Connecting discrete and continuous lookback or hindsight options in exponential Lévy models

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CONNECTING DISCRETE AND CONTINUOUS LOOKBACK OR HINDSIGHT OPTIONS IN EXPONENTIAL LÉVY MODELS

EL HADJ ALY DIA* AND DAMIEN LAMBERTON†

Abstract. Motivated by the pricing of lookback options in exponential Lévy models, we study the difference between the continuous and discrete supremum of Lévy processes. In particular, we extend the results of Broadie et al. (1999) to *jump diffusion* models. We also derive bounds for general exponential Lévy models.

Key words. Exponential Lévy model, Lookback option, Continuity correction, Spitzer identity

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JEL classification. C02, G13

1. Introduction.

The payoff of a lookback option typically depends on the maximum or the minimum of the underlying stock price. The maximum can be evaluated in continuous or discrete time depending on the contract. In the Black-Scholes setting, Broadie, Glasserman and Kou (1999 and 1997) derived a number of results relating discrete and continuous path-dependent options. In particular, they obtained continuity correction formulas for lookback, barrier and hindsight options. The purpose of this paper is to establish similar results for exponential Lévy models. We will focus on lookback or hindsight options, leaving the treatment of barrier options to another paper.

Our results are based on the analysis of the difference between the discrete and continuous maximum of a Lévy process. In the case of a Lévy process with finite activity and a non zero Brownian part, we extend (see Theorem 4.2) the theorem of Asmussen, Glynn and Pitman (1995) which is the key to the continuity correction formulas for lookback options in Broadie, Glasserman and Kou (1999). This allows us to extend these formulas to jump-diffusion models. We also establish estimates for the L_1-norm of the difference of the continuous and discrete maximum of a general Lévy process. These estimates are based on Spitzer's identity, which relates the expectation of the supremum of sums of iid random variables to a weighted sum of the expectations of the positive parts of the partial sums. In the case of Lévy processes with finite activity, we derive an expansion up to te order $o(1/n)$, where n is the number of dates in the discrete supremum, see Theorem 3.5. In the case of infinite activity, we have precise upper bounds (see Theorem 3.9). We also derive an expansion in the case of Lévy processes with finite variation (see Theorem 3.12).

The paper is organized as follows. In the next section, we recall some basic facts about real Lévy processes. In section 3, we state Spitzer's identity for Lévy Processes and use it to analyse the expectation of the difference of the continuous and discrete maximum of a general Lévy process. Section 4 is devoted to the extension of the theorem of Asmussen et al. The last two sections are devoted to financial applications. In Section 5, we derive continuity corrections for lookback options in

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jump-diffusion models, and in Section 6, we give upper bounds for the case of general exponential Lévy models.

2. Preliminaries. A real Lévy process X is characterized by its generating triplet (γ, σ^2, ν), where $(\gamma, \sigma) \in \mathbb{R} \times \mathbb{R}^+$, and ν is a Radon measure on $\mathbb{R} \backslash \{0\}$ satisfying

$$\int_{\mathbb{R}} \left(1 \wedge x^2\right) \nu(dx) < \infty.$$

By the Lévy-Itô decomposition, X can be written in the form

$$X_t = \gamma t + \sigma B_t + X_t^l + \lim_{\epsilon \downarrow 0} \widetilde{X}_t^\epsilon, \tag{2.1}$$

with

$$X_t^l = \int_{|x|>1, s\in[0,t]} x J_X(dx \times ds) \equiv \sum_{0 \leq s \leq t}^{|\Delta X_s| \geq 1} \Delta X_s$$

$$\widetilde{X}_t^\epsilon = \int_{\epsilon \leq |x| \leq 1, s \in [0,t]} x \widetilde{J}_X(dx \times ds) \equiv \sum_{0 \leq s \leq t}^{\epsilon \leq |\Delta X_s| < 1} \Delta X_s - t \int_{\epsilon \leq |x| \leq 1} x \nu(dx).$$

Here J is a Poisson measure on $\mathbb{R} \times [0, \infty)$ with intensity $\nu(dx)dt$, $\widetilde{J}_X(dx \times ds) = J_X(dx \times ds) - \nu(dx)ds$ and B is a standard Brownian motion. We also have the Lévy-Khinchine formula for the characteristic function of X_t. Namely

$$\mathbb{E} e^{iuX_t} = e^{t\varphi(u)}, \quad u \in \mathbb{R},$$

where φ is given by

$$\varphi(u) = i\gamma u - \frac{\sigma^2 u^2}{2} + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbb{1}_{|x| \leq 1}) \nu(dx). \tag{2.2}$$

We say that X has finite activity if the Lévy measure ν is finite ($\nu(\mathbb{R}) < \infty$). We then have

$$X_t = \gamma_0 t + \sigma B_t + \sum_{i=1}^{N_t} Y_i, \tag{2.3}$$

where N is a Poisson process with rate $\lambda = \nu(\mathbb{R})$, $(Y_i)_{i \geq 1}$ are i.i.d. random variables with common distribution $\frac{\nu(dx)}{\nu(\mathbb{R})}$ and

$$\gamma_0 = \gamma - \int_{|x| \leq 1} x \nu(dx). \tag{2.4}$$

This is a *jump-diffusion* process. If the jump part of X has finite variation (which is equivalent to $\int_{|x| \leq 1} |x| \nu(dx) < \infty$), then

$$X_t = \gamma_0 t + \sigma B_t + \int_{x \in \mathbb{R}, s \in [0,t]} x J_X(dx \times ds), \tag{2.5}$$

with γ_0 given by (2.4). Note that X is a finite variation Lévy process if and only if $\sigma = 0$ and $\int_{|x| \leq 1} |x| \nu(dx) < \infty$. Moreover, X is integrable if only if $\int_{|x|>1} |x| \nu(dx) < \infty$.

3. Spitzer's identity and applications. In this section we will first state Spitzer's identity for Lévy processes (we refer to [1], Proposition 4.5, p. 177 for the classical form of Spitzer's identity). Then we will use this result to derive expansions for the error between the continuous and discrete supremum of Lévy processes.

DEFINITION 3.1. *We define*

$$M_t^X = \sup_{0\le s\le t} X_s, \quad M_t^{X,n} = \max_{0\le k\le n} X_{\frac{kt}{n}}.$$

When there is no ambiguity we can remove the super index X.

REMARK 3.2. Note that M_t is integrable for all $t>0$ if and only if $\int_{x>1} x\nu(dx)$ is finite. We also have, for all $\alpha>0$, $\mathbb{E}e^{\alpha M_t}<\infty$ if only if $\int_{x>1} e^{\alpha x}\nu(dx)$ is finite.

In the setting of Lévy processes, we have the following version of Spitzer's identity.

PROPOSITION 3.3. *If X is a Lévy process with generating triplet (γ,σ^2,ν) satisfying $\int_{x>1} x\nu(dx)<\infty$, then*

$$\mathbb{E}M_t^n = \sum_{k=1}^{n} \frac{\mathbb{E}X^+_{k\frac{t}{n}}}{k}, \quad \mathbb{E}M_t = \int_0^t \frac{\mathbb{E}X_s^+}{s} ds.$$

For the proof of the above result, we need some estimates for $\mathbb{E}M_t$ with respect to t.

PROPOSITION 3.4. *Let X a Lévy process with generating triplet (γ,σ^2,ν) satisfying $\int_{x>1} x\nu(dx)<\infty$, then*

$$\mathbb{E}M_t \le \left(\gamma^+ + \int_{x>1} x\nu(dx)\right) t + \left(\sigma\sqrt{\frac{2}{\pi}} + 2\sqrt{\int_{|x|\le 1} x^2\nu(dx)}\right)\sqrt{t}.$$

If in addition $\int_{|x|\le1}|x|\nu(dx)<\infty$, then

$$\mathbb{E}M_t \le \left(\gamma_0^+ + \int_{\mathbb{R}^+} x\nu(dx)\right) t + \sigma\sqrt{\frac{2}{\pi}}\sqrt{t}.$$

Proof of proposition 3.4. We will first prove the second result of the proposition. We have (see (2.5))

$$\begin{aligned}\sup_{0\le s\le t} X_s &= \sup_{0\le s\le t}\left(\gamma_0 s + \sigma B_s + \int_{x\in\mathbb{R},\tau\in[0,s]} xJ_X(dx\times d\tau)\right)\\ &\le \gamma_0^+ t + \sigma \sup_{0\le s\le t} B_s + \int_{x\in\mathbb{R}^+,\tau\in[0,t]} xJ_X(dx\times d\tau).\end{aligned}$$

So

$$\mathbb{E}\sup_{0\le s\le t} X_s \le \gamma_0^+ t + \sigma\mathbb{E}\sup_{0\le s\le t} B_s + t\int_{\mathbb{R}^+} x\nu(dx).$$

By the reflexion theorem, we know that $\sup_{0\le s\le t} B_s$ has the same distribution as $|B_t|$. Therefore

$$\mathbb{E}\sup_{0\le s\le t} B_s = \mathbb{E}|B_t| = \sqrt{\frac{2}{\pi}}\sqrt{t}.$$

Hence

$$\mathbb{E}\left(\sup_{0\le s\le t} X_s\right) \le \left(\gamma_0^+ + \int_{\mathbb{R}^+} x\nu(dx)\right) t + \sigma\sqrt{\frac{2}{\pi}}\sqrt{t}.$$

Consider now the general case. We define the process $(R_t)_{t\ge 0}$ by

$$R_t = \lim_{\epsilon\downarrow 0} \widetilde{X}_t^\epsilon = \lim_{\epsilon\downarrow 0}\int_{\epsilon\le|x|\le 1, s\in[0,t]} x\widetilde{J}_X(dx\times ds).$$

We have, using (2.1),

$$\mathbb{E}\left(\sup_{0\le s\le t} X_s\right) \le \mathbb{E}\sup_{0\le s\le t}\left(\gamma s + \sigma B_s + X_s^l\right) + \mathbb{E}\sup_{0\le s\le t}\left(R_s\right).$$

The process $\left(\gamma s + \sigma B_s + X_s^l\right)_{t\ge 0}$ has finite activity and the support of its Lévy measure does not intersect $[-1,1]$, so

$$\mathbb{E}\sup_{0\le s\le t}\left(\gamma s + \sigma B_s + X_s^l\right) \le \left(\gamma^+ + \int_{x>1} x\nu(dx)\right) t + \sigma\sqrt{\frac{2}{\pi}}\sqrt{t}.$$

Besides, using the Cauchy-Schwarz and Doob inequalities (note that R is a martingale) we get

$$\mathbb{E}\sup_{0\le s\le t}\left(R_s\right) \le 2\sqrt{t\int_{|x|\le 1} x^2\nu(dx)}.$$

Hence

$$\mathbb{E}\left(\sup_{0\le s\le t} X_s\right) \le \left(\gamma^+ + \int_{x>1} x\nu(dx)\right) t + \left(\sigma\sqrt{\frac{2}{\pi}} + 2\sqrt{\int_{|x|\le 1} x^2\nu(dx)}\right)\sqrt{t}.$$

$\diamond$

Proof of proposition 3.3. By Proposition 3.4 we have

$$\exists c_1, c_2 > 0, \quad \forall t\ge 0, \quad \mathbb{E}\sup_{0\le s\le t} X_s \le c_1 t + c_2\sqrt{t}. \tag{3.1}$$

Thus

$$\begin{aligned}\frac{\mathbb{E}X_s^+}{s} &\le \frac{\mathbb{E}\sup_{0\le\tau\le s} X_\tau}{s}\\ &\le c_1 + \frac{c_2}{\sqrt{s}}.\end{aligned}$$

Since $s\to\frac{1}{\sqrt{s}}$ is integrable on $[0,t]$, so is $s\to\frac{\mathbb{E}X_s^+}{s}$. For $s\in(0,t]$, define

$$\begin{aligned}f(s) &= \frac{\mathbb{E}X_s^+}{s}\\ f_n(s) &= \sum_{k=1}^n \mathbb{1}_{\left(\frac{(k-1)t}{n},\frac{kt}{n}\right]}(s) f\left(\frac{kt}{n}\right),\end{aligned}$$

so that

$$\sum_{k=1}^{n} \frac{\mathbb{E}X^+_{k\frac{t}{n}}}{k} = \frac{t}{n}\sum_{k=1}^{n} f\left(\frac{kt}{n}\right) = \int_0^t f_n(s)ds.$$

We can prove that f is continuous on $(0,t]$. We deduce that $\lim_{n\to+\infty} f_n = f$ *a.e.* We also have for any $s \in (0,t]$

$$\begin{aligned}
|f_n(s)| &\leq \sum_{k=1}^{n} \mathbb{1}_{\left(\frac{(k-1)t}{n},\frac{kt}{n}\right]}(s)\left|f\left(\frac{kt}{n}\right)\right| \\
&\leq \sum_{k=1}^{n} \mathbb{1}_{\left(\frac{(k-1)t}{n},\frac{kt}{n}\right]}(s)\left(c_1 + \frac{c_2}{\sqrt{\frac{kt}{n}}}\right) \\
&\leq c_1 + \frac{c_2}{\sqrt{s}}.
\end{aligned}$$

