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ERROR BOUNDS FOR SMALL JUMPS OF LÉVY PROCESSES

EL HADJ ALY DIA*

Abstract. The pricing of options in exponential Lévy models amounts to the computation of expectations of functionals of Lévy processes. In many situations, Monte-Carlo methods are used. However, the simulation of a Lévy process with infinite Lévy measure generally requires either to truncate small jumps or to replace them by a Brownian motion with the same variance. We will derive bounds for the errors generated by these two types of approximation.

Key words. Approximation of small jumps, Lévy processes, Skorokhod embedding, Spitzer identity

AMS subject classifications. 60G51, 65N15

JEL classification. C02, C15

1. Introduction. In the recent years, the use of general Lévy processes in financial models has grown extensively (see [2, 5, 11]). A variety of numerical methods have been subsequently developed, in particular methods based on Fourier analysis (see [4, 12, 13, 15]). Nonetheless, in many situations, Monte-Carlo methods have to be used. The simulation of a Lévy process with infinite Lévy measure is not straightforward, except in some special cases like the Gamma or Inverse Gaussian models. In practise, the small jumps of the Lévy process are either just truncated or replaced by a Brownian motion with the same variance (see [1, 7, 8, 16, 18]). The latter approach was introduced by Asmussen and Rosinsky [1], who showed that, under suitable conditions, the normalized cumulated small jumps asymptotically behave like a Brownian motion.

The purpose of this article is to derive bounds for the errors generated by these two methods of approximation in the computation of functions of Lévy processes at a fixed time or functionals of the whole path of Lévy processes. We also derive bounds for the cumulative distribution functions. These bounds can be used to determine which type of approximations to use, since replacing small jumps by Brownian is more time-consuming (if we use Monte Carlo methods). Our bounds can be applied to derive approximation errors for lookback, barrier, American or Asian options. But this latter point will not be developed, and is left to another paper.

The characteristic function of a real Lévy process X with generating triplet (γ, b^2, ν) is given by

$$\mathbb{E}e^{iuX_t} = \exp \left\{ t \left(i\gamma u - \frac{b^2 u^2}{2} + \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux \mathbb{1}_{|x| \leq 1}) \nu(dx) \right) \right\},$$

where $(\gamma, b) \in \mathbb{R} \times \mathbb{R}^+$, and ν is a Lévy measure. The process X is the independent sum of a drift term γt , a Brownian component bB_t , and a compensated jump part with Lévy measure ν . The process X has finite (resp. infinite) activity if $\nu(\mathbb{R}) < \infty$ (resp. $\nu(\mathbb{R}) = +\infty$).

For $0 < \epsilon \leq 1$, the process X^ϵ is defined by

$$X_t^\epsilon = \gamma t + bB_t + \sum_{0 \leq s \leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s| > \epsilon\}} - \int_{\epsilon < |x| \leq 1} x \nu(dx).$$

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The process X^ϵ is obtained (from X) by subtracting the compensated sum of jumps not exceeding ϵ in absolute value. Let

$$R^\epsilon = X - X^\epsilon. \quad (1.1)$$

The process R^ϵ is a Lévy process with characteristic function

$$\mathbb{E}e^{iuR_t^\epsilon} = \exp \left\{ t \int_{|x| \leq \epsilon} (e^{iux} - 1 - iux \mathbf{1}_{|x| \leq 1}) \nu(dx) \right\}.$$

Note that, we have

$$\mathbb{E}R_t^\epsilon = 0, \quad \text{Var}(R_t^\epsilon) = \sigma(\epsilon)^2 t,$$

where $\sigma(\epsilon) = \sqrt{\int_{|x| \leq \epsilon} x^2 \nu(dx)}$ and $\lim_{\epsilon \rightarrow 0} \sigma(\epsilon) = 0$. The behavior of $\sigma(\epsilon)$ when ϵ goes to 0 is known for classical models (VG, NIG, CGMY...). As noted in Example 2.3 of [1], if

$$\nu(dx) = |x|^{-1-\alpha} L(x) dx,$$

where L is slowly varying at 0 and $0 \leq \alpha < 2$, then it holds

$$\sigma(\epsilon) \sim \sqrt{\frac{L(-\epsilon) + L(\epsilon)}{2 - \alpha}} \epsilon^{1-\frac{\alpha}{2}}.$$

Note that, for $\alpha > 0$, we get $\lim_{\epsilon \rightarrow 0} \sigma(\epsilon)/\epsilon = +\infty$.

