 3 is devoted to given some results in the case $G \backslash S_n \neq \emptyset$. Results in the case when $G$ is a subgroup of $\mathcal{S}_n$ are given in Section 4. In Section 5, we prove Theorem 1.1, Corollaries 1.2, 1.3, 1.4, 1.6, and 1.8. In Section 7, we give four examples.

2. Preliminaries Lemmas

Recall that $\text{Fix}(f) := \{x \in \mathbb{R}^n : f(x) = x\}$, for every $f \in \mathcal{H}(n, \mathbb{R})$. So

$\text{Fix}(f) := \begin{cases} \emptyset, & \text{if } f \in \mathcal{T}_n \\ \{a\}, & \text{if } f = (a, \varepsilon) \in \mathcal{S}_n \backslash \mathcal{T}_n \\ \{a\}, & \text{if } f = (a, \lambda) \in \mathcal{H}(n, \mathbb{R}) \backslash \mathcal{S}_n \end{cases}$

**Lemma 2.1.** Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$. The set $\Lambda_G$ is a subgroup of $\mathbb{R}^\mathbb{R}$. Moreover, if $G \backslash S_n \neq \emptyset$, then $0 \in \overline{\Lambda_G}$. 
Proof. Since \( id_{\mathbb{R}^n} \in G \), so \( 1 \in \Lambda_G \). Let \( \lambda, \mu \in \Lambda_G \) and \( f, g \in G \) defined by \( f : x \mapsto \lambda x + a, \) and \( g : x \mapsto \mu x + b, \) \( x \in \mathbb{R}^n \), so \( f \circ g^{-1}(x) = f \left( \frac{x - b}{\mu} \right) = \frac{\lambda}{\mu} x - \frac{\lambda b}{\mu} + a \). Hence \( \frac{\lambda}{\mu} \in \Lambda_G \). Moreover, if \( G \backslash S_n \neq \emptyset, \Gamma_G \backslash \{-1,1\} \neq \emptyset \). So \( \lim_{m \to \pm \infty} \lambda^m = 0 \), for any \( \lambda \in \Gamma_G \). It follows that \( 0 \in \overline{\Lambda}_G \). This proves the Lemma. \( \square \)

**Lemma 2.2.**

(i) Let \( f = (a, \alpha), \ g = (b, \beta) \in \mathcal{H}(n, \mathbb{R}) \backslash S_n \) then \( f \circ g = g \circ f \) if and only if \( a = b \) or \( \alpha = 1 \) or \( \beta = 1 \).

(ii) If \( \text{Fix}(f) = \text{Fix}(g) \) then \( f \circ g = g \circ f \).

(iii) Let \( G \) be a non abelian subgroup of \( \mathcal{H}(n, \mathbb{R}) \) such that \( G \backslash S_n \neq \emptyset \), then there exist \( f = (a, \alpha), \ g = (b, \beta) \in G \) such that \( a \neq b \). Moreover, there exist \( a, b \in \Gamma_G \) such that \( a \neq b \).

**Proof.** (i) If \( f \circ g(x) = g \circ f(x), \) for every \( x \in \mathbb{R}^n \), then

\[
\begin{aligned}
\lambda(\mu(x - b) + b) + a &= \mu(\lambda(x - a) + a - b) + b, \\
\text{so} \quad -\lambda \mu b + \lambda(b - a) + a &= -\mu \lambda a + \mu(a - b) + b,
\end{aligned}
\]

thus \((a - b)(\lambda \mu - \lambda - \mu + 1) = 0\). Hence \((a - b)(\lambda - 1)(\mu - 1) = 0\). This proves the lemma.

(ii) There are two cases:

- If \( \text{Fix}(f) = \text{Fix}(g) = \emptyset \) then \( f = T_a \) and \( g = T_b \) for some \( a, b \in \mathbb{R}^n \), so \( f \circ g = g \circ f \).
- If \( \text{Fix}(f) = \text{Fix}(g) = a \neq \emptyset \), so \( f, g \in \mathcal{H}(n, \mathbb{R}) \backslash T_n \), there are four cases:
  - \( f = (a, \lambda), \ g = (a, \mu) \in \mathcal{H}(n, \mathbb{R}) \backslash S_n \), then \( f \circ g(x) = \lambda(\mu(x - a) + a - a) + a = \lambda \mu(x - a) + a = g \circ f(x), \) for every \( x \in \mathbb{R}^n \).
  - \( f = g = (2a, -1) \in S_n \backslash T_n \), so \( f \circ g = g \circ f \).
  - \( f(2a, -1) \in S_n \backslash T_n \) and \( g = (a, \mu) \in \mathcal{H}(n, \mathbb{R}) \backslash S_n \), so for every \( x \in \mathbb{R}^n \),
    \[
    f \circ g(x) = -(\mu(x - a) + a) + 2a
    \]
    and \( g \circ f(x) = \mu(-x + 2a - a) + a \) so \( f \circ g(x) = g \circ f(x) = -\mu x + (1 + \mu)a \)
    hence \( f \circ g = g \circ f \).
- \( f = (a, \lambda) \in \mathcal{H}(n, \mathbb{R}) \backslash S_n \) and \( g(2a, -1) \in S_n \backslash T_n \), so as above \( f \circ g = g \circ f \). This completes the proof.

(iii) Since \( G \) is non abelian then the proof of (iii) results from (ii).

Moreover, we have \( \Gamma_G \neq \emptyset \) and let \( h = (c, \lambda) \in G \backslash S_n \). Since \( G \) is non abelian then there exists \( h' \in G \) such that \( h \circ h' \neq h' \circ h \). By (ii), \( \text{Fix}(h) \neq \text{Fix}(h') \).
so $h'(c) \neq c$. Then $h' \circ h \circ h'^{-1} = (h'(c), \lambda) \in G \setminus S_n$, hence $h'(c), c \in \Gamma_G$.

Lemma 2.3. Let $B = (a_1, \ldots, a_n)$ be a basis of $\mathbb{R}^n$. Then the smaller affine subspace $Aff(B)$ of $\mathbb{R}^n$ containing $\{a_1, \ldots, a_n\}$ is defined by

$$Aff(B) := \left\{ x = \sum_{k=1}^{n} a_k a_k : a_k \in \mathbb{R}, \sum_{k=1}^{n} a_k = 1 \right\}.$$

Proof. Let $E := T_{-a_1}(Aff(B))$. Then $E$ is a vector subspace of $\mathbb{R}^n$ generated by $\{a_2 - a_1, \ldots, a_n - a_1\}$. Since $E$ is the smaller vector space containing $0, a_2 - a_1, \ldots, a_n - a_1$, so $T_{a_1}(E)$ is the smaller affine subspace of $\mathbb{R}^n$ containing $\{a_1, \ldots, a_n\}$.

Remark 2.4. As consequence of Lemma 2.3, if $E_G$ contains $a_1, a_2, \ldots, a_n$ such that $(a_1, \ldots, a_n)$ is a basis of $\mathbb{R}^n$, and $a = \sum_{k=1}^{n} a_k a_k$ with $\sum_{k=1}^{n} a_k \neq 1$. Then $E_G = \mathbb{R}^n$.

Lemma 2.5. Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G \setminus S_n \neq \emptyset$. Then for every $x \in \mathbb{R}^n$ we have $\Gamma_G \subset \overline{G(x)}$.

Proof. Let $x \in \mathbb{R}^n$ and $a \in \Gamma_G$. Since $G \setminus S_n \neq \emptyset$ then there exists $f = (a, \lambda) \in G \setminus S_n$, so $|\lambda| \neq 1$. Suppose that $|\lambda| > 1$ and so

$$\lim_{k \to -\infty} f^k(x) = \lim_{k \to -\infty} \lambda^k(x-a) + a = a.$$

Hence $a \in \overline{G(x)}$. It follows that $\Gamma_G \subset \overline{G(x)}$.

Lemma 2.6. Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G \setminus S_n \neq \emptyset$. Then:

(i) if $E_G$ is a vector space, there exist $a_1, \ldots, a_p \in \Gamma_G$ such that $(a_1, \ldots, a_p)$ is a basis of $E_G$.

(ii) if $G' = T_{-a} \circ G \circ T_a$ for some $a \in \Gamma_G$, then $E_{G'} = T_{-a}(E_G)$ and $\Lambda_{G'} = \Lambda_G$.