So, by dominated convergence, we have $\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{\mathbb{E}X^+_{k\frac{t}{n}}}{k} = \int_0^t \frac{\mathbb{E}X^+_s}{s}ds$. On the other hand

$$\begin{aligned}
\max_{k=0,\ldots,n} X_{k\frac{t}{n}} &= \max\left(0, X_{\frac{t}{n}}, X_{2\frac{t}{n}}, \ldots, X_t\right) \\
&= \max\left(X^+_{\frac{t}{n}}, X^+_{2\frac{t}{n}}, \ldots, X^+_t\right).
\end{aligned}$$

Note that, for $k \geq 1$, we have $X_{k\frac{t}{n}} = \sum_{j=1}^{k}\left(X_{j\frac{t}{n}} - X_{(j-1)\frac{t}{n}}\right)$ and the random variables $\left(X_{j\frac{t}{n}} - X_{(j-1)\frac{t}{n}}\right)_{j\geq 1}$ are i.i.d. So by Spitzer's identity, we have

$$\mathbb{E}\max_{k=0,\ldots,n} X_{k\frac{t}{n}} = \sum_{k=1}^{n}\frac{1}{k}\mathbb{E}X^+_{k\frac{t}{n}}.$$

The sequence $\left(\max_{k=0,\ldots,n} X_{k\frac{t}{n}}\right)_{n\geq 0}$ is dominated by $\sup_{0\leq s\leq t} X_s$, so by using the dominated convergence theorem, we get

$$\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq t} X_s &= \mathbb{E}\lim_{n\to+\infty}\max_{k=0,\ldots,n} X_{k\frac{t}{n}} \\
&= \lim_{n\to+\infty}\mathbb{E}\max_{k=0,\ldots,n} X_{k\frac{t}{n}} \\
&= \lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{k}\mathbb{E}X^+_{k\frac{t}{n}} \\
&= \int_0^t \frac{\mathbb{E}X^+_s}{s}ds.
\end{aligned}$$

◇

3.1. Case of finite activity Lévy processes. The use of proposition 3.3 in the finite activity case, leads to the following theorem.

THEOREM 3.5. *Let X be a finite activity Lévy process satisfying $\int_{x>1} x\nu(dx) < \infty$, $t > 0$ and $n \in \mathbb{N}$.*

1. *If $\sigma > 0$, we have, for $n \to +\infty$,*

$$\mathbb{E}\left(M_t - M_t^n\right) = \frac{1}{2n}\left(\frac{\gamma_0 t}{2} + \lambda t \mathbb{E} Y_1^+ - \sigma\sqrt{t}\mathbb{E}\phi\left(\frac{\gamma_0}{\sigma}\sqrt{t} + \frac{\sum_{i=1}^{N_t} Y_i}{\sigma\sqrt{t}}\right)\right)$$
$$-\frac{1}{2n}\mathbb{E}\left(\gamma_0 t + \sum_{i=1}^{N_t} Y_i\right)\Phi\left(\frac{\gamma_0}{\sigma}\sqrt{t} + \frac{\sum_{i=1}^{N_t} Y_i}{\sigma\sqrt{t}}\right)$$
$$-\frac{\sigma\sqrt{t}\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi n}} + o\left(\frac{1}{n}\right).$$

Here, ζ is the zeta Riemann function and ϕ and Φ are the probality density function and the cumulative distribution function of the standard normal distribution.

2. *If $\sigma = 0$, then $s \to \frac{\mathbb{E}X_s^+}{s}$ is absolutely continuous on $[0, t]$ and we have*

$$\mathbb{E}\left(M_t - M_t^n\right) = \frac{1}{2n}\left(\gamma_0^+ t + \lambda t \mathbb{E} Y_1^+ - \mathbb{E} X_t^+\right) + o\left(\frac{1}{n}\right)$$

when $n \to +\infty$.

Recall that in the case of Brownian motion, Broadie Glasserman and Kou prove in [4] (cf. lemma 3) a result similar to the first point of the above theorem. In the case $\sigma = 0$, if Y_1 have a continuous density function or $\gamma_0 = 0$, the error $o\left(\frac{1}{n}\right)$ is in fact $O\left(\frac{1}{n^2}\right)$ (see [7]). To prove Theorem 3.5, we need the following more or less elementary lemmas.

Lemma 3.6. *Let $f \in C^2[0, t]$. Then*

$$\int_0^t \frac{1}{\sqrt{x}} f(\sqrt{x})dx = \frac{t}{n}\sum_{k=1}^{n} \frac{1}{\sqrt{\frac{kt}{n}}} f\left(\sqrt{\frac{kt}{n}}\right) - \frac{\sqrt{t}\zeta\left(\frac{1}{2}\right) f(0)}{\sqrt{n}}$$
$$-\frac{\sqrt{t}f(\sqrt{t}) - t f'(0)}{2n} + o\left(\frac{1}{n}\right).$$

Lemma 3.7. *Let f be an absolutely continuous function on $[0, t]$, then we have*

$$\int_0^t f(s)ds - \frac{t}{n}\sum_{k=1}^{n} f\left(\frac{kt}{n}\right) = \frac{t}{2n}\left(f(0) - f(t)\right) + o\left(\frac{1}{n}\right).$$

The proof of the previous lemma is based on the following result.

Lemma 3.8. *Let $h \in L^1([0, t])$, we define the sequence $(I_m(h))_{m \geq 1}$ by*

$$I_m(h) = \sum_{k=1}^{m} \int_{(k-1)\frac{t}{m}}^{k\frac{t}{m}} h(u)\left(u - (k-1)\frac{t}{m}\right)du.$$

Then we have

$$\lim_{m \to +\infty} m I_m(h) = \frac{t}{2}\int_0^t h(u)du.$$

Proof of lemma 3.8. Consider first the case where $h \in C([0,t])$. By the variable substitutions $v = u - (k-1)\frac{t}{m}$, then $w = mv$ we get

$$\begin{aligned} I_m(h) &= \sum_{k=1}^{m} \int_0^{\frac{t}{m}} h\left(v + (k-1)\frac{t}{m}\right) v dv \\ &= \sum_{k=1}^{m} \int_0^{t} h\left(\frac{w}{m} + (k-1)\frac{t}{m}\right) \frac{w}{m}\frac{dw}{m} \\ &= \frac{1}{m}\int_0^t \frac{1}{m}\sum_{k=1}^{m} h\left(\frac{w}{m} + (k-1)\frac{t}{m}\right) w dw. \end{aligned}$$

But h is continuous and for $w \in [0,t]$ we have $\frac{w}{m} + (k-1)\frac{t}{m} \in \left[(k-1)\frac{t}{m}, k\frac{t}{m}\right]$, so

$$\lim_{m\to\ +\infty} \frac{t}{m}\sum_{k=1}^{m} h\left(\frac{w}{m} + (k-1)\frac{t}{m}\right) = \int_0^t h(s)ds.$$

Hence

$$\begin{aligned} \lim_{m\to\ +\infty} mI_m(h) &= \int_0^t \left(\frac{1}{t}\int_0^t h(s)ds\right) w dw \\ &= \frac{t}{2}\int_0^t h(s)ds. \end{aligned}$$

Consider now the case where h is integrable on $[0,t]$. Then there exists a sequence of functions $(h_n)_{n\geq 0}$ in $C([0,t])$ such that

$$\lim_{n\to\ +\infty} \int_0^t |h(u) - h_n(u)| du = 0.$$

So we have

$$\begin{aligned} u_m^n :&= \left| mI_m(h_n) - m\sum_{k=1}^{m} \int_{(k-1)\frac{t}{m}}^{k\frac{t}{m}} h(u)\left(u - (k-1)\frac{t}{m}\right) du \right| \\ &= \left| m\sum_{k=1}^{m} \int_{(k-1)\frac{t}{m}}^{k\frac{t}{m}} (h_n(u) - h(u))\left(u - (k-1)\frac{t}{m}\right) du \right| \\ &\leq m\sum_{k=1}^{m} \int_{(k-1)\frac{t}{m}}^{k\frac{t}{m}} |h_n(u) - h(u)| \left| u - (k-1)\frac{t}{m} \right| du \\ &\leq t\sum_{k=1}^{m} \int_{(k-1)\frac{t}{m}}^{k\frac{t}{m}} |h_n(u) - h(u)| du \\ &\leq t\int_0^t |h_n(u) - h(u)| du. \end{aligned}$$

The convergence (with respect to m) of $mI_m(h_n)$ is uniform. Hence by the limits inversion theorem

$$\begin{aligned}
&\lim_{m\to+\infty}\lim_{n\to+\infty} mI_m(h_n)=\lim_{n\to+\infty}\lim_{m\to+\infty} mI_m(h_n)\\
&\Rightarrow \lim_{m\to+\infty} mI_m(h)=\lim_{n\to+\infty}\frac{t}{2}\int_0^t h_n(u)du\\
&\Rightarrow \lim_{m\to+\infty} mI_m(h)=\frac{t}{2}\int_0^t h(u)du.
\end{aligned}$$

◇

Proof of lemma 3.7. Let h be the a.e. derivative of f. We have

$$\begin{aligned}
\int_0^t f(s)ds-\frac{t}{n}\sum_{k=1}^n f\left(\frac{kt}{n}\right)&=\sum_{k=1}^n\int_{(k-1)\frac{t}{n}}^{k\frac{t}{n}}\left(f(s)-f\left(\frac{kt}{n}\right)\right)ds\\
&=-\sum_{k=1}^n\int_{(k-1)\frac{t}{n}}^{k\frac{t}{n}}\int_s^{k\frac{t}{n}} h(u)duds\\
&=-\sum_{k=1}^n\int_{(k-1)\frac{t}{n}}^{k\frac{t}{n}}\int_{(k-1)\frac{t}{n}}^{u} h(u)dsdu,\ \text{by Fubini.}
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^t f(s)ds-\frac{t}{n}\sum_{k=1}^n f\left(\frac{kt}{n}\right)&=-\sum_{k=1}^n\int_{(k-1)\frac{t}{n}}^{k\frac{t}{n}} h(u)\left(u-(k-1)\frac{t}{n}\right)du\\
&=-\frac{t}{2n}\int_0^t h(u)du+o\left(\frac{1}{n}\right),\ \text{by lemma 3.8}\\
&=-\frac{t}{2n}\left(f(t)-f(0)\right)+o\left(\frac{1}{n}\right)\\
&=\frac{t}{2n}\left(f(0)-f(t)\right)+o\left(\frac{1}{n}\right).
\end{aligned}$$

◇

Proof of lemma 3.6. We consider first the case $t=1$. The case $t\neq 1$ will be deduced by a variable substitution. We have

$$\frac{1}{\sqrt{x}}f(\sqrt{x})=\frac{f(0)}{\sqrt{x}}+\frac{f\left(\sqrt{x}\right)-f(0)}{\sqrt{x}}.$$

Set

$$g(x)=\frac{f\left(\sqrt{x}\right)-f(0)}{\sqrt{x}}.$$

The function g can be extended to a continuous function on $[0,1]$, and $\lim_{x\to 0} g(x)=f'(0)$. Furthermore g is differentiable on $(0,1]$ and

$$g'(x)=\frac{f(0)-f\left(\sqrt{x}\right)+\sqrt{x}f'\left(\sqrt{x}\right)}{2x^{\frac{3}{2}}}.$$

The function g' is integrable on $[0,1]$, so g is absolutely continuous. Thus

$$
\begin{aligned}
\epsilon_n(f) &= \int_0^1 \frac{f(0)}{\sqrt{x}}dx + \int_0^1 g(x)dx - \frac{1}{n}\sum_{k=1}^{n} \frac{f(0)}{\sqrt{\frac{k}{n}}} - \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) \\
&= f(0)\left(\int_0^1 \frac{1}{\sqrt{x}}dx - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{\frac{k}{n}}}\right) + \left(\int_0^1 g(x)dx - \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\right).
\end{aligned}
$$

By using [8] (see p.538) and lemma 3.7, we get

$$
\begin{aligned}
\epsilon_n(f) &= f(0)\left(-\frac{\zeta\left(\frac{1}{2}\right)}{\sqrt{n}} - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right) + \frac{g(0)}{2n} - \frac{g(1)}{2n} + o\left(\frac{1}{n}\right) \\
&= -\frac{\zeta\left(\frac{1}{2}\right)}{\sqrt{n}} f(0) - \frac{f(0)}{2n} - \frac{f(1) - f'(0) - f(0)}{2n} + o\left(\frac{1}{n}\right) \\
&= -\frac{\zeta\left(\frac{1}{2}\right) f(0)}{\sqrt{n}} - \frac{f(1) - f'(0)}{2n} + o\left(\frac{1}{n}\right).
\end{aligned}
$$

$\diamond$

Proof of theorem 3.5. We know by theorem 3.3 that

$$
\mathbb{E}\left(\sup_{0\le s\le t} X_s - \max_{k=0,\dots,n} X_{k\frac{t}{n}}\right) = \int_0^t \frac{\mathbb{E}X_s^+}{s}ds - \frac{t}{n}\sum_{k=1}^{n}\frac{\mathbb{E}X^+_{k\frac{t}{n}}}{\frac{kt}{n}}.
$$

So we need to study the smoothness of the function $s \mapsto \mathbb{E}X_s^+/s$ and conclude with lemmas 3.6 and 3.7.

