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Analysis of a turbulence model related to the $k-\varepsilon$ one for stationary and compressible flows

P. Dreyfuss*

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Abstract

We shall study a turbulence model arising in compressible fluid mechanics. The model called $\theta-\varphi$ we study is closely related to the $k-\varepsilon$ model. We shall establish existence, positivity and regularity results in a very general framework.

Keywords: turbulence modelling, $k-\varepsilon$ model, compressible flows, Stampacchia estimates.
2010 MSC: 35J70, 35J75, 76F50, 76F60

1 Introduction

We shall first recall some basic ideas concerning the modelisation of the turbulent fluids, the reader can consult [23, 27] for a more detailled introduction.

Let $\mathbf{u}, p, \rho, T$ be the velocity, pressure, density and temperature of a newtonien compressible fluid. Let also $\tilde{\Omega} \subset \mathbb{R}^3$ a domain which is asummed to be bounded. Then the motion of the flow in $\tilde{\Omega}$ and at a time $t \in \mathbb{R}^+$ can be described by the compressible Navier Stokes equations (see system (C) page 8 in [18]). It is well known that direct simulation based on such a model is harder or even impossible at high reynolds numbers. The reason is that too many points of discretization are necessary and so only very simple configurations can be handled.

Thus engineers and physicists have proposed new sets of equations to describe the average of a turbulent flow. The most famous one is the $k-\varepsilon$ model, introduced by Kolmogorov [13]. We shall briefly present its basic principles in the following. Let v denote a generic physical quantity subject to be turbulent

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(i.e. unpredictable at the macroscopic scale), we introduce its mean part (or its esperance) $\langle v \rangle$ by setting:

$$\langle v(x,t) \rangle = \int_{\mathcal{P}} v(\omega, x, t) d\mathcal{P}(\omega),$$

where the integral is taken in a probalistic context which we shall not detail any more here. Note however that the operation $\langle . \rangle$ is more generally called a filter. The probalistic meaning is one but not the only possible filter (see for instance [23] chap.3). We shall then consider the decomposition: $v = \langle v \rangle + v'$, where v' is refered to the non computable or the non relevant part and $\langle v \rangle$ is called the mean part (i.e. the macroscopic part).

The principle of the $k - \varepsilon$ model is to describe the mean flow in terms of the mean quantities $\langle \mathbf{u} \rangle, \langle p \rangle, \langle \rho \rangle, \langle T \rangle$ together with two scalar functions k and ε, which contains relevant informations about the small scales processes (or the turbulent processes). The variable k (SI: $[\frac{m^2}{s^2}]$) is called the turbulent kinetic energy and ε $[\frac{m^2}{s^3}]$ is the rate of dissipation of the kinetic energy. They are defined by:

$$k = \frac{1}{2}|\mathbf{u}'|^2 \qquad \varepsilon = \frac{\nu}{2}\langle |\nabla \mathbf{u}' + (\nabla \mathbf{u}')^T|^2 \rangle, \tag{1}$$

where ν is the molecular viscosity of the fluid. The model is then constructed by averaging (i.e. by appling the operator $\langle . \rangle$ on) the Navier-Stokes equations. Under appriopriate assumptions (i.e. the Reynolds hypothesis in the incompressible case, and the Favre average in the compressible case) we obtain a closed system of equations for the variables $\langle \mathbf{u} \rangle, \langle p \rangle, \langle \rho \rangle, \langle T \rangle, k$ and ε (see [23] pages 61-62 for the incompressible case, and pages 116-117 in the compressible situation).

Here we shall focus on the equations for k and ε and we consider that the others quantities $\langle \mathbf{u} \rangle, \langle p \rangle, \langle \rho \rangle, \langle T \rangle$ are known. Moreover, in order to simplify the readability we do not use the notation $\langle . \rangle$ in the following, i.e in the sequel we will write $\mathbf{u}$ instead of $\langle \mathbf{u} \rangle$ and ρ instead $\langle \rho \rangle$ to represent the mean velocity and density of the fluid. The equations for k and ε are of convection-diffusion-reaction type:

$$\partial_t k + \mathbf{u} \cdot \nabla k - \frac{c_\nu}{\rho} \text{div } ((\nu + \rho \frac{k^2}{\varepsilon}) \nabla k) = c_\nu \frac{k^2}{\varepsilon} F - \frac{2}{3} k D - \varepsilon, \tag{2}$$

$$\partial_t \varepsilon + \mathbf{u} \cdot \nabla \varepsilon - \frac{c_\varepsilon}{\rho} \text{div } ((\nu + \rho \frac{k^2}{\varepsilon}) \nabla \varepsilon) = c_1 k F - \frac{2c_1}{3c_\nu} \varepsilon D - c_2 \frac{\varepsilon^2}{k}, \tag{3}$$

where $D(x,t) := \text{div } \mathbf{u}(x,t)$, $F(x,t) := \frac{1}{2}|\nabla \mathbf{u} + (\nabla \mathbf{u})^T|^2 - \frac{2}{3} D(x,t)^2 \geq 0$ (see subsection 6.2 in the Appendix) and $c_\nu, c_\varepsilon, c_1, c_2$ are generally taken as positive constants (see (70) in the Appendix).

Note that equations (2)-(3) are only valid sufficiently far from the walls. In fact, in the vicinity of the walls of the domain $\tilde{\Omega}$, there is a thin domain Σ, called logarithmic layer in which the modulus of the velocity goes from 0 to $\mathcal{O}(1)$. In this layer we can use some wall law or a one equation model (see [23] chap.1 and [24]) instead of (2)-(3). Note however that the equations (2)-(3)

can be considered even in the logarithmic layer if we allow the coefficients $c_\nu, c_\varepsilon, c_1$ and c_2 to depend appriopriatly on some local Reynolds numbers (see [23] pages 59-60 and page 115). In this last situation the system is called Low-Reynolds number k-ε model.

In the following we focus on the study in the domain $\Omega := \tilde{\Omega} \setminus \Sigma$ and we assume that its boundary $\partial\Omega$ is Lipschitz[1]. We denote by $\mathbf{n}(x)$ the outward normal defined for all points $x \in \partial\Omega$. The boundary conditions for k and ε on $\partial\Omega$ are well understood. We have:

$$k = k_0 \quad \text{and} \quad \varepsilon = \varepsilon_0 \quad \text{on } \partial\Omega, \tag{4}$$

where k_0 and ε_0 are strictly positive functions which can be calculated by using a wall law (see [23] p.59) or a one equation model (see [24]). In the following we assume that k_0 and ε_0 are given. Moreover we can assume (see again [23] p.59) that:

$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{5}$$

We shall concentrate in this paper on a reduced system called $\theta - \varphi$ model. The new variables θ [s] and φ $[m^{-2}]$ are obtained from k and ε by the formulas:

$$\theta = \frac{k}{\varepsilon} \qquad \varphi = \frac{\varepsilon^2}{k^3}. \tag{6}$$

These variables have a physical meaning (see [27]): θ represents a characteristic time of turbulence and $L = \varphi^{-1/2}$ is a characteristic turbulent length scale. By using this change of variable in the equations (2)-(3) and after considering some modelisation arguments for the diffusion processes (see the Annexe) we obtain:

$$(P) \quad \begin{cases} \partial_t\theta + \mathbf{u}\cdot\nabla\theta - \frac{1}{\rho}\operatorname{div}\Big((\nu + \frac{c_\theta\rho}{\theta\varphi})\nabla\theta\Big) = -c_3\theta^2 F + c_4\theta D + c_5 \\ \partial_t\varphi + \mathbf{u}\cdot\nabla\varphi - \frac{1}{\rho}\operatorname{div}\Big((\nu + \frac{c_\varphi\rho}{\theta\varphi})\nabla\varphi\Big) = -\varphi\big(c_6\theta F - c_7 D + c_8\theta^{-1}\big) \end{cases}$$

where the coefficients c_θ, c_φ and c_i are all positives.

Problem (P) is known as the $\theta - \varphi$ model. It differs to the $k - \varepsilon$ one only by the diffusive parts and it is attractive by some stronger mathematical properties. Another model closely related to these systems, and having some popularity, is the $k - \omega$ one (with $\omega = \theta^{-1}$, see for instance [27]).
In the papers [16, 20] the authors have established the existence of a weak solution for (P) and a property of positiveness. This last feature takes the model useful in practice: it can be used directly or also as a intermediary stabilization procedure to the $k - \varepsilon$ one (see [22]). Another important property attempted for a turbulence model is its capability to predict the possible steady states. In the previous works (unless in [21]. However in this paper only the incompressible situation, with moreover pertubated viscosities were studied), only the evolutive version of (P) was studied and the results obtained cannot predict the existence or non-existence of steady states.

Hence in this paper we shall study the stationary version of (P) on a bounded domain $\Omega \subset \mathbb{R}^N$, N=2 or 3, on which we impose the boundary

[1]see [9] p. 127

conditions $\theta = a, \varphi = b$ on $\partial\Omega$. Remark that by using (4) together with (6) we obtain

$$a = \frac{k_0}{\varepsilon_0} \quad b = \frac{\varepsilon_0^2}{k_0^3}. \tag{7}$$

Hence we can assume that a and b are strictly positive given functions.

We shall establish existence, positivity and regularity results in a very general framework. In [8] we establish existence and regularity results for a turbulent circulation model involving $\mathbf{u}$ and k as unknowns. The reader interested for recent references concerning the numerical simulation of turbulent fluid can consult [25].

2 Main results

2.1 Assumptions and notations

Let (Q) denote the stationary system associated to (P). For simplicity we introduce the new parameters $C_{\text{ind}} := \rho c_{\text{ind}}$ where the subscript 'ind' takes the integer values 3,4,5,6,7,8 or the letters θ and φ. Then our main model (Q) has the following form:

$$\text{(Q)} \quad \begin{cases} \rho\mathbf{u}\cdot\nabla\theta - \operatorname{div}\Big((\nu + \frac{C_\theta}{\theta\varphi})\nabla\theta\Big) = -C_3F\theta^2 + C_4\theta D + C_5 \quad \text{in } \Omega \\ \rho\mathbf{u}\cdot\nabla\varphi - \operatorname{div}\Big((\nu + \frac{C_\varphi}{\theta\varphi})\nabla\varphi\Big) = -\varphi(C_6\theta F - C_7D + C_8\theta^{-1}) \quad \text{in } \Omega \\ \theta = a, \varphi = b \quad \text{on } \partial\Omega \end{cases}$$

For physical reasons we are only interested by positive solutions (θ, φ) for (Q). Note however that even with this rectriction, the problem (Q) may be singular (i.e. the viscosities $\nu + \frac{C_\theta}{\theta\varphi}$ and $\nu + \frac{C_\varphi}{\theta\varphi}$ may be unbounded). Moreover, because we allow $\nu \equiv 0$ the equations may degenerate (i.e. the viscosities may vanish). Hence without additional restriction there may be various non equivalent notions of weak solution (see for instance [8]).

In fact a good compromise between respect of the physics, simplification of the mathematical study and obtention of significative results, is to restrict θ and φ to be within the classe $\mathcal{S}$ defined by:

$$\mathcal{S} = \big\{f : \Omega \to \mathbb{R}^+ \text{ such that } f \in H^1(\Omega) \cap L^\infty(\Omega),\ f^{-1} \in L^\infty(\Omega)\big\}.$$

In particular, if the parameters appearing in (Q) are sufficiently regular and if we restrict θ and φ to be within the classe $\mathcal{S}$, then the notion of a weak solution for (Q) is univocally defined: it is a distributional solution (θ, φ) that satisfies the boundary conditions in the sense of the trace.
In this last situation we will tell that (θ, φ) is a **weak solution of (Q) in the class $\mathcal{S}$**.

In the following we shall precise what are the sufficient conditions of regularity to be imposed on the data in order to can consider and prove the existence of a weak solution for (Q).

Let N=2 or 3 denote the dimension of the domain Ω, and r be a fixed number such that:

$$r > \frac{N}{2}. \tag{8}$$

We then have the following continuous injection (see lemma 5):

$$L^r(\Omega) \subset W^{-1,\beta}(\Omega), \quad \text{where } \beta = r^* > N. \tag{9}$$

Recall that $D = \text{div}(\mathbf{u})$, we will consider the following assumptions:

- Assumptions on Ω:

$$\Omega \subset \mathbb{R}^N \quad \text{is open, bounded and it has a Lipschitz boundary } \partial\Omega. \tag{10}$$

- Assumptions on the flow data [2] $\mathbf{u}, F, D, \rho$ and ν:

$$\nu \geq 0, \tag{11}$$
$$F, \rho : \ \Omega \to \mathbb{R}^+, \quad \rho, \rho^{-1} \in L^\infty(\Omega), \quad F \in L^r(\Omega), \tag{12}$$
$$\mathbf{u} \in (L^3(\Omega))^N, D \in L^r(\Omega), \text{div}(\rho\mathbf{u}) = 0, \tag{13}$$
$$\mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega. \tag{14}$$

- Assumptions on the turbulent quantities on the boundary:

$$\begin{aligned} &a, b \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega), \\ &a(x), b(x) \geq \delta > 0 \text{ a.e. } x \in \partial\Omega, \end{aligned} \tag{15}$$

  where $\delta > 0$ is a fixed number.