Proof. (i) Since $E_G$ is a vector subspace of $\mathbb{R}^n$ with dimension $p$, so $E_G = \text{vect}(\Omega_G)$ where $\Omega_G = \Gamma_G \cup \gamma_G$. As $G \setminus S_n \neq \emptyset$ then $\Gamma_G \neq \emptyset$. Let $a_1, \ldots, a_k \in \Gamma_G$ and $b_{k+1}, \ldots, b_p \in \gamma_G$ such that $B_1 = (a_1, \ldots, a_k, b_{k+1}, \ldots, b_p)$ is a basis of $E_G$ and $(a_1, \ldots, a_k)$ is a basis of $\text{vect}(\Gamma_G)$. For every $k+1 \leq i \leq p$ there exists $g_i \in S_n \cap G$ such that $g_i(0) = b_i$. Since $a_1 \in \Gamma_G$ then there exists $f \in G \setminus S_n$ with $f = (a_1, \lambda)$. Write $f_i = g_i \circ f \circ g_i^{-1}$, for every $k+1 \leq i \leq p$. We have $f_i = (g_i(a_1), \lambda) \in G \setminus S_n$. See that $g_i(a_1) = \varepsilon_i a_1 + b_i$, with $\varepsilon_i \in \{-1, 1\}$, and
Let $B = \{a_1, \ldots, a_k\}$, for every $k + 1 \leq i \leq p$.

Let’s show that $B_2 = (a_1, \ldots, a_k, g_{k+1}(a_1), \ldots, g_p(a_1))$ is a basis of $E_G$.

Let

$$M = \begin{bmatrix} I_k & A \\ 0 & I_{p-k} \end{bmatrix},$$

with $A = \begin{bmatrix} \varepsilon_{k+1} & \cdots & \varepsilon_p \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$,

and $I_k, I_{p-k}$ are respectively the identity matrix of $M_k(\mathbb{R})$ and $M_{p-k}(\mathbb{R})$.

So $M$ is invertible and $M(B_1) = B_2$. So $B_2$ is a basis of $E_G$ contained in $\Gamma_G$, a contradiction. We conclude that $k = p$.

(ii) Second, suppose that $E_G$ is an affine subspace of $\mathbb{R}^n$ with dimension $p$. Let $a \in \Gamma_G$ and $G' = T_a \circ G \circ T_a$. Set $f = (a, \lambda) \in G' \setminus S_n$, then $T_{-a} \circ f \circ T_a = (0, \lambda) \in G' \setminus S_n$, so $0 \in \Gamma_{G'} \subset E_{G'}$, hence $E_{G'}$ is a vector space. By (i) there exists a basis $(a'_1, \ldots, a'_p)$ of $E_{G'}$ contained in $\Gamma_{G'}$. Since $\Gamma_{G'} = T_{-a}(\Gamma_G)$, we let $a_k = T_a(a'_k)$, $1 \leq k \leq p$, so $a_1, \ldots, a_p \in \Gamma_G$. We have $\Gamma_{G'} = T_{-a}(\Gamma_G) \subset T_{-a}(E_G)$ and $T_{-a}(E_G)$ is a vector subspace of $\mathbb{R}^n$ with dimension $p$, containing $a'_1, \ldots, a'_p$. So $E_{G'} = T_{-a}(E_G)$.

See that for every $f = (b, \lambda) \in G' \setminus S_n$, $T_{-a} \circ f \circ T_a = (b - a, \lambda)$, so $\Lambda_{G'} = \Lambda_G$. 

\[\square\]

**Lemma 2.7.** Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G \setminus S_n \neq \emptyset$ and $E_G$ is a vector space. If $a_1, \ldots, a_p \in \Gamma_G$ such that $B_1 = (a_1, \ldots, a_p)$ is a bases of $E_G$ then there exists $a \in \Gamma_G$ such that $B_2 = (a_1 - a, \ldots, a_p - a)$ is also a basis of $E_G$.

**Proof.** Since $a_{p-1} \in \Gamma_G$, then there exists $\lambda \in \Lambda_G \setminus \{-1, 1\}$ such that $f_{p-1} = (a_{p-1}, \lambda) \in G \setminus S_n$. Let $a = f_{p-1}(a_p) = \lambda(a_p - a_{p-1}) + a_{p-1}$. Then $a = \lambda a_p + (1 - \lambda)a_{p-1}$. By Lemma 2.4(ii), $\Gamma_G$ is $G$-invariant, so $a \in \Gamma_G$.

We have $P(B_1) = B_2$ where

$$P = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & 0 \\ \lambda - 1 & \cdots & \cdots & \lambda - 1 & \lambda & \lambda - 1 \\ -\lambda & \cdots & \cdots & -\lambda & \lambda & 1 - \lambda \end{bmatrix}.$$  

Since $\det(P) = 2\lambda(1 - \lambda) \neq 0$ then $P$ is invertible and so $B_2$ is a basis of $\mathbb{R}^n$.

\[\square\]
Lemma 2.8. Let $G$ be the subgroup of $\mathcal{H}(n, \mathbb{R})$ generated by $f_1 = (a_1, \lambda_1), \ldots, f_p = (a_p, \lambda_p) \in \mathcal{H}(n, \mathbb{R}) \backslash S_n$. Then $E_G = Aff\{(a_1, \ldots, a_p)\}$.

Proof. Since $E_G = Aff(\Omega_G)$, it suffices to show that $\Omega_G \subset Aff\{(a_1, \ldots, a_p)\}$.

(i) First, suppose that $E_G$ is a vector space. We will prove that $\Omega_G \subset \text{vect}(\{a_1, \ldots, a_p\})$:

Let $f \in G$, so $f = f_{i_1}^{n_1} \circ \cdots \circ f_{i_q}^{n_q}$ for some $q \in \mathbb{N}^*$, $n_1, \ldots, n_q \in \mathbb{Z}$ and $i_1, \ldots, i_q \in \{1, \ldots, p\}$. So for every $x \in \mathbb{R}^n$,

$$f(x) = f_{i_1}^{n_1} \circ \cdots \circ f_{i_q}^{n_q}(x)$$

$$= \lambda_{i_1}^{n_1} \lambda_{i_2}^{n_2} \cdots (\lambda_{i_{q-1}}^{n_{q-1}} (\lambda_{i_q}^{n_q} (x - a_{i_q}) + a_{i_q} - a_{i_{q-1}}) + a_{i_{q-1}}) \cdots (x - a_{i_1}) + a_{i_1}$$

$$= \lambda_{i_1}^{n_1} \cdots \lambda_{i_q}^{n_q} (x - a_{i_q}) + \sum_{k=1}^{q-1} \lambda_{i_1}^{n_1} \cdots \lambda_{i_{k-1}}^{n_{k-1}} (\lambda_{i_{k-1}}^{n_{k-1}} (a_{i_{k-1}} - a_{i_k}) + a_{i_k})$$

$$= \lambda_{i_1}^{n_1} \cdots \lambda_{i_q}^{n_q} x + \left( \sum_{k=1}^{q-1} \lambda_{i_1}^{n_1} \cdots \lambda_{i_{k-1}}^{n_{k-1}} a_{i_{k-1}+1} - \sum_{k=1}^{q} \lambda_{i_1}^{n_1} \cdots \lambda_{i_{k-1}}^{n_{k-1}} a_{i_k} + a_{i_1} \right)$$

$$= \lambda x + a$$

where

\[
\begin{align*}
\lambda &= \lambda_{i_1}^{n_1} \cdots \lambda_{i_q}^{n_q} \\
a &= \sum_{k=1}^{q-1} \lambda_{i_1}^{n_1} \cdots \lambda_{i_{k-1}}^{n_{k-1}} a_{i_{k-1}+1} - \sum_{k=1}^{q} \lambda_{i_1}^{n_1} \cdots \lambda_{i_{k-1}}^{n_{k-1}} a_{i_k} + a_{i_1}
\end{align*}
\]

- If $|\lambda| \neq 1$, then $f(x) = \lambda x + a = \lambda \left( x - \frac{a}{1-\lambda} \right) + \frac{a}{1-\lambda}$, $x \in \mathbb{R}^n$, so $f = \left( \frac{a}{1-\lambda}, \lambda \right)$, with $\frac{a}{1-\lambda} \in \text{vect}(\{a_1, \ldots, a_p\})$, so $\Gamma_G \subset \text{vect}(a_1, \ldots, a_p)$.

- If $|\lambda| = 1$, then $f = \left( a, \frac{\lambda}{|\lambda|} \right)$, with $a = f(0) \in \text{vect}(a_1, \ldots, a_p)$, so $\gamma_G \subset \text{vect}(a_1, \ldots, a_p)$.

It follows that $\Omega_G \subset \text{vect}(\{a_1, \ldots, a_p\})$.

