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**ON THE FREE VIBRATION OF SANDWICH PANELS WITH A
TRANSVERSELY FLEXIBLE AND TEMPERATURE DEPENDENT
CORE MATERIAL – PART I: MATHEMATICAL FORMULATION**

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ABSTRACT

The free vibration analysis of sandwich panels with a core that is flexible and compliant in the vertical direction and with temperature dependent mechanical properties is presented in two parts. The first part presents the mathematical formulation while the second deals numerically with the effects of the degrading properties of the core on the free vibration response. The analysis is based on the high-order sandwich panel theory approach (HSAPT), and the equations of motions along with the appropriate boundary conditions are derived using the Hamilton's principle. The study investigates the role of increasing temperature, through the degradation of the mechanical properties of the core, on the free vibration response of structural sandwich panels. The mathematical formulation uses two types of computational models. At first, following the HSAPT approach, the unknowns include the displacements of the face sheets as well as the shear stress in the core. Secondly, it is assumed that the through-thickness distributions of the vertical and horizontal core displacements can be represented as polynomials, following the results of the HSAPT static case, and the effect of the variable mechanical properties are implemented directly.

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INTRODUCTION

Sandwich structures are being used extensively in aerospace, naval, transportation and civil engineering industries. Structural sandwich panels can be considered as a special type of composite laminate where two thin, stiff, strong and relatively dense face sheets, which are often by themselves composite laminates, are separated by and bonded to a thick, lightweight and compliant core material. Such sandwich structures are being used as primary and secondary structural members due to their superior qualities in terms of: high strength and stiffness to weight ratios, ease of manufacturing, acoustic and thermal insulation, and flexibility in design. Sandwich structures panels are often subjected to aggressive service conditions which may include elevated temperatures, which lead to thermally induced deformation loads and degradation of the mechanical properties.

The material properties of the constituents of a sandwich structure generally depend on the temperature field imposed. However, this dependency is usually ignored in the design process, even for applications where the induced temperature field causes the material properties to degrade significantly as the temperature increases. In many modern sandwich panel applications, the core material is made of a polymer foam where significant changes in the properties may occur in the operating range of the temperatures. A PMI type of foam such as Rohacell® loses its heat distortion resistance at about 200°C, see Rohacell Data (2004), while a PVC foam core such as Divinycell loses its strength at about 80-100°C, see DIAB reports (2003 and 2005). Hence, it becomes extremely important to understand how the degradation of the core properties affects the mechanical response including the load and the free vibration response.

The load response including non-linear thermo-mechanical interactions and buckling (global and local) was recently treated by Frostig and Thomsen (2007a and 2007b). However, the vibration response and its dependency of temperature has hitherto no been considered in open literature. The free vibration response of a sandwich structure with a compliant core, is associated with both overall and through-thickness modes (the latter being denoted as local modes). In general the overall or global modes correspond to lower eigenfrequencies, and the local modes to higher eigenfrequencies.

However, as the mechanical properties of the core degrade as a result of the elevated temperature, the rigidity of the core is reduced and the local modes may shift from higher frequencies into lower ones. The principal objective of this investigation is to investigate how the eigenfrequencies and the corresponding eigenmodes change as the temperature is increased and the mechanical core properties degrade.

The approaches used by many researchers for the analysis of sandwich structures can be describe following one of two categories; the core is considered as vertically incompressible, or the core is considered as compressible or compliant in the through-thickness direction. Many of the classical works on sandwich structures assume that the core material is vertically incompressible, see for Allen (1969), Plantema (1966), Zenkert (1995) and Vinson (1999). Usually, the models adopted for predicting the response of such sandwich structures are based on the “equivalent single layer” approach (ESL), where the layered sandwich panel (beam, plate or shell type) is replaced by an equivalent single layer with equivalent mechanical properties, see Mindlin’s (1951) first-order shear deformation theory, and Reddy’s high-order theories (1984). Recently, Carrera and Demasi (2003), and Carrera and Ciuffreda (2005) and Carrea (2005) have presented ESL and layer-wise models with various plate theories for the analysis of sandwich panels with and without vertical normal strain. The classical and the ESL models, in general, usually disregard the changes in the height of the core (i.e. the vertical compressibility) when the panel is deformed. Hence, when using these approaches, for the free vibration response, the through-thickness modes of the core cannot be detected. Examples of research following these approaches include Kant and Mallikarjuna (1989), Senthilnathan et al. (1988) and Kant and Swaminathan (2001) who used a high-order model but with an incompressible core, and Meunier and Shenoï (2001) and Nayak et al. (2002); Bardell et al. (1997) and Lee and Fan (1996) who used different finite elements analysis approaches adopting various pre-assumed displacement distributions.

An extensive literature search reveals that only a limited number of research works are available in open literature that take into account the temperature dependent core properties in their analyses. Moreover, the available research works are based on the assumption of an incompressible

core, and they adopt the ESL approach along with various finite element analysis formulations. Examples of such research works include Pradeep and Ganesan (2007), and Pradeep *et al.* (2006) who assumed a viscoelastic core and used a finite element formulation, Shiau and Kuo (2006) who used the splitted rigidity approach due to Allen (1969) and Plantema (1966), Ibrahim *et al.* (2006) discuss the case of a sandwich panel made of a functionally graded material (FGM), Kim (2005) dealt with an FGM panel, Hao and Rao (2005) assumed a core made of a pressure sensitive adhesive (PSA), and Duan *et al.* (2000) dealt with a sandwich panel utilizing shape memory alloys (SMA) at elevated temperatures. Generally, the classical sandwich theories, based on the ESL and high-order models, mentioned above, disregard the changes in the height of the core (vertical compressibility) during the deformation of the sandwich panel. Accordingly, they are unable to detect the high-order modes that are associated with deformations through the thickness of the core.