- Assumptions on the model coefficients:

$$C_i : \Omega \to \mathbb{R}^+, \ C_5, C_8 \in L^r(\Omega), \ \text{for } i \neq 5, 8 : C_i \in L^\infty(\Omega) \tag{16}$$
$$C_{\text{ind}} : \Omega \times (\mathbb{R}^+)^2 \to \mathbb{R}^+ \text{ are Caratheodory} \tag{17}$$
$$C_{\text{ind}}(x, v, w) \geq \alpha_{\text{ind}} > 0 \quad \forall v, w \in \mathbb{R}^+ \text{ and for a.a. } x \in \Omega \tag{18}$$
$$\forall v, w \in \mathbb{R}^+, \ x \to C_{\text{ind}}(x, v, w) \in L^\infty(\Omega), \tag{19}$$

  where in (17)-(19) C_{ind} means C_θ or C_φ. The assumption (17) signifies that $\forall v, w \in \mathbb{R}^+$, $x \to C_{\text{ind}}(x, u, v)$ is measurable and for a.a. $x \in \Omega$: $(v, w) \to C_{\text{ind}}(x, v, w)$ is continuous. This ensures that $C_{\text{ind}}(x, \theta, \varphi)$ is measurable when θ and φ are measurables. The conditions (18) means that C_{ind} is uniformly positive, whereas (19) tells that $C_{\text{ind}}(x, \theta, \varphi)$ remains bounded if θ and φ are bounded.

We will study problem (Q) under the **main assumption**:

$$(A_0): \quad (8) - (19) \text{ are satisfied.}$$

Note that in the main situation (A_0) the assumption (11) made for ν allows the possibilty $\nu \equiv 0$. On other words the molecular viscosity ν can be neglected in the model. This is often choosen in practice because the eddy viscosities $\frac{C_\theta}{\theta\varphi}$ and $\frac{C_\varphi}{\theta\varphi}$ are dominants in the physical situations (see [23, 24]).

[2] when $N = 2$ one assumption in (13) can be relaxed: $\mathbf{u} \in (L^p(\Omega))^2$ with $p > 2$ (instead of $p = 3$) is sufficient, but this would not improve any result significantly

Remark also that the coefficients C_i are allowed to depend on x, and the viscosity parameters C_θ, C_φ may depend on x, θ, φ.

For a given function $f : \Omega \to \mathbb{R}$, we shall use the notations f^+ and f^- to represent the positive and negative parts of f, that is:

$$f = f^+ + f^-, \quad f^+(x) = \max(f(x), 0) \geq 0, \quad f^-(x) = \min(f(x), 0) \leq 0.$$

We will also consider some assumption of low compressibility of the form:

$$\|D^+\|_{L^r(\Omega)} \leq \tau, \tag{20}$$

for some $\tau > 0$ that will be precised.
This last kind of condition seems to be necessary (see the Appendix) in order to obtain a weak solution for (Q) in the three dimensional case, whereas when $N = 2$ we shall use some particularities of the situation to obtain a weak solution without any assumption of low compressibility. Nevertheless in this case we will assume that in addition to (A_0) the following condition is satisfied:

$$(A_1): \quad \frac{C_4^2}{C_3} \in L^r, \quad c_6 = 0.014, \ c_7 = 0.104, \ c_8 = 0.84.$$

In this last condition the values for c_6, c_7 and c_8 are in fact their classical constant values (see (74) in the Appendix)

In the sequel we denote by $DATA$ some quantity depending only on the data under the assumption (A_0), i.e.:

$$DATA = Const\big(\Omega, a, b, \|\mathbf{u}\|_{(L^3)^N}, \alpha_\theta, \alpha_\varphi, (\|C_j\|_{L^r})_{j=5,8}, \|F\|_{L^r}, \\ (\|C_i\|_{L^\infty})_{i=3,4,6,7}\big). \tag{21}$$

Note that $DATA$ does not depends on D and ν.
The exact form of the dependency (i.e. the function $Const$) is allowed to change from one part of the text to another.

2.2 Main results

We shall establish two theorems of existence. The first one apply if N=2 or 3 and the second one is limited to the case N=2.

Theorem 1 *Assume that (A_0) holds. Then there exists $\tau > 0$ such that if $\|D^+\|_{L^r(\Omega)} \leq \tau$ then problem (Q) admits at lest one weak solution (θ, φ) in the class $\mathcal{S}$.*

Theorem 2 *Assume that $N = 2$ and (A_0),(A_1) hold. Then problem (Q) admits at lest one weak solution (θ, φ) in the class $\mathcal{S}$.*

In all the situations we have the following regularity result:

Theorem 3 *Let (θ, φ) be a weak solution of (Q) in the class $\mathcal{S}$ and assume that (A_0) is satisfied. We have:*

i) If $\mathbf{u} \in (L^\infty(\Omega))^N$ and a, b are Hölder continuous, then $(\theta, \varphi) \in (\mathcal{C}^{0,\alpha}(\overline{\Omega}))^2$, for some $\alpha > 0$

ii) Assume that in addition, the following conditions are satisfied:

$$\partial\Omega, a \text{ and } b \text{ are of class } \mathcal{C}^{2,\alpha}, \quad \rho\mathbf{u} \in (\mathcal{C}^{1,\alpha}(\overline{\Omega}))^N$$
$$F \in \mathcal{C}^{0,\alpha}(\overline{\Omega}), \quad \forall i = 3,..,8: \ C_i \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$$
$$C_{ind}(x, v, w) \in \mathcal{C}^{1,\alpha}(\overline{\Omega} \times (\mathbb{R}^+)^2)$$

Then $(\theta, \varphi) \in (\mathcal{C}^{2,\alpha}(\overline{\Omega}))^2$ *and it is a classical solution of (Q).*

2.3 Discussion on the results

The major feature of our work is to treat the compressible situation in a general framework. The compressible case is interesting for several applications (see for instance [22, 5]). However the additional terms of the model arising from the compressibility effects induce important complications for its analysis. Roughly speeking: these additional terms are responsibles of an increase or decrease (depending on the signe of D) of the turbulence. Then the balance between the increase/decrease of the source terms of turbulence and the possible explosion/vanishing of the turbulent viscosity is difficult to control. Compared to previous works (see [16, 15, 23, 28]) our basic assumption (A_0) made in theorem 1 is very general, and we can remark that:

i) We do not impose a free divergence condition on $\mathbf{u}$ and the mean density ρ is not supposed constant. Hence our analysis can handle with compressible turbulent flows. The condition of the form (20) assumed for $\operatorname{div}\mathbf{u}$ is much weaker than the corresponding assumption made in [15]. In dimension two, under the additional condition (A_1) we obtain in theorem 2 an existence result without any additional assumption of low compressibility. The necessarity of a condition of the form (20) when N=3 is discuted in the Appendix.

ii) We allow the viscosity parameters C_θ, C_φ to depend on x, θ, φ whereas in the previous works these parameters were taken constants (unless in [20, 21] where however an artificial regularization was made, and only the incompressible situation were studied).

iii) We only assume weak regularity for the mean flow, i.e: $\mathbf{u} \in (L^3(\Omega))^N$ and $\operatorname{div}\mathbf{u} \in L^r(\Omega)$, with some $r > 3/2$, whereas in the previous works it was assumed $\mathbf{u} \in (L^\infty(\Omega))^N$ and $\operatorname{div}\mathbf{u} \in L^\infty(\Omega)$. Our assumption is more interesting from a mathematical point of view because it is satisfied when $\mathbf{u}$ is a weak solution of the Navier Stokes equations. Hence our work would be used for a future analysis of the full coupled system Navier-Stokes plus (Q).

iv) All the coefficients C_i are allowed to depend on $x \in \Omega$. Hence our study is also a step for the analysis of the Low-Reynolds $k - \varepsilon$ model (in [28] the Low-Reynolds $k - \varepsilon$ model is studied only in the incompressible and unsteady situation). Finally the boundary values for θ and φ are not taken constants contrarily to the previous works.

Note also that under the additional assumption $\mathbf{u} \in (L^\infty(\Omega))^N$ we give a Hölder continuity result for k and θ. Moreover, we establish an existence result for a classical solution under some smoothness assumptions on the data.

These regularity results seem to be completly new.

Another feature of our work is to point out that the choice $\varphi = \frac{\varepsilon^2}{k^3}$ is indicated even in the compressible situation: this makes the dynamic of the φ equation stable when $N = 2$ and stable under additional conditions when $N = 3$ (see the Appendix). These results improve the study of the model presented in [23], chap. 9.

In order to establish our theorems we establish intermediate results (see proposition 9), concerning the existence, positivity and regularity properties for a weak solution of an elliptic scalar problem (possibly degenerate and singular) on the form:

$$\text{(S)} \quad \begin{cases} \rho \mathbf{u} \cdot \nabla \zeta - \operatorname{div}\left((\nu + \frac{\kappa}{\zeta})\nabla\zeta\right) = g(x,\zeta) \quad \text{in } \Omega \\ \zeta = \zeta_0 \quad \text{on } \partial\Omega, \end{cases}$$

These results have also an independent mathematical interest.

2.4 Organization of the paper

• In section 3 we shall recall some results concerning: the truncature at a fixed level and the Stampacchia's estimates. This last technique takes an important role in our analysis, moreover we shall need a precise control of the estimates. Hence we shall present it with some details and developments.
• In section 4 we introduce a sequence (Q_n) of problems which approximate (Q). For n fixed (Q_n) is a PDE system of two scalar equations of the form (S): one equation for the unknown θ_{n+1} and one for φ_{n+1}. The point is that the unknowns θ_{n+1} and φ_{n+1} are only weakly coupled. The coupling of the two equations is essentially realized through the quantities θ_n and φ_n calculated at the previous step. Hence we shall firstly study carefully the problem (S). The major tool used here are the Stampacchia's estimates. We next prove that (Q_n) is well posed. Hence we obtain an approximate sequence of solutions (θ_n, φ_n) for problem (Q). Moreover, we prove that θ_n and φ_n are uniformly bounded from above and below, which are the key estimates.
• In section 5 we use the uniform bounds established in section 4, in order to extract a converging subsequence from (θ_n, φ_n). We then prove that the limit (θ, φ) is a weak solution of (Q) in the class $\mathcal{S}$.
Under the additional assumption $\mathbf{u} \in (L^\infty(\Omega))^3$ we are able to use the De Giorgi-Nash Theorem and we obtain an Hölder continuity result for θ, φ. By assuming in addition some smothness properties for the data we can iterate the Schauder estimates and prove Theorem 3.
• In section 6 (Appendix) we present the derivation of the $\theta - \varphi$ model from the $k - \varepsilon$ one. Moreover we justify that the choice $\varphi = \frac{\varepsilon^2}{k^3}$ is valid even in the compressible situation. The justification uses in particular a property of positivity of the function F. We also dicuss briefly the necessarity of the low compressibility assumption when $N = 3$. Finally we recall a generalized version of the chain rule for $G(u)$ where G is a Lipchitz function and u a Sobolev one.

3 Mathematical background

In this section we shall recall some results concerning: the truncature at a fixed level and the Stampacchia's estimates. This last technique takes an important role in our analysis, moreover we shall need a precise control of the estimates. Hence we shall present here the technique with some details and developments. As in the rest on the paper we denote by $\Omega \subset \mathbb{R}^N$ a bounded open Lipschitz domain. These properties for Ω are always implicity assumed if they are not precised.