(ii) Second, suppose that $E_G$ is an affine space. We will prove that $\Omega_G \subset Aff\{(a_1, \ldots, a_p)\}$: Let $G' = T_{-a_1} \circ G \circ T_{a_1}$, so $G'$ is generated by $f_k' = T_{-a_1} \circ f_k \circ T_{a_1} = (a_k - a_1, \lambda_k)$, $k = 1, \ldots, p$. By Lemma 3.12 (ii), $E_{G'} = T_{-a_1}(E_G)$ is a vector space and by (i) we have $E_{G'} \subset \text{vect}(\{a_2 - a_1, \ldots, a_p - a_1\})$, so $E_G = T_{a_1}(E_{G'}) \subset T_{a_1}(\text{vect}(\{a_2 - a_1, \ldots, a_p - a_1\})) = Aff\{(a_1, a_2, \ldots, a_p)\}$, so the proof is complete. \qed

3. Some results in the case $G \backslash S_n \neq \emptyset$

In this case, $G$ contains an affine homothety having a ratio with module different to 1. In the following, we give some Lemmas and propositions, will be used to prove Theorem 1.1.
Lemma 3.1. Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$, then:

(i) If $E_G$ is a vector subspace of $\mathbb{R}^n$, then $\{f(0), f \in G\} \subset E_G$.

(ii) If $\Gamma_G \neq \emptyset$, then $\Gamma_G$ and $E_G$ are $G$-invariant.

Proof. (i) Let $f \in G$, there are two cases:

- If $f \in G \setminus S_n$, then $f = (a, \lambda)$, for some $\lambda \in \Lambda_G$ and $a \in \Gamma_G \subset E_G$. Therefore $f(x) = \lambda(x - a) + a, x \in \mathbb{R}^n$ and $f(0) = (1 - \lambda)a$, so $f(0) \in E_G$ since $E_G$ is a vector space.

- If $f \in S_n$, then $f = (a, \varepsilon)$, with $|\varepsilon| = 1$ and so $f(0) = a \in E_G$.

(ii) Suppose that $\Gamma_G \neq \emptyset$:

$\Gamma_G$ is $G$-invariant: Let $a \in \Gamma_G$ and $g \in G$ then there exists $\lambda \in \mathbb{R}\{\pm 1\}$ such that $f = (a, \lambda) \in G \setminus S_n$. We let $h = g \circ f \circ g^{-1} \in G$. We obtain $h = (a', \lambda) \in G \setminus S_n$ with $a' = g(a)$. It follows that $g(a) \in \Gamma_G$ and so $\Gamma_G$ is $G$-invariant.

$E_G$ is $G$-invariant: Let $a \in \Gamma_G$ and $G_a = T_{-a} \circ G \circ T_{a}$. We have $G_a$ is a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ and $E_{G_a} = T{-a}(E_G)$ is a vector subspace of $\mathbb{R}^n$. Let $f \in G_a$ having the form $f(x) = \lambda x + b, x \in \mathbb{R}^n$. By (i), $b = f(0) \in \Gamma_G \subset E_{G_a}$. So for every $x \in E_{G_a}, f(x) \in E_{G_a}$, hence $E_{G_a}$ is $G_a$-invariant. By Lemma 2.6(ii) one has $E_{G_a} = T{-a}(E_{G'})$, so it is $G$-invariant. \hfill \Box

Lemma 3.2. Let $\lambda > 1$ and $H^\lambda := \{q\lambda^p(1 - \lambda^p), \ p, q \in \mathbb{Z}\}$. Then $H^\lambda$ is dense in $\mathbb{R}$.

Proof. Let $x, y \in \mathbb{R}^*_+, $ such that $x < y$. Since $\lim_{p \to -\infty} \frac{\lambda^p - x}{\lambda^p(1 - \lambda^p)} = +\infty$ then there exists $p \in \mathbb{Z}^*_+$ such that $\frac{\lambda^p - x}{\lambda^p(1 - \lambda^p)} > 1$. Therefore there exists $q \in \mathbb{Z}$ such that $\frac{\lambda^q}{\lambda^q(1 - \lambda^q)} < q < \frac{\lambda^p}{\lambda^p(1 - \lambda^p)}$. Since $\lambda > 1$ and $p \neq 0$ then $1 - \lambda^p > 0$ and so $x < q\lambda^p(1 - \lambda^p) < y$ and $-y < -q\lambda^p(1 - \lambda^p) < -x$. Hence $\mathbb{R}^*_+ \subset \overline{H^\lambda}$ and $\mathbb{R}^*_+ \subset \overline{H^\lambda}$. It follows that $H^\lambda$ is dense in $\mathbb{R}$. \hfill \Box

Lemma 3.3. Let $\lambda > 1$, $a \in \mathbb{R}^n \setminus \{0\}$ and $H_a^\lambda := \{q\lambda^p(1 - \lambda^p)a + a, \ p, q \in \mathbb{Z}\}$. If $G$ is the group generated by $f = (a, \lambda) \in \mathcal{H}(n, \mathbb{R}) \setminus S_n$ and $h = \lambda.id_{\mathbb{R}^n}$, then $H_a^\lambda \subset G(a) \subset \mathbb{R}a$. Moreover, $T_{(1-\lambda^m)a} \in G$, for every $m \in \mathbb{Z}$.
Proof. Let \( p, q \in \mathbb{Z} \). For every \( z \in \mathbb{R}^n \) we have
\[
h^{-p} \circ f^p(z) = \lambda^{-p}(\lambda^p(z - a) + a) = z + (\lambda^{-p} - 1)a,
\]
so
\[
(h^{-p} \circ f^p)^{(q-1)}(z) = z + (q-1)(\lambda^{-p} - 1)a.
\]
For \( z = h^{-p}(a) \) we have
\[
(h^{-p} \circ f^p)^{(q-1)}(h^{-p}(a)) = \lambda^{-p}a + (q-1)(\lambda^{-p} - 1)a = q\lambda^{-p}a - (q-1)a.
\]

Then
\[
f^{2p} \circ (h^{-p} \circ f^p)^{(q-1)} \circ h^{-p}(a) = \lambda^{2p}\left(q\lambda^{-p}a - (q-1)a - a\right) + a.
\]
\[
= q\lambda^p(1 - \lambda^p)a + a
\]
It follows that \( q\lambda^p(1 - \lambda^p)a + a \in G(a) \) and so \( H^\lambda \subseteq G(a) \). Since \( h(\mathbb{R}a) = \mathbb{R}a \) and \( f(aa) = \lambda(aa - a) + a = (\lambda a - \lambda + 1)a \) then \( \mathbb{R}a \) is \( G \)-invariant, so \( G(a) \subseteq \mathbb{R}a \).

Moreover, by taking \( p = -m \) in (1), for some \( m \in \mathbb{Z} \), we obtain
\[
 h^m \circ f^m(z) = z + (\lambda^m - 1)a, \quad z \in \mathbb{R}^n.
\]
Then \( T_{(\lambda^m - 1)a} \in G \) and so \( T_{(1 - \lambda^m)a} = T_{(\lambda^m - 1)a}^{-1} \in G \). The proof is complete.

\( \square \)

Lemma 3.4. Let \( \lambda > 1 \), \( a, b \in \mathbb{R}^n \) with \( a \neq b \). If \( G \) is the group generated by \( f = (a, \lambda) \) and \( g = (b, \lambda) \) then \( G(a) = \mathbb{R}(b - a) + a \).

Proof. Let \( \lambda > 1 \), \( a, b \in \mathbb{R}^n \) with \( a \neq b \) and \( G \) be the group generated by \( f = (a, \lambda) \) and \( g = (b, \lambda) \). Denote by \( G' = T_{-b} \circ G \circ T_b \), then \( G' \) is a subgroup of \( H(n, \mathbb{R}) \) and it is generated by \( h = T_{-b} \circ f \circ T_b \) and \( g' = T_{-b} \circ g \circ T_b \). We obtain \( h = \lambda \cdot \text{id}_{\mathbb{R}^n} \) and \( g' = (a - b, \lambda) \).