A different approach that models the sandwich panel as a layered structure made of two face sheets and a core layer that are interconnected through fulfillment of equilibrium and compatibility conditions, and thus accounts for the vertical core flexibility, has been implemented into the so-called High-Order Sandwich Panel Theory (HSAPT), see Frostig *et al.* (1992). It has been successfully used by the authors and by others for the analysis of various linear and non-linear applications; Frostig and Baruch (1994) for high-order vibration of sandwich panels; Bozhevolnaya and Frostig (2001) dealt with the vibration of curved sandwich panels; Frostig and Thomsen (2004) treated the vibration of sandwich plates; Yang and Qiao (2005) and Qiao and Yang (2007) used the HSAPT model and its modification for impact problems; Schwarts-Givli *et al.* (2007a, 2007b, 2007c and 2007d) dealt with free and forced vibrations of delaminated sandwich panels; and recently Frostig and Thomsen (2007a and 2007b) treated the non-linear response of sandwich panel with temperature-dependent properties.

In this two parts paper, the dynamic governing equations, including rotary inertia and the required boundary conditions are derived explicitly using the Hamilton's principle. The mathematical formulation follows the high-order theory (HSAPT) approach, and it incorporates the effects of the flexible core into the equations of motion and the boundary conditions as well as the temperature-dependent mechanical properties of the core into the governing equations. The sandwich panel is

assumed to be linear elastic with small displacements and consists of a core and two thin face sheets. The face sheets are modeled as beams or panels with in-plane and flexural rigidities and negligible shear strain. The core is assumed to possess vertical normal and shear stiffnesses, whereas the in-plane and flexural rigidities are assumed to be negligible. Moreover, perfect bonding is assumed for the interfaces between the core and the face sheets, i.e. the core/face interfaces can resist shear and vertical normal stresses.

The free vibration analysis of a unidirectional sandwich panel with a compliant temperature-dependent core is investigated using two well known computational models, see Frostig and Thomsen (2004). It is assumed that the core is subjected to a non-uniform temperature field, which yields a core with coordinate dependent moduli as a result of the temperature-dependent mechanical core properties,

The first model adopts the approach of the HSAPT model, see Frostig and Baruch (1994) and Bozhevolnaya and Frostig (2001), where the unknowns consist of the displacements of the face sheets and the shear stresses in the core. This is denoted as a mixed formulation, and according to this approach the stress and the displacements fields of the non-uniform core are derived in the form of a closed-form solution using a least-square curve-fitting procedure. The second model assumes that the distribution of the in-plane and vertical core displacements can be represented as cubic and quadratic polynomials, respectively, following the results of the HSAPT model (for core displacements see Frostig *et al.* (1992)). The unknowns in this model consist of the displacements of the face sheets and the coefficients of the polynomials in the core. In this formulation, the effects of the non-uniform stiffness of the core are implemented following a straight forward approach. Here, the high-order stress resultants of the non-uniform core are determined using a direct integration process. It should be noticed that this approach involves higher-order core stress resultants that have no physical interpretation, and the model yields higher-order modes that involve vibrations through the depth of the core that the HSAPT model cannot detect.

The first part manuscript outlines the mathematical formulation including the derivation of the temperature-dependent governing equations of motion along with the associated boundary conditions for the two computational models.

MATHEMATICAL FORMULATION

The mathematical formulation consists of a general section that presents the dynamic equations that are applicable to two computational models followed by the field/governing equations of motions along with the boundary conditions for each model.

The equations of motions of the free vibration response are derived through the Hamilton principle which extremizes the Lagrangian that consists of the kinetic and the internal potential energy as follows:

$$\int_{t_1}^{t_2} (-\delta T + \delta U) dt = 0 \quad (1)$$

where T is the kinetic energy, t is the time coordinate that varies between the times t_1 and t_2 ; and U and V are the internal and external potential energies.

The first variation of the kinetic energy for the sandwich panel reads:

$$\delta T = \int_{t_1}^{t_2} \left[\int_{V_t} \rho_t u_{t,t} \delta u_{t,t} + \rho_t w_{t,t} \delta w_{t,t} dV + \int_{V_b} \rho_b u_{b,t} \delta u_{b,t} + \rho_b w_{b,t} \delta w_{b,t} dV + \int_{V_c} \rho_c u_{c,t} \delta u_{c,t} + \rho_c w_{c,t} \delta w_{c,t} dV \right] dt \quad (2)$$

where ρ_j ($j=t,b,c$) is the density of the upper and lower face sheets and the core, respectively; $u_{j,t}$ and $w_{j,t}$, ($j=t,b,c$) are the velocities in the longitudinal and vertical directions, respectively, of the sandwich panel constituents; $f_{,t} = \frac{\partial f}{\partial t}$ is the first derivative of the function f with respect to the time

coordinate; V_j ($j=t,b,c$) is the volume of upper and lower face sheets and the core, respectively, and dV is the volume of a differential segment.