3.1 Truncatures and related properties

The technique of Stampacchia is based on the use of special test functions which are constructed by using some truncatures. We shall recall some basic properties of the truncatures used in the paper. An important tool is the generalized chain rule (see Theorem 13 in the Annexe).
Let $l > 0$ we denote by T_l the truncature function $T_l : \mathbb{R} \to \mathbb{R}$ defined by

$$T_l(s) = \begin{cases} l & \text{if } s > l \\ s & \text{if } -l \leq s \leq l \\ -l & \text{if } s < -l \end{cases} \tag{22}$$

Let $v \in H^1(\Omega)$. By applying the Theorem 13 we see that $T_l(v) \in H^1(\Omega) \cap L^\infty(\Omega)$. Moreover, if we denote by $\Omega_{v,l}$ the set $\{x \in \Omega \text{ s.t. } |v(x)| \leq l\}$ then we have:

$$\nabla T_l(v) = \begin{cases} \nabla v & \text{in } \Omega_{v,l} \\ 0 & \text{elsewhere} \end{cases}$$

Note that $T_l(.)$ truncates both the positive and the negative large values. In some cases we need only to truncate in one side. For this reason we introduce the semi-truncatures $T_{l,+}$ and $T_{l,-}$ defined by:

$$T_{l,+}(t) = \begin{cases} l & \text{if } t > l \\ t & \text{elsewhere} \end{cases} \qquad T_{l,-}(t) = \begin{cases} -l & \text{if } t < -l \\ t & \text{elsewhere} \end{cases}$$

We then have the decomposition: $T_l = T_{l,+} \circ T_{l,-} = T_{l,-} \circ T_{l,+}$.
For $v_l \in H^1(\Omega)$ and $s > 0$ be given we shall also consider

$$\psi_s(v) = v_l - T_s(v_l) = \begin{cases} v_l - s & \text{in } A^+_{s,l} \\ v_l + s & \text{in } A^-_{s,l} \\ 0 & \text{elsewhere} \end{cases} \tag{23}$$

where we have used the notations:

$$A^+_{s,l} := \{v_l \geq s\},\ A^-_{s,l} := \{v_l \leq -s\} \text{ and } A_{s,l} = A^+_{s,l} \cup A^-_{s,l}. \tag{24}$$

Let also $\psi_{s,\pm}$ be the functions defined by replacing T_l by $T_{l,+}$ or by $T_{l,-}$ in (23). It is easy to verify that $\psi_{s,+}$ (resp. $\psi_{s,+}$) is in fact the positive (resp. the negative) part of ψ_s. On other words, we have :

$$\psi_{s,+}(v_l) = \psi^+_s(v_l) = (v_l - s)\mathbf{1}_{A^+_{s,l}} \geq 0, \quad \psi_{s,-}(v_l) = \psi^-_s(v_l) = (v_l + s)\mathbf{1}_{A^-_{s,l}} \leq 0.$$

$$\psi_s = \psi^+_s + \psi^-_s.$$

The function ψ_s has the following properties:

Lemma 4

i) $\psi_s \in H^1(\Omega)$ *and* $\nabla\psi_s = \mathbf{1}_{A_{s,l}}\nabla v_l$

ii) if $\gamma v_l \in L^\infty(\partial\Omega)$ *then for all* $s > \|\gamma v_l\|_{L^\infty(\partial\Omega)}$ *we have* $\psi_s \in H_0^1(\Omega)$,

where $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ *denotes the trace function.*

Proof
the point i) is a direct consequence of the Theorem 13. The property ii) is proved in [14], lemma 3.3 p.53 □

3.2 The Stampacchia estimates

The Stampacchia estimates is a general method which allows one to obtain an L^∞-estimate for the weak solution of a large class of elliptic PDEs of the second order. The L^∞-estimate presented in the original work [26] depends on various quantities related to the PDE problem studied, but the exact dependency is not established. In our analysis we need a precise control of the L^∞-estimates with respect to some quantities (in particular with the diffusion coefficient of the PDEs). Hence in the following we take over and detail the technique in order to obtain a more precise L^∞-estimates.

The Stampacchia estimates are established by using the test functions ψ_s (or $\psi_s^\pm$) defined previously, where in this case v (resp. v_l) is a weak solution of the problem (resp. the sequence of problems) considered.
For technical reasons we need a classical result concerning some relationship between L^r functions and linear form on Sobolev spaces:

Lemma 5 *Let* $1 \le r < \infty$ *and* $E \in L^r(\Omega)$. *Then*

$$E \in W^{-1,\beta}(\Omega), \text{ with } \beta = r^* = \frac{Nr}{N-r}, \tag{25}$$

and there exists $\tilde{E} \in (L^\beta(\Omega))^3$ *such that:*

$$\int_\Omega E\varphi = \int_\Omega \tilde{E}\cdot\nabla\varphi \quad \forall\varphi \in \mathcal{D}(\Omega), \quad \|\tilde{E}\|_{(L^\beta(\Omega))^3} \le \|E\|_{L^r(\Omega)}. \tag{26}$$

Moreover we have:

$$r > \frac{N}{2} \Rightarrow \beta > N. \tag{27}$$

Proof
The property (25) is easy to prove by using the Sobolev injection Theorem together with the Hölder inequality: $\varphi \to \int_\Omega E\varphi$ is a linear form on $W_0^{1,p}$ if $p^* := \frac{Np}{N-p} \ge \frac{r}{r-1}$. This last condition holds true for $p = P := \frac{Nr}{r(N+1)-N}$. Hence $P' = p/(p-1) = \frac{Nr}{N-r} = r^*$.
We next obtain (26) by using a classical result (see [2] Proposition IX.20).
Finally, if we assume that $r > \frac{N}{2}$ then $\beta = r^* > \frac{N^2/2}{N-N/2} = N$. □

The technique of Stampacchia works within two steps: the first one is dependent on the problem (or the sequence of problems) studied and the second

one is independent of it. Here the purpose is to present the key ingredients of these two steps. Because the first one is dependent of the problem studied we shall not present it here in all generallity, but we will consider a simple problem which contains the main technical points (in fact this introductive presentation will be useful to treat a more complicated class of problems in section 4). Let $v_l \subset H_0^1(\Omega)$ be a sequence of functions satisfying:

$$\int_\Omega \nu_l \nabla v_l \nabla \varphi = \int_\Omega g_l \varphi, \quad \forall \varphi \in H_0^1, \tag{28}$$

where (ν_l) is a given sequence of strictly positive bounded functions and $(g_l) \subset L^r(\Omega)$, with $r > \frac{N}{2}$. Let also m_l, M_l denote the bounds from above and below for ν_l, that is:

$$0 < m_l \leq \nu_l \leq M_l < \infty, \quad \text{a.e. in } \Omega.$$

-*Step 1*

By testing (28) with $\psi_s = \psi_s(v_l)$, we obtain:

$$m_l \int_\Omega |\nabla \psi_s|^2 = \int_\Omega g_l \psi_s \overset{\text{by lemma 5}}{=} \int_\Omega E_l \nabla \psi_s, \tag{29}$$

with $E_l \in L^\beta, \|E_l\|_{L^\beta} \leq \|g_l\|_{L^r}$ for some $\beta = \beta(r) > N$.

Recall also that Supp $\psi_s(v_l) \subset A_{l,s}$. Hence by using the Hölder inequality we obtain:

$$\begin{aligned}\int_\Omega E_l \nabla \psi_s &\leq \|\nabla \psi_s\|_{L^2} (\int_{A_{l,s}} |E_l|^2)^{\frac{1}{2}} \leq \|\nabla \psi_s\|_{L^2} \|E_l\|_{L^\beta} (\int_{A_{l,s}} 1)^{\frac{1}{2(\frac{\beta}{2})'}} \\ &\leq \|\nabla \psi_s\|_{L^2} \|E_l\|_{L^\beta} |A_{l,s}|^{\frac{\beta-2}{2\beta}}.\end{aligned}$$

Consequently (29) leads to:

$$\|\psi_s\|^2_{H_0^1(\Omega)^2} \leq C|A_{l,s}|^\Phi, \quad \text{whith } C > 0, \Phi > \frac{2}{2^*} = \frac{N-2}{N}. \tag{30}$$

This is the key estimate needed to pass at the second step which is independent of the problem studied.

Note that with the particular sequence of problems (28) choosen here, the constants C and Φ are:

$$C = \frac{\|g_l\|^2_{L^r}}{m_l^2}, \quad \Phi = \frac{\beta - 2}{\beta}. \tag{31}$$

Hence Φ does not depend on l, s. Moreover if we assume that (g_l) is uniformly bounded in the L^r-norm and that (m_l) is uniformly bounded from above by a striclty positive constant, then C is also independent on l, s.

This is an important point because as we will see it hereafter, an estimate (30) with C and Φ independant on l, s leads to a uniform L^∞ bound for (v_l).

-*Step 2*

Assume that we have obtained (30). We can obtain an L^∞-estimate for v_l as follows.

Let $2^* = \frac{2N}{N-2}$ be the Sobolev exponent associated to 2 in dimension N. By using the Poincaré-Sobolev inequality we have:

$$\Big(\int_{A_{l,s}} |\psi_s|^{2^*}\Big)^{2/2^*} \leq C_1 \|\psi_s\|^2_{H_0^1(\Omega)}, \quad C_1 = C_1(|\Omega|) \tag{32}$$

Let now $t > s$. It is clear that $A_{l,t} \subset A_{l,s}$ and consequently

$$\Big(\int_{A_{l,s}} |\psi_s|^{2^*}\Big)^{2/2^*} \geq \Big(\int_{A_{l,t}} |\psi_s|^{2^*}\Big)^{2/2^*} \geq \Big(\int_{A_{l,t}} |t-s|^{2^*}\Big)^{2/2^*} \geq |t-s|^2 \big|A_{l,t}\big|^{2/2^*} \tag{33}$$

We set

$$\chi(s) := \big|A_{l,s}\big|, \quad \forall s \geq 0. \tag{34}$$

For fixed l, χ is a decreasing function, and from the estimates (30)-(33), we obtain

$$\chi(t) \leq C_2 |\chi(s)|^{2^*\Phi/2} (t-s)^{-2^*} \quad \forall t > s \geq 0, \quad C_2 = (C_1 C)^{2^*/2}.$$

Recall that we have assumed in (30) that $\Phi > \frac{2}{2^*}$. Hence $2^*\Phi/2 > 1$ and by using the lemma 4.1 in [26] we obtain:

$$\chi(d) = 0, \quad d = 2^{\frac{2^*\Phi}{2^*\Phi-2}} C_2^{1/2^*} |\Omega|^{\frac{\Phi}{2}-\frac{1}{2^*}} < \infty. \tag{35}$$

This property tells exactly that:

$$\|v_l\|_{L^\infty(\Omega)} \leq d, \quad d = C_3(\Phi, N, |\Omega|)\sqrt{C}. \tag{36}$$

In particular d does not depend on l if the constants C and Φ appearing in (30) are independents on l. For instance with the particular sequence of problems (28) the constants C and Φ are given by (31), and if we assume that $\|g_l\|_{L^r} \leq K,\ m_l \geq m > 0$ we obtain:

$$d = \tilde{C}\frac{K}{m}, \quad \tilde{C} = \tilde{C}(\Omega, N, r).$$

Remark 6

- *If you are only interested in obtaining an uniform majoration or minoration for v_l then instead of (30) it suffices to have*

$$\|\psi_s^\pm\|^2_{H_0^1(\Omega)^2} \leq C^\pm |A_{l,s}^\pm|^{\Phi^\pm}, \quad \textit{with } C^\pm > 0, \Phi^\pm > \frac{N-2}{N}. \tag{37}$$

  In fact in this case we consider $\chi_\pm(s) := \big|A_{l,s}^\pm\big|$ instead of (34). This function is decreasing and we obtain again (35). But now this property tells:

$$\pm v_l(x) \leq d^\pm \ \textit{a.e. in } \Omega, \quad d^\pm = \tilde{C}^\pm(\Phi^\pm, N, |\Omega|)\sqrt{C^\pm}.$$

- *Let again (v_l) be a sequence of functions satisfying (28) and assume that $m_l \geq m > 0$. Then we have:*

$$g_l \leq h_l \ \textit{and } H := \sup_l \|h_l\|_{L^r} < \infty \Rightarrow v_l \leq d_1 \ \textit{a.e. in } \Omega, \tag{38}$$

$$g_l \geq k_l \ \textit{and } K := \sup_l \|k_l\|_{L^r} < \infty \Rightarrow v_l \geq -d_2 \ \textit{a.e. in } \Omega, \tag{39}$$

$$\textit{with } d_1, d_2 > 0,\ d_1 = \tilde{C}\frac{H}{m},\ d_2 = \tilde{C}\frac{K}{m}, \quad \tilde{C} = \tilde{C}(\Phi, N, |\Omega|).$$

The proof of (38) and (39) are obtained by taking over the first step of the technique of Stampacchia, but with using $\psi_s^{\pm}$ as test function instead of ψ_s.
In fact, the function ψ_s^+ is positive. Hence instead of (29) we have:

$$m\int_\Omega |\nabla\psi_s^+|^2 = \int_\Omega g_l\psi_s^+ \le \int_\Omega h_l\psi_s^+ = \int_\Omega E_{l,+}\nabla\psi_s^+,$$

with $E_{l,+} \in L^\beta, \|E_{l,+}\|_{L^\beta} \le \|h_l\|_{L^r} \le H$. Consequently, in this case we obtain (37) for the function ψ_s^+ , with $C^+ = \frac{H^2}{m^2}$, and the uniform majoration (38) follows.
The relation (39) can be proven by using ψ_s^- as test function in (28). In fact we remark now that ψ_s^- is negative. Hence instead of (29) we obtain:

$$m\int_\Omega |\nabla\psi_s^-|^2 = \int_\Omega g_l\psi_s^- \le \int_\Omega k_l\psi_s^- = \int_\Omega E_{l,-}\nabla\psi_s^-,$$

with $E_{l,-} \in L^\beta, \|E_{l,-}\|_{L^\beta} \le \|k_l\|_{L^r} \le K$. Consequently we now obtain (37) for the function ψ_s^-, with $C^- = \frac{K^2}{m^2}$. This implies: $-v_l \le d_2$ and consequently $v_l \ge -d_2$.