Since \( \lambda > 1 \), \( g' \in G_\lambda \cdot S_n \), so by Lemma 3.3, we have \( H^\lambda_{a-b} \subseteq G'(a-b) \), where \( H^\lambda_{a-b} := \{ q\lambda^p(1 - \lambda^p)(a-b) + a-b, \; p, q \in \mathbb{Z} \} \). Since \( a-b \neq 0 \) then \( H^\lambda_{a-b} \) and \( H^\lambda := \{ q\lambda^p(1 - \lambda^p), \; p, q \in \mathbb{Z} \} \) are homeomorphic. By Lemma 3.2, we have \( H^\lambda \) is dense in \( \mathbb{R} \) so \( H^\lambda_{a-b} \) is dense in \( \mathbb{R}(a-b) \). Since \( H^\lambda_{a-b} \subseteq G'(a-b) \) and by Lemma 3.3, \( G'(a-b) \subseteq \mathbb{R}(a-b) \), so \( G'(a-b) = \mathbb{R}(a-b) \). We conclude that \( G(a) = T_b(\mathbb{R}(a-b)) = \mathbb{R}(a-b) + b \). As \( \mathbb{R}(a-b) + b = \mathbb{R}(a-b) + (a-b) + b = \mathbb{R}(b-a) + a \). It follows that \( G(a) = \mathbb{R}(b-a) + a \). \( \square \)

Lemma 3.5. Let \( \lambda > 1 \), \( \mu \in \mathbb{R} \setminus \{0, 1\} \), \( a, b \in \mathbb{R}^n \) with \( a \neq b \). If \( G \) is the group generated by \( f = (a, \lambda) \) and \( g = (b, \mu) \) then \( G(a) = \mathbb{R}(b-a) + a \).
Proof. Let $\lambda > 1$, $\mu \in \mathbb{R}\setminus\{0, 1\}$, $a, b \in \mathbb{R}^n$ with $a \neq b$ and $G$ is the group generated by $f = (a, \lambda)$ and $g = (b, \mu)$, then by Lemma 2.2.(i), $G$ is non-abelian.

(i) First, we will show that $\mathbb{R}(b - a) + a$ is $G$-invariant:

Let $\alpha \in \mathbb{R}$, and $x = \alpha(b - a) + a$ we have

$$f(x) = \lambda(\alpha(b - a) + a - a) + a$$

and

$$f(x) = \mu(\alpha(b - a) + a - b) + b$$

$$= \mu(\alpha - 1)(b - a) + b - a + a.$$ 

So $f(x), g(x) \in \mathbb{R}(b - a) + a$.

(ii) Second, we let $g' = g \circ f \circ g^{-1}$, we have $g' = (g(a), \lambda) \in G\setminus\mathcal{S}_n$. Since $a \neq b$ and $\mu \neq 1$ then

$$g(a) - a = \mu(a - b) + b - a = (1 - \mu)(b - a) \neq 0.$$ 

If $G'$ is the subgroup of $G$ generated by $f$ and $g'$ then by Lemma 3.3 we have $G'(a) = \mathbb{R}(g(a) - a) + a$. Since $g(a) = \mu(a - b) + b$ then

$$\mathbb{R}(g(a) - a) + a = \mathbb{R}(\mu(a - b) + b - a) + a$$

$$= \mathbb{R}(1 - \mu)(b - a) + a$$

$$= \mathbb{R}(b - a) + a.$$ 

By (i), we have $\mathbb{R}(b - a) + a$ is $G$-invariant so $\overline{G'(a)} \subset \overline{G(a)} \subset \mathbb{R}(b - a) + a$, hence $\overline{G(a)} = \mathbb{R}(b - a) + a$. \hfill \Box

Proposition 3.6. Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G\setminus\mathcal{S}_n \neq \emptyset$. Then for every $x \in E_G$, we have $G(x) = E_G$.

To prove the above Proposition, we need the following Lemmas:

Lemma 3.7. Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$. Let $f \in G$, $u, v \in \mathbb{R}^n$ then $f(\mathbb{R}u + v) = \mathbb{R}u + f(v)$.

Proof. Every $f \in G$ has the form $f(x) = \lambda x + a, x \in \mathbb{R}^n$. Let $\alpha \in \mathbb{R}$ then $f(\alpha u + v) = \lambda(\alpha u + v) + a = \lambda\alpha u + (\lambda u + v) = \lambda\alpha u + f(v)$. So $f(\mathbb{R}u + v) = \mathbb{R}u + f(v)$. \hfill \Box
Lemma 3.8. Let \( G \) be a non abelian subgroup of \( \mathcal{H}(n, \mathbb{R}) \) such that \( E_G \) is a vector subspace of \( \mathbb{R}^n \) and \( \Gamma_G \neq \emptyset \). Let \( a, a_1, \ldots, a_p \in \Gamma_G \) such that \((a_1, \ldots, a_p)\) and \((a_1 - a, \ldots, a_p - a)\) are two basis of \( E_G \) and let \( D_k = \mathbb{R}(a_k - a) + a, \ 1 \leq k \leq p \). If \( D_k \subseteq G(a) \) for every \( 1 \leq k \leq p \), then \( G(a) = E_G \).

Proof. The proof is done by induction on \( \dim(E_G) = p \geq 1 \).

For \( p = 1 \), by Lemma 2.2(iii) there exist \( a, b \in \Gamma_G \) with \( a \neq b \), since \( G \) is non abelian and \( \Gamma_G \neq \emptyset \). In this case \( D_1 = \mathbb{R}(b - a) + a = \mathbb{R} = E_G \), then if \( D_1 \subseteq G(a) \) so \( G(a) = E_G \). Suppose that \( \mathcal{H}(n, \mathbb{R}) \) with \( \Gamma_G \neq \emptyset \) and let \( a, a_1, \ldots, a_p \in \Gamma_G \) such that \((a_1, \ldots, a_p)\) is a basis of \( E_G \). Suppose that \( D_k \subseteq G(a) \) for every \( 1 \leq k \leq p \).

Denote by \( H \) the vector subspace of \( E_G \) generated by \((a_1 - a), \ldots, (a_p - a)\) and \( \Delta_{p-1} = T_a(H) \). We have \( \Delta_{p-1} \) is an affine subspace of \( E_G \) and it contains \( a, a_1, \ldots, a_{p-1} \).

Set \( \lambda, \lambda_k \in \Gamma_G, \ 1 \leq k \leq p - 1 \) such that \( f = (a, \lambda), f_k = (a_k, \lambda_k) \in G \setminus \mathcal{S}_n \).

Suppose that \( \lambda > 1 \) and \( \lambda_k > 1 \), for every \( 1 \leq k \leq p - 1 \) (leaving to replace \( f \) and \( f_k \) respectively by \( f^2 \) or \( f^{-2} \) and by \( f_k^2 \) or \( f_k^{-2} \)). Let \( G_k \) be the group generated by \( f \) and \( f_k, \ 1 \leq k \leq p - 1 \). By Lemma 2.8 we have \( G_k(a) \subseteq D_k \).

Let \( G' \) be the subgroup of \( G \) generated by \( f, f_1, \ldots, f_{p-1} \), then \( D_k \subseteq G'(a) \) for every \( 1 \leq k \leq p - 1 \).

By Lemma 2.8 we have \( E_G' = \Delta_{p-1} \). Let \( G'' = T_{-a} \circ G' \circ T_a \), by Lemma 2.4(ii) we have \( E_{G''} = T_{-a}(\Delta_{p-1}) = H \) and \( D_k' = T_{-a}(D_k) \subseteq G''(a) \) for every \( 1 \leq k \leq p - 1 \). By induction hypothesis applied to \( G'' \) we have \( G''(a) = \Delta_{p-1} \).

Since \( G'(a) \subseteq G(a) \), then

\[ \Delta_{p-1} \subseteq G(a) \] (1).

Let \( x \in E_G \setminus \Delta_{p-1} \) and \( D = \mathbb{R}(a_p - a) + x \). Since \((a_1 - a, \ldots, a_p - a)\) is a basis of \( E_G \), so \( H \oplus \mathbb{R}(a_p - a) = E_G \) with \( x, a \in E_G \), then \( x - a = z + \alpha(a_p - a) \) with \( z \in H \) and \( \alpha \in \mathbb{R} \). Let \( y = z + a \), as \( H + a = \Delta_{p-1} \) we have \( y \in \Delta_{p-1} \), and

\[ y = -\alpha(a_p - a) + x \in D \]

\[ = z + a \]

\[ = x - a - \alpha(a_p - a) + a \]

\[ = -\alpha(a_p - a) + x \in D \]

Hence \( y \in \Delta_{p-1} \cap D \).