The internal potential energy reads:

$$\delta U = \int_{V_t} \sigma_{xxt} \delta \epsilon_{xxt} dV + \int_{V_b} \sigma_{xxb} \delta \epsilon_{xxb} dV + \int_{V_c} \tau_{xz} \delta \gamma_{xz} + \sigma_{zz} \delta \epsilon_{zz} dV \quad (3)$$

where σ_{xxj} and ε_{xxj} ($j=t,b$) are the in-plane normal stresses and strains in the upper and the lower face sheets; τ_{xz} and γ_{xz} are the vertical shear stresses and strains in the core; and σ_{zz} and ε_{zz} are the normal stresses and strains in the vertical direction of the core.

The displacements pattern of the face sheets ($j=t,b$) following the classical Bernoulli assumptions, the kinematic relations corresponding to small linear displacements of the face sheets and the core, and the compatibility conditions at the upper and the lower face-core interfaces equal:

$$u_j(x, z_j) = u_{oj}(x) - z_j w_{j,x}(x) \quad (4)$$

$$\varepsilon_{xxj} = u_{oj,x}(x) - z_j w_{j,xx}(x) - \alpha_j T_j(x, z_j) \quad (5)$$

$$\varepsilon_{zzc} = w_{c,z}(x, z_c) - \alpha_t T_c(x, z_c), \quad \gamma_{xzc} = u_{c,z}(x, z_c) + w_{c,x}(x, z_c) \quad (6)$$

$$u_c \left(x, z_{c_I} = 0 \text{ or } z_{c_{II}} = -\frac{1}{2} c \right) = u_{ot}(x) - \frac{1}{2} d_t w_{t,x}(x), \quad w_c \left(x, z_{c_I} = 0 \text{ or } z_{c_{II}} = -\frac{1}{2} c \right) = w_t(x) \quad (7)$$

$$u_c \left(x, z_{c_I} = c \text{ or } z_{c_{II}} = \frac{1}{2} c \right) = u_{ob}(x) + \frac{1}{2} d_b w_{b,x}(x), \quad w_c \left(x, z_{c_I} = c \text{ or } z_{c_{II}} = \frac{1}{2} c \right) = w_b(x) \quad (8)$$

where z_j ($j=t,b$) are the vertical coordinates of each face sheet (measured downwards from the centroid of each face); $u_c(x, z_c)$ and $w_c(x, z_c)$ are the in-plane and vertical displacements of the core, respectively; c is the height of the core, d_j ($j=t,b$) are the thicknesses of the upper and the lower face sheets, respectively; z_{c_k} ($k=I,II$) is the vertical coordinate of the core (measured downwards from upper core-face interface for the first model (I), and from mid-height of core for the second model (II)) (see Fig. 1a), α_k and $T_k(x, z_k)$ ($k=t,b,c$) are the coefficients of thermal expansion (CTE's) and the temperature fields of the face sheets and the core respectively (see Fig. 1b), and finally $(\cdot)_{,k}$ ($k=x$ or z_c) denotes a partial derivative with respect to an independent variable k .

The first variation of the kinetic energy, assuming homogeneous initial conditions and after integration by parts with respect to the time, coordinate reads:

$$\delta T = - \int_{t_1}^{t_2} \left[\int_{V_t} \rho_t u_{t,tt} \delta u_t + \rho_t w_{t,tt} \delta w_t dV + \int_{V_b} \rho_b u_{b,tt} \delta u_b + \rho_b w_{b,tt} \delta w_b dV + \int_{V_c} \rho_c u_{c,tt} \delta u_c + \rho_c w_{c,tt} \delta w_c dV \right] dt \quad (9)$$

where $u_{j,tt}$ and $w_{j,tt}$ ($j=t,b,c$) are the accelerations in the longitudinal and vertical directions, respectively, of the sandwich panel constituents, and $f_{,tt} = \frac{\partial^2 f}{\partial t^2}$ denotes the second derivative of the function f with respect to the time coordinate.

The differences between the two sandwich models are a result of the description of the accelerations and the displacements in the core, as well as the solutions for the core stress and the displacements fields.

HSAPT Computational Model – Mixed Formulation (Model I)

The core, in this model, is regarded as a medium that transfers its inertia loads to the face sheets rather than resisting them by itself. Hence, the distributions of the accelerations through the depth of the core are assumed to follow the distributions of the static displacements under fully distributed loads, see Frostig *et al.* (1992). This approach is commonly used in many dynamic analyses of ordinary beams, plates and shells, see Shames and Dym (1973). Thus, the dynamic stress and displacements fields of the core are assumed to be identical with the static stress and displacements fields without inertia loads terms. The distributions of the static displacements through the depth of the core are non-linear; quadratic for the vertical displacement, see Frostig *et al.* (1992), and cubic for the in-plane displacements when the mechanical properties of the core are uniform through its depth. These (non-linear) polynomial patterns (through the depth of the core) are usually a result of gradients of the vertical shear stresses, and they differ significantly from the linear distributions when localized loads or restrictive constrains are imposed on the sandwich structure. However, when fully distributed loads, such as inertia loads, are applied to the face sheets, the localized effects diminish leading to displacements distributions with small non-linearities. Hence, it