4 Approximate sequence and estimates

Let $g \in H^{1/2}(\partial\Omega)$, we denote by $\mathcal{R}g$ its harmonic lifting, that is:

$$\mathcal{R}g \in H^1(\Omega), \mathcal{R}g = g \text{ on } \partial\Omega \text{ and } \Delta Rg = 0 \text{ on } \Omega.$$

We define the functions θ_0 and φ_0 by the formula:

$$\theta_0 := \mathcal{R}a \quad \varphi_0 := \mathcal{R}b. \tag{40}$$

Hence, by the using the maximum principle (see [2] p.189 and [11]) together with the condition (15) we obtain:

$$0 < \delta \le \theta_0 \le \|a\|_{L^\infty(\partial\Omega)}, \quad \delta \le \varphi_0 \le \|b\|_{L^\infty(\partial\Omega)}. \tag{41}$$

Let now $n \ge 0$, (θ_n, φ_n) be given and

$$C_\theta^{(n)}(.) := C_\theta(., \theta_n(.), \varphi_n(.)), \quad C_\varphi^{(n)}(.) := C_\varphi(., \theta_n(.), \varphi_n(.)).$$

In order to construct an approximate solution $(\theta_{n+1}, \varphi_{n+1})$ for problem (Q), we introduce the following system:

$$(Q_n) \quad \begin{cases} \rho\mathbf{u}\cdot\nabla\theta_{n+1} - \operatorname{div}\left((\nu + \frac{C_\theta^{(n)}}{\theta_{n+1}\varphi_n})\nabla\theta_{n+1}\right) = g_\theta^{(n)} & \text{in } \Omega, \\ \rho\mathbf{u}\cdot\nabla\varphi_{n+1} - \operatorname{div}\left((\nu + \frac{C_\varphi^{(n)}}{\theta_{n+1}\varphi_{n+1}})\nabla\varphi_{n+1}\right) = g_\varphi^{(n)} & \text{in } \Omega, \\ \theta_{n+1} = a \text{ and } \varphi_{n+1} = b \quad \text{on } \partial\Omega, \end{cases}$$

where we are used the notations:

$$g_\theta^{(n)} := C_5 - C_3F\theta_{n+1}^2 + C_4\theta_{n+1}D, \quad g_\varphi^{(n)} := -\varphi_n(C_8\theta_{n+1}^{-1} + C_6\theta_{n+1}F - C_7D).$$

For $n \in \mathbb{N}$ we denote by (H_n) the following condition:

$$(H_n) \quad \begin{cases} \theta_n, \varphi_n, \theta_n^{-1}, \varphi_n^{-1} \in L^\infty(\Omega), \\ \theta_n, \varphi_n \geq 0. \end{cases}$$

Let $\varphi_{\max}$ be a real number fixed such that:

$$\varphi_{\max} > \|b\|_{L^\infty(\partial\Omega)}. \tag{42}$$

We shall also consider the condition: $(K_n) := (H_n) \ + \ (\varphi_n \leq \varphi_{\max})$.

Note that (41) shows that the condition (K_n) is statisfied for $n = 0$. We will prove in the sequel that under condition (K_n) we can obtain a weak solution $(\theta_{n+1}, \varphi_{n+1})$ for problem (Q_n), with moreover $(\theta_{n+1}, \varphi_{n+1})$ satisfying the condition (K_{n+1}). This last property ensures the right definition of an approximate sequence. More precisely, we have:

Proposition 7 *Let $n \in \mathbb{N}$ be given and assume that (A_0) is satisfied. Let also (θ_n, φ_n) be given and verifying the condition (K_n). There exists $\tau > 0$ depending only on $DATA$ such that if $\|D^+\|_{L^r(\Omega)} \leq \tau$ then problem (Q_n) admits at least one weak solution $(\theta_{n+1}, \varphi_{n+1}) \subset \left(H^1(\Omega) \times L^\infty(\Omega)\right)^2$.*
Moreover $(\theta_{n+1}, \varphi_{n+1})$ satifies the condition (K_{n+1}) and the estimates:

$$0 < \theta_{min} \leq \quad \theta_{n+1} \quad \leq \theta_{max}, \tag{43}$$

$$0 < \varphi_{min} \leq \quad \varphi_{n+1} \quad \leq \varphi_{max}, \tag{44}$$

where φ_{max} was fixed in (42) and $\varphi_{min}, \theta_{min}$, θ_{max} are positives numbers depending on $DATA$, but not on n.

Remark 8 *The proposition 7 is the key result that will be used latter on to prove the theorem 1, whereas for the theorem 2 we shall establish and use a more simple version of this proposition (see subsection 5.2).*

In order to prove the proposition we establish intermediate results.

4.1 Auxilliary results

Let $n \in \mathbb{N}$ and (θ_n, φ_n) be given and satisfying (K_n). We want to obtain $(\theta_{n+1}, \varphi_{n+1})$ by solving (Q_n) and in order to can iterate the algorithm we also want that $(\theta_{n+1}, \varphi_{n+1})$ satisfies (K_{n+1}).
Remark that the system (Q_n) is composed by two coupled scalar elliptic equation on divergence form, with a possible singular and degenerate structure. Hence the goal of this subsection is to study this last kind of scalar problem.

In order to do this, we firstly introduce a weight $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ which is assumed to be mesurable and satisfying:

$$0 < \kappa_0 \leq \kappa \leq \kappa_1 \quad \text{a.e. in } \Omega, \tag{45}$$

where κ_0 and κ_1 are two given reals.
Let also $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be a Caratheodory function (i.e. $\forall u \in \mathbb{R}^+ :$ $x \to g(x, u)$ is measurable, and for a.a. $x \in \Omega :$ $u \to g(x, u)$ is continuous).

Let consider the scalar problem:

$$(\mathrm{S}) \quad \begin{cases} \rho \mathbf{u} \cdot \nabla \zeta - \operatorname{div}\Big((\nu + \frac{\kappa}{\zeta})\nabla \zeta\Big) = g(x,\zeta) & \text{in } \Omega \\ \zeta = \zeta_0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\zeta_0 \in H^{1/2}(\partial\Omega) \cap L^\infty(\partial\Omega)$, $\zeta_0 \geq \delta > 0$ a.e. in $\delta\Omega$, is given. We always assume that $\rho, \mathbf{u}, \nu, \Omega$ which appear in (S) satisfy their corresponding conditions in (A_0).

Recall that we allow $\nu \equiv 0$ in (A_0). Hence problem (S) may degenerate (i.e. the viscosity $\nu + \frac{\kappa}{\zeta}$ may vanish) when $\zeta \to \infty$. Moreover (S) is singular (i.e. the viscosity tends to infinity) when $\zeta \to 0$.

We want now to find sufficient additional conditions for g that guarantee the existence of a bounded positive weak solution for problem (S). Hence we shall consider:

$$\gamma(x) := \sup_{u \in [0,1]} |g^-(x,u)| \in L^r(\Omega), \tag{46}$$

$$g^+(x,u) \leq \gamma_1(x) + \gamma_2(x) h(u), \tag{47}$$

where $\gamma_1, \gamma_2 \in L^r(\Omega)$ and $h : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. In fact, more than establishing only the existence of a bounded positive solution for (S), we are interested to obtain some uniform (with respect to κ_1) bounds from above and below and some regularity results. We have:

Proposition 9

i) Let κ verifying (45) and $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be a Caratheodory function satisfying (46), (47). There exists a real $\tau > 0$ depending on κ_0, h such that if $\|\gamma_2\|_{L^r(\Omega)} \leq \tau$ then there exists a weak solution $\zeta \in H^1(\Omega) \cap L^\infty(\Omega)$ for the problem (S). Moreover we have:

$$0 < \zeta_{min} \leq \zeta \leq \zeta_{max}, \quad \zeta_{min} = e^{-\frac{C}{\kappa_0}}, \zeta_{max} = e^{\frac{C}{\kappa_0}}, \tag{48}$$

where C depend only on $\gamma, \gamma_1, \Omega, r, N, \zeta_0$. In particular ζ_{min} and ζ_{max} are independents on κ_1.
In addition, the following extended [3] maximum principle holds:

$$\|\zeta\|_{L^\infty(\Omega)} \leq \|\zeta_0\|_{L^\infty(\partial\Omega)} + \tilde{C}\frac{\|g^+\|_{L^r}}{\kappa_0}\|\zeta\|_{L^\infty(\Omega)}, \ \tilde{C} = \tilde{C}(\Omega, N, r, \gamma_1). \tag{49}$$

ii) Assume that in addition $\mathbf{u} \in (L^\infty(\Omega))^N$ and ζ_0 is Hölder continuous. Then $\zeta \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Moreover if $\partial\Omega$ is of class $\mathcal{C}^{2,\alpha}$, $g \in \mathcal{C}^{0,\alpha}(\overline{\Omega} \times \mathbb{R}^+), \rho\mathbf{u} \in (\mathcal{C}^{1,\alpha}(\overline{\Omega}))^N, \kappa \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ and $\zeta_0 \in \mathcal{C}^{2,\alpha}(\partial\Omega)$, then $\zeta \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ and it is a classical solution of (S).

Before proving the proposition 9 we establish an intermediary result. In a first step we consider the change of variable $v = \ln \zeta$ in (S), and for $l \in \mathbb{N}$ we introduce a truncated version (S_l) of the system obtained:

$$(S_l) \quad \begin{cases} \rho\mathbf{u}\nabla e^{T_l(v)} - \operatorname{div}((\nu e^{T_l(v)} + \kappa)\nabla v) = g(x, e^{T_l(v)}) & \text{in } \Omega \\ v = \ln \zeta_0 \quad \text{on } \partial\Omega \end{cases}$$

We then establish:

[3] when $g^+ \equiv 0$ it is a maximum principle

Lemma 10

i) Let κ satisfying (45) and $g : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ be a Caratheodory function satisfying (46), (47). Then, for any $l \in \mathbb{N}$ there exists a weak solution $v = v_l \in H^1(\Omega) \cap L^\infty$ for the problem (S_l).

ii) Let (v_l) be the sequence given in i) and l be a fixed integer greater or equal to one. Then there exists $\tau_l > 0$ such that if the function γ_2 in (46) satisfies $\|\gamma_2\|_{L^r} \leq \tau_l$ then we have:

$$\|v_l\|_{L^\infty(\Omega)} \leq C\frac{\|\gamma + \gamma_1\|_{L^r}}{\kappa_0}, \quad C = C(DATA, \zeta_0). \tag{50}$$

In particular C is independent on κ, ν and l.