By (1) we have \( y \in G(a) \). Then there exists a sequence \((f_m)_{m \in \mathbb{N}} \) in \( G \) such that \( \lim_{m \to +\infty} f_m(a) = y \). For every \( m \in \mathbb{N} \) denote by \( f_m = (b_m, \lambda_m) \).
By Lemma 3.7 we have \( f_m(D_p) = f_m(\mathbb{R}(a_p - a) + a) = \mathbb{R}(a_p - a) + f_m(a) \). Since \( \lim_{m \to +\infty} f_m(a) = y \) then

\[
\lim_{m \to +\infty} f_m(D_p) = \mathbb{R}(a_p - a) + y
\]

As \( y \in D \) then \( y - x \in \mathbb{R}(a_p - a) \), thus \( \mathbb{R}(a_p - a) + y = \mathbb{R}(a_p - a) + x = D \) and so \( \lim_{m \to +\infty} f_m(D_p) = D \).

Since \( D_p \subset G(a) \) then \( D \subset \overline{G(a)} \), so \( x \in \overline{G(a)} \), hence \( E_G \Delta_{p-1} \subset \overline{G(a)} \). By (1) we obtain \( E_G \subset \overline{G(a)} \). Since \( \Gamma_G \neq \emptyset \) then by Lemma 3.1(ii), we have \( E_G \) is \( G \)-invariant, so \( G(a) \subset E_G \) since \( a \in E_G \). It follows that \( G(a) = E_G \).  

**Proof of Proposition 3.6.** Let \( G \) be a non abelian subgroup of \( \mathcal{H}(n, \mathbb{R}) \). Since \( G \setminus S_n \neq \emptyset \) then \( \Gamma_G \neq \emptyset \) and suppose that \( E_G \) is a vector subspace of \( \mathbb{R}^n \), (one can replace \( G \) by \( G' = T_{-a} \circ G \circ T_a \), for some \( a \in \Gamma_G \)).

First, we will prove that there exists \( a \in \Gamma_G \) such that \( \overline{G(a)} = E_G \). By Lemmas 2.4(i) and 2.7, there exists \( a, a_1, \ldots, a_p \in \Gamma_G \) such that \((a_1, \ldots, a_p) \) and \((a_1 - a, \ldots, a_p - a) \) are two basis of \( E_G \). Denote by \( D_k = \mathbb{R}(a_k - a) + a, \) \( 1 \leq k \leq p \). Since \( a \in \Gamma_G \), then there exists \( f \in G \) such that \( f = (a, \lambda) \). Suppose that \( \lambda > 1 \) (one can replace \( f \) by \( f^2 \) or \( f^{-2} \)). By Lemma 3.3, \( D_k \subset \overline{G(a)} \), for every \( 1 \leq k \leq p \). By Lemma 3.8, we have \( \overline{G(a)} = E_G \).

Second, let \( x \in E_G \) and by Lemma 2.3 we have \( \Gamma_G \subset \overline{G(x)} \) and by Lemma 3.1(ii), \( \Gamma_G \) is \( G \)-invariant. Since \( a \in \Gamma_G \) then

\[
E_G = \overline{G(a)} \subset \overline{\Gamma_G} \subset \overline{G(x)}.
\]

It follows that \( \overline{G(x)} = E_G \) since \( E_G \) is \( G \)-invariant (Lemma 3.1(ii)). The proof is complete.

**Proposition 3.9.** Let \( G \) be a non abelian subgroup of \( \mathcal{H}(n, \mathbb{R}) \). Suppose that \( G \setminus S_n \neq \emptyset \) and \( E_G \) is a vector space. Then for every \( x \in \mathbb{R}^n \setminus E_G \), we have \( \overline{G(x)} = \Lambda_G x + E_G \).

To prove the above Proposition, we need the following Lemma:

**Lemma 3.10.** Let \( G \) be a non abelian subgroup of \( \mathcal{H}(n, \mathbb{R}) \) such that \( G \setminus S_n \neq \emptyset \). For every \( \lambda \in \Lambda_G \setminus \{-1, 1\} \) and for every \( b \in E_G \), there exists a sequence \( (f_m)_{m \in \mathbb{N}} \) in \( G \) such that \( \lim_{m \to +\infty} f_m = f \), with \( f = (b, \lambda) \in \mathcal{H}(n, \mathbb{R}) \setminus S_n \).

**Proof.** Let \( \lambda \in \Lambda_G \setminus \{-1, 1\} \) and \( b \in E_G \). Given \( g = (a, \lambda) \in G \setminus S_n \), so \( a \in \Gamma_G \subset E_G \). By Proposition 3.6 we have \( \overline{G(a)} = E_G \). Then there exists a sequence \( (g_m)_{m \in \mathbb{N}} \) in \( G \) such that \( \lim_{m \to +\infty} g_m(a) = b \). For every \( m \in \mathbb{N} \),
Lemma 3.11. Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $E_G$ is a vector space and $G \setminus S_n \neq \emptyset$. Then:

(i) For every $b \in E_G$ there exists a sequence $(T_{b_m})_{m \in \mathbb{Z}}$ in $G \cap T_n$ such that $\lim_{m \to -\infty} T_{b_m} = T_b$.

(ii) Suppose that $-1 \in \Lambda_G$, then for every $b \in E_G$ there exists a sequence $(S_m)_{m \in \mathbb{N}}$ in $G \cap (S_n \setminus T_n)$ such that $\lim_{m \to +\infty} S_m = S = (b, -1)$.

Proof. (i) • First, suppose that $b \in \Gamma_G$, then there exists $f \in G \setminus S_n$ with $f = (b, \lambda)$. Since $G$ is non abelian set $g \in G$ such that $f \circ g \neq f \circ f$. Set $h = g \circ f \circ g^{-1}$, so $h = (g(b), \lambda)$. Let $G' = T_{-g(b)} \circ T_{g(b)}$, $f' = T_{-g(b)} \circ f \circ T_{g(b)}$ and $h' = T_{-g(b)} \circ h \circ T_{g(b)}$, so $f' = (b - g(b), \lambda)$ and $h' = (0, \lambda) = \lambda \cdot \mathbb{I}$. By Lemma 3.4 for every $m \in \mathbb{Z}$, $T_m = T_{(1 - \lambda^m)(b - g(b))} \in G'$. Write $T_{b_m} = T_g(b) \circ T_m' \circ T_{-g(b)}$, so

$$b_m = (1 - \lambda^m)(b - g(b)) + g(b) = (1 - \lambda^m)b + \lambda^m g(b), \quad m \in \mathbb{Z}.$$ 

Since $|\lambda| \neq 1$, suppose that $\lambda > 1$, so $\lim_{m \to -\infty} (1 - \lambda^m)a + \lambda^m g(b) = b$. It follows that the sequence $(T_{b_m})_m \in G \cap T_n$ and $\lim_{m \to -\infty} T_{b_m} = T_b$.

• Now, suppose that $b \in E_G$ and let $a \in \Gamma_G$. By Proposition 3.6, $G(a) = E_G$, so there exists a sequence $(g_k)_k$ in $G$ such that $\lim_{k \to +\infty} g_k(a) = b$. By above state, there exists a sequence $(T_{a_m})_m \in G \cap T_n$ such that $\lim_{m \to +\infty} T_{a_m} = T_a$.

Set $T_{b_{m,k}} = g_k \circ T_{a_m} \circ g_k^{-1}$, one has $T_{b_{m,k}} = (g_k(a_m), 1) \in G \cap T_n$. We have $\lim_{m \to -\infty} b_{m,k} = \lim_{m \to -\infty} g_k(a_m) = g_k(a)$, so

$$\lim_{\|k - m\| \to +\infty} b_{m,k} = \lim_{k \to +\infty} g_k(a) = b.$$ 

So $\lim_{\|k - m\| \to +\infty} T_{b_{m,k}} = T_b$. This complete the proof of (i).

(ii) Suppose that $-1 \in \Lambda_G$ and let $b \in E_G$. Then there exists $f = (a, -1) \in G \cap S_n$. By Lemma 3.1 (i), $a = f(0) \in E_G$, so $b - a \in E_G$, since $E_G$ is a vector space. By (i), there exists a sequence $(T_{a_m})_{m \in \mathbb{Z}}$ in $G \cap T_n$ such that $\lim_{m \to -\infty} T_{a_m} = T_{b-a}$. Set $S_m = T_{a_m} \circ f$. We have $S_m = (a + a_m, -1) \in G \cap S_n$. Since $\lim_{m \to +\infty} a_m = b - a$, then $\lim_{m \to +\infty} a + a_m = b$, so $\lim_{m \to +\infty} S_m = S = (b, -1)$.

The proof is complete. □
Proof of Proposition 3.9. Let $G$ be a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G \setminus S_n \neq \emptyset$ and $E_G$ is a vector space. Let $x \in U = \mathbb{R}^n \setminus E_G$.