can be rationally assumed that these distributions are linear rather than non-linear. Thus, the distributions of the accelerations, through the depth of the core, are assumed to be linear as follows:

$$\begin{aligned} u_{c,tt}(x, z_c, t) &= u_{t,tt}(x, z_t = d_t/2, t)(1 - \frac{z_c}{c}) + u_{b,tt}(x, z_b = -d_b/2, t)\frac{z_c}{c} \\ w_{c,tt}(x, z_c, t) &= w_{t,tt}(x, z_t, t)(1 - \frac{z_c}{c}) + w_{b,tt}(x, z_b, t)\frac{z_c}{c} \end{aligned} \quad (10)$$

Notice that this simplification is applied to the kinetic inertia terms only.

The equations of motion are derived using the Hamilton's principle, Eq. (1); the expression for the kinetic energy, see Eq. (9), along with the presumed accelerations with linear through-thickness distributions, Eqs. (10); the expression for the internal potential energy, Eq. (3), along with the kinematic relations of the face sheets and the core, Eqs. (5) and (6); and finally the compatibility conditions at the face-core interfaces, see Eqs. (7) and (8). The results are similar to those that appear in Frostig and Baruch (1994), except that herein the vertical normal stresses are undefined. Hence, the equations for the face sheets read:

$$\begin{aligned} \left(\frac{1}{3} u_{ot,tt}(x, t) + \frac{1}{12} \left(\frac{\partial}{\partial x} w_{b,tt}(x, t) \right) d_b - \frac{1}{6} \left(\frac{\partial}{\partial x} w_{t,tt}(x, t) \right) d_t \right. \\ \left. + \frac{1}{6} u_{ob,tt}(x, t) \right) M_c + u_{ot,tt}(x, t) M_t - \left(\frac{\partial}{\partial x} N_{xxt}(x, t) \right) - \tau(x, t) b_w = 0 \end{aligned} \quad (11.1)$$

$$\begin{aligned} M_t w_{t,tt}(x, t) - \left(\frac{\partial^2}{\partial x^2} w_{t,tt}(x, t) \right) I_{mt} + \left(\frac{1}{24} \left(\frac{\partial^2}{\partial x^2} w_{b,tt}(x, t) \right) d_b d_t \right. \\ \left. + \frac{1}{6} \left(\frac{\partial}{\partial x} u_{ot,tt}(x, t) \right) d_t + \frac{1}{3} w_{t,tt}(x, t) - \frac{1}{12} \left(\frac{\partial^2}{\partial x^2} w_{t,tt}(x, t) \right) d_t^2 \right. \end{aligned} \quad (11.2)$$

$$\begin{aligned} \left. + \frac{1}{12} \left(\frac{\partial}{\partial x} u_{ob,tt}(x, t) \right) d_t + \frac{1}{6} w_{b,tt}(x, t) \right) M_c - \sigma_{zzt}(x, t) b_w \\ - \frac{1}{2} b_w d_t \left(\frac{\partial}{\partial x} \tau(x, t) \right) - \left(\frac{\partial^2}{\partial x^2} M_{xxt}(x, t) \right) = 0 \end{aligned} \quad (11.3)$$

$$\begin{aligned} \left(\frac{1}{3} u_{ob,tt}(x, t) - \frac{1}{12} \left(\frac{\partial}{\partial x} w_{t,tt}(x, t) \right) d_t + \frac{1}{6} \left(\frac{\partial}{\partial x} w_{b,tt}(x, t) \right) d_b \right. \\ \left. + \frac{1}{6} u_{ot,tt}(x, t) \right) M_c + u_{ob,tt}(x, t) M_b - \left(\frac{\partial}{\partial x} N_{xxb}(x, t) \right) + \tau(x, t) b_w = 0 \end{aligned}$$

(11.4)

$$\begin{aligned}
& M_b w_{b,tt}(x,t) - \left(\frac{\partial^2}{\partial x^2} w_{b,tt}(x,t) \right) I_{mb} + \left(\frac{1}{24} \left(\frac{\partial^2}{\partial x^2} w_{t,tt}(x,t) \right) d_t d_b \right. \\
& - \frac{1}{6} \left(\frac{\partial}{\partial x} u_{ob,tt}(x,t) \right) d_b + \frac{1}{6} w_{t,tt}(x,t) - \frac{1}{12} \left(\frac{\partial^2}{\partial x^2} w_{b,tt}(x,t) \right) d_b^2 \\
& - \frac{1}{12} \left(\frac{\partial}{\partial x} u_{ot,tt}(x,t) \right) d_b + \frac{1}{3} w_{b,tt}(x,t) \left. \right) M_c + \sigma_{zzb}(x,t) b_w \\
& - \frac{1}{2} b_w d_b \left(\frac{\partial}{\partial x} \tau(x,t) \right) - \left(\frac{\partial^2}{\partial x^2} M_{xzb}(x,t) \right) = 0
\end{aligned}$$

where M_j, I_{mj} ($j=t,b$) are the mass and the rotary inertia per area unit length, respectively, of the upper and the lower face sheets; M_c is the mass per area unit length of the core; $\tau(x,t)$ is the shear stress in the core; $\sigma_{zj}(x,t)$, ($j=i,t$) are the vertical normal stresses at the upper and the face-core interfaces, respectively, and $(\cdot)_{,it}$ denotes a partial derivative with respect to i and t where the indices refer to the longitudinal coordinate of the panel and the time coordinate. For sign conventions see Fig. 1. Notice that all stress resultants and displacements are function of the spatial coordinates and time.