Case 1 : $\sigma > 0$ and $\mathbb{E}Y_1^+ < \infty$.

Let U be a normal r.v. with mean γ and variance σ^2. By an easy computation we get

$$
\mathbb{E}U^+ = \sigma\phi\left(\frac{\gamma}{\sigma}\right) + \gamma\Phi\left(\frac{\gamma}{\sigma}\right).
$$

So, for any $s > 0$, we have, by conditionning with respect to the jump part of the process X,

$$
\mathbb{E}\frac{X_s^+}{s} = \mathbb{E}\frac{\sigma}{\sqrt{s}}\phi\left(\frac{\gamma_0}{\sigma}\sqrt{s} + \frac{\sum_{i=1}^{N_s} Y_i}{\sigma\sqrt{s}}\right) + \mathbb{E}\left(\gamma_0 + \frac{\sum_{i=1}^{N_s} Y_i}{s}\right)\Phi\left(\frac{\gamma_0}{\sigma}\sqrt{s} + \frac{\sum_{i=1}^{N_s} Y_i}{\sigma\sqrt{s}}\right).
$$

Let f and g be the functions defined by

$$
\begin{aligned}
f(s) &= \mathbb{E}\phi\left(\frac{\gamma_0}{\sigma}s + \frac{\sum_{i=1}^{N_{s^2}} Y_i}{\sigma s}\right) \\
g(s) &= \mathbb{E}\left(\frac{\gamma_0}{\sigma}s + \frac{\sum_{i=1}^{N_{s^2}} Y_i}{\sigma s}\right)\Phi\left(\frac{\gamma_0}{\sigma}s + \frac{\sum_{i=1}^{N_{s^2}} Y_i}{\sigma s}\right),
\end{aligned}
$$

so that

$$
\mathbb{E}\frac{X_s^+}{s} = \frac{\sigma}{\sqrt{s}}f\left(\sqrt{s}\right) + \frac{\sigma}{\sqrt{s}}g\left(\sqrt{s}\right).
$$

If f and g can be extended as C^2 functions on $[0,t]$ then, using lemma 3.6, we get the first part of the theorem. By [6], proposition 9.5, we have

$$f(s) = \mathbb{E}s^{2N_1}e^{-\lambda(s^2-1)}\phi\left(\frac{\gamma_0}{\sigma}s + \frac{\sum_{i=1}^{N_1} Y_i}{\sigma s}\right).$$

So, the function f has the same regularity as $\tilde{f}$ defined by

$$\tilde{f}(s) = \mathbb{E}s^{2N_1}\phi\left(\mu s + \frac{\sum_{i=1}^{N_1} Y_i}{\sigma s}\right),$$

where $\mu = \frac{\gamma_0}{\sigma}$. For $x \in \mathbb{R}$, we define the function

$$s \mapsto h(s,x) = \phi\left(\mu s + \frac{x}{s}\right).$$

We then have

$$\tilde{f}(s) = \mathbb{E}s^{2N_1}h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right).$$

Note that

$$0 \le h\left(s,x\right) \le \frac{1}{\sqrt{2\pi}},$$

and

$$\begin{aligned} h\left(s,x\right) &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left(\mu s + \frac{x}{s}\right)^2\right) \\ &= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{\mu^2 s^2}{2}\right)\exp\left(-\mu x - \frac{x^2}{2s^2}\right). \end{aligned}$$

Using the inequality $-\mu x \le \mu^2 s^2 + \frac{x^2}{4s^2}$, we get

$$h\left(s,x\right) \le \frac{1}{\sqrt{2\pi}}\left(e^{\frac{\mu^2 s^2}{2}}e^{-\frac{x^2}{4s^2}} \wedge 1\right). \tag{3.2}$$

Moreover, we have

$$\frac{\partial}{\partial s}h\left(s,x\right) = \left(\frac{x^2}{s^3} - \mu^2 s\right)h(s,x)$$

and

$$\frac{\partial^2}{\partial s^2}h\left(s,x\right) = \left(-\frac{3x^2}{s^4} - \mu^2\right)\phi\left(\mu s + \frac{x}{s}\right) + \left(\frac{x^2}{s^3} - \mu^2 s\right)^2\phi\left(\mu\sqrt{s} + \frac{x}{\sqrt{s}}\right).$$

Using (3.2), we get

$$\begin{aligned} \left|\frac{\partial}{\partial s}h\left(s,x\right)\right| &\le \frac{\mu^2 s}{2\sqrt{2\pi}} + \frac{x^2}{s^3\sqrt{2\pi}}e^{\frac{\mu^2 s^2}{2}}e^{-\frac{x^2}{4s^2}} \\ &\le \frac{\mu^2 s}{2\sqrt{2\pi}} + \frac{C_1\mathbb{1}_{\{x\neq 0\}}}{s}e^{\frac{\mu^2 s}{2}}, \end{aligned}$$

where $C_1 = \sup_{y>0}\left(\frac{y^2 e^{-\frac{y^2}{4}}}{\sqrt{2\pi}}\right)$. Using (3.2) again and the fact that $\left(\frac{x^2}{s^3} - \mu^2 s\right)^2 \leq 2\left(\frac{x^4}{s^6} + \mu^4 s^2\right)$, we obtain

$$\begin{aligned}\left|\frac{\partial^2}{\partial s^2} h\left(s,x\right)\right| &\leq \left(\mu^2 + 2\mu^4 s^2\right) h(s,x) + \left(\frac{3x^2}{s^4} + 2\frac{x^4}{s^6}\right) h(s,x)\\ &\leq \frac{\mu^2 + 2\mu^4 s^2}{\sqrt{2\pi}} + \frac{C_2 \mathbb{1}_{x\neq 0}}{s^2} e^{\frac{\mu^2 s^2}{2}},\end{aligned}$$

where $C_2 = \sup_{y>0}\left(\frac{3y^2+2y^4}{\sqrt{2\pi}} e^{-\frac{y^2}{4}}\right)$. Hence

$$\begin{aligned}\frac{\partial}{\partial s}\left(s^{2N_1} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\right) &= 2N_1 s^{2N_1-1} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\\ &\quad + s^{2N_1}\frac{\partial}{\partial s} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right).\end{aligned}$$

Thus

$$\left|\frac{\partial}{\partial s}\left(s^{2N_1} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\right)\right| \leq \frac{2N_1 s^{2N_1-1}}{\sqrt{2\pi}} + \frac{\mu^2 s^{2N_1-1}}{\sqrt{2\pi}} + C_1 \mathbb{1}_{\{N_1>0\}} s^{2N_1-1} e^{\frac{\mu^2 s^2}{2}}.$$

We deduce that $\tilde{f}$ is continuously differentiable, and

$$\tilde{f}'(s) = \mathbb{E}\left[2N_1 s^{2N_1-1} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right) + s^{2N_1}\frac{\partial}{\partial s} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\right].$$

Similarly,

$$\begin{aligned}\frac{\partial^2}{\partial s^2}\left(s^{2N_1} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\right) &= 2N_1\left(2N_1 - 1\right) s^{2N_1-2} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\\ &\quad + 4N_1 s^{2N_1-1}\frac{\partial}{\partial s} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right) + s^{2N_1}\frac{\partial^2}{\partial s^2} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right).\end{aligned}$$

But

$$\begin{aligned}\left|2N_1\left(2N_1 - 1\right) s^{2N_1-2} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\right| &\leq \frac{2N_1\left(2N_1-1\right) s^{2N_1-2}}{\sqrt{2\pi}}\\ \left|42N_1 s^{2N_1-1}\frac{\partial}{\partial s}\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\right| &\leq \frac{4N_1\left(2N_1-1\right) s^{N_1}}{\sqrt{2\pi}}\\ &\quad + 4N_1\left(2N_1-1\right) s^{2N_1-2} C_1 \mathbb{1}_{\{N_1>0\}} e^{\frac{\mu^2 s^2}{2}}\\ \left|s^{2N_1}\frac{\partial^2}{\partial s^2} h\left(s, \frac{\sum_{i=1}^{N_1} Y_i}{\sigma}\right)\right| &\leq \frac{\mu^2 + 2\mu^4 s^2}{\sqrt{2\pi}} s^{2N_1} + C_2 \mathbb{1}_{N_1>0} s^{2N_1-2} e^{\frac{\mu^2 s^2}{2}}.\end{aligned}$$

We deduce that $\tilde{f}$ is twice differentiable on $[0,t]$ and

$$\tilde{f}''(s) = \mathbb{E}\left[2N_1\left(2N_1-1\right)s^{2N_1-2}h\left(s,\frac{\sum_{i=1}^{N_1}Y_i}{\sigma}\right)+4N_1s^{2N_1-1}\frac{\partial}{\partial s}h\left(s,\frac{\sum_{i=1}^{N_1}Y_i}{\sigma}\right)\right]$$
$$+\mathbb{E}\left[s^{2N_1}\frac{\partial^2}{\partial s^2}h\left(s,\frac{\sum_{i=1}^{N_1}Y_i}{\sigma}\right)\right].$$

Hence f is in $C^2[0,t]$ and we verify that $f(0)=\frac{1}{\sqrt{2\pi}}$ and $f'(0)=0$. On the other hand the function g can be written in the following form (see [6], proposition 9.5)

$$g(s)=\mathbb{E}s^{2N_1}e^{-\lambda(s^2-1)}\left(\frac{\gamma_0}{\sigma}s+\frac{\sum_{i=1}^{N_1}Y_i}{\sigma s}\right)\Phi\left(\frac{\gamma_0}{\sigma}s+\frac{\sum_{i=1}^{N_1}Y_i}{\sigma s}\right).$$

With the same reasoning we could prove that g is in $C^2[0,t]$, and satisfies $g(0)=0$ and $g'(0)=\frac{\lambda\mathbb{E}Y_1^+}{\sigma}+\frac{\gamma_0}{2\sigma}$. This proves the first part of the theorem.

Case 2: $\sigma=0$ and $\mathbb{E}Y_1^+<\infty$.

We have

$$\frac{\mathbb{E}X_s^+}{s}=\gamma_0^+e^{-\lambda s}+e^{-\lambda s}\sum_{n=1}^{+\infty}\frac{\lambda^n s^{n-1}}{n!}\mathbb{E}\left(\gamma_0 s+\sum_{i=1}^n Y_i\right)^+.$$

Observe that, for any positive integer n, the function $s\mapsto\mathbb{E}\left(\gamma_0 s+\sum_{i=1}^n Y_i\right)^+$ is absolutely continuous. So is $s\mapsto\frac{\lambda^n s^{n-1}}{n!}\mathbb{E}\left(\gamma_0 s+\sum_{i=1}^n Y_i\right)^+$. If we call h_n its a.e. derivative, then, for any $n\geq 2$,

$$h_n(s)=\gamma_0\frac{\lambda^n s^{n-1}}{n!}\mathbb{P}\left(\gamma_0 s+\sum_{i=1}^n Y_i\geq 0\right)+\frac{n-1}{n!}\lambda^n s^{n-2}\mathbb{E}\left(\gamma_0 s+\sum_{i=1}^n Y_i\right)^+,$$

so that, for $s\in[0,t]$,

$$|h_n(s)|\leq|\gamma_0|\frac{\lambda^n t^{n-1}}{n!}+\frac{n-1}{n!}\lambda^n t^{n-2}\left(|\gamma_0|t+n\mathbb{E}Y_1^+\right).$$

Hence the normal convergence of $\sum h_n$ on $[0,t]$, and thus the absolute continuity of $\frac{\mathbb{E}X_s^+}{s}$ on $[0,t]$. So, by proposition 3.3 and lemma 3.7,

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq s\leq t}X_s-\max_{k=0,\ldots,n}X_{k\frac{t}{n}}\right)&=\int_0^t\frac{\mathbb{E}X_s^+}{s}ds-\frac{t}{n}\sum_{k=1}^n\frac{\mathbb{E}X_{k\frac{t}{n}}^+}{\frac{kt}{n}}\\&=\frac{t}{2n}\left(\lim_{s\to 0^+}\frac{\mathbb{E}X_s^+}{s}-\frac{\mathbb{E}X_t^+}{t}\right)+o\left(\frac{1}{n}\right)\\&=\frac{1}{2n}\left(\left(\gamma_0^++\lambda\mathbb{E}Y_1^+\right)t-EX_t^+\right)+o\left(\frac{1}{n}\right)\\&=\frac{1}{2n}\left(\gamma_0^+t+\lambda t\mathbb{E}Y_1^+-EX_t^+\right)+o\left(\frac{1}{n}\right).\end{aligned}$$

◇

3.2. Case of infinite activity Lévy processes. In the case of Lévy processes with infinite activity, we cannot use (2.3). So the method used in theorem 3.5 does not work anymore and we must use another approach.

THEOREM 3.9. *Let X be an integrable Lévy process with generating triplet (γ, σ^2, ν). Then*

1. *If $\sigma > 0$*

$$\mathbb{E}\left(M_t - M_t^n\right) = O\left(\frac{1}{\sqrt{n}}\right).$$

2. *If $\sigma = 0$*

$$\mathbb{E}\left(M_t - M_t^n\right) = o\left(\frac{1}{\sqrt{n}}\right).$$

3. *If $\sigma = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$*

$$\mathbb{E}\left(M_t - M_t^n\right) = O\left(\frac{\log(n)}{n}\right).$$

To prove the result 2 of theorem 3.9, we will use the lemma below.