Lest’ prove that $\Lambda_G \cdot x + E_G \subset \overline{G(x)}$: Let $\alpha \in \Lambda_G$ and $a \in E_G$.

- Suppose that $\alpha \in \Lambda_G \setminus \{1, -1\}$. Since $E_G$ is a vector space, $a' = \frac{a}{1 - \alpha} \in E_G$.

By Lemma 3.10 there exists a sequence $(f_m)_m$ in $G$ such that $\lim_{m \to +\infty} f_m = f = (a', \alpha) \in G \setminus S_n$. Then

$$f(x) = \alpha(x - a') + a' = \alpha x + (1 - \alpha)a' = \alpha x + a \in \overline{G(x)},$$

so

$$(\Lambda_G \setminus \{1, 1\}) \cdot x + E_G \subset \overline{G(x)}.$$

- Suppose that $\alpha \in \Lambda_G \cap \{1, -1\}$.

  - If $\alpha = 1$, by Lemma 3.11(i), there exists a sequence $(T_m)_m$ in $G$ such that $\lim_{m \to +\infty} T_m = T_n$. So $T_n(x) = x + a \in \overline{G(x)}$.

  - If $\alpha = 1$, by Lemma 3.11(i), there exists a sequence $(S_m)_m$ in $G \cap (S_n \setminus T_n)$ such that $\lim_{m \to +\infty} S_m = S = (a, -1)$. So $S(x) = -x + a \in \overline{G(x)}$.

It follows that $\alpha x + a \in \overline{G(x)}$ and so

$$(\Lambda_G \cap \{1, -1\}) \cdot x + E_G \subset \overline{G(x)}.$$  

This proves that $\Lambda_G \cdot x + E_G \subset \overline{G(x)}$.

Conversely, let’s prove that $G(x) \subset \Lambda_G \cdot x + E_G$. Let $f \in G$.

- Suppose that $f = (a, \lambda) \in G \setminus S_n$. By Lemma 3.11(i), $f(0) = (1 - \lambda)a \in E_G$ since $E_G$ is a vector space.

Then $f(x) = \lambda(x - a) + a = \lambda x + (1 - \lambda)a \in \Lambda_G \cdot x + E_G$.

- Suppose that $f = (a, \varepsilon) \in G \cap S_n$, so $f(x) = \varepsilon x + a \in \Lambda_G \cdot x + E_G$, since by Lemma 3.11(i), $f(0) = a \in E_G$.

It follows that $G(x) \subset \Lambda_G \cdot x + E_G$. Therefore $\overline{G(x)} \subset \overline{\Lambda_G \cdot x + E_G}$. Hence $\overline{G(x)} = \overline{\Lambda_G \cdot x + E_G}$.

\[\square\]

Lemma 3.12. If there exists $\lambda, \mu \in \Lambda_G$ such that $\lambda \mu < 0$ and $\frac{\log |\lambda|}{\log |\mu|} \notin \mathbb{Q}$, then $\overline{\Lambda_G} = \mathbb{R}$.

Proof. Suppose that $\lambda < 0 < \mu$. Let $H_+: = \{\lambda^{2p} \mu^{2q}, p, q \in \mathbb{Z}\}$ and $H_-: = \lambda H_+$. See that $H_- \subset \Lambda_G$ and so $H_+ \cup H_- \subset \Lambda_G$. Set $f : [0, +\infty[ \to \mathbb{R}$, the homeomorphism defined by $f(x) = \log x$, so $f(H_+) = \mathbb{Z} + \frac{\log |\lambda|}{\log |\mu|} \mathbb{Z}$. As $\frac{\log |\lambda|}{\log |\mu|} \notin \mathbb{Q}$ then $f(H_+)$ is dense in $\mathbb{R}$, so $H_+$ and $H_-$ are dense respectively in $[0, +\infty]$ and in $]-\infty, 0]$. We deduce that $\overline{\Lambda_G} = \mathbb{R}$.

\[\square\]
4. Some results for non abelian subgroup of $S_n$

In this case, $G$ is a non abelian subgroup of $S_n$, then it contains necessarily an affine symmetry. In the following, recall that $G_1 = G \cap T_n$ and every $f \in S_n$ is denoted by $f = (a, \varepsilon)$, where $f : x \mapsto \varepsilon x + a$. Denote by $\delta_G := \{f(0), f \in G \cap (S_n \setminus T_n)\}$.

We use the following lemmas and propositions to prove Theorem 1.1 and above Corollaries:

**Lemma 4.1.** Let $G$ be a non abelian subgroup of $S_n$. Then:

(i) $G_1(0)$ is an additif subgroup of $\mathbb{R}^n$.

(ii) $\delta_G \neq \emptyset$.

(iii) For every $f \in G \setminus T_n$, we have $f(G_1(0)) = \delta_G$ and $f(\delta_G) = G_1(0)$.

**Proof.** The proof of (i) is obvious.

(ii) If $\delta_G = \emptyset$ then $G \cap (S_n \setminus T_n) = \emptyset$, so $G$ is a subgroup of $T_n(\mathbb{R})$, hence $G$ is abelian, a contradiction.

(iii) Let $f \in G \setminus T_n$, $b \in G_1(0)$ and $g = (b, 1) \in G_1$. Then for every $x \in \mathbb{R}^n$ we have $f \circ g(x) = f(x + b) = -x - b + a$, so $f \circ g = (-b + a, -1)$. Hence $f(b) = f \circ g(0) = -b + a \in \delta_G$.

Conversely, let $b \in \delta_G$ and $g = (b, -1) \in G \setminus T_n$ such that $g(0) = b$. We have $f \circ g(x) = f(-x + b) = x - b + a$, so $f \circ g = (-b + a, 1) \in G_1$, thus $c = -b + a \in G_1(0)$. Hence $b = -c + a = f(c)$ and so $b \in f(G_1(0))$. It follows that $f(G_1(0)) = \delta_G$. As $f^{-1} = f$ so $f(\delta_G) = G_1(0)$. $\square$

**Proposition 4.2.** Let $G$ be a non abelian subgroup of $S_n$, $a \in \delta_G$ and $x \in \mathbb{R}^n$. Then:

$$G(x) = (x + G_1(0)) \cup (-x + a + G_1(0)).$$

**Proof.** Let $G$ be a non abelian subgroup of $S_n$ and $x \in \mathbb{R}^n$. We have

$$G(x) = \{g(x) = \varepsilon x + b, g = (b, \varepsilon) \in G\} = (x + G_1(0)) \cup (-x + \delta_G).$$

So

$$G(x) = \left(x + G_1(0)\right) \cup (-x + \overline{\delta_G}). \quad (1)$$

Since $G$ is non abelian then $G \setminus T_n \neq \emptyset$, so let $f = (a, -1) \in G \setminus T_n$ with $a \in \delta_G$. By Lemma 4.1(iii) we have $\delta_G = f(G_1(0))$. By Lemma 4.1(i), $G_1(0)$ is an additif subgroup of $\mathbb{R}^n$ then $f(G_1(0)) = G_1(0) + a$, so $\delta_G = G_1(0) + a$. Hence $-x + \overline{\delta_G} = -x + G_1(0) + a$. By (1) we conclude that

$$G(x) = \left(x + G_1(0)\right) \cup \left(-x + a + G_1(0)\right).$$

$\square$
5. Proof of main results

Proof of Theorem 1.4. Let \( a \in E_G \) and \( G' = T_{-a} \circ G \circ T_a \). By Lemma 2.6(ii), \( E_{G'} = T_{-a}(E_G) \), so \( E_{G'} \) is a vector subspace of \( \mathbb{R}^n \). Then:

- **Proof of (1).**(i): One has \( \Gamma_G \neq \emptyset \) since \( G \\backslash S_n \neq \emptyset \). Then by Lemma 2.1, \( 0 \in \Lambda_G \) and by Lemma 3.1(ii), \( E_G \) is \( G \)-invariant. As \( G \\backslash S_n \neq \emptyset \), there exist \( b, c \in \Lambda_G \subset E_G \), with \( b \neq c \) (Lemma 2.3(iii)), so \( \dim(E_G) \geq 1 \).

- **Proof of (1).**(ii): By Proposition 3.6, \( G'(x - a) = E_{G'} \), for every \( x \in E_G \). So \( T_{-a}(G(x)) = E_{G'} \), it follows that \( G(x) = T_a(E_{G'}) = E_G \). So the proof of (1)(i) is complete.