The equations of motion for the core coincide with those of the static case, see Frostig *et al.* (1992) and Frostig and Thomsen (2007b) and they yield that the shear stress through the depth of the core are uniform, and the vertical normal stresses are linear.

In order to achieve an explicit description of the equations of motion of the face sheets in terms of the unknowns of the core the fields of the core must be determined first. The fields of the core are determined assuming that the core is orthotropic with the following constitutive relations:

$$\begin{aligned}
\frac{\sigma_{zz}(x, z_c)}{E_{zc}(T_c(x, z_c))} &= \frac{\partial}{\partial z_c} w_c(x, z_c) - \alpha_c T_c(x, z_c) \\
\frac{\tau_c(x, z_c)}{G_{xzc}(T_c(x, z_c))} &= \frac{\partial}{\partial z_c} u_c(x, z_c) + \frac{\partial}{\partial x} w_c(x, z_c)
\end{aligned} \tag{12}$$

where $E_{zc}(T(x, z_c))$ and $G_{xzc}(T(x, z_c))$ are the vertical modulus of elasticity and the shear modulus of the core that are known functions of the temperature distribution.

The description of the stress and the displacements fields of the core appear in Frostig and Thomsen (2007b) using special integrals. It should be noticed, that the non-uniform moduli of the core yield a closed-form analytical solution only when these moduli have a linear or quadratic distribution. In other cases a different approach that is based on the approach that appears in Frostig and Thomsen (2007b) is required. In addition, the temperature distribution through the depth of the core is assumed to be linear with a gradient between the two face sheets.

The governing equations are derived by substitution of the force-displacements relations of the isotropic face sheets into the equations of motions that consist of Eqs. (11) and the compatibility equations, see Frostig and Thomsen (2007b). The force-displacements relation of the isotropic face sheets ($j=t,b$) read:

$$\begin{aligned} N_{xyj}(x) &= EA_j \left(\frac{d}{dx} u_{oj}(x) \right) - N_{Tj}(x) \\ M_{xyj}(x) &= -EI_j \left(\frac{d^2}{dx^2} w_j(x) \right) - M_{Tj}(x) \end{aligned} \quad (13)$$

where

$$\left[N_{Tj}(x), M_{Tj}(x) \right] = \int_{-\frac{1}{2}d_j}^{\frac{1}{2}d_j} E_j \alpha_{Tj}(x, z_j) T_j(x, z_j) b_w \{1, z_j\} dz_j$$

where EA_j and EI_j ($j=t,b$) are the axial and the flexural rigidity of each face sheet, respectively, $N_{Tj}(x)$ and $M_{Tj}(x)$ are the equivalent axial force and moment due to temperature and E_j is the modulus of elasticity of the face sheets.

The governing equations consist of a set of partial differential equations expressed in terms of space coordinates and time of the order of fourteen. Notice that due to the thermal terms in the face sheet equations, Eqs. (13), and the vertical normal stresses at the face-core interfaces the panel undergoes static displacements, and the free vibration oscillations occur with respect to this static deformed configuration. The solution of the set of governing equations can be achieved numerically for general boundary conditions and external dynamic loads, or analytically in closed-form for the particular case of a simply-supported unidirectional panel.

Displacements Formulation - Model II

The second computational model is used to investigate the accuracy of the results of the first model, which uses the linear distributions of the accelerations through the depth of the core, see Eqs. (10), as compared with the formulation that uses the quadratic and cubic polynomials to describe the accelerations. Hence, this formulation describes a sandwich panel that is more flexible

then that based on the first model. The advantage of this formulation is that the dynamic loads as well as its degrading mechanical properties are directly included in the equations of motion of the core, and that they are not a result of the interaction with the upper and the lower face sheets. It should be noticed that this is achieved at the expense of predicting higher-order bending moments and shear stress resultants in the core that lack any physical interpretation. In addition, this means that any constraint imposed on the core can be fulfilled only in the global sense, and not in the differential sense as is the case with the first model. Thus, Model I is overall more physically consistent than Model II.

The formulation of Model II follows the same steps as Model I, using the same basic equations, Eqns. (1) to (9), but here the unknowns are the displacements of the face sheets and the core. In order to achieve this goal, the displacements fields of the core are assumed *a priori*, using the quadratic and cubic polynomial distribution of the static displacement fields of a core with uniform properties, see Frostig *et. al.* (1992). Here, the coefficients of these polynomials are the unknowns, and they are determined through the variational principle.