Proof

i) By using the divergence formula we obtain, for all $w \in H^1(\Omega)$:

$$\begin{aligned}\int_\Omega \rho\mathbf{u}\nabla e^{T_l(v)}.w &= -\textstyle\int_\Omega e^{T_l(v)} \operatorname{div}(\rho\mathbf{u}w) + \underbrace{\int_{\partial\Omega} e^{T_l(v)}\rho w\mathbf{u}\cdot\mathbf{n}d\sigma}_{=0} \\ &= -\textstyle\int_\Omega e^{T_l(v)}\rho\mathbf{u}\cdot\nabla w - \int_\Omega e^{T_l(v)} \underbrace{\operatorname{div}(\rho\mathbf{u})}_{=0} w.\end{aligned}$$

Let $v_0 := \ln(\mathcal{R}\zeta_0)$ and consider the change of variable $\tilde{v} := v - v_0$. Then the problem (S_l) is equivalent to find $\tilde{v} \in H_0^1(\Omega)$ such that:

$$-\operatorname{div}\sigma(x, \tilde{v}, \nabla\tilde{v}) = f(x, \tilde{v}) \quad \text{in } \mathcal{D}'(\Omega), \tag{51}$$

where $\sigma : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ are defined by:

$$\begin{aligned}\sigma(x, w, \mathbf{G}) &= (\nu e^{T_l(w+v_0)} + \kappa(x))\mathbf{G} - \rho(x)\mathbf{u}(x)e^{T_l(w+v_0(x))}, \\ f(x, w) &= g(x, e^{T_l(w+v_0(x))}).\end{aligned}$$

We now remark that (51) is a quasilinear equation in divergence form. Moreover, it is easy to see that f, σ satisfies the classical growth assumptions and σ satisfies also the classical coercivity condition. Note that:

$$\begin{aligned}\langle\sigma(x, w, \mathbf{G}) - \sigma(x, w, \mathbf{G}'), \mathbf{G} - \mathbf{G}'\rangle &= (\nu e^{T_l(w+v_0)} + \kappa)|\mathbf{G} - \mathbf{G}'|^2 \\ &\geq \kappa_0|\mathbf{G} - \mathbf{G}'|^2.\end{aligned}$$

Hence σ is strictly monotone in the third variable. We can then conclude (see for instance [1] Theorem 1.5, or [14] Theorem 8.8 page 311) that there exists a weak solution $\tilde{v} \in H_0^1(\Omega)$ for (51).
Consequently $v_l := \tilde{v} + v_0$ is a weak solution for (S_l), that is $\forall w \in H_0^1(\Omega)$:

$$\int_\Omega (\nu e^{T_l(v_l)} + \kappa)\nabla v_l \nabla w + \int_\Omega \rho\mathbf{u}\nabla e^{T_l(v_l)} w = \int_\Omega g(x, e^{T_l(v_l)})w. \tag{52}$$

By applying the Theorem 4.2 page 108 in [26] we obtain: $v_l \in L^\infty(\Omega)$.

ii)

$$\text{Let } \tau_l := \frac{1}{h(e^l)} > 0 \text{ and assume now that } \|\gamma_2\|_{L^r} \leq \tau_l. \tag{53}$$

With this additional assumption, we are able to obtain an useful estimation for $\|v_l\|_{L^\infty}$.Technically we will detail a technique due to Stampacchia (see subsection 3.2 for the notations and for an introduction of the method. Here only the first step of the method will be developed further). Let $s > |\ln \|\zeta_0\|_{L^\infty(\partial\Omega)}|$, we consider the function $\psi_s = v_l - T_s(v_l)$. We have (see lemma 4) $\psi_s \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and by testing (52) with ψ_s we obtain:

$$\int_\Omega (\nu e^{T_l(v_l)} + \kappa)|\nabla\psi_s|^2 + \underbrace{\int_\Omega \rho\mathbf{u}\cdot\nabla e^{T_l(v_l)}\psi_s}_{=:I} = \underbrace{\int_\Omega g(x, e^{T_l(v_l)})\psi_s}_{=:II}. \tag{54}$$

• We will now evaluate the terms I and II.
The term I is firstly simplified by writing one of its integrand factors, namely $\nabla e^{T_l(v_l)}\psi_s$ as a gradient. More precisely we have $\nabla e^{T_l(v_l)}\psi_s = \nabla\zeta_l$, with $\zeta_l \in H^1(\Omega) \cap L^\infty$ (see the lemma 14 in the Appendix). Hence by applying the divergence formula we see that I vanishes:

$$I = \int_\Omega \rho\mathbf{u}\cdot\nabla\zeta_l \overset{\text{div. formula}}{=} -\int_\Omega \zeta_l \underbrace{\operatorname{div}\rho\mathbf{u}}_{=0} + \underbrace{\int_{\partial\Omega} \rho\zeta_l\mathbf{u}\cdot\mathbf{n}d\sigma}_{=0 \text{ by } (14)} = 0. \tag{55}$$

We next estimate the term II:

$$II = \int_\Omega g(x, e^{T_l(v_l)})\psi_s \leq \underbrace{\int_{A_{s,l}^+} g^+(x, e^{T_l(v_l)})\psi_s}_{:=II_1} + \underbrace{\int_{A_{s,l}^-} g^-(x, e^{T_l(v_l)})\psi_s}_{:=II_2}.$$

Remark now that on $A_{s,l}^-$ we have $v_l \leq -s \leq 0$, which implies that $e^{T_l(v_l)} \leq 1$. Consequently by using the assumption (46) we obtain:

$$II_2 \leq \int_\Omega \gamma\psi_s \mathbf{1}_{A_{s,l}^-}. \tag{56}$$

The term II_1 is majorated as follows:

$$II_1 \underset{\text{by (47)}}{\leq} \int_{A_{s,l}^+} (\gamma_1 + \gamma_2 h(e^l))\psi_s \underset{\text{by (53)}}{\leq} \int_\Omega (\gamma_1 + 1)\psi_s \mathbf{1}_{A_{s,l}^+} \tag{57}$$

• At this point by using the estimates (55), (56) and (57) together with (54) and (45), we obtain:

$$\kappa_0 \int_\Omega |\nabla\psi_s|^2 \leq \int_\Omega \underbrace{(\gamma\mathbf{1}_{A_{s,l}^-} + (\gamma_1 + 1)\mathbf{1}_{A_{s,l}^+})}_{=:E}\psi_s. \tag{58}$$

Note that $E \in L^r(\Omega)$ and: $\|E\|_{L^r} \leq C_0$, $\quad C_0 = \|\gamma\|_{L^r} + \|\gamma_1 + 1\|_{L^r}$. On the other hand (see lemma 5) there exists $\tilde{E} \in (L^\beta(\Omega))^3$ satisfying: $\|\tilde{E}\|_{(L^\beta)^N} \leq \|E\|_{L^r}$, and $\int_\Omega E\varphi = \int_\Omega \tilde{E}\nabla\varphi \quad \forall\varphi \in H_0^1$. Recall also that

we have assumed in (8) that $r > \frac{N}{2}$ which implies that $\beta > N$. By using again the Hölder inequality we obtain for $\varphi \in H_0^1$:

$$\int_\Omega \tilde{E}\nabla\varphi \leq \|\tilde{E}\|_{(L^\beta)^N}\|\varphi\|_{H_0^1}|\text{Supp }\varphi|^{\frac{\beta-2}{2\beta}}.$$

Consequently (58) leads to:

$$\begin{aligned}\kappa_0\|\psi_s\|_{H_0^1}^2 &\leq \int_\Omega \tilde{E}\nabla\psi_s \leq C_0|A_{s,l}|^{\frac{\beta-2}{2\beta}}\|\psi_s\|_{H_0^1} \\ &\overset{\text{Young ineq.}}{\leq} \frac{\kappa_0}{2}\|\psi_s\|_{H_0^1}^2 + \frac{C_0^2}{2\kappa_0}|A_{s,l}|^{\frac{\beta-2}{\beta}}.\end{aligned}$$

Let $\Phi := \frac{\beta-2}{\beta} > \frac{N-2}{N}$. We have obtained the estimate:

$$\|\psi_s\|_{H_0^1}^2 \leq \tilde{C}_0|A_{s,l}|^\Phi, \quad \tilde{C}_0 = \frac{C_0^2}{4\kappa_0^2}.$$

By using now the Stampacchia estimates (see subsection 3.2) we obtain the existence of a real Λ independent of l such that $|A_{\Lambda,l}| = 0$. Hence: $\|\tilde{v}_l\|_{L^\infty(\Omega)} \leq \Lambda, \quad \Lambda = C(\|\zeta_0\|_{L^\infty(\partial\Omega)}, |\Omega|, N, r)\frac{C_0}{\kappa_0}$

□

Proof of Proposition 9

i) Existence and estimates.
Let $(v_l)_{l\geq 1}$ be the sequence given in lemma 10. Let also $l \geq 1$ be given and $\tau_l := \frac{1}{h(e^l)}$. We assume that $\|\gamma_2\|_{L^r} \leq \tau_l$. It follows from lemma 10.ii) that $\|v_l\|_{L^\infty} \leq K$, where $K = K(\kappa_0, DATA)$ (K independent on l) is the integer defined by $K = [\Lambda] + 1$.
Let now $\tau := \tau_K$ and assume that $\|\gamma_2\|_{L^r} \leq \tau$. Then we have: $\|v_K\|_{L^\infty} \leq K$. Hence $T_K(v_K) = v_K$. On the other hand v_K satisfies (S_K), that is:

$$\begin{cases} \rho\mathbf{u}\nabla e^{v_K} - \text{div}((\nu e^{v_K} + \kappa)\nabla v_K) = g(x, e^{v_K}) & \text{in } \Omega \\ v_K = \ln\zeta_0 \quad \text{on } \partial\Omega \end{cases}$$

Let $\zeta := e^{v_K}$. We have (see Theorem 13 in the Appendix) $\zeta \in H^1(\Omega) \cap L^\infty(\Omega)$ and $\nabla\zeta = \zeta\nabla v_K$. Consequently ζ is a solution of problem (S). Moreover $\|\zeta\|_{L^\infty(\Omega)} \leq e^K = C(\kappa_0, DATA)$.
On the other hand $v_K \geq -K$ almost everywhere, implies that $\zeta \geq e^{-K}$ a.e. in Ω. Hence we obtain (48) by setting $\zeta_{\min} = e^{-K}$ and $\zeta_{\max} = e^K$.
• The estimation (49) can be obtained as follows.
By using the test function ψ_s^+ instead of ψ_s we obtain:

$$\zeta_{\max} \leq \Lambda_1 := e^{C\frac{\|\gamma_1\|_{L^r}}{\kappa_0}}. \tag{59}$$

This last estimation is only a first step in order to obtain the majoration for $\zeta_{\max}$ announced in (49).
In fact, let $\tilde{\kappa} := \nu + \frac{\kappa}{\zeta}$. We have $0 < \nu + \frac{\kappa_0}{\Lambda_1} =: \tilde{\kappa}_0 \leq \tilde{\kappa} \leq \nu + \frac{\kappa_0}{\zeta_{\min}} < \infty$, and

$$\begin{cases} \rho\mathbf{u}\cdot\nabla\zeta - \text{div}(\tilde{\kappa}\nabla\zeta) = g^- + g^+ & \text{in } \Omega \\ \zeta = \zeta_0 \quad \text{on } \partial\Omega \end{cases}$$

we can then consider the decomposition $\zeta = \zeta_1 + \zeta_2$, where ζ_1 (resp. ζ_2) $\in H^1(\Omega) \cap L^\infty(\Omega)$ satisfy the following problem (S_1) (resp. (S_2)) :

$$(S_1) \quad \begin{cases} \rho \mathbf{u} \cdot \nabla \zeta_1 - \operatorname{div}(\tilde{\kappa} \nabla \zeta_1) = g^- & \text{in } \Omega \\ \zeta_1 = \zeta_0 \quad \text{on } \partial\Omega \end{cases}$$

$$(S_2) \quad \begin{cases} \rho \mathbf{u} \cdot \nabla \zeta_2 - \operatorname{div}(\tilde{\kappa} \nabla \zeta_2) = g^+ & \text{in } \Omega \\ \zeta_2 = 0 \quad \text{on } \partial\Omega \end{cases}$$

Note that the second member in the PDE in (S_1) is negative. Hence, by the maximum principle (see [26] p.80 or [2] p.191 for a simplified situation) we obtain: $\zeta_1 \leq \|\zeta_0\|_{L^\infty(\partial\Omega)}$ a.e. in Ω.
By using next the technique of Stampacchia (see again subsection 3.2, remark 6) we can major the function ζ_2 as follows:

$$\zeta_2 \leq C' \frac{\|g^+\|_{L^r}}{\tilde{\kappa}_0} \underset{\text{by (59)}}{\leq} e^{C \frac{\|\gamma_1\|_{L^r}}{\kappa_0}} \frac{\|g^+\|_{L^r}}{k_0}, \quad C = C(DATA).$$

This leads to the majoration (49).

ii) Regularity results.
• If we assume that $\mathbf{u} \in (L^\infty(\Omega))^N$ then $\rho \mathbf{u} \in (L^\infty(\Omega))^N$. Moreover by using the estimates (48) we see that the diffusion coefficient $\nu + \frac{\kappa}{\zeta}$ is bounded from above and below and $g(x, \zeta) \in L^r(\Omega)$ with $r > 3/2$ fixed. Hence (see lemma 5): $g(x, \zeta) \in W^{-1,\beta}(\Omega)$ with $\beta > N$. Consequently, by using the De Giorgi-Nash Theorem (see [12] Th. 8.22) we obtain: $\zeta \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$, for some $\alpha > 0$.
• Assume that in addition we have:

$$\partial\Omega, \zeta_0 \text{ are of class } \mathcal{C}^{2,\alpha}, \quad g \in \mathcal{C}^{0,\alpha}(\overline{\Omega} \times \mathbb{R}^+),$$
$$\rho \mathbf{u} \in (\mathcal{C}^{1,\alpha}(\overline{\Omega}))^N, \quad \kappa \in \mathcal{C}^{1,\alpha}(\overline{\Omega}).$$

We have proved in the previous point that $\zeta \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$. We can now iterate the Schauder estimates as follows. In a first step we see that $g(x, \zeta)$ and $\nu + \frac{\kappa}{\zeta}$ are in $\mathcal{C}^{0,\alpha}(\overline{\Omega})$, and by applying the Theorem 2.7 in [4] we obtain that $\zeta \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$. Consequently we now obtain (see Appendix B in [8]) $g(x, \zeta), \nu + \frac{\kappa}{\zeta} \in \mathcal{C}^{1,\alpha}(\overline{\Omega})$ and by using the Theorem 2.8 in [4] we finally obtain $\zeta \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$. Hence ζ is a classical solution of (S).