- **Proof of (1).**(iii): By Proposition 3.3, \( G'(x - a) = \Lambda_{G'}(x - a) + E' \), for every \( x \in U \). So by Lemma 2.6(ii), \( E_{G'} = T_{-a}(E_G) \) is a vector space. Let \( x \in U \) and \( y \in G(x) \cap U \). By Theorem 1.4(1), there exists \( a \in E_G \) such that \( G(x) = \Lambda_G x + E_G \). Since \( E_G \) is a vector space and \( a \in E_G \), then \( G(x) = \Lambda_G x + E_G \). In the same way,

\[
G(y) = \Lambda_G y + E_G, \quad (1)
\]

See that \( G(x) \cap U = (\Lambda_G \setminus \{0\})x + E_G \). Write \( y = \alpha x + b \), where \( \alpha \in \Lambda_G \setminus \{0\} \) and \( b \in E_G \). So by (1),

\[
G(y) = \Lambda_G y + E_G = \Lambda_G (\alpha x + b) + E_G = \alpha \Lambda_G x + E_G.
\]

Since \( \alpha \in \Lambda_G \) and by Lemma 2.3, \( \Lambda_G \setminus \{0\} \) is a subgroup of \( \mathbb{R}^n \), then \( \alpha \Lambda_G = \Lambda_G \). Therefore \( G(y) = \Lambda_G x + E_G = G(x) \). \( \Box \)

Lemma 5.1. Let \( G \) be a non abelian subgroup of \( \mathcal{H}(n, \mathbb{R}) \) with \( G \\backslash S_n \neq \emptyset \), then for every \( x \in U \) we have \( G(x) = G(y) \).

Proof. Suppose that \( E_G \) is a vector space (leaving to replace \( G \) by \( G' = T_{-a} \circ G \circ T_a \) for some \( a \in E_G \), and by Lemma 2.6(ii), \( E_{G'} = T_{-a}(E_G) \) is a vector space). Let \( x \in U \) and \( y \in G(x) \cap U \). By Theorem 1.4(1), there exists \( a \in E_G \) such that \( G(x) = \Lambda_G x + E_G \). Since \( E_G \) is a vector space and \( a \in E_G \) then \( G(x) = \Lambda_G x + E_G \). In the same way,

\[
\alpha \Lambda_G x + E_G = \alpha \Lambda_G x + E_G.
\]

Since \( \alpha \in \Lambda_G \) and by Lemma 2.3, \( \Lambda_G \setminus \{0\} \) is a subgroup of \( \mathbb{R}^n \), then \( \alpha \Lambda_G = \Lambda_G \). Therefore \( G(y) = \Lambda_G x + E_G = G(x) \). \( \Box \)

Lemma 5.2. Let \( G \) be a non abelian subgroup of \( \mathcal{H}(n, \mathbb{R}) \) such that \( E_G \) is a vector subspace of \( \mathbb{R}^n \). Let \( x \in U \) then the vector subspace \( H_x = \mathbb{R}x \oplus E_G \) of \( \mathbb{R}^n \) is \( G \)-invariant.

Proof. Let \( x \in \mathbb{R}^n \setminus E_G \) and \( H_x = \mathbb{R}x + E_G \). Let \( f \in G \) having the form \( f(z) = \lambda z + a, \ z \in \mathbb{R}^n \), then by Lemma 1.2(i), \( a = f(0) \in E_G \). For every \( \alpha \in \mathbb{R}, \ b \in E_G \), we have \( f(\alpha x + b) = \lambda (\alpha x + b) + a = \lambda \alpha x + \lambda b + a \). Since \( E_G \) is a vector space, then \( \lambda b + a \in E_G \) and so \( f(\alpha x + b) \in H_x \). \( \Box \)
Proof of Corollary 1.3.
- The proof of (i): From Corollary 1.2.(ii), the closure of every orbit of $G$ contains $E_G$ and by Theorem 1.1.(1), we have $\dim(E_G) \geq 1$, so $G$ has no periodic orbit. Moreover, if $G$ is countable then every orbit $O$ is also countable, hence $O$ can not be closed.
- The proof of (ii): Let $x \in \mathbb{R}^n$ and $y \in \overline{G(x)}$. By Proposition 1.2 we have $\overline{G(x)} = \bigl(\overline{x + G_1(0)}\bigr) \cup \bigl(-\overline{x + a + G_1(0)}\bigr)$. Suppose that $y \in (x + \overline{G_1(0)})$ then $y = x + b$ for some $b \in \overline{G_1(0)}$. By Lemma 4.1.(i), $G_1(0)$ is an additive group, so $b + G_1(0) = G_1(0)$. Therefore, by Proposition 1.2 we have

$$\overline{G(y)} = \bigl(x + b + \overline{G_1(0)}\bigr) \cup \bigl(-\overline{x + a + b + \overline{G_1(0)}}\bigr)$$

$$= \bigl(x + \overline{G_1(0)}\bigr) \cup \bigl(-\overline{x + a + \overline{G_1(0)}}\bigr) = \overline{G(x)}.$$  

The same proof is used if $y \in (x + a + \overline{G_1(0)})$. 

Proof of Corollary 1.4. Let $G$ is a non abelian subgroup of $\mathcal{H}(n, \mathbb{R})$ such that $G \setminus S_n \neq \emptyset$. Suppose that $E_G$ is a vector subspace of $\mathbb{R}^n$ (leaving, by Lemma 2.6.(ii), to replace $G$ by $T_{-a} \circ G \circ T_a$, for some $a \in E_G$.)

- Let’s prove that (1) and (2) are equivalent: if $\overline{G(x)} = \mathbb{R}^n$, for some $x \in \mathbb{R}^n$, so $x \in U$. Let $y \in U$, then by Corollary 1.2.(1), $\overline{G(y)} \cap U = \overline{G(x)} \cap U = U$. Since $U$ is dense in $\mathbb{R}^n$, $\overline{G(y)} = \mathbb{R}^n$. Conversely, the proof is obvious.

- (3).(i) $\implies$ (1): If $E_G = \mathbb{R}^n$ then by Theorem 1.1.(1).(ii) we have $\overline{G(x)} = E_G$, for every $x \in E_G$. So $G$ has a dense orbit.

- (3).(ii) $\implies$ (1): If $\Lambda_G$ is dense in $\mathbb{R}$ then by Theorem 1.1.(1).(ii) we
have \( G(x) = \Lambda_G x + E_G \), for every \( x \in U \). So \( G \) has a dense orbit.

- \( (1) \implies (3) \): Suppose that \( G \) has a dense orbit \( G(x) \), for some \( x \in \mathbb{R}^n \).

  There are two cases:
  - If \( E_G = \mathbb{R}^n \), then we obtain (3).(i).
  - If \( E_G \neq \mathbb{R}^n \) then \( \dim(E_G) \leq n - 1 \), so and \( U \neq \emptyset \), hence \( x \in U \). By Theorem 1.1.(1).(iii) we have \( G(x) = \mathbb{R} x + E_G \), so \( \dim(E_G) = n - 1 \) and \( \Lambda_G \) is dense in \( \mathbb{R} \). Then (3).(ii) follows.

We use the following Lemma to prove Corollary 1.6.

**Lemma 5.3.** Let \( H \) be an additive subgroup of \( \mathbb{R}^n \). Then

\[
\mathcal{H} \neq \emptyset \quad \text{if and only if} \quad \mathcal{H} = \mathbb{R}^n
\]

**Proof.** Suppose that \( \mathcal{H} \neq \emptyset \) and let \( a \in \mathcal{H} \neq \emptyset \). Then there exists \( \varepsilon > 0 \) such that \( B_{(a,\varepsilon)} \subset \mathcal{H} \neq \emptyset \), where \( B_{(a,\varepsilon)} = \{ x \in \mathbb{R}^n : \| x - a \| < \varepsilon \} \) and \( \| . \| \) is the euclidian norm. Since \( \mathcal{H} \neq \emptyset \) is an additive group, it follows that \( B_{(0,\varepsilon)} = T_a(B_{(a,\varepsilon)}) \subset \mathcal{H} \). Moreover, we also have \( B_{(0,\varepsilon)} = mB_{(0,\varepsilon)} \subset \mathcal{H} \), for every \( m \in \mathbb{N}^* \). As \( \mathbb{R}^n = \bigcup_{m \in \mathbb{N}^*} B_{(0,\varepsilon)} \subset \mathcal{H} \), it follows that \( \mathcal{H} = \mathbb{R}^n \).

Conversely, the proof is obvious. \( \square \)

**Proof of Corollary 1.6.** Let \( G \) be a non abelian subgroup of \( S_n \). By Lemma 5.3.1(i), one has \( \delta_G \neq \emptyset \). Let \( a \in \delta_G \) and \( f = (a, -1) \in G \).