LEMMA 3.10. *Let X be an integrable Lévy process with generating triplet $(\gamma, 0, \nu)$. Then we have*

$$\mathbb{E}X_t^+ = o\left(\sqrt{t}\right)$$

when $t \to 0$.

The proof of this lemma is quite standard, and is left to the reader. For more details, see [7].

Proof of theorem 3.9. With the notation $\delta = \frac{t}{n}$, we have, using proposition 3.3,

$$\begin{aligned}
\mathbb{E}\left(M_t - M_t^n\right) &= \int_0^t \frac{\mathbb{E}X_s^+}{s} ds - \sum_{k=1}^n \frac{\mathbb{E}X_{k\delta}^+}{k} \\
&= \sum_{k=1}^n \int_{(k-1)\delta}^{k\delta} \frac{\mathbb{E}X_s^+}{s} - \sum_{k=1}^n \int_{(k-1)\delta}^{k\delta} \frac{\mathbb{E}X_{k\delta}^+}{k\delta} ds \\
&= \int_0^\delta \left(\frac{\mathbb{E}X_s^+}{s} - \frac{\mathbb{E}X_\delta^+}{\delta}\right) ds + \sum_{k=2}^n \int_{(k-1)\delta}^{k\delta} \left(\frac{\mathbb{E}X_s^+}{s} - \frac{\mathbb{E}X_{k\delta}^+}{k\delta}\right) ds.
\end{aligned}$$

We call $u(\delta)$ (respectively $v(\delta)$) the first (respectively the second) term on the right of the last equality. We easily deduce from Proposition 3.4 that, if $\sigma > 0$, $u(\delta) = O(\sqrt{\delta})$ and, if $\sigma = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, $u(\delta) = O(\delta)$. We also have

$$\begin{aligned}
\frac{u(\delta)}{\sqrt{\delta}} &= \int_0^\delta \frac{\mathbb{E}X_s^+}{s\sqrt{\delta}} ds - \frac{\mathbb{E}X_\delta^+}{\sqrt{\delta}} \\
&= \int_0^1 \frac{1}{\sqrt{s}} \frac{\mathbb{E}X_{s\delta}^+}{\sqrt{s\delta}} - \frac{\mathbb{E}X_\delta^+}{\sqrt{\delta}},
\end{aligned}$$

and we easily deduce from Lemma 3.10 that, if $\sigma = 0$, $u(\delta) = o(\sqrt{\delta})$.

We now study $v(\delta)$. For $s \geq 0$, let $\tilde{X}_s = X_s - \alpha s$, where $\alpha = \mathbb{E}X_1$. Then, $\tilde{X}$ is a martingale and, for a fixed $s \geq 0$, $\left(\tilde{X}_\tau + \alpha s\right)^+_{\tau \geq 0}$ is a submartingale, because $x \to x^+$ is a convex function. So, for $s \in [(k-1)\delta, \delta]$,

$$\mathbb{E}X_s^+ = \mathbb{E}\left(\tilde{X}_s + \alpha s\right)^+ \leq \mathbb{E}\left(\tilde{X}_{k\delta} + \alpha s\right)^+.$$

Hence

$$\begin{aligned}
v(\delta) &= \sum_{k=2}^{n} \int_{(k-1)\delta}^{k\delta} \left(\frac{\mathbb{E}X_s^+}{s} - \frac{\mathbb{E}X_{k\delta}^+}{k\delta} \right) ds \\
&\leq \sum_{k=2}^{n} \int_{(k-1)\delta}^{k\delta} \left(\frac{\mathbb{E}\left(\tilde{X}_{k\delta} + \alpha s\right)^+}{s} - \frac{\mathbb{E}\left(\tilde{X}_{k\delta} + \alpha k\delta\right)^+}{k\delta} \right) ds \\
&= \sum_{k=2}^{n} \int_{(k-1)\delta}^{k\delta} \mathbb{E}\left(\tilde{X}_{k\delta} + \alpha k\delta\right)^+ \left(\frac{1}{s} - \frac{1}{k\delta} \right) ds \\
&\quad + \sum_{k=2}^{n} \int_{(k-1)\delta}^{k\delta} \frac{\mathbb{E}\left(\tilde{X}_{k\delta} + \alpha s\right)^+ - E\left(\tilde{X}_{k\delta} + \alpha k\delta\right)^+}{s} ds.
\end{aligned}$$

Using the inequality $|x^+ - y^+| \leq |x - y|$, we get

$$\begin{aligned}
v(\delta) &\leq \sum_{k=2}^{n} \mathbb{E}X_{k\delta}^+ \left(\log\left(\frac{k}{k-1}\right) - \frac{1}{k} \right) + \sum_{k=2}^{n} \int_{(k-1)\delta}^{k\delta} \frac{|\alpha|\,(k\delta - s)}{s} ds \\
&= \sum_{k=2}^{n} \mathbb{E}X_{k\delta}^+ \left(\log\left(1 + \frac{1}{k-1}\right) - \frac{1}{k} \right) ds + \sum_{k=2}^{n} \int_{(k-1)\delta}^{k\delta} |\alpha| \left(\frac{k\delta}{s} - 1 \right) ds \\
&\leq \sum_{k=2}^{n} \mathbb{E}X_{k\delta}^+ \left(\frac{1}{k-1} - \frac{1}{k} \right) + \sum_{k=2}^{n} |\alpha|\delta \left(k \log\left(\frac{k}{k-1}\right) - 1 \right) \\
&\leq \sum_{k=2}^{n} \mathbb{E}X_{k\delta}^+ \frac{1}{k(k-1)} + |\alpha|\delta \sum_{k=2}^{n} \left(\frac{k}{k-1} - 1 \right) \\
&= \sum_{k=2}^{n} \mathbb{E}X_{k\delta}^+ \frac{1}{k(k-1)} + |\alpha|\delta \sum_{k=2}^{n} \frac{1}{k-1}.
\end{aligned}$$

Now, if $\sigma = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, we know from Proposition 3.4 that $\mathbb{E}X_{k\delta}^+ \leq Ck\delta$ for some $C > 0$, so that

$$\begin{aligned}
v(\delta) &\leq C\delta \sum_{k=2}^{n} \frac{1}{k-1} + |\alpha|\delta \sum_{k=2}^{n} \frac{1}{k-1} \\
&\leq (C\delta + |\alpha|)\,(1 + \log(n-1)) \\
&= O\left(\frac{\log(n)}{n} \right),
\end{aligned}$$

so that the last statement of the Theorem is proved.

For the other cases, let $f(s) = \mathbb{E}(X_s^+)/\sqrt{s}$, so that

$$\sum_{k=2}^{n} \mathbb{E}X_{k\delta}^+ \frac{1}{k(k-1)} = \sqrt{\delta} \sum_{k=2}^{n} f(k\delta) \frac{1}{\sqrt{k}(k-1)}.$$

We know from Proposition 3.4 that f is bounded on $[0,t]$, so that the first statement of Theorem 3.9 now follows from the convergence of the series $\sum 1/k^{3/2}$.

In order to prove the second statement (i.e. the case $\sigma = 0$), we observe that $\sum_{k=2}^{n} f(k\delta)\frac{1}{\sqrt{k}(k-1)}$ goes to 0 as $n \to \infty$, as follows easily from $\lim_{s\to 0} f(s) = 0$ (cf. Lemma 3.10). $\diamond$

Remark 3.11. The second result of theorem 3.9 is optimal in the following sense: for any $\epsilon > 0$, there exists a Lévy process X satisfying $\sigma = 0$, such that

$$\lim_{n\to+\infty} n^{\frac{1}{2}+\epsilon}\mathbb{E}\left(M_t - M_t^n\right) = +\infty.$$

More precisely, if X is a stable process of order α, with $\alpha \in (1,2)$, we have

$$\lim_{n\to\infty} n^{1/\alpha}\mathbb{E}\left(M_t - M_t^n\right) = -t^{1/\alpha}\zeta\left(1-\frac{1}{\alpha}\right)\mathbb{E}X_1^+.$$

The proof can be found in [7].

In the finite variation case, with a stronger assumption, we extend the results on compound Poisson processes which we get in the previous section, to infinite activity case.

Theorem 3.12. *Let X be an integrable Lévy process with generating triplet $(\gamma, 0, \nu)$. Suppose that* $\int_{|x|\le 1} |x|\,|\log(|x|)|\,\nu(dx) < \infty$ *and* $\nu(\mathbb{R}) = +\infty$, *then*

$$\mathbb{E}\left(M_t - M_t^n\right) = \left(\left(\gamma_0^+ + \int_{\mathbb{R}} x^+\nu(dx)\right)t - \mathbb{E}X_t^+\right)\frac{1}{2n} + o\left(\frac{1}{n}\right).$$

Lemma 3.13. *If X is a finite variation Lévy process with infinite activity and $\gamma_0 \neq 0$, then*

$$\int_0^t ds \int_0^1 du \frac{1}{s}\left|\mathbb{P}\left[X_s \ge 0\right] - \mathbb{P}\left[X_{su} \ge 0\right]\right| < \infty. \tag{3.3}$$

Proof of lemma 3.13. We first consider the case $\gamma_0 < 0$. Recall that, since X has finite variation, we have, with probability one, $\lim_{t\to 0}\frac{X_t}{t} = \gamma_0$, therefore $\mathbb{P}(R_0 > 0) = 1$, where

$$R_0 = \inf\{t > 0 \mid X_t > 0\},$$

and $\int_0^t s^{-1}\mathbb{P}(X_s > 0)ds < \infty$ (see [9], Section 47, especially Theorem 47.2). Set

$$I = \int_0^t ds \int_0^1 du \frac{1}{s}\left|\mathbb{P}\left[X_s \ge 0\right] - \mathbb{P}\left[X_{su} \ge 0\right]\right|$$

Note that, since X has infinite activity, we have $\mathbb{P}(X_s = 0) = 0$, for all $s > 0$ (see [9], Theorem 27.4), so that

$$\begin{aligned} I &\le \int_0^t \frac{1}{s}\mathbb{P}\left[X_s \ge 0\right]ds + \int_0^t ds \int_0^1 du \frac{1}{s}\mathbb{P}\left[X_{su} \ge 0\right] \\ &= \int_0^t \frac{1}{s}\mathbb{P}\left(X_s > 0\right)ds + \int_0^t ds \int_0^1 du \frac{1}{s}\mathbb{P}\left(X_{su} > 0\right). \end{aligned}$$

So, we need to prove that $\int_0^t ds \int_0^1 du s^{-1}\mathbb{P}\left(X_{su}>0\right)<\infty$. We have

$$\begin{aligned}
\int_0^t ds \int_0^1 du \frac{1}{s}\mathbb{P}\left[X_{su}>0\right] &= \int_0^t ds \int_0^s du \frac{1}{s^2}\mathbb{P}\left[X_u>0\right] \\
&= \int_0^t \frac{1}{s^2}\left(\int_0^s \mathbb{P}\left[X_u>0\right] du\right) ds \\
&= \left[-\frac{1}{s}\left(\int_0^s \mathbb{P}\left[X_u>0\right] du\right)\right]_0^t + \int_0^t \frac{1}{s}\mathbb{P}\left[X_s>0\right] ds.
\end{aligned}$$

But, for any $s>0$,

$$\left|\frac{1}{s}\left(\int_0^s \mathbb{P}\left[X_u \geq 0\right] du\right)\right| \leq 1$$

So, using again $\int_0^t s^{-1}\mathbb{P}(X_s>0)ds<\infty$, we conclude that

$$\int_0^t ds \int_0^1 du \frac{1}{s}\mathbb{P}\left[X_{su}\geq 0\right]<\infty.$$

Consider now $\gamma_0>0$. Let $\tilde{X}$ be the dual process of X (e.g. $\tilde{X}=-X$). Then $\gamma_0^{\tilde{X}}=-\gamma_0$, and so $\gamma_0^{\tilde{X}}<0$. Thus

$$\begin{aligned}
I &= \int_0^t ds \int_0^1 du \frac{1}{s}\left|\mathbb{P}\left[X_s<0\right]-\mathbb{P}\left[X_{su}<0\right]\right| \\
&= \int_0^t ds \int_0^1 du \frac{1}{s}\left|\mathbb{P}\left[\tilde{X}_s\geq 0\right]-\mathbb{P}\left[\tilde{X}_{su}\geq 0\right]\right| \\
&< \infty.
\end{aligned}$$

$\diamond$

Proof of theorem 3.12. By proposition 3.3, we have

$$\mathbb{E}\left(M_t-M_t^n\right)=\int_0^t \frac{\mathbb{E}X_s^+}{s}ds-\sum_{k=1}^n \frac{\mathbb{E}X_{k\delta}^+}{k}.$$