The pre-assumed displacement fields of the core read:

$$\begin{aligned} u_c(x, z_c, t) &= u_o(x, t) + u_1(x, t) z_c + u_2(x, t) z_c^2 + u_3(x, t) z_c^3 \\ w_c(x, z_c, t) &= w_o(x, t) + w_1(x, t) z_c + w_2(x, t) z_c^2 \end{aligned} \quad (14)$$

where u_k ($k=0,1,2,3$) are the unknowns of the in-plane displacements of the core, and w_l ($l=0,1,2$) are the unknowns of the core vertical displacements, respectively. The distributions of the accelerations and the velocities of the core are assumed to follow the static displacements, similar to the general approach that ignores the effect of the inertia terms, see Dym and Shames (1973), and they equal their second and first derivatives with respect to the time coordinate. The compatibility conditions at the upper and the lower face-core interfaces, see Eqs. (7) to (8), are enforced through the use of four Lagrange multipliers. Thus, the variation of the potential energy, see Eq. (3), reads:

$$\begin{aligned}
 \delta U = & \int_{V_t} \sigma_{xxt} \delta \varepsilon_{xxt} dV + \int_{V_b} \sigma_{xxb} \delta \varepsilon_{xxb} dV + \int_{V_c} \tau_{xz} \delta \gamma_{xz} + \sigma_{zz} \delta \varepsilon_{zz} dV + \\
 & \delta \left[\int_0^L \lambda_{xt} \left(u_t \left(z_t = \frac{1}{2} d_t \right) - u_c \left(z_c = -\frac{1}{2} c \right) \right) + \lambda_{zt} \left(w_t - w_c \left(z_c = \right. \right. \right. \\
 & \left. \left. \left. -\frac{1}{2} c \right) \right) + \lambda_{xb} \left(u_c \left(z_c = \frac{1}{2} c \right) - u_b \left(z_b = -\frac{1}{2} d_b \right) \right) \right. \\
 & \left. + \lambda_{zb} \left(w_c \left(z_c = \frac{1}{2} c \right) - w_b \right) dx \right]
 \end{aligned} \tag{15}$$

where λ_{kj} ($k=x,z$ and $j=t,b$) are the Lagrange multiplier in the horizontal and vertical directions at the upper and the lower face–core interfaces, respectively. For sign conventions see Fig. 1a.

The equations of motion and the boundary conditions are derived using the Hamilton's principle, Eqs. (1), the expression of the kinetic energy, Eq. (9) along with the acceleration distribution, see Eqs. (14); the expression of the internal potential energy, Eq. (15); the displacements distributions of the face sheets, see Eqs. (4); the kinematic relations of small deformations in the faces and the core, see Eqs. (5) and (6); the compatibility conditions corresponding to perfect bonding at the face-core interfaces, Eqs. (7) and (8); the stress resultants of the face sheets, see Fig. 1c, and the high-order stress resultants of the core. Hence, after integration by parts and some algebraic manipulation equations of motion read:

For the face sheets:

$$\begin{aligned}
 & u_{ot,u}(x,t) M_t - \left(\frac{\partial}{\partial x} N_{xxt}(x,t) \right) - \lambda_{xt}(x,t) = 0 \\
 & M_t w_{t,u}(x,t) - \left(\frac{\partial^2}{\partial x^2} w_{t,u}(x,t) \right) I_{mt} - \lambda_{zt}(x,t) - \left(\frac{\partial^2}{\partial x^2} M_{xxt}(x,t) \right) \\
 & - \frac{1}{2} \left(\frac{\partial}{\partial x} \lambda_{xt}(x,t) \right) d_t = 0 \\
 & u_{ob,u}(x,t) M_b - \left(\frac{\partial}{\partial x} N_{xxb}(x,t) \right) + \lambda_{xb}(x,t) = 0 \\
 & M_b w_{b,u}(x,t) - \left(\frac{\partial^2}{\partial x^2} w_{b,u}(x,t) \right) I_{mb} + \lambda_{zb}(x,t) - \left(\frac{\partial^2}{\partial x^2} M_{xxb}(x,t) \right) \\
 & - \frac{1}{2} \left(\frac{\partial}{\partial x} \lambda_{xb}(x,t) \right) d_b = 0
 \end{aligned} \tag{16}$$

For the core:

$$\left(\frac{1}{12} c^3 u_{2,u}(x,t) + c u_{o,u}(x,t) \right) M_c - \lambda_{xb}(x,t) + \lambda_{xt}(x,t) = 0 \tag{17}$$

$$\begin{aligned}
 & \left(\frac{1}{80} c^5 u_{3,uu}(x,t) + \frac{1}{12} c^3 u_{1,uu}(x,t) \right) M_c + Q_c(x,t) - \frac{1}{2} \lambda_{xt}(x,t) c \\
 & \quad - \frac{1}{2} \lambda_{xb}(x,t) c = 0 \\
 & \left(\frac{1}{80} c^5 u_{2,uu}(x,t) + \frac{1}{12} c^3 u_{o,uu}(x,t) \right) M_c + 2 M_{QIx}(x,t) + \frac{1}{4} \lambda_{xt}(x,t) c^2 \\
 & \quad - \frac{1}{4} \lambda_{xb}(x,t) c^2 = 0 \\
 & \left(\frac{1}{448} u_{3,uu}(x,t) c^7 + \frac{1}{80} c^5 u_{1,uu}(x,t) \right) M_c + 3 M_{Q2x}(x,t) - \frac{1}{8} \lambda_{xt}(x,t) c^3 \\
 & \quad - \frac{1}{8} \lambda_{xb}(x,t) c^3 = 0 \\
 & \left(\frac{1}{12} c^3 w_{2,uu}(x,t) + c w_{o,uu}(x,t) \right) M_c + \lambda_{zt}(x,t) - \lambda_{zb}(x,t) - \left(\frac{\partial}{\partial x} Q_c(x,t) \right) \\
 & \quad = 0 \\
 & \frac{1}{12} M_c c^3 w_{1,uu}(x,t) + R_{zc}(x,t) - \frac{1}{2} \lambda_{zb}(x,t) c - \frac{1}{2} \lambda_{zt}(x,t) c \\
 & \quad - \left(\frac{\partial}{\partial x} M_{QIx}(x,t) \right) = 0 \\
 & \left(\frac{1}{80} c^5 w_{2,uu}(x,t) + \frac{1}{12} c^3 w_{o,uu}(x,t) \right) M_c - \left(\frac{\partial}{\partial x} M_{Q2x}(x,t) \right) + 2 M_{zc}(x,t) \\
 & \quad - \frac{1}{4} \lambda_{zb}(x,t) c^2 + \frac{1}{4} \lambda_{zt}(x,t) c^2 = 0
 \end{aligned}$$