- First, by Corollary 1.3.(ii) we prove that (i), (ii) and (iii) are equivalent.
- Second, let’s prove that (iii) and (iv) are equivalent: Suppose that \( G(0) = \mathbb{R}^n \). By Proposition 4.2 we have \( \overline{G(0)} = \overline{G_1(0)} \cup (a + \overline{G_1(0)}) \). Since \( \overline{G(0)} \neq \emptyset \) then \( \overline{G_1(0)} \neq \emptyset \). By Lemma 4.1.(i), \( H_G = G_1(0) \) is an additive subgroup of \( \mathbb{R}^n \) then by Lemma 5.3, \( \mathcal{H}_G = \mathbb{R}^n \). Conversely, if \( \mathcal{H}_G = \mathbb{R}^n \) then by Proposition 4.2 we have \( G(0) = \overline{G_1(0)} \cup (a + \overline{G_1(0)}) = \mathbb{R}^n \).

**Proof of Corollary 1.8.** For \( n = 1 \), \( G \) is a non abelian group of affine maps of \( \mathbb{R} \).

- The proof of (i): If \( G \backslash S_1 \neq \emptyset \), then by Theorem 1.1.(1), \( E_G \) is a \( G \)-invariant affine subspace of \( \mathbb{R} \) with dimension \( p = 1 \) such that every orbit of \( E_G \) is dense in it. In this case \( E_G = \mathbb{R} \).
- The proof of (ii): If \( G \subset S_1 \), then by Theorem 1.1.(2), \( H_G \) is a \( G \)-invariant closed subgroup of \( \mathbb{R} \) and there exists \( a \in E_G \) such that for every \( x \in \mathbb{R} \), we have \( G(x) = (x + H_G) \cup (-x - a + H_G) \). Then there are two cases:
  - If \( H_G \) is dense in \( \mathbb{R} \), so every orbit of \( G \) is dense in \( \mathbb{R} \).
  - If \( H_G \) is discrete then every orbit is closed and discrete.
6. Examples

Example 6.1. Let $G$ be a subgroup of $\mathcal{H}(2, \mathbb{R})$ generated by $f_1 = (a_1, \alpha_1)$ and $f_2 = (a_2, \alpha_2)$ and $f_3 = (a_3, \alpha_3)$, where $\alpha_k \neq 1$, for every $1 \leq k \leq 3$ and $a_1 = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $a_3 = \begin{bmatrix} -\sqrt{3} \\ -\sqrt{2} \end{bmatrix}$. Then every orbit of $G$ is dense in $\mathbb{R}$.

Indeed, by Lemma 2.2(i), $G$ is non abelian. By Proposition 3.6, for every $x \in E_G$, we have $G(x) = E_G$. In this case, by Remark 2.4, $E_G = \mathbb{R}^2$ so every orbit of $G$ is dense in $\mathbb{R}^2$.

Example 6.2. Let $(a_1, \ldots, a_n)$ be a basis of $\mathbb{R}^n$ and $a = \sum_{k=1}^{n} \alpha_k a_k$, with $\sum_{k=1}^{n} \alpha_k a_k \neq 1$, then for every $t > 1$, the subgroup $G$ of $\mathcal{H}(n, \mathbb{R})$ generated by \{ $f = (a, t)$, $T_{a_k}, 2 \leq k \leq n$ \} is minimal. (i.e. every orbit of $G$ is dense in $\mathbb{R}^n$).

Indeed: By Remark 2.4, we have $E_G = \mathbb{R}^n$ and by Proposition 3.6 every orbit of $G$ is dense in $\mathbb{R}^n$.

Example 6.3. Let $(a_1, \ldots, a_n)$ be a basis of $\mathbb{R}^n$ and $\lambda \in \mathbb{R}\setminus\{0, 1\}$. Then every orbit of the group generated by $T_{a_1}, \ldots, T_{a_n}, \lambda Id$ is dense in $\mathbb{R}^n$.

Indeed, By Remark 2.4 we have $E_G = \mathbb{R}^n$ and by Proposition 3.6 every orbit of $G$ is dense in $\mathbb{R}^n$.

Example 6.4. Let $a \in \mathbb{R}^n$ and $G$ be the group generated by $f = T_a$, $g = (a, -1)$ and $h = T_{\sqrt{2}a}$. Then for every $x \in \mathbb{R}^n \setminus Ra$, we have $G(0)$ and $G(x)$ are not homeomorphic.

Proof. Remark that for every $\varphi \in G_1$, there exist $n_1, m_1, p_1, \ldots, n_r, m_r, p_r \in \mathbb{Z}$ such that $\varphi = (f^{m_1} \circ g^{m_1} \circ h^{p_1}) \circ \cdots \circ (f^{m_r} \circ g^{m_r} \circ h^{p_r})$, for some $r \in \mathbb{N}^*$.

- First, let’s show by induction on $r \geq 1$ that $\varphi(0) \in (\mathbb{Z} + \sqrt{2}\mathbb{Z})a \quad (i)$. 

For $r = 1$, we have $\varphi(0) = f^{m_1} \circ g^{m_1} \circ h^{p_1}(0)$

$= -p_1\sqrt{2}a + m_1a + n_1a$

$= (m_1 + n_1 + \sqrt{2}p_1)a$. 

So \( \varphi(0) \in (\mathbb{Z} + \sqrt{2}\mathbb{Z})a \).

Suppose property (i) is true up to order \( r - 1 \). If

\[
\varphi = (f^{n_1} \circ g^{m_1} \circ h^{p_1}) \circ \cdots \circ (f^{n_r} \circ g^{m_r} \circ h^{p_r}),
\]

then by induction property there exists \( p, q \in \mathbb{Z} \) such that

\[
f^{n_2} \circ g^{m_2} \circ h^{p_2} \circ \cdots \circ f^{n_r} \circ g^{m_r} \circ h^{p_r}(0) = (p + \sqrt{2}q)a.
\]

So \( \varphi(0) = f^{n_1} \circ g^{m_1} \circ h^{p_1}((p + \sqrt{2}q)a) \), thus

\[
\varphi(0) = \begin{cases}
(p + \sqrt{2}q)a + p_1a + a + n_1a, & \text{if } m_1 \text{ is odd}, \\
(p + \sqrt{2}q)a + p_1a + n_1a, & \text{if } m_1 \text{ is even}.
\end{cases}
\]

Hence, \( \varphi(0) \in (\mathbb{Z} + \sqrt{2}\mathbb{Z})a \).

It follows that

\[
G_1(0) \subset (\mathbb{Z} + \sqrt{2}\mathbb{Z})a \quad (1)
\]

- Second, we will proof that \( G_1(0) = (\mathbb{Z} + \sqrt{2}\mathbb{Z})a \). Let \( p, q \in \mathbb{Z} \), we have \( f^p \circ h^q = T_{(p + \sqrt{2}q)a} \); thus \( f^p \circ h^q(0) = (p + \sqrt{2}q)a \in G_1(0) \). It follows by (1) that \( G_1(0) = (\mathbb{Z} + \sqrt{2}\mathbb{Z})a \). With the same proof we can show that \( \delta_G = G_1(0) \).

- Thirdly, by Proposition 4.2 for every \( x \in \mathbb{R}^n \), we have \( \overline{G(x)} = (x + G_1(0)) \cup (-x + G_1(0)) \). Therefore \( \overline{G(0)} = G_1(0) = (\mathbb{Z} + \sqrt{2}\mathbb{Z})a = \mathbb{R}a \) and it is connected. But \( G(x) = (x + \mathbb{R}a) \cup (-x + \mathbb{R}a) \), is not connected for every \( x \in \mathbb{R}^n \setminus \mathbb{R}a \). Hence \( G(0) \) and \( G(x) \) can not be homeomorphic. \( \square \)

**Remark 6.5.** Remark that the form of \( \varphi \) used in the proof of Example 6.4 is general of every \( \varphi \in G \) and the order \((f^{n_k} \circ g^{m_k} \circ h^{p_k})\) is not particular of \( \varphi \). For example, if \( \varphi = g^m \circ f^n \circ h^p \), we write

\[
\varphi = (f^{n_1} \circ g^{m_1} \circ h^{p_1}) \circ (f^{n_2} \circ g^{m_2} \circ h^{p_2}) \circ (f^{n_3} \circ g^{m_3} \circ h^{p_3})
\]

with \( n_1 = p_1 = m_2 = p_2 = n_3 = m_3 = 0, m_1 = m, n_2 = n \) and \( p_3 = p \).

**References**

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