Define

$$h(s)=\frac{\mathbb{E}X_s^+}{s}, \quad s\in[0,t].$$

In order to prove the theorem we need to show that h is absolutely continuous (cf. Lemma 3.7). We will first show that the derivative (in the sense of distributions) of $s\mapsto \mathbb{E}X_s^+$ is given by the function

$$\frac{d}{ds}\mathbb{E}(X_s)^+=\gamma_0\mathbb{P}\left[X_s\geq 0\right]+\int_{\mathbb{R}}\mathbb{E}\left((X_s+y)^+-(X_s)^+\right)\nu(dy), \quad s\in(0,t).$$

We first consider a continuously differentiable function f with bounded derivative. Since X is a finite variation process, Itô's formula reduces to

$$f(X_s)=f(0)+\gamma_0\int_0^s f'\left(X_\tau\right)d\tau+\sum_{0\leq\tau\leq s}\left(f\left(X_\tau\right)-f\left(X_{\tau^-}\right)\right),$$

so that

$$\mathbb{E}f(X_s) = f(0) + \gamma_0 \mathbb{E} \int_0^s f'(X_\tau)\, d\tau + \mathbb{E} \sum_{0 \le \tau \le s} \left(f(X_\tau) - f(X_{\tau^-}) \right).$$

The compensation formula (see [3], preliminaries) yields that, if

$$\mathbb{E}\left[\int_0^s d\tau \int_{\mathbb{R}} \left| f(X_\tau + y) - f(X_\tau) \right| \nu(dy) \right] < \infty, \tag{3.4}$$

then

$$\mathbb{E} \sum_{0 \le \tau \le s} \left(f(X_\tau) - f(X_{\tau^-}) \right) = \mathbb{E}\left[\int_0^s ds \int_{\mathbb{R}} \left(f(X_\tau + y) - f(X_\tau) \right) \nu(dy) \right].$$

Since f is a Lipschitz function and X is integrable, the condition (3.4) is satisfied and we have

$$\begin{aligned} \mathbb{E}f(X_s) &= f(0) + \gamma_0 \mathbb{E} \int_0^s f'(X_\tau)\, d\tau + \mathbb{E}\left[\int_0^s ds \int_{\mathbb{R}} \left(f(X_\tau + y) - f(X_\tau) \right) \nu(dy) \right] \\ &= f(0) + \mathbb{E}\left[\gamma_0 \int_0^s f'(X_\tau)\, d\tau + \int_0^s ds \int_{\mathbb{R}} \left(f(X_\tau + y) - f(X_\tau) \right) \nu(dy) \right]. \end{aligned}$$

Now, for $\epsilon > 0$, define

$$f_\epsilon(x) = \frac{x}{2} + \frac{\sqrt{\epsilon + x^2}}{2}, \quad x \in \mathbb{R}.$$

Note that f_ϵ is continuously differentiable and

$$f'_\epsilon(x) = \frac{1}{2} + \frac{x}{2\sqrt{\epsilon + x^2}}, \quad x \in \mathbb{R},$$

so that $\|f'_\epsilon\|_\infty \le 1$. We can write

$$\mathbb{E}f_\epsilon(X_s) = \frac{1}{2} + \mathbb{E}\left[\gamma_0 \int_0^s f'_\epsilon(X_\tau)\, d\tau + \int_0^s ds \int_{\mathbb{R}} \left(f_\epsilon(X_\tau + y) - f_\epsilon(X_\tau) \right) \nu(dy) \right].$$

Note that the function f_ϵ converges uniformly to $x \to x^+$ when ϵ goes to 0. And, for any $x \neq 0$,

$$\lim_{\epsilon \to 0} f'_\epsilon(x) = \mathbb{1}_{x \ge 0}.$$

Moreover, for any $\tau > 0$, $\mathbb{P}(X_\tau \neq 0) = 1$ (because X have infinite activity), and, for any $x \in \mathbb{R}$,

$$x^+ \le f_\epsilon(x) \le \frac{x}{2} + \frac{\sqrt{\epsilon} + |x|}{2} \le x^+ + \frac{\sqrt{\epsilon}}{2}.$$

By dominated convergence, we get

$$\begin{aligned} \mathbb{E}(X_s)^+ &= \frac{1}{2} + \mathbb{E}\left[\gamma_0 \int_0^s \mathbb{1}_{\{X_\tau \ge 0\}} d\tau + \int_0^s ds \int_{\mathbb{R}} \left((X_\tau + y)^+ - (X_\tau) \right)^+ \nu(dy) \right] \\ &= \frac{1}{2} + \gamma_0 \int_0^s \mathbb{P}\left[X_\tau \ge 0 \right] d\tau + \int_0^s ds \int_{\mathbb{R}} \mathbb{E}\left((X_\tau + y)^+ - (X_\tau)^+ \right) \nu(dy). \end{aligned}$$

Hence

$$\frac{d}{ds}\mathbb{E}(X_s)^+ = \gamma_0 \mathbb{P}\left[X_s \geq 0\right] + \int_{\mathbb{R}} \mathbb{E}\left((X_s+y)^+ - (X_s)^+\right)\nu(dy).$$

Now, we have

$$\begin{aligned}
h(s) - \int_{\mathbb{R}} y^+\nu(dy) &= \frac{\mathbb{E}(X_s)^+}{s} - \int_{\mathbb{R}} y^+\nu(dy) \\
= \frac{1}{s}\int_0^s \left(\gamma_0\mathbb{P}\left[X_u \geq 0\right] + \int_{\mathbb{R}} \mathbb{E}\left((X_u+y)^+ - X_u^+\right)\nu(dy)\right)du - \int_{\mathbb{R}} y^+\nu(dy) \\
= \frac{\gamma_0}{s}\int_0^s \mathbb{P}\left[X_u \geq 0\right]du + \frac{1}{s}\int_0^s\int_{\mathbb{R}} \mathbb{E}\left((X_u+y)^+ - X_u^+ - y^+\right)\nu(dy)du.
\end{aligned}$$

But

$$\begin{aligned}
(X_u+y)^+ - X_u^+ - y^+ &= (X_u+y)\,\mathbb{1}_{\{X_u+y>0\}} - X_u\mathbb{1}_{\{X_u>0\}} - y\mathbb{1}_{\{y>0\}} \\
&= (X_u+y)\,\mathbb{1}_{\{X_u+y>0\}} - X_u\mathbb{1}_{\{X_u>0\}} - y\mathbb{1}_{\{y>0\}} \\
&= X_u\left(\mathbb{1}_{\{X_u+y>0\}} - X_u\mathbb{1}_{\{X_u>0\}}\right) \\
&\quad -y\left(\mathbb{1}_{\{X_u+y>0\}} - X_u\mathbb{1}_{\{y>0\}}\right) \\
&= -|X_u|\mathbb{1}_{\{yX_u<0,|y|>|X_u|\}} - |y|\mathbb{1}_{\{yX_u<0,|y|\leq|X_u|\}} \\
&= -|X_u| \wedge |y|\mathbb{1}_{\{yX_u<0\}}.
\end{aligned}$$

So

$$h(s) - \int_{\mathbb{R}} y^+\nu(dy) = \frac{\gamma_0}{s}\int_0^s \mathbb{P}\left[X_u \geq 0\right]du - \frac{1}{s}\int_0^s\int_{\mathbb{R}} \mathbb{E}|X_u| \wedge |y|\mathbb{1}_{\{yX_u<0\}}\nu(dy)du.$$

It is now clear that h is continuous on $(0,+\infty)$, and that its derivative is given by

$$h'(s) = u_s + v_s + w_s,$$

where

$$\begin{aligned}
u_s &= \frac{\gamma_0}{s}\mathbb{P}\left[X_s \geq 0\right] - \frac{\gamma_0}{s^2}\int_0^s \mathbb{P}\left[X_u \geq 0\right]du, \\
v_s &= -\frac{1}{s}\int_{\mathbb{R}} \mathbb{E}\left[|X_s| \wedge |y|\mathbb{1}_{\{yX_s<0\}}\right]\nu(dy), \\
w_s &= \frac{1}{s^2}\int_0^s\int_{\mathbb{R}} \mathbb{E}\left[|X_u| \wedge |y|\mathbb{1}_{\{yX_u<0\}}\right]\nu(dy)du.
\end{aligned}$$

We will now show that

$$\int_0^t |h'(s)|ds < \infty$$

We have $u_s = 0$ if $\gamma_0 = 0$, and, for $\gamma_0 \neq 0$, we can write

$$\begin{aligned}
|u_s| &= \left|\frac{\gamma_0}{s}\mathbb{P}\left[X_s \geq 0\right] - \frac{\gamma_0}{s^2}\int_0^s \mathbb{P}\left[X_u \geq 0\right]du\right| \\
&\leq \frac{|\gamma_0|}{s}\int_0^1 \left|\mathbb{P}\left[X_s \geq 0\right] - \mathbb{P}\left[X_{su} \geq 0\right]\right|du.
\end{aligned}$$

Hence, by lemma 3.13,

$$\int_0^t |u_s| ds < \infty.$$

Besides, using the concavity of the function $x \in \mathbb{R}^+ \to x \wedge |y|$ and Proposition 3.4, we get

$$\begin{aligned}
|v_s| &\le \frac{1}{s} \int_{\mathbb{R}} \mathbb{E}\left(|X_s| \wedge |y|\right) \mathbb{1}_{yX_s<0} \nu(dy) \\
&\le \frac{1}{s} \int_{\mathbb{R}} \mathbb{E}\left(|X_s| \wedge |y|\right) \nu(dy) \\
&\le \frac{1}{s} \int_{\mathbb{R}} \left(\mathbb{E}|X_s|\right) \wedge |y| \nu(dy) \\
&\le \frac{1}{s} \int_{\mathbb{R}} (cs) \wedge |y| \nu(dy),
\end{aligned}$$

where the positive constant c comes from Proposition 3.4. Now, let $\hat{v}_s = \frac{1}{s}\int_{\mathbb{R}} (cs) \wedge |y|\nu(dy)$. Using Fubini's theorem, we have

$$\begin{aligned}
\int_0^t |\hat{v}_s| ds &= \int_{\mathbb{R}} \nu(dy) \int_0^t \frac{ds}{s} (cs) \wedge |y| \\
&\le c \int_{\mathbb{R}} \int_0^{\frac{|y|}{c}} ds\nu(dy) + \int_{\mathbb{R}} \int_{\frac{|y|}{c}}^t \frac{1}{s} |y| \mathbb{1}_{\{|y| \le ct\}} ds \nu(dy) \\
&= \int_{\mathbb{R}} |y| \nu(dy) + \int_{\mathbb{R}} \log\left(\frac{ct}{|y|}\right) |y| \mathbb{1}_{\{|y|\le ct\}} \nu(dy) \\
&= \int_{\mathbb{R}} |y| \nu(dy) + \int_{|y|\le ct} \log\left(\frac{ct}{|y|}\right) |y| \nu(dy) \\
&< \infty.
\end{aligned}$$

Note that the last integral is finite, due to the assumption on the Lévy measure. For the term w_s, we have

$$\begin{aligned}
|w_s| &\le \frac{1}{s^2} \int_0^s \int_{\mathbb{R}} (cu) \wedge |y| \nu(dy) du \\
&\le \frac{1}{s^2} \int_0^s \int_{\mathbb{R}} (cs) \wedge |y| \nu(dy) du \\
&= \frac{1}{s} \int_{\mathbb{R}} (cs) \wedge |y| \nu(dy) = \hat{v}_s.
\end{aligned}$$

We deduce that

$$\int_0^t |w_s| ds < \infty.$$

Therefore, we have proved that h is absolutely continuous. Using lemma 3.7 and theorem 3.3 we complete the proof. $\diamond$

4. Extension of the Asmussen-Glynn-Pitman Theorem. The continuity correction results of Broadie Glasserman and Kou for lookback options within the Black-Scholes model are based on a result due to Asmussen, Glynn and Pitman, about the weak convergence of the normalized difference between the continuous and discrete maximum of Brownian motion(see [2], Theorem 1). In this section, we extend this result to Lévy processes with finite activity and a non-trivial Brownian component, i.e. a Lévy process with generating triplet (γ, σ^2, ν), where $\sigma^2 > 0$ and ν is a finite measure.

The following statement is a reformulation of the Asmussen-Glynn-Pitman Theorem. It can be deduced from a careful reading of the proof of Theorem 1 in [2] (see particularly pages 879 to 883, and Remark 2).

Theorem 4.1. *Consider four real numbers a, b, x and y, with $0 \leq a < b$. Let $\beta = (\beta_t)_{a \leq t \leq b}$ be a Browian bridge from x to y over the time interval $[a, b]$ (so that $\beta_a = x$ and $\beta_b = y$) and let t be a fixed positive number. Denote by M the supremum of β and, for any positive integer n, by M^n the discrete supremum associated with a mesh of size $\frac{t}{n}$, so that*

$$M = \sup_{a \leq t \leq b} \beta_t \quad \text{and} \quad M^n = \sup_{k \in I_n} \beta_{\frac{kt}{n}}, \text{ where } I_n = \left\{ k \in \mathbb{N} \mid \frac{kt}{n} \in [a, b] \right\}.$$

Then, as n goes to infinity, the pair $(\sqrt{n}\,(M - M^n)\,, \beta)$ converges in distribution to the pair $(\sqrt{t}W, \beta)$ where W is independent of β and can be written as

$$W = \min_{\{j \in \mathbb{Z}\}} \check{R}(U + j). \tag{4.1}$$

Here $(\check{R}(t))_{t \in \mathbb{R}}$ is a two sided three dimensional Bessel process (i.e. $\check{R}(t) = R_1(t)$ for $t \geq 0$ and $\check{R}(t) = R_2(-t)$ for $t < 0$, where R_1 and R_2 are independent copies of the usual three dimensional Bessel process, starting from 0) and U is uniformly distributed on $[0, 1]$ and independent of $\check{R}$. We can now state and prove the main result of this section.