The compatibility conditions corresponding to perfect bonding bond equal:

$$\begin{aligned}
 & -u_{ot}(x,t) + \frac{1}{2} \left(\frac{\partial}{\partial x} w_t(x,t) \right) d_t + u_o(x,t) - \frac{1}{2} u_1(x,t) c + \frac{1}{4} u_2(x,t) c^2 \\
 & \quad - \frac{1}{8} u_3(x,t) c^3 = 0 \\
 & u_{ob}(x,t) + \frac{1}{2} \left(\frac{\partial}{\partial x} w_b(x,t) \right) d_b - u_o(x,t) - \frac{1}{2} u_1(x,t) c - \frac{1}{4} u_2(x,t) c^2 \\
 & \quad - \frac{1}{8} u_3(x,t) c^3 = 0 \\
 & -w_t(x,t) + w_o(x,t) - \frac{1}{2} w_1(x,t) c + \frac{1}{4} w_2(x,t) c^2 = 0 \\
 & w_b(x,t) - w_o(x,t) - \frac{1}{2} w_1(x,t) c - \frac{1}{4} w_2(x,t) c^2 = 0
 \end{aligned} \tag{18}$$

where the high-order stress resultants in the core equal:

$$\begin{aligned}
 \{ Q_{xc}(x,t), M_{QIx}(x,t), M_{Q2xc}(x,t) \} &= \int_{-\frac{c}{2}}^{\frac{c}{2}} (1, z_c, z_c^2) \tau(x,t) dz_c \\
 \{ R_{zc}(x,t), M_{zc}(x,t) \} &= \int_{-\frac{c}{2}}^{\frac{c}{2}} (1, z_c) \sigma_{zz}(x,t) dz_c
 \end{aligned} \tag{19}$$

The number of equations including the compatibility equations is fifteen. The set of governing equations consists of four equilibrium equations for the face sheets, Eqs. (16), seven equations for the core, Eqs. (17), and four compatibility equations, Eqs. (18). It should be noticed that some of the equations of the core are algebraic, which means that this set of equations is a DAEs (Differential-Algebraic Equations) set and not an ordinary PDEs set. The solution procedure consists of isolating four of the unknowns in the displacements of the core using the compatibility equations, Eqs. (18), followed by isolating the additional four Lagrange multipliers using four equations of the core, Eqs. (17). The isolated unknowns are substituted into the remaining seven equations.

The necessary and sufficient number of boundary conditions is nine, but two of the conditions are imposed on high-order displacement conditions, which yields that the corresponding natural conditions are null. Hence, the full set of boundary conditions here reads:

For the face sheets ($j=t,b$):

$$\begin{aligned}
 \lambda N_{xxj}(x_e, t) - N_{ej}(t) = 0 & \quad \text{or} \quad u_{oj}(x_e, t) = u_{ej}(t) \\
 -M_{ej}(t) - \lambda M_{xxi}(x_e, t) = 0 & \quad \text{or} \quad w_{j,x}(x_e, t) = Dw_{ej}(t) \\
 \lambda V_{xzi}(x_e, t) - P_{ej}(t) = 0 & \quad \text{or} \quad w_j(x_e, t) = w_{ej}(t)
 \end{aligned} \tag{20}$$

where $\lambda=1$ when $x_e=L$ and -1 when $x_e=0$, and the shear stress resultants equals:

$$V_{xzi}(x_e, t) = D_1(w_{j,tt})(x_e, t) I_{mj} + \frac{1}{2} \lambda_{xj}(x_e, t) d_j + D_1(M_{xxj})(x_e, t) \text{ where } D_1 f = df(x,t)/dx.$$

For the core:

$$\begin{aligned}
 Q_c(x_e) = 0 & \quad \text{or} \quad w_o(x_e, t) - w_{eo}(t) = 0 \\
 M_{Q1c}(x_e, t) = 0 & \quad \text{or} \quad w_1(x_e, t) - w_{e1}(t) = 0 \\
 M_{Q2c}(x_e, t) = 0 & \quad \text{or} \quad w_2(x_e, t) - w_{e2}(t) = 0
 \end{aligned} \tag{21}$$

Where w_{ek} ($k=0,1,2$) are the imposed vertical displacement, its slope and curvature at the mid-height of the core. Notice that only the vertical displacement at mid-height of the core can be imposed. This yields that the first and the second higher-order moments of the shear stresses are null thus,

$$M_{Q1c}(x_e, t) = M_{Q2c}(x_e, t) = 0.$$

In order to determine the governing equations of motion the high-order stress resultant terms of the core must be defined first in terms of the displacements. The stress fields and the high-order terms are derived assuming that the core is isotropic, see Eqs. (12), with variable elastic moduli, using the pre-assumed displacements patterns, Eqs. (14), and the high-order terms, Eqs. (19).