□

4.2 Proof of Proposition 7

Let $n \in \mathbb{N}$, and θ_n, φ_n be given. We assume that the condition (K_n) is satified. Recall that this implies in particular: $\varphi_n \leq \varphi_{\max}$, where $\varphi_{\max} > \|b\|_{L^\infty(\partial\Omega)}$ was fixed in (42). Hence, let $\varepsilon := \varphi_{\max} - \|b\|_{L^\infty(\partial\Omega)} > 0$.
- Step 1: We introduce

$$\kappa^{(n)} := \frac{C_\theta^{(n)}}{\varphi_n}, \quad g_\theta(x, u) := C_5 - C_3 F u^2 + C_4 D u.$$

Hence the first subproblem in (Q_n) reads as:

$$(Q_n.1) \quad \begin{cases} \rho \mathbf{u} \cdot \nabla \theta_{n+1} - \operatorname{div}\Big((\nu + \frac{\kappa^{(n)}}{\theta_{n+1}})\nabla \theta_{n+1}\Big) = g_\theta(x, \theta_{n+1}) & \text{in } \Omega \\ \theta_{n+1} = a \quad \text{on } \partial\Omega \end{cases}$$

Moreover, it is easy to verify that:

$$0 < \underbrace{\frac{\alpha_\theta}{\varphi_{\max}}}_{=:\kappa_0} \leq \kappa^{(n)} \leq \|\frac{C_\theta^{(n)}}{\varphi_n}\|_{L^\infty} < \infty$$
$$g_\theta^+(x,u) = C_5 + C_4 D^+ u, \quad g_\theta^-(x,u) = -C_3 F u^2 + C_4 D^- u.$$

Note also that κ_0 is independent of n. Hence we can apply the proposition 9.i) (take $\zeta = \theta_{n+1}, \kappa = \kappa^{(n)}, g = g_\theta, \gamma_1 = C_5, \gamma_2 = C_4 D^+, h(t) = t, \gamma = C_3 F + C_4 |D^-|, \zeta_0 = a$). We obtain the existence of $\tau_0 > 0$ independent of n such that if $\|D^+\|_{L^r} \leq \tau_0$ then the problem $(Q_n.1)$ admits at least one weak solution $\theta_{n+1} \in H^1(\Omega) \cap L^\infty(\Omega)$. Moreover we have

$$0 < e^{-C\varphi_{\max}} \leq \theta_{n+1} \leq e^{C\varphi_{\max}} < \infty, \quad C = C(DATA). \tag{60}$$

- Step 2: Let now

$$\tilde{\kappa}^{(n)} := \frac{C_\varphi^{(n)}}{\theta_{n+1}}, \quad g_\varphi^{(n)}(x,u) = g_\varphi^{(n)}(x) = -\varphi_n(C_8 \theta_{n+1}^{-1} + C_6 \theta_{n+1} F - C_7 D).$$

With these notations, the second subproblem in (Q_n) reads as:

$$(Q_n.2) \quad \begin{cases} \rho \mathbf{u} \cdot \nabla \varphi_{n+1} - \operatorname{div}\Big((\nu + \frac{\tilde{\kappa}^{(n)}}{\varphi_{n+1}})\nabla \varphi_{n+1}\Big) = g_\varphi^{(n)} & \text{in } \Omega \\ \varphi_{n+1} = b \quad \text{on } \partial\Omega \end{cases}$$

We can verify that:

$$0 < \underbrace{\frac{\alpha_\varphi}{\theta_{\max}}}_{=:\tilde{\kappa}_0} \leq \tilde{\kappa}^{(n)} \leq \frac{\|C_\varphi^{(n)}\|_{L^\infty}}{\theta_{\min}} < \infty, \quad g_\varphi^+ = \varphi_n C_7 D^+ \leq \varphi_{\max} C_7 D^+,$$

$$g_\varphi^- = -\varphi_n(C_8 \theta_{n+1}^{-1} + C_6 \theta_{n+1} F - C_7 D^-), \quad |g_\varphi^-| \overset{by(61)}{\leq} \tilde{C}(DATA, \varphi_{\max}).$$

Hence we can apply the proposition 9.i) (take now $\zeta = \varphi_{n+1}, \kappa = \tilde{\kappa}^{(n)}, g = g_\varphi^{(n)}, \gamma_1 = \varphi_{\max} C_7 D^+, h = 0, \gamma = \tilde{C}, \zeta_0 = b$). Then we obtain the existence of a weak solution[4] $\varphi_{n+1} \in H^1(\Omega) \cap L^\infty(\Omega)$ for problem $(Q_n.2)$. Moreover we have

$$0 < e^{-C' e^{C\varphi_{\max}}} \leq \varphi_{n+1} \leq e^{C' e^{C\varphi_{\max}}} < \infty, \tag{61}$$

where C,C' depend on DATA but not on n. Moreover, by using (49) we have:

$$\begin{aligned} \|\varphi_{n+1}\|_{L^\infty(\Omega)} &\leq \|b\|_{L^\infty(\partial\Omega)} + C'' \|g_\varphi^+\|_{L^r} e^{C\varphi_{\max}} \|\varphi_{n+1}\|_{L^\infty(\Omega)}, \quad C'' = C''(DATA), \\ &\leq \|b\|_{L^\infty(\partial\Omega)} + \underbrace{C'' \|C_7\|_{L^\infty} e^{C\varphi_{\max}}}_{:=K(\varphi_{\max}, DATA)} \|D^+\|_{L^r} \|\varphi_{n+1}\|_{L^\infty(\Omega)}. \end{aligned} \tag{62}$$

[4]at this stage there is anymore additional condition needed for $\|D^+\|_{L^r}$ because $h = 0$

Assume now that

$$\|D^+\|_{L^r} \leq \tau := \min(\tau_0, \frac{\varepsilon}{\varphi_{\max} K}).$$

Then (62) leads to

$$\|\varphi_{n+1}\|_{L^\infty(\Omega)} \leq \|b\|_{L^\infty(\partial\Omega)} + \frac{\varepsilon}{\varphi_{\max}} \|\varphi_{n+1}\|_{L^\infty(\Omega)},$$

and it follows:

$$\|\varphi_{n+1}\|_{L^\infty(\Omega)} \leq \frac{\|b\|_{L^\infty(\partial\Omega)}}{1 - \frac{\varepsilon}{\varphi_{\max}}} \leq \varphi_{\max}.$$

- Final Step: if we assume that $\|D^+\|_{L^r} \leq \tau$ then by using the results established in the previous two steps, we can conclude that there exists a solution $(\theta_{n+1}, \varphi_{n+1}) \in (H^1 \cap L^\infty)^2$ for problem (Q_n). Moreover this solution satisfies (K_{n+1}) and (43),(44) hold □

5 Proofs of the theorems

We begin by a lemma:

Lemma 11 *Under the assumptions of proposition 7, we can extract a subsequence (still denoted by (θ_n, φ_n)) such that*

$$\theta_n \overset{*}{\rightharpoonup} \theta, \varphi_n \overset{*}{\rightharpoonup} \varphi \ \ \text{in } L^\infty(\Omega), \qquad \theta_n \rightharpoonup \theta, \varphi_n \rightharpoonup \varphi \ \text{in } H^1(\Omega), \tag{63}$$

$$\frac{1}{\theta_n \varphi_n} \to \frac{1}{\theta\varphi}, C_\theta^{(n)} \to C_\theta(x,\theta,\varphi), C_\varphi^{(n)} \to C_\varphi(x,\theta,\varphi) \ \text{in } L^p(\Omega), \ p < \infty \tag{64}$$

Proof

The first properties in (63) follow directly from proposition 7. By using next $\theta_n - \theta_0$ as test function in $(Q_n.1)$ and $\varphi_n - \varphi_0$ as test function in $(Q_n.2)$ we obtain a uniform bound for (θ_n) and (φ_n) in the H^1-norm. Hence the second properties in (63) follow. Finally the property (64) can be obtained by using the dominated convergence theorem. In fact we have:

$$\frac{1}{\theta_n\varphi_n} \to \frac{1}{\theta\varphi}, \ C_\theta^{(n)} \to C_\theta(x,\theta,\varphi), \ C_\varphi^{(n)} \to C_\varphi(x,\theta,\varphi) \quad \text{a.e in } \Omega,$$

$$|\frac{1}{\theta_n\varphi_n}| \leq \frac{1}{\theta_{\min}\varphi_{\min}}, \ |C_\theta^{(n)}| \leq \sup_{(v,w)\in\mathcal{K}} C_\theta(x,v,w), \ |C_\varphi^{(n)}| \leq \sup_{(v,w)\in\mathcal{K}} C_\varphi(x,v,w),$$

where $\mathcal{K} = [0, \theta_{\max}] \times [0, \varphi_{\max}]$. □

5.1 Proof of the theorem 1

By using (63) together with (64) we obtain:

$$\begin{aligned}
\rho\mathbf{u}.\nabla\theta_{n+1} &\rightharpoonup \rho\mathbf{u}.\nabla\theta \ \text{ in } L^1(\Omega) \\
(\nu + \frac{C_\theta^{(n)}}{\theta_{n+1}\varphi_n})\nabla\theta_{n+1} &\rightharpoonup (\nu + \frac{C_\theta}{\theta\varphi})\nabla\theta \ \text{ in } L^q(\Omega), \ q < 2 \\
\rho\mathbf{u}.\nabla\varphi_{n+1} &\rightharpoonup \rho\mathbf{u}.\nabla\varphi \ \text{ in } L^1(\Omega) \\
(\nu + \frac{C_\varphi^{(n)}}{\theta_{n+1}\varphi_{n+1}})\nabla\varphi_{n+1} &\rightharpoonup (\nu + \frac{C_\varphi}{\theta\varphi})\nabla\varphi \ \text{ in } L^q(\Omega), \ q < 2.
\end{aligned}$$

Moreover by using (63) together with the property $\theta_{n+1} \geq \theta_{\min} > 0$ we obtain:

$$\theta_{n+1}^2 \to \theta^2,\ \theta_{n+1} \to \theta,\ \theta_{n+1}^{-1} \to \theta^{-1} \quad \text{in } L^p(\Omega),\ p < \infty.$$

Hence we can pass to the limit in the approximate problems (Q_n). We obtain a weak solution (θ, φ) for problem (Q). That is for all $\psi \in \mathcal{D}(\Omega)$:

$$\int_\Omega \rho \mathbf{u} \cdot \nabla\theta\ \psi + \int_\Omega (\nu + \frac{C_\theta}{\theta\varphi}) \nabla\theta \cdot \nabla\psi = \int_\Omega (C_5 - C_3 F\theta^2 + C_4\theta D)\psi$$
$$\int_\Omega \rho \mathbf{u} \cdot \nabla\varphi\ \psi + \int_\Omega (\nu + \frac{C_\varphi}{\theta\varphi}) \nabla\varphi \cdot \nabla\psi = \int_\Omega -\varphi(C_8\theta^{-1} + C_6\theta F - C_7 D)\psi$$
$$\theta = a, \varphi = b \quad \text{on } \partial\Omega$$

Moreover this solution verifies:

$$\theta, \varphi \in H^1(\Omega) \cap L^\infty(\Omega), \quad \theta, \varphi \geq \min(\theta_{\min}, \varphi_{\min}) > 0 \ \text{ a.e. in } \Omega.$$