Theorem 4.2. *Let $X = (X_t)_{t \geq 0}$ be a finite activity Lévy process with generating triplet (γ, σ^2, ν) satisfying $\sigma^2 > 0$. For a fixed positive real number t, consider the continuous supremum of X over $[0, t]$ and, for any positive integer n, the discrete supremum associated with a mesh of size $\frac{t}{n}$, that is*

$$M_t = \sup_{0 \leq s \leq t} X_s \quad \text{and} \quad M_t^n = \sup_{k = 0, 1, \ldots, n} X_{\frac{kt}{n}}.$$

Then, as n goes to infinity, the pair $\left(\sqrt{n}\,(M_t - M_t^n)\,, X^{(t)} = (X_s)_{0 \leq s \leq t}\right)$ converges in distribution to the pair $(\sigma\sqrt{t}W, X^{(t)})$ where W is independent of $X^{(t)}$ and given by (4.1). Note that, in the above statement, $X^{(t)}$ is viewed as a random variable with values in the space of càd-làg functions defined on the interval $[0, t]$, which can be endowed with the Skorohod topology.

Proof of theorem 4.2. We will prove that for any bounded and continuous function f and for any bounded random variable Z which is measurable with respect to the σ-algebra generated by the random variables X_s, $0 \leq s \leq t$, we have

$$\lim_{n \to \infty} \mathbb{E}\left(f\left(\sqrt{n}(M_t - M_t^n)\right) Z\right) = \mathbb{E}\left(f\left(\sigma\sqrt{t}W\right)\right)\mathbb{E}(Z). \tag{4.2}$$

Since X is a finite activity process, it admits the following representation

$$X_s = \gamma_0 s + \sigma B_s + \sum_{j=1}^{N_s} Y_j, \quad s \geq 0,$$

where B is a standard Brownian motion, N is a Poisson process with intensity $\lambda = \nu(\mathbb{R})$, and the random variables Y_j are iid with distribution $\frac{\nu}{\nu(\mathbb{R})}$. Note that B, N and the Y_j's are independent.

By conditioning with respect to N_t, we have

$$\mathbb{E}\left(f\left(\sqrt{n}(M_t - M_t^n)\right) Z\right) = \sum_{m=0}^{\infty} \mathbb{E}\left(f\left(\sqrt{n}(M_t - M_t^n)\right) Z \mid N_t = m\right) \mathbb{P}(N_t = m).$$

Note that, conditionally on $\{N_t = 0, X_t = y\}$, the process $\frac{X^{(t)}}{\sigma}$ is a Brownian bridge from 0 to $\frac{y}{\sigma}$ so that, using Theorem 4.1,

$$\lim_{n\to+\infty} \mathbb{E}\left(f\left(\sqrt{n}(M_t - M_t^n)\right) Z \mid N_t = 0\right) = \mathbb{E}\left(f\left(\sigma\sqrt{t}W\right)\right) \mathbb{E}\left(Z \mid N_t = 0\right).$$

For the conditional expectation given $\{N_t = m\}$, $m \geq 1$, we condition further with respect to the jump times, to the values of X and to the values of the left-hand limits at the jump times. Denote by T_1, T_2,..., T_j,... the jump times of the Poisson process N. For any numbers $0 < t_1 < t_2 < \ldots < t_m < t$, x_1,..., x_m, y_1,...,y_m, y_{m+1}, let

$$A_m = \left\{N_t = m, T_i = t_i, X_{T_i^-} = x_i, X_{T_i} = y_i, i = 1, \ldots, m, X_t = y_{m+1}\right\}.$$

We observe that, conditionally on A_m, the random processes β^0, ..., β^m defined by

$$\beta_s^j = \begin{cases} \dfrac{1}{\sigma} X_s & \text{if } s \in [t_j, t_{j+1}), \\[2ex] \dfrac{1}{\sigma} X_{t_{j+1}^-} & \text{if } s = t_{j+1}, \end{cases}$$

with $t_0 = 0$ and $t_{m+1} = t$, are independent Brownian bridges over the intervals $[t_j, t_{j+1}]$. Introduce the random variables

$$M^j = \sup_{t_j \leq s \leq t_{j+1}} \beta_s^j, \quad M^{j,n} = \sup_{k \in I_n^j} \beta_{\frac{kt}{n}}^j,$$

where $I_n^j = \left\{k \in \mathbb{N} \mid t_j \leq \frac{kt}{n} \leq t_{j+1}\right\}$. Conditionally on A_m, the random variables M^j are independent and each of them admits a density. Therefore, with probability one, one of them has to be strictly larger than the others. For $j = 0$,..., m, set

$$A_m^j = \{M^j > M^i \text{ for } i \neq j\}.$$

Conditionally on A_m, we have

$$f\left(\sqrt{n}(M_t - M_t^n)\right) Z = \sum_{j=0}^{m} \mathbf{1}_{A_m^j} f\left(\sqrt{n}(\sigma M^j - M_t^n)\right) G_j(\beta^0, \ldots, \beta^m),$$

for some bounded Borel functions G_j defined on the space $\prod_{j=0}^m C([t_j, t_{j+1}])$. Now, on the set A_m^j, we have, for n large enough, $M_t^n = \sigma M^{j,n}$. This follows from the fact that the maximum of β^j is attained at an interior point of the interval (t_j, t_{j+1}) and the fact that for n large enough, some elements of I_n^j are arbitrarily close to this point. Therefore, for n large enough, we have

$$f\left(\sqrt{n}(M_t - M_t^n)\right) Z = \sum_{j=0}^{m} \mathbf{1}_{A_m^j} f\left(\sigma \epsilon_n^j\right) G_j(\beta^0, \ldots, \beta^m),$$

with $\epsilon_n^j = \sqrt{n}(M^j - M_t^{j,n})$. We deduce from Theorem 4.1 and the independence of the Brownian bridges that

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left(f\left(\sqrt{n}(M_t - M_t^n)\right) Z \mid A_m\right) &= \sum_{j=0}^{m} \lim_{n\to\infty} \mathbb{E}\left(\mathbf{1}_{A_m^j} f\left(\sigma \epsilon_n^j\right) G_j(\beta^0, \ldots, \beta^m) \mid A_m\right) \\
&= \sum_{j=0}^{m} \mathbb{E}\left(f\left(\sigma\sqrt{t}W\right)\right) \mathbb{E}\left(\mathbf{1}_{A_m^j} G_j(\beta^0, \ldots, \beta^m) \mid A_m\right) \\
&= \mathbb{E}\left(f\left(\sigma\sqrt{t}W\right)\right) \mathbb{E}(Z \mid A_m).
\end{aligned}$$

Hence, for all $m \geq 1$,

$$\lim_{n\to\infty} \mathbb{E}\left(f\left(\sqrt{n}(M_t - M_t^n)\right) Z \mid N_t = m\right) = \mathbb{E}\left(f\left(\sigma\sqrt{t}W\right)\right) \mathbb{E}(Z \mid N_t = m),$$

so that (4.2) follows easily. ◇

In order to use the convergence in distribution above, we sometimes need to switch between limit and expected value. For that purpose, the following result of uniform integrability will be useful.

LEMMA 4.3. *Let X be a finite activity Lévy process with generating triplet (γ, σ^2, ν), satisfying $\sigma > 0$. Fix $t > 0$ and set $\epsilon_n = M_t - M_t^n$. Then the sequence $(\sqrt{n}\epsilon_n e^{-M_t})_{n\geq 1}$ is uniformly integrable. If in addition $\mathbb{E}e^{qM_t} < \infty$ for some $q > 2$, then the sequence $(\sqrt{n}\epsilon_n e^{M_t})_{n\geq 1}$ is uniformly integrable.*

Proof of lemma 4.3. We will prove that $(\sqrt{n}\epsilon_n e^{M_t})_{n\geq 1}$ is uniformly integrable. The other case can be easily deduced. We will use the same notations as in the proof of Theorem 4.2. Note that on the set $\{N_t = 0\}$, we have $X_s = \gamma_0 s + \sigma B_s$ for $0 \leq s \leq t$, so that the uniform integrability of the sequence $(\sqrt{n}\epsilon_n e^{M_t}\mathbf{1}_{\{N_t=0\}})_{n\geq 1}$ follows from Lemma 6 in [2]. On the event $\{N_t \geq 1\}$, we will need to rule out the case when there is no jump between two mesh-points. So, we introduce the event

$$\Lambda_n = \{N_t \geq 1 \text{ and } \exists j \in \{1, \ldots, N_t\}\ T_j - T_{j-1} \leq t/n\} \cup \{t - T_{N_t} \leq t/n\}.$$

Note that

$$\begin{aligned}
\mathbb{P}(\Lambda_n) &\leq \mathbb{P}(t - T_{N_t} \leq t/n) + \mathbb{E}\sum_{j=1}^{N_t} \mathbf{1}_{\{T_j - T_{j-1} \leq t/n\}} \\
&\leq \mathbb{E}N_t(N_t + 1)/n,
\end{aligned}$$

where we have used the inequalities $\mathbb{P}(t - T_{N_t} \leq t/n \mid N_t = l) \leq l/n$ and $\mathbb{P}(T_j - T_{j-1} \leq t/n \mid N_t = l) \leq l/n$ (cf. [7], Proposition 5.5).

Therefore, we have, using $\epsilon_n \leq M_t$ and Hölder's inequality,

$$\mathbb{E}\left(\sqrt{n}\epsilon_n e^{M_t}\mathbf{1}_{\Lambda_n}\right) \leq \sqrt{n}\left(\mathbb{E}M_t^p e^{pM_t}\right)^{\frac{1}{p}} \left(\mathbb{P}(\Lambda_n)\right)^{1-\frac{1}{p}},$$

for every $p > 1$. Since $\mathbb{E}e^{qM_t} < \infty$ for some $q > 2$, we can choose $p > 2$. Hence

$$\lim_{n\to\infty} \mathbb{E}\left(\sqrt{n}\epsilon_n e^{M_t}\mathbf{1}_{\Lambda_n}\right) = 0.$$

Now, we want to prove that the sequence $(\sqrt{n}\epsilon_n e^{M_t}\mathbf{1}_{\{N_t\geq 1\}\cap\Lambda_n^c})_{n\geq 1}$ is uniformly integrable.

Fix $m \geq 1$ and $t_1,\ldots,\ t_m$ satisfying $0 < t_1 < \ldots < t_m < t$. Conditionaly on $\{N_t = m, T_1 = t_1, \ldots, T_m = t_m\} \cap \Lambda_n^c$, we have, with probability one,

$$\epsilon_n = \sum_{j=0}^{m} \left(M^j - M_t^n\right)\mathbf{1}_{\{M^j > \max_{i\neq j} M^i\}},$$

where $M^j = \sup_{t_j\leq s<t_{j+1}} X_s$, $t_0 = 0$ and $t_{m+1} = t$. Moreover, due to the definition of Λ_n, each subinterval $[t_j, t_{j+1})$ contains at least one mesh point. Denote

$$\begin{aligned} k_j &= \min\{k \in \{0,1,\ldots,n\} \mid kt/n \geq t_j\} \\ l_j &= \max\{k \in \{0,1,\ldots,n\} \mid kt/n \leq t_{j+1}\}, \end{aligned}$$

and let s^* be a point at which the supremum of X_s over $[t_j, t_{j+1})$ is attained. If $s^* \in (t_j, k_jt/n)$, we can write $M^j - M_t^n \leq \sup_{s\in(t_j,k_jt/n)}(X_s - X_{k_jt/n})$. If $s^* \in (l_jt/n, t_{j+1})$, we have $M^j - M_t^n \leq \sup_{s\in(l_jt/n,t_{j+1})}(X_s - X_{l_jt/n})$. Hence

$$M^j - M_t^n \leq \delta_{n,j} + \epsilon_{n,j} + \eta_{n,j},$$

where

$$\delta_{n,j} = \sup_{s\in(t_j,k_jt/n)}(X_s - X_{k_jt/n}), \quad \eta_{n,j} = \sup_{s\in(l_jt/n,t_{j+1})}(X_s - X_{l_jt/n}),$$

and

$$\epsilon_{n,j} = \sup_{k_jt/n\leq s\leq l_jt/n} X_s - \max_{k_j\leq k\leq l_j} X_{kt/n}.$$

Observe that

$$\begin{aligned} \delta_{n,j} &= \sup_{s\in(t_j,k_jt/n)}\left[\gamma_0 s + \sigma B_s - \left(\gamma_0\frac{k_jt}{n} + \sigma B_{k_jt/n}\right)\right] \\ &\leq |\gamma_0|\frac{t}{n} + \sigma \sup_{s\in(t_j,k_jt/n)}\left|B_s - B_{k_jt/n}\right|. \end{aligned} \tag{4.3}$$