Hence, the core stresses are expressed in the form:

$$\begin{aligned}\sigma_{zzc}(x, t, z_c) &= E_{zc}(T_c(x, z_c)) \left(w_1(x, t) + 2 w_2(x, t) z_c - \alpha_c T_c(x, z_c) \right) \\ \tau(x, z_c, t) &= G_{xzc}(T_c(x, z_c)) \left(u_1(x, t) + 2 u_2(x, t) z_c + 3 u_3(x, t) z_c^2 \right. \\ &\quad \left. + \frac{\partial}{\partial x} w_o(x, t) + \left(\frac{\partial}{\partial x} w_1(x, t) \right) z_c + \left(\frac{\partial}{\partial x} w_2(x, t) \right) z_c^2 \right)\end{aligned}\quad (32)$$

The core stress resultants are derived by substitution of Eq. (22) in Eqs. (19). They are not presented for the sake of brevity.

The governing equations of motion are formulated in terms of the following fifteen unknowns: the longitudinal in-plane and vertical displacements of the face sheets, the four Lagrange multipliers and the eleven polynomial coefficients of the core. They are derived by substituting: the force-displacement relations of the face sheets, Eqs. (13), into the governing equations for the face sheets, Eqs. (16), and the high-order stress resultants into the core equations, Eqs. (17). Also here the temperature induced deformation causes the sandwich panel to deform, and it oscillates with respect to the static deformed shape. Notice that this set of governing equations does not have a general closed-form analytical solution. However, for the particular case of a simply-supported sandwich panel a closed-form solution exists.

CONCLUSIONS

A rigorous systematic analysis of the problem of free vibrations of sandwich panels with compliant and temperature dependent core properties has been presented in this part. More specifically, the effect of degrading material properties with increasing temperature has been investigated. Two basic well known computational models for the analysis of sandwich structures have been discussed: The first one uses the HSAPT approach where the unknowns are the displacements of the face sheets and the shear stress of the core, and it is assumed that the

acceleration distribution is linear through the depth of the core. In this model the stress resultants can be attributed a clear physical interpretation, and any conditions on the core are imposed in the strict differential sense. The second model assumes that the through-the thickness displacements distributions of the core are quadratic and cubic for the vertical and horizontal displacements and accelerations, respectively. Hence, the unknowns in this model consist of the coefficients of these polynomial together with the face sheet displacements. This model implicates the existence of higher-order stress resultants in the core, which cannot be associated with any meaningful physical interpretation. Moreover, any constraints that are impose on the core may be defined in the overall sense only. The advantage of this formulation is the straight forward implementation of the variable stiffness of the core without resorting to an equivalent distribution of the mechanical stiffness properties as used in mixed formulation.

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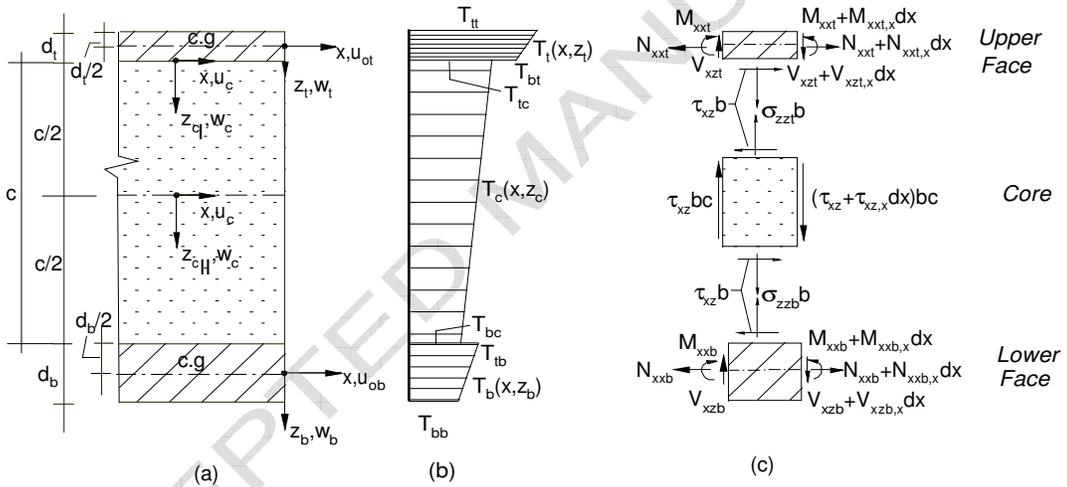


Fig. 1: Geometry, temperature and stress resultants: (a) Geometry (model I and II); (b) Temperature Distribution; (c) Stress resultants (model I).