5.2 Proof of the theorem 2

When $N = 2$ the function F has a stronger property of positivity (see lemma 12 in the Appendix):

$$F \geq \frac{D^2}{3}. \tag{65}$$

We will see that this last property allows to obtain a weak solution for problem (Q) under the assumptions (A_0) and (A_1) but without have to assume a low compressibility condition of the form (20).
In order to prove this result, we take over the proof of proposition 7 with slight modifications: if (θ_n, φ_n) is given and satisfies (H_n) (it is not useful to consider (K_n) here) then the problem (Q_n) has at least one solution $(\theta_{n+1}, \varphi_{n+1})$ satisfying in addition (H_{n+1}) and the estimates (43),(44).
- Step 1: By using the property (65) we can majore g_θ^+ as follows:

$$g_\theta^+(x,u) = C_5 + C_4 D^+ u - C_3 F u^2 \leq C_5 + C_4 D^+ u - \frac{C_3}{3}(D^+u)^2$$
$$\leq C_5 + \underbrace{D^+u}_{\geq 0} \underbrace{(C_4 - \frac{C_3}{3}(D^+u)}_{\leq 0 \text{ if } D^+u \geq \frac{3C_4}{C_3}} \leq C_5 + \frac{3C_4^2}{C_3}.$$

Hence we have here estimated g_θ^+ independtly to the second variable. We can then apply the proposition 9.i), but now we take $\gamma_1 = C_5 + \frac{3C_4^2}{C_3}$ instead of C_5 and $h(t) = 0$ instead of $h(t) = t$. It follow that there exists (without any condition on $\|D^+\|_{L^r}$ because $h = 0$) a weak solution $\theta_{n+1} \in H^1(\Omega) \cap L^\infty(\Omega)$ for problem $(Q_n.1)$, with the estimate:

$$0 < e^{-C\|\varphi_n\|_{L^\infty}} \leq \theta_{n+1} \leq e^{C\|\varphi_n\|_{L^\infty}} < \infty, \quad C = C(DATA). \tag{66}$$

- Step 2: By taking over the arguments presented in the proof of the proposition 7 we see that problem $(Q_n.2)$ has at least one positive solution $\varphi_{n+1} \in H^1(\Omega) \cap L^\infty(\Omega)$.
Hence at this point we have yet obtained a weak solution $(\theta_{n+1}, \varphi_{n+1})$ for

(Q_n) satisfying in addition (H_{n+1}). It remains to prove that the estimates (43),(44) hold. We have make a first step in this direction by proving (66). We will now prove:

$$0 < e^{-C\|\theta_{n+1}\|_{L^\infty}} \leq \varphi_{n+1} \leq \|b\|_{L^\infty(\partial\Omega)}, \quad C = C(DATA). \tag{67}$$

In fact, by using the additional assumption (A_1) we can majore the function $g_\varphi^{(n)}$ as follows:

$$g_\varphi^{(n)}(x) = -\varphi_n(C_8\theta_{n+1}^{-1} + C_6\theta_{n+1}F - C_7D) \overset{\text{by (65)}}{\leq} -\varphi_n(C_8\theta_{n+1}^{-1} + \frac{C_6}{3}\theta_{n+1}D^2 - C_7D)$$
$$\leq -\frac{\rho\varphi_n}{\theta_{n+1}}(c_8 + \frac{c_6}{3}(\theta_{n+1}D)^2 - c_7(\theta_{n+1}D)) = -\frac{\rho\varphi_n}{\theta_{n+1}}\mathcal{P}(\theta_{n+1}D),$$

with $\mathcal{P}(X) := \frac{c_6}{3}X^2 - c_7X + c_8$. We remark that the discriminant Δ of $\mathcal{P}$ is negative: $\Delta = c_7^2 - \frac{4}{3}c_6c_8 = -4.864 * 10^{-3} < 0$. It follows that $\mathcal{P}$ is positive and consequently $g_\varphi^{(n)}$ is negative. Hence by applying the proposition 9.i) with now $g^+ \equiv 0$ we obtain (67).
- *F*inal Step: By using (66) together with (67) we obtain the estimates (43) and (44). Hence we have recovered the conclusions of the proposition 7. The rest of the proof for theorem 3 is exactly the same as for theorem 1: we can extract a subsequence with the properties (63)-(63). These properties are sufficients to pass to the limite $n \to \infty$ in (Q_n) and theorem 3 follows.

5.3 Proof of the theorem 3

Let (θ, φ) be a weak solution for (Q) in the class $\mathcal{S}$ and consider the notations:

$$g_\theta(x,u) := C_5 - C_3Fu^2 + C_4Du, \quad g_\varphi(x,u) := -u(C_8\theta^{-1} + C_6\theta F - C_7D).$$

i) It suffices to remark that the coefficients $\frac{C_\theta}{\varphi}$ and $\frac{C_\varphi}{\theta}$ are bounded from above and below, and g_θ, g_φ are Caratheodory functions satisfying (46) and (47). Hence we can apply the first point in proposition 9-ii) in each equation of (Q). We obtain: $\theta, \varphi \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$, for some $\alpha > 0$.

ii) Assume that in addition we have:

$$\partial\Omega, a, b \text{ are of class } \mathcal{C}^{2,\alpha}, \quad F \in \mathcal{C}^{0,\alpha}(\overline{\Omega}),$$
$$\rho\mathbf{u} \in (\mathcal{C}^{1,\alpha}(\overline{\Omega}))^N, \quad C_\theta, C_\varphi \in \mathcal{C}^{1,\alpha}(\overline{\Omega} \times (\mathbb{R}^+)^2).$$

We remark now that the conditions in the second part of proposition 9-ii) are satisfied for each equation of (Q). Hence $\theta, \varphi \in \mathcal{C}^{2,\alpha}(\overline{\Omega})$ and it is a classical solution of (Q).

6 Appendix

6.1 Derivation of the $\theta - \varphi$ model.

The model is constructed from the $k - \varepsilon$ one which takes the form:

$$\partial_t k + \mathbf{u} \cdot \nabla k - \frac{c_\nu}{\rho}\text{div } ((\nu + \rho\frac{k^2}{\varepsilon})\nabla k) = c_\nu\frac{k^2}{\varepsilon}F - \frac{2}{3}kD - \varepsilon, \tag{68}$$

$$\partial_t \varepsilon + \mathbf{u} \cdot \nabla \varepsilon - \frac{c_\varepsilon}{\rho}\text{div } ((\nu + \rho\frac{k^2}{\varepsilon})\nabla \varepsilon) = c_1kF - \frac{2c_1}{3c_\nu}\varepsilon D - c_2\frac{\varepsilon^2}{k}, \tag{69}$$

where $D(x,t) := \operatorname{div} \mathbf{u}(x,t)$, $F(x,t) := \frac{1}{2}|\nabla \mathbf{u} + (\nabla \mathbf{u})^T|^2 - \frac{2}{3}D(x,t)^2 \geq 0$ (see the next subsection) and $c_\nu, c_\varepsilon, c_1, c_2$ are generally taken as positive constants. Their usual values are (see [23] p.122):

$$c_\nu = 0.09,\ c_\varepsilon = 0.07,\ c_1 = 0.128,\ c_2 = 1.92. \tag{70}$$

We then consider the new variables

$$\theta = \frac{k}{\varepsilon}, \quad \varphi = k^\alpha \varepsilon^\beta, \tag{71}$$

with α and β to be choosen appriopriatly. Let D_t denote the total derivative operator. By using (68) together with (69) we obtain an equation for θ:

$$\begin{aligned} D_t\theta = \frac{\partial \theta}{\partial t} + \mathbf{u}\cdot\nabla\theta = \frac{1}{\varepsilon}D_t k - \frac{k}{\varepsilon^2}D_t\varepsilon = -c_3\theta^2 F + c_4\theta D + c_5 + \mathrm{Diff}_\theta, \\ c_3 = c_1 - c_\nu,\ c_4 = \frac{2}{3}(\frac{c_1}{c_\nu} - 1),\ c_5 = c_2 - 1, \end{aligned} \tag{72}$$

where Diff_θ denotes the collected terms coming from the viscous one in the k and ε equations. The equation for φ is obtained on the same way:

$$\begin{aligned} D_t\varphi &= \alpha k^{\alpha-1}\varepsilon^\beta D_t k + \beta k^\alpha \varepsilon^{\beta-1} D_t\varepsilon = \alpha k^{\alpha-1}\varepsilon^\beta(c_\nu\frac{k^2}{\varepsilon}F - \frac{2}{3}kD - \varepsilon) + \mathrm{Diff1}_\varphi \\ &+ \beta k^\alpha\varepsilon^{\beta-1}(c_1 kF - \frac{2c_1}{3c_\nu}\varepsilon D - c_2\frac{\varepsilon^2}{k}) + \mathrm{Diff2}_\varphi = Fk^{\alpha+1}\varepsilon^{\beta-1}(\alpha c_\nu + \beta c_1) \\ &- k^\alpha\varepsilon^\beta D\frac{2}{3}(\alpha + \beta\frac{c_1}{c_\nu}) - k^{\alpha-1}\varepsilon^{\beta+1}(\alpha + \beta c_2) + \mathrm{Diff}_\varphi, \\ &= -\varphi(c_6\theta F - c_7 D + c_8\theta^{-1}) + \mathrm{Diff}_\varphi, \\ c_6 &= -\alpha c_\nu - \beta c_1,\ c_7 = -\frac{2}{3}(\alpha + \beta\frac{c_1}{c_\nu}),\ c_8 = \alpha + \beta c_2, \end{aligned} \tag{73}$$

where $\mathrm{Diff}_\varphi = \mathrm{Diff1}_\varphi + \mathrm{Diff2}_\varphi$ is the sum of the terms coming from the viscous one in the k and ε equations.

The usual constant values for the parameters c_3, c_4, c_5 appearing in the model are obtained by replacing the values (70) in the expressions (72). This leads to:

$$c_3 = 0.038, c_4 = 0.2815, c_5 = 0.92.$$

At this stage it remains to choose appriopriatly α and β in (71), and to model the terms Diff_θ and Diff_φ.
It is schown in [23], page 67, that a good choice in the incompressible situation (i.e. when $D = 0$) is for instance $\alpha = -3, \beta = 2$. This leads to the following constant values:

$$c_6 = 0.014,\ c_7 = 0.104,\ c_8 = 0.84, \tag{74}$$

and this makes the dynamic stable for the equation in φ. I.e. in the abscence of the viscous part Diff_φ, we have: $D_t\varphi \leq 0$.
In the compressible situation the authors suggest in [23], page 125 to consider another choice: $\alpha = -2\frac{c_1}{c_\nu} \approx -2.88$ and $\beta = 2$ which make again the dynamic stable.
Nevertheless in this last situation, the variable φ does not have a clear physical meaning (whereas when $\alpha = -3, \beta = 2$ we have $\varphi = \frac{\varepsilon^2}{k^3}$ and $L := \varphi^{-1/2}$

represents a length scale of turbulence (see [27])). Moreover, a carefully estimation schows (see lemma 12 in the next subsection) that when $N = 2$ we have $F \geq \frac{1}{3}D^2$. This leads to:

$$-\varphi(c_6\theta F - c_7 D + c_8\theta^{-1}) \leq -\frac{\varphi}{\theta}(\frac{c_6}{3}(\theta D)^2 - c_7(\theta D) + c_8) = -\frac{\varphi}{\theta}\mathcal{P}(\theta D),$$

with $\mathcal{P}(X) := \frac{c_6}{3}X^2 - c_7 X + c_8$. Hence the choice $\alpha = -3, \beta = 2$ makes again the dynamic stable when $N = 2$. In fact, in this case c_6, c_7, c_8 take the values (74) and the discriminant Δ of $\mathcal{P}$ is $\Delta = c_7^2 - \frac{4}{3}c_6 c_8 = -4.864 * 10^{-3} < 0$. Consequently $\mathcal{P}(X) \geq 0$ and $D_t\varphi \leq 0$ in the absence of the viscous terms.

In consequence we point out that the choice $\alpha = -3, \beta = 2$ is also interesting in the compressible situation. We shall make this choice in all the situations. Our analysis (see theorem 1) schows that this leads to a well posed model even when $N = 3$ under an additional assumption of low compressibility of the flow.