Similarly,

$$\eta_{n,j} \leq |\gamma_0|\frac{t}{n} + \sigma \sup_{s\in(l_jt/n,t_{j+1})}\left|B_s - B_{l_jt/n}\right|. \tag{4.4}$$

Note that $|t_j - k_jt/n| \leq t/n$ and $t_{j+1} - l_jt/n \leq t/n$. Therefore, we easily deduce from (4.3) (resp. (4.4)) that the conditional expectation of any power of $\sqrt{n}\delta_{n,j}$ (resp. $\sqrt{n}\eta_{n,j}$) is bounded by a constant which is independent of the conditioning. We also have

$$\epsilon_{n,j} = \sup_{0\leq s\leq(l_j-k_j)t/n} \beta_s^j - \max_{0\leq k\leq l_j-k_j} \beta_{kt/n}^j,$$

where $\beta_s^j = \gamma_0 s + \sigma(B_{s+k_j t/n} - B_{k_j t/n})$. Using Lemma 6 of [2], we see that the conditional expectation of any power of $\sqrt{n}\epsilon_{n,j}$ is bounded by a constant which is independent of the conditioning. We conclude from this discussion that, for any $p > 1$,

$$\mathbb{E}\left[\left(\sqrt{n}\epsilon_n \mathbf{1}_{\Lambda_n^c \cap \{N_t \geq 1\}}\right)^p \mid N_t\right] \leq C_p N_t^p,$$

where C_p is a deterministic constant which depends only on p, γ_0, σ and t. The uniform integrability of $\sqrt{n}\epsilon_n e^{M_t}$ follows easily. $\diamond$

5. Continuity correction. In this section, we extend the results of Broadie-Glasserman-Kou (1999) on lookback and hindsight options to the jump-diffusion model. Let $(S_t)_{t\in[0,T]}$ be the price of a security modeled as a stochastic process on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$. The σ-algebra $\mathcal{F}_t$ represents the historical information on the price until time t. Under the exponential Lévy model, the process S behaves as the exponential of a Lévy process

$$S_t = S_0 e^{X_t},$$

where X is a Lévy process with generating triplet (γ, σ^2, ν). The considered probability is a risk-neutral probability, under which the process $\left(e^{-(r-\delta)t} S_t\right)_{t\in[0,T]}$ is a martingale. The parameter r is the risk-free interest rate, and δ is the dividend rate. The options we will consider in the sequel will have as underlying the asset with price S. We will denote by K the strike price of the option (in the case of hindsight options). Figure 5.1 gives the payoffs of lookback and hindsight options. The corresponding prices are the expected values of the discounted payoffs.

Option	continuous	discrete
Lookback call	$S_T - S_0 e^{m_T}$	$S_T - S_0 e^{m_T^n}$
Lookback put	$S_0 e^{M_T} - S_T$	$S_0 e^{M_T^n} - S_T$
Hindsight call	$\left(S_0 e^{M_T} - K\right)^+$	$\left(S_0 e^{M_T^n} - K\right)^+$
Hindsight put	$(K - S_0 e^{m_T})^+$	$\left(K - S_0 e^{m_T^n}\right)^+$

FIG. 5.1. *The payoffs of lookback and hindsight options.*

The r.v. m_T and m_T^n in table 5.1 satisfy

$$m_T = \inf_{0\leq s\leq T} X_s, \; m_T^n = \min_{0\leq k\leq n} X_{k\Delta t},$$

where $\Delta t = \frac{T}{n}$. The results we are going to show depend on the assumptions made on the process X. That is why we need to introduce the following assumptions:

H1 X is an integrable Lévy process with finite activity, satisfying $\sigma > 0$ and there exists $q > 2$ such that $\mathbb{E}e^{qM_T} < \infty$;

H2 X is an integrable Lévy process with finite activity, satisfying $\sigma > 0$.

Let W be the r.v. defined in theorem 4.1. We set $\beta_1 = \mathbb{E}W = -\frac{\zeta\left(\frac{1}{2}\right)}{\sqrt{2\pi}}$, where ζ is the Riemann zeta function.

At a given time $t \in [0, T)$, the value of the continuous lookback put is given by

$$V(S_+) = e^{-r(T-t)}\mathbb{E}\max\left(S_+, \max_{t\leq u\leq T} S_u\right) - S_t e^{-\delta(T-T)},$$

where $S_+ = \max_{0 \le u \le t} S_u$ is the predetermined maximum. The continuous value of the lookback call will depend similarly on $S_- = \min_{0 \le u \le t} S_u$ (the predetermined minimum) and on $\min_{t \le u \le T} S_u$. The price of the discrete lookback put at the k-th fixing date is given by

$$V_n\left(S_+\right) = e^{-r\Delta(n-k)}\mathbb{E}\max\left(S_+, \max_{k \le j \le n} S_{j\Delta t}\right) - S_{k\Delta t}e^{-\delta(n-k)\Delta t},$$

where $S_+ = \max_{0 \le j \le k} S_{j\Delta t}$. The discrete call value will depend similarly on $S_- = \min_{0 \le j \le k} S_{j\Delta t}$ and on $\min_{k \le j \le n} S_{j\Delta t}$.

Proposition 5.1. *The price of a discrete lookback option at the k-th fixing date and the price of the continuous lookback option at $k\Delta t$ satisfy*

$$V_n\left(S_\pm\right) = e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}V\left(S_\pm e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}\right) \pm \left(e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}} - 1\right)e^{-\delta(T-t)}S_t + o\left(\frac{1}{\sqrt{n}}\right)$$

$$V\left(S_\pm\right) = e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}V_n\left(S_\pm e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}\right) \pm \left(e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}} - 1\right)e^{-\delta(T-t)}S_t + o\left(\frac{1}{\sqrt{n}}\right),$$

where in $\pm$ and $\mp$, the top case applies for puts and the bottom case for calls. The relations for the put are true under $H1$, and those for the call under $H2$. These formulas are the same as those found by Broadie, Glasserman and Kou (1999) for the Black-Scholes model.

Proof of proposition 5.1. Since we have theorem 4.2 and lemma 4.3, the proofs of the above proposition is similar to the proof of theorem 3 of [4]. For example to relate discrete lookback put with respect to continuous lookback put, we need to prove that for $x \in \mathbb{R}$

$$\mathbb{E}\left(e^{M_T^n} - x\right)^+ = e^{-\beta_1\sigma\sqrt{\frac{T}{n}}}\mathbb{E}\left(e^{M_T} - e^{\beta_1\sigma\sqrt{\frac{T}{n}}}x\right)^+ + o\left(\frac{1}{\sqrt{n}}\right).$$

In fact we have to show first that

$$\mathbb{E}\left(e^{M_T} - x\right)^+ = \mathbb{E}\left(e^{M_T} - e^{M_T^n}\right)\mathbb{1}_{\left\{e^{M_T} > x\right\}} + \mathbb{E}\left(e^{M_T^n} - x\right)^+ + \mathbb{E}\left(e^{M_T^n} - x\right)\mathbb{1}_{\left\{e^{M_T^n} \le x < e^{M_T}\right\}}.$$

So

$$\mathbb{E}\left(e^{M_T^n} - x\right)^+ = \mathbb{E}\left(e^{M_T} - x\right)^+ - \mathbb{E}\left(e^{M_T} - e^{M_T^n}\right)\mathbb{1}_{\left\{e^{M_T} > x\right\}} - \mathbb{E}\left(e^{M_T^n} - x\right)\mathbb{1}_{\left\{e^{M_T^n} \le x < e^{M_T}\right\}}.$$

But

$$\begin{aligned}\mathbb{E}\left|e^{M_T^n} - x\right|\mathbb{1}_{\left\{e^{M_T^n} \le x < e^{M_T}\right\}} &\le \mathbb{E}\left(e^{M_T} - e^{M_T^n}\right)\mathbb{1}_{\left\{e^{M_T^n} \le x < e^{M_T}\right\}} \\ &\le \mathbb{E}\left(M_T - M_T^n\right)e^{M_T}\mathbb{1}_{\left\{e^{M_T^n} \le x < e^{M_T}\right\}}.\end{aligned}$$

Moreover the sequence

$$\left(\sqrt{n}\left(M_T - M_T^n\right)e^{M_T}\mathbb{1}_{\left\{e^{M_T^n} \le x < e^{M_T}\right\}}\right)_{n \ge 1}$$

is uniformly integrable (by lemma 4.3). So

$$\lim_{n\to+\infty} \mathbb{E}\sqrt{n}\left(M_T - M_T^n\right) e^{M_T} \mathbb{1}_{\left\{e^{M_T^n} \leq x < e^{M_T}\right\}} = 0.$$

On the other hand, using theorem 4.2 and lemma 4.3, we get

$$E\left(e^{M_T} - e^{M_T^n}\right) \mathbb{1}_{\left\{e^{M_T} > x\right\}} = \sigma\beta_1\sqrt{\frac{T}{n}}\mathbb{E}e^{M_T}\mathbb{1}_{\left\{e^{M_T} > x\right\}} + o\left(\frac{1}{\sqrt{n}}\right).$$

Thus

$$\begin{aligned}
\mathbb{E}\left(e^{M_T^n} - x\right)^+ &= \mathbb{E}\left(e^{M_T} - x\right)^+ - \sigma\beta_1\sqrt{\frac{T}{n}}\mathbb{E}e^{M_T}\mathbb{1}_{\left\{e^{M_T} > x\right\}} + o\left(\frac{1}{\sqrt{n}}\right) \\
&= e^{-\sigma\beta_1\sqrt{\frac{T}{n}}}\mathbb{E}\left(e^{M_T} - xe^{\sigma\beta_1\sqrt{\frac{T}{n}}}\right)\mathbb{1}_{\left\{e^{M_T} > x\right\}} + o\left(\frac{1}{\sqrt{n}}\right) \\
&= e^{-\sigma\beta_1\sqrt{\frac{T}{n}}}\mathbb{E}\left(e^{M_T} - xe^{\sigma\beta_1\sqrt{\frac{T}{n}}}\right)\mathbb{1}_{\left\{x < e^{M_T} \leq xe^{\sigma\beta_1\sqrt{\frac{T}{n}}}\right\}} \\
&\quad + e^{-\sigma\beta_1\sqrt{\frac{T}{n}}}\mathbb{E}\left(e^{M_T} - xe^{\sigma\beta_1\sqrt{\frac{T}{n}}}\right)^+ + o\left(\frac{1}{\sqrt{n}}\right).
\end{aligned}$$

But, we can show that

$$\mathbb{E}\left(e^{M_T} - xe^{\sigma\beta_1\sqrt{\frac{T}{n}}}\right)\mathbb{1}_{\left\{x < e^{M_T} \leq xe^{\sigma\beta_1\sqrt{\frac{T}{n}}}\right\}} = o\left(\frac{1}{\sqrt{n}}\right).$$

Hence

$$\mathbb{E}\left(e^{M_T^n} - x\right)^+ = e^{-\sigma\beta_1\sqrt{\frac{T}{n}}}\mathbb{E}\left(e^{M_T} - xe^{\sigma\beta_1\sqrt{\frac{T}{n}}}\right)^+ + o\left(\frac{1}{\sqrt{n}}\right).$$

The others cases can be derived in the same way. Detailed proofs are given in [7]. $\diamond$

For hindsight options, we have similar results as for the lookback case. The price of a continuous hindsight call option at time t with a predetermined maximum S_+ and strike K is

$$V\left(S_+, K\right) = e^{-r(T-t)}\mathbb{E}\left(\max\left(S_+, \max_{t\leq u\leq T} S_u\right) - K\right)^+.$$

Similarly, for the put we have

$$V\left(S_-, K\right) = e^{-r(T-t)}\mathbb{E}\left(K - \min\left(S_-, \min_{t\leq u\leq T} S_u\right)\right)^+.$$

The discrete versions at the k-th fixing date are

$$V_n\left(S_+, K\right) = e^{-r\Delta t(n-k)}\mathbb{E}\left(\max\left(S_+, \max_{k\leq j\leq n} S_{j\Delta t}\right) - K\right)^+$$

and

$$V_n\left(S_-, K\right) = e^{-r\Delta t(n-k)}\mathbb{E}\left(K - \min\left(S_-, \min_{k\leq j\leq n} S_{j\Delta t}\right)\right)^+.$$

Proposition 5.2. *The prices of a discrete hindsight option at the k-th fixing date and its continuous version at $k\Delta t$, satisfy*

$$V_n\left(S_\pm, K\right) = e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}} V\left(S_\pm e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}, Ke^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}}\right) + o\left(\frac{1}{\sqrt{n}}\right)$$

and

$$V\left(S_\pm, K\right) = e^{\pm\beta_1\sigma\sqrt{\frac{T}{n}}} V_n\left(S_\pm e^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}, Ke^{\mp\beta_1\sigma\sqrt{\frac{T}{n}}}\right) + o\left(\frac{1}{\sqrt{n}}\right),$$

where in $\pm$ and $\mp$, the top case applies for calls and the bottom for puts. The relations for the calls are true under $H1$, and those for the put under $H2$. To explain the above proposition one can say that, in order to price a continuous (resp. discrete) hindsight option using a discrete (resp. continuous) one, we must shift the predetermined extremum and the strike. Proposition 5.2 can be deduced from proposition 5.1, thanks to the relations between lookback and hindsight options.

Remark 5.3. If the process X is an integrable Lévy process with generating triplet $(\gamma, 0, \nu)$, satisfying $\nu(\mathbb{R}) < \infty$, then the price of a discrete lookback option and its continuous version at time $k\Delta t$ satisfy

1. for the call

$$V_n\left(S_-\right) = V\left(S_-\right) + \frac{\alpha}{n} + o\left(\frac{1}{n}\right),$$

where the constant α can be derived explicitly,

2. for the put, if there exists $\beta > 1$ such that $\mathbb{E}e^{\beta M_T} < \infty$, then

$$V_n\left(S_+\right) = V\left(S_+\right) + o\left(\frac{1}{n^{\frac{\beta-1}{\beta}}}\right).$$

The proof of these results can be found in [7].

6. Upper bounds. In the infinite activity case and if there is no Brownian part, the prices of the discrete and continuous calls are close to each other. The following proposition is a consequence of theorems 3.9 and 3.12.

Proposition 6.1. *Suppose that X is an integrable infinite activity Lévy process with generating triplet $(\gamma, 0, \nu)$. Then the prices of a discrete call option at the k^{th} fixing date and its continuous version at $k\Delta t$ satisfy*

1.

$$V_n\left(S_-\right) = V\left(S_-\right) + o\left(\frac{1}{\sqrt{n}}\right).$$

2. If $\int_{|x|\leq 1} |x|\nu(dx) < \infty$,

$$V_n\left(S_-\right) = V\left(S_-\right) + O\left(\frac{\log(n)}{n}\right).$$

3. If $\int_{|x|\leq 1} |x|\log(|x|)\nu(dx) < \infty$,

$$V_n\left(S_-\right) = V\left(S_-\right) + O\left(\frac{1}{n}\right).$$

In the put case, the error between continuous and discrete prices depends on the integrability of the exponential of the supremum of the Lévy process driving the underlying asset.

THEOREM 6.2. *Suppose that X is an infinite activity Lévy process with generating triplet $(\gamma, 0, \nu)$ and there exists $\beta > 1$ such that $\mathbb{E}e^{\beta M_T} < \infty$. Then the price of a discrete put option at the k-th fixing date and its continuous version at $k\Delta t$, satisfy*

1. *We have, for any $\epsilon > 0$,*

$$V_n(S_+) = V(S_+) + O\left(\frac{1}{n^{\frac{\beta-1}{2\beta}-\epsilon}}\right).$$

2. *If $\int_{|x|\le 1}|x|\nu(dx) < \infty$, we have, for any $\epsilon > 0$,*

$$V_n(S_+) = V(S_+) + O\left(\left(\frac{\log(n)}{n}\right)^{\frac{\beta-1}{\beta}-\epsilon}\right).$$

3. *If $\int_{|x|\le 1}|x|\log(|x|)\nu(dx) < \infty$, we have, for any $\epsilon > 0$,*

$$V_n(S_+) = V(S_+) + O\left(\frac{1}{n^{\frac{\beta-1}{\beta}-\epsilon}}\right).$$

The main technical difficulty for the proof of theorem 6.2 consists of deducing an estimate of $\mathbb{E}\left(e^{M_T} - e^{M_T^n}\right)$ from an estimate of $\mathbb{E}\left(M_T - M_T^n\right)$. In fact, the theorem can be deduced from the following lemma.

LEMMA 6.3. *Assume that X is an infinite activity Lévy process with generating triplet $(\gamma, 0, \nu)$ and there exists $\beta > 1$ such that $\mathbb{E}e^{\beta M_T} < \infty$. Then for any $\epsilon > 0$*

$$\mathbb{E}\left(e^{M_T} - e^{M_T^n}\right) \le C\left(\mathbb{E}\left(M_T - M_T^n\right)\right)^{\frac{\beta-1}{\beta}-\epsilon},$$

where C is a positive constant.

Proof of lemma 6.3. By the convexity of the exponential function, we have

$$e^{M_T} - e^{M_T^n} \le \left(M_T - M_T^n\right)e^{M_T}.$$

So, by Hölder's inequality,

$$\mathbb{E}\left(e^{M_T} - e^{M_T^n}\right) \le \left(\mathbb{E}e^{\beta M_T}\right)^{\frac{1}{\beta}}\left(\mathbb{E}\left(M_T - M_T^n\right)^{\frac{\beta}{\beta-1}}\right)^{\frac{\beta-1}{\beta}}.$$

Note that $\mathbb{E}e^{\beta M_T} < \infty$ implies that $\mathbb{E}M_T^q < \infty$ for any $q > 0$. Let $\rho \in]0, 1[$, we have

$$\begin{aligned}
\mathbb{E}\left(M_T - M_T^n\right)^{\frac{\beta}{\beta-1}} &= \mathbb{E}\left(M_T - M_T^n\right)^{\rho}\left(M_T - M_T^n\right)^{\frac{\beta}{\beta-1}-\rho} \\
&= \mathbb{E}\left(M_T - M_T^n\right)^{\rho}\left(M_T - M_T^n\right)^{\frac{\beta(1-\rho)+\rho}{\beta-1}} \\
&\le \left(\mathbb{E}\left(M_T - M_T^n\right)\right)^{\rho}\left(\mathbb{E}\left(M_T - M_T^n\right)^{\frac{\beta(1-\rho)+\rho}{(\beta-1)(1-\rho)}}\right)^{1-\rho}.
\end{aligned}$$

Hence, from the fact that $\lim_{n\to+\infty}\mathbb{E}\left(M_T - M_T^n\right)^{\frac{\beta(1-\rho)+\rho}{(\beta-1)(1-\rho)}} = 0$, there exists a constant $C > 0$ such that

$$\begin{aligned}
\mathbb{E}\left(e^{M_T} - e^{M_T^n}\right) &\le C\left(\mathbb{E}\left(M_T - M_T^n\right)\right)^{\rho\frac{\beta-1}{\beta}} \\
&= C\left(\mathbb{E}\left(M_T - M_T^n\right)\right)^{\frac{\beta-1}{\beta}-(1-\rho)\frac{\beta-1}{\beta}}.
\end{aligned}$$

Then for any $\epsilon > 0$, there exists a constant $C > 0$ such that

$$\mathbb{E}\left(e^{M_T} - e^{M_T^n}\right) \leq C\left(\mathbb{E}\left(M_T - M_T^n\right)\right)^{\frac{\beta-1}{\beta}-\epsilon}.$$

◇

When the Lévy process driving the underlying asset has no positive jumps, we get tighter estimates.

PROPOSITION 6.4. *Let X be a Lévy process with generating triplet (γ, σ^2, ν). We assume that X has no positive jump ($\nu(0, +\infty) = 0$), that $\int_{-1\leq x<0} |x|\nu(dx) < \infty$ and that there exists $\beta > 1$ such that $\mathbb{E}e^{\beta M_T} < \infty$. Then, the price of a discrete put lookback at the k-th fixing date and its continuous version at time $k\Delta t$, satisfy*

1. *if $\sigma = 0$*

$$V_n(S_+) = V(S_+) + O\left(\frac{1}{n}\right).$$

2. *if $\sigma > 0$*

$$V_n(S_+) = V(S_+) + O\left(\frac{\log(n)}{\sqrt{n}}\right).$$

Proposition 6.4 is based on the estimation of the moments of $M_T - M_T^n$, which can be performed when there are no positive jumps.

LEMMA 6.5. *Let X be a Lévy process with generating triplet (γ, σ^2, ν), satisfying $\int_{|x|\leq 1} |x|\nu(dx) < \infty$. We suppose that X has no positive jumps, then for any $\beta > 1$, we have*

1. *if $\sigma = 0$,*

$$\mathbb{E}\left(M_T - M_T^n\right)^{\beta} = O\left(\frac{1}{n^{\beta}}\right).$$

2. *if $\sigma > 0$,*

$$\mathbb{E}\left(M_T - M_T^n\right)^{\beta} = O\left(\left(\frac{\log(n)}{\sqrt{n}}\right)^{\beta}\right).$$

Proof of lemma 6.5. We have

$$\begin{aligned}
M_T - M_T^n &= \sup_{0\leq s\leq T} X_s - \max_{0\leq k\leq n} X_{\frac{kT}{n}} \\
&= \max_{1\leq k\leq n} \sup_{\frac{(k-1)T}{n}\leq s\leq \frac{kT}{n}} X_s - \max_{0\leq k\leq n} X_{\frac{kT}{n}} \\
&\leq \max_{1\leq k\leq n} \sup_{\frac{(k-1)T}{n}\leq s\leq \frac{kT}{n}} X_s - \max_{1\leq k\leq n} X_{\frac{(k-1)T}{n}} \\
&\leq \max_{1\leq k\leq n} \left(\sup_{\frac{(k-1)T}{n}\leq s\leq \frac{kT}{n}} X_s - X_{\frac{(k-1)T}{n}} \right),
\end{aligned}$$

where the random variables $\left(\sup_{\frac{(k-1)T}{n}\leq s\leq \frac{kT}{n}} X_s - X_{\frac{(k-1)T}{n}}\right)_{1\leq k\leq n}$ are i.i.d., with the same distribution as $\sup_{0\leq s\leq \frac{T}{n}} X_s$. But, since X has no positive jumps, we have (see

(2.5))

$$\begin{aligned}\sup_{0\leq s\leq \frac{T}{n}} X_s &\leq \sup_{0\leq s\leq \frac{T}{n}} (\gamma_0 s+\sigma B_s)\\ &\leq \frac{|\gamma_0|T}{n}+\sigma \sup_{0\leq s\leq \frac{T}{n}} B_s.\end{aligned}$$

We can easily deduce the first result of the lemma ($\sigma=0$). In the case $\sigma>0$, we have

$$\begin{aligned}\sup_{0\leq s\leq \frac{T}{n}} X_s &\leq \frac{1}{\sqrt{n}}\left(\frac{|\gamma_0|T}{\sqrt{n}}+\sigma\sqrt{n}\sup_{0\leq s\leq \frac{T}{n}} B_s\right)\\ &\leq \frac{1}{\sqrt{n}}\left(|\gamma_0|T+\sigma\sqrt{n}\sup_{0\leq s\leq \frac{T}{n}} B_s\right)\\ &=^d \frac{1}{\sqrt{n}}\left(|\gamma_0|T+\sigma\sup_{0\leq s\leq T} B_s\right).\end{aligned}$$

Let $(V_k)_{1\leq k\leq n}$ be i.i.d. r.v. with the same distribution as $|\gamma_0|T+\sigma\sup_{0\leq s\leq T} B_s$. Then we have

$$\mathbb{E}\left(M_T-M_T^n\right)^\beta \leq \left(\frac{1}{\sqrt{n}}\right)^\beta \mathbb{E}\max_{1\leq k\leq n} V_k^\beta.$$

Let g be the function defined as follows

$$g(x)=(\log(x))^\beta,\quad x>1.$$

The function g is concave and non-decreasing on the set $[e^{\beta-1},+\infty)$. So we have

$$\begin{aligned}\mathbb{E}\sup_{1\leq k\leq n} V_k^\beta &= \mathbb{E}\sup_{1\leq k\leq n} g\left(e^{V_k}\right)\\ &=\mathbb{E}g\left(\sup_{1\leq k\leq n} e^{V_k}\right),\ \text{because g is non-decreasing}\\ &\leq \mathbb{E}g\left(\sup_{1\leq k\leq n} e^{\max(V_k,\beta-1)}\right),\ \text{because g is non-decreasing}\\ &\leq g\left(\mathbb{E}\sup_{1\leq k\leq n} e^{\max(V_k,\beta-1)}\right),\ \text{by Jensen}\\ &\leq g\left(\mathbb{E}\sum_{k=1}^n e^{\max(V_k,\beta-1)}\right),\ \text{because g is non-decreasing}\\ &\leq g\left(n\mathbb{E}e^{\max(V_1,\beta-1)}\right).\end{aligned}$$

Note that we have $\mathbb{E}e^{\max(V_1,\beta-1)}<\infty$. Hence the second result of the lemma. ◇

Proof of proposition 6.4. To prove proposition 6.4, we need to show that

$$\mathbb{E}\left(e^{M_T}-e^{M_T^n}\right)=\begin{cases} O\left(\frac{1}{n}\right) & if\ \sigma=0\\ O\left(\frac{\log(n)}{\sqrt{n}}\right) & if\ \sigma>0\end{cases}$$

But by the convexity of the exponential function, we have

$$e^{M_T} - e^{M_T^n} \le e^{M_T} \left(M_T - M_T^n\right).$$

So using Hölder's inequality, we get

$$\mathbb{E}\left(e^{M_T} - e^{M_T^n}\right) \le \left(\mathbb{E}e^{\beta M_T}\right)^{\frac{1}{\beta}} \left(\mathbb{E}\left(M_T - M_T^n\right)^{\frac{\beta}{\beta-1}}\right)^{\frac{\beta-1}{\beta}}.$$

We conclude by lemma 6.5. ◇

Results for hindsight options are similar to those for lookback options. This is simply due to the relations between lookback and hindsight options.

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