The terms Diff_θ and Diff_φ are modelled (see [16]) by:

$$\text{Diff}_\theta = \frac{1}{\rho}\,\text{div}((\nu + c_\theta\nu_t)\nabla\theta), \quad \text{Diff}_\varphi = \frac{1}{\rho}\,\text{div}((\nu + c_\varphi\nu_t)\nabla\theta), \tag{75}$$

where $\nu_t := \rho c_\nu \frac{k^2}{\varepsilon} = \rho\frac{c_\nu}{\theta\varphi}$ is the turbulent viscosity coming from the equation of k, and c_θ, c_φ are two new parameters for the model.
The determination of the parameters c_θ and c_φ can be realized on the same spirit than for the determination of the coefficients arising in the $k - \varepsilon$ model (see [23]). In [22] a constant value for both c_θ and c_φ were numerically tuned from a simulation of a Poiseuil flow. However better results are obtained if we allow c_θ, c_φ to be some positive functions (see [24]). In our analysis we allow the coefficients to be on a very general form, in particular c_θ, c_φ may depend on x, θ and φ. We only assume that they are Caratheodory functions and that they satisfy some positivity and boundedness properties (see (17)-(19), where $C_\theta = \rho c_\theta$ and $C_\varphi = \rho c_\varphi$).

6.2 Positivity of the function F

In this paragraph we will establish some properties of positivity for the function F appearing in the models.
Let $\mathcal{M}_N(\mathbb{R})$ denote the vectorial space of the N-square matrix with real coefficients, equipped with the scalar product:

$$A : B = \sum_i \sum_j a_{ij} b_{ij}, \quad \forall A = (a_{ij}), B = (b_{ij}) \in \mathcal{M}_N(\mathbb{R}).$$

Hence $|A|^2 := \sqrt{A : A}$ defines a norm on $\mathcal{M}_N(\mathbb{R})$.

For a vectorial field $\mathbf{u} : \Omega \to \mathbb{R}^N$ we classically defines the gradient $\nabla\mathbf{u}$ and the divergence $\text{div}\,\mathbf{u}$ (=:D) by:

$$\nabla\mathbf{u} : \ \Omega \to \mathcal{M}_N(\mathbb{R}), \quad (\nabla\mathbf{u}(x))_{ij} = \frac{\partial u_i(x)}{\partial x_j},$$

$$D : \ \Omega \to \mathbb{R}, \quad D(x) = \sum_i \frac{\partial u_i(x)}{\partial x_i} = \text{Tr}(\nabla\mathbf{u}).$$

Recall that the function F were defined by the formula:

$$F(x) := \frac{1}{2}|\nabla \mathbf{u} + (\nabla \mathbf{u})^T|^2 - \frac{2}{3}D(x)^2, \tag{76}$$

and by an easy calculation we obtain:

$$F(x) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^t) : \nabla \mathbf{u} - \frac{2}{3}D(x)^2. \tag{77}$$

This last expression is sometimes choosen (for instance in [23]) to equivalently define F.
The important fact is that we always have $F \geq 0$ but moreover, when $N = 2$ the stronger estimate: $F \geq \frac{1}{3}D^2$ holds. These properties are established in the following lemma:

Lemma 12 *The function F satisfies the estimates:*

$$\begin{aligned} F &= \frac{2}{3}\left((\partial_1 u_1 - \partial_2 u_2)^2 + (\partial_1 u_1 - \partial_3 u_3)^2 + (\partial_2 u_2 - \partial_3 u_3)^2\right) \\ &\quad + (\partial_2 u_1 + \partial_1 u_2)^2 + (\partial_3 u_1 + \partial_1 u_3)^2 + (\partial_3 u_2 + \partial_2 u_3)^2 \geq 0, \quad \textit{when } N = 3. \\ F &= (\partial_1 u_1 - \partial_2 u_2)^2 + (\partial_2 u_1 + \partial_1 u_2)^2 + \frac{1}{3}D^2 \geq \frac{1}{3}D^2, \quad \textit{when } N = 2. \end{aligned}$$

Proof
Let $N = 2$ or 3, and $M := \nabla \mathbf{u} + (\nabla \mathbf{u})^t$. Then $M_{ij} = \partial_j u_i + \partial_i u_j$ and we obtain:

$$\begin{aligned} |M|^2 &= \sum_i \left((2\partial_i u_i)^2 + \sum_{j \neq i}(\partial_j u_i + \partial_i u_j)^2\right) = 4\sum_i (\partial_i u_i)^2 + 2\sum_i \sum_{j>i}(\partial_j u_i + \partial_i u_j)^2, \\ F &= \frac{1}{2}|M|^2 - \frac{2}{3}D^2 = \underbrace{2\sum_i (\partial_i u_i)^2 - \frac{2}{3}(\sum_i \partial_i u_i)^2}_{:=A} + \sum_i \sum_{j>i}(\partial_j u_i + \partial_i u_j)^2 \end{aligned}$$

The term A is evaluated separately in the case $N = 2$ and $N = 3$.
We remark that $2(a^2+b^2+c^2) - \frac{2}{3}(a+b+c)^2 = \frac{2}{3}((a-b)^2 + (a-c)^2 + (b-c)^2)$.
Hence, when $N = 3$ we have

$$A = \frac{2}{3}\left((\partial_1 u_1 - \partial_2 u_2)^2 + (\partial_1 u_1 - \partial_3 u_3)^2 + (\partial_2 u_2 - \partial_3 u_3)^2\right),$$

and we obtain the expression announced for F.
On the same way, we remark that

$$\begin{aligned} 2(a^2 + b^2) - \frac{2}{3}(a+b)^2 &= \frac{2}{3}(a-b)^2 + \frac{2}{3}(a^2+b^2) \\ &= \frac{2}{3}(a-b)^2 + \frac{1}{3}(a+b)^2 + \frac{1}{3}(a-b)^2 = (a-b)^2 + \frac{1}{3}(a+b)^2. \end{aligned}$$

Hence, when $N = 2$ we obtain:

$$A = (\partial_1 u_1 - \partial_2 u_2)^2 + \frac{1}{3}\underbrace{(\partial_1 u_1 + \partial_2 u_2)^2}_{=D^2},$$

and the expression for F follows. □

6.3 On the low compressibility assumption

We will schow here that without any assumption of low compressibility of the form (20), the problem (Q) may be very hard to analyze when N=3, and singular solutions or non existence of weak solution may occur.

When the dimension equals two we have seen in theorem 3 that a condition of low compressibility is not necessary. The reason is related to the fact that a stronger property of positivity for F holds in this case, i.e. we have: $F \geq \frac{D^2}{3}$. When the dimension equals 3 such a property does not holds in general.

In fact, let for instance $\Omega = B_{\mathbb{R}^3}(0,1)$ and

$$\rho = (\prod_{i=1}^{3}(x_i+4))^{-1}, \quad \mathbf{u} = (u_1, u_2, u_3)^t, \ u_i = x_i + 4.$$

Then a simple calculation gives: $D = 3$ and $F = 0$. Hence (Q) reads has:

$$\begin{cases} \rho\mathbf{u}\cdot\nabla\theta - \operatorname{div}\Big((\nu + \frac{C_\theta}{\theta\varphi})\nabla\theta\Big) = 3C_4\theta + C_5 \quad \text{in } \Omega \\ \rho\mathbf{u}\cdot\nabla\varphi - \operatorname{div}\Big((\nu + \frac{C_\varphi}{\theta\varphi})\nabla\varphi\Big) = \varphi(3C_7 - C_8\theta^{-1}) \quad \text{in } \Omega \\ \theta = a, \varphi = b \quad \text{on } \partial\Omega \end{cases}$$

In this situation the problem becomes hard to analyse. Assume however that we have obtained a solution (θ, φ) in the class $\mathcal{S}$. Then the equation satisfied by θ is closely related to:

$$(R) \quad \begin{cases} -\operatorname{div}(\eta\nabla\theta) = 3C_4e^\theta + C_5 \quad \text{in } \Omega, \\ \theta = a \quad \text{on } \partial\Omega, \end{cases}$$

with η bounded from above and below. Hence a contradiction can occur because the problem (R) may not have any weak solution (see for instance [3, 19]). Note that in the exemple considered ρ and $\mathbf{u}$ satify all the conditions needed in (A_0), unless $\mathbf{u}\cdot\mathbf{n} = 0$ on $\partial\Omega$ but this is not restrictive for the purpose here. In fact we can consider the domain $\Omega_1 = B_{\mathbb{R}^3}(0,2)$ which contains Ω and we can extend $\rho, \mathbf{u}$ in Ω_1 in such a way that all the conditions in (A_0) are satisfyed. Hence we obtain an exemple within the main situation of the study but the problems evocated remain the same.

6.4 A generalized chain rule

Let $u \in W^{1,p}(\Omega)$ and $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. We recall here some useful properties of the composed function $G(u)$. In particular we schall see that $G(u) \in W^{1,p}(\Omega)$. Moreover in some situations we also have $\frac{\partial}{\partial x_i}G(u) = G'(u)\frac{\partial u}{\partial x_i}$.

The main result we have in mind is the theorem 13 which is due to Stampacchia. In particular we point out that the additional assumption $G(0) = 0$ for the Lipschitz function G is only necessary if Ω is unbounded and $p \neq \infty$, or if we want that $G(u)$ has a vanishing trace on $\partial\Omega$ when u has it (this last situation was in fact the case of interest of Stampacchia).

Theorem 13 *Let G be a Lipschitz real function, $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz domain, and $u \in W^{1,p}(\Omega)$, with $p \in [1;\infty]$. We have:*

i) $G(u) \in W^{1,p}(\Omega)$. *Moreover if* $u \in W_0^{1,p}(\Omega)$ *and* $G(0) = 0$ *then* $G(u) \in W_0^{1,p}(\Omega)$.

ii) If G' has a finite number of discontinuity[5]*. Then the weak derivatives of $G(u)$ are given by the formula:*

$$\frac{\partial}{\partial x_i} G(u) = G'(u) \frac{\partial u}{\partial x_i} \quad a.e. \ in \ \Omega. \tag{78}$$

Proof
see the appendix in [26] for the original proof or [12] theorem 7.8 and [9] theorem 4 for alternative proofs and additional comments.
We also recall that the formula (78) may be interpreted in some critical points. In fact, let $(t_i)_{i=1,..,n}$ denotes the points of discontinuity of G' and $E_i := \{x \in \Omega : \psi(x) = t_i\}$ be the associated level sets for the function ψ. Let $1 \leq i \leq n$ be a fixed integer. If $|E_i| > 0$ then the formula (78) has a priori no sense in this last set which is not negligible. Nevertheless it can be shown (see [26]) that $\frac{\partial \psi}{\partial x_i} = 0$ on such a set. Hence we interpret the right hand side of (78) as zero in the critical set E_i. □

We now establish some technical results used in the proof of lemma 10. Let $v_l \in H^1(\Omega)$ be given and consider the function $h_\pm \in L^{\mathbf{1}}_{loc}(\mathbb{R})$ defined by:

$$h_\pm(y) := e^y (y - T_{s,\pm}(y)). \tag{79}$$

We introduce the functions

$$\begin{aligned} g_\pm(t) &:= \int_0^{T_l(t)} h_\pm(y) dy, \quad t \in \mathbb{R} \\ \zeta_{l,\pm}(x) &:= g_\pm(v_l(x)) \quad \text{a.e. } x \in \Omega. \end{aligned}$$

We have:

Lemma 14 *The function $\zeta_{l,\pm}$ has the properties:*

$$\begin{aligned} &\zeta_{l,\pm} \in H^1(\Omega) \cap L^\infty(\Omega), \\ &\nabla \zeta_{l,\pm} = \nabla e^{T_l(v_l)} \psi_s^\pm. \end{aligned}$$

Proof
A simple majoration gives

$$|\zeta_{l,\pm}(x)| \leq \int_0^l |h_\pm(y)| dy, \quad \text{a.e. in } \Omega,$$

hence $\zeta_{l,\pm} \in L^\infty(\Omega)$.
We remark next that $g_\pm$ is a Lipschitz function and its classical derivative is given by:

$$g'_\pm(t) = h_\pm(T_l(t)) \mathbf{1}_{\{|t| \leq l\}} \quad \forall t \neq l, -l.$$

[5] the derivative G' of G takes here the classical sense

Hence by using Theorem 13 we obtain $\zeta_{l,\pm} \in H^1(\Omega)$ and

$$\begin{aligned}\nabla\zeta_{l,\pm} &= h_\pm(T_l(v_l))\mathbf{1}_{\{|v_l|\le l\}}\nabla v_l = h_\pm(v_l)\mathbf{1}_{\{|v_l|\le l\}}\nabla v_l \\ &= e^{v_l}\nabla v_l\mathbf{1}_{\{|v_l|\le l\}}\psi_s^\pm(v_l) = \nabla e^{T_l(v_l)}\psi_s^\pm.\end{aligned}$$

□

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