We also define the processes \hat{X}^ϵ by

$$\hat{X}_t^\epsilon = X_t^\epsilon + \sigma(\epsilon) \hat{W}_t, \quad t \geq 0,$$

where \hat{W} is a standard Brownian motion independent of X . We aim to study the behavior of the errors made by replacing X by X^ϵ or \hat{X}^ϵ , with respect to the level ϵ . These errors are studied for the process X at a fixed date and for its running supremum. Set, for any $t \geq 0$

$$M_t^X = \sup_{0 \leq s \leq t} X_s, \quad M_t^{\epsilon, X} = \sup_{0 \leq s \leq t} X_s^\epsilon, \quad \hat{M}_t^{\epsilon, X} = \sup_{0 \leq s \leq t} \hat{X}_s^\epsilon.$$

When there is no ambiguity we can remove the super index X . Moreover, unless stated otherwise, X is a Lévy process with generating triplet (γ, b^2, ν) .

The paper is organized as follows. In the next section, we will study the errors resulting from the truncation of the compensated sum of small jumps. The results of this section are based on estimates for the moments of R^ϵ . We also derive an estimate for the expectation of the difference of the supremum processes (M and M^ϵ), by using Sptzer's identity. The errors resulting from Brownian approximation are studied in section 3. The process X will be approximated by the process \hat{X}^ϵ . The main result of this section is Theorem 3.5, which states an error bound for the expectation of a function of the supremum. The proof of this result relies on the Skorohod embedding theorem.

2. Truncation of the compensated sum of small jumps. In this section, we will study the errors resulting from the approximation of X by X^ϵ . These errors are related to the moments of R^ϵ . Define

$$\sigma_0(\epsilon) = \max(\sigma(\epsilon), \epsilon). \quad (2.1)$$

The next result will be useful for many proofs in this paper.

PROPOSITION 2.1. *Let X be a Lévy process and R^ϵ defined in (1.1). Then*

$$\mathbb{E} |R_t^\epsilon|^4 = t \int_{|x| \leq \epsilon} x^4 \nu(dx) + 3(t\sigma(\epsilon)^2)^2,$$

and for any real $q > 0$

$$\mathbb{E} |R_t^\epsilon|^q \leq K_{q,t} \sigma_0(\epsilon)^q,$$

where $K_{q,t}$ is a positive constant which depends only on q and t .

Proof. We have

$$\mathbb{E} e^{iuR_t^\epsilon} = e^{\Psi_\epsilon(u)},$$

where $\Psi_\epsilon(u) = t \int_{|y| \leq \epsilon} (e^{iuy} - 1 - iuy) \nu(dy)$. Set

$$c_n(\epsilon) = \frac{1}{i^n} \frac{\partial^n \Psi_\epsilon}{\partial u^n}(0).$$

Using Proposition 3.13 of [7], we get

$$c_n(\epsilon) = t \int_{|x| \leq \epsilon} x^n \nu(dx), \quad \forall n \geq 2.$$

Note that $c_1(\epsilon) = \mathbb{E} R_t^\epsilon = 0$. Furthermore, by Theorem 2 of [14], we have

$$\mathbb{E} (R_t^\epsilon)^n = \sum_{k=0}^{n-1} C_{n-1}^k c_{n-k}(\epsilon) \mathbb{E} (R_t^\epsilon)^k. \quad (2.2)$$

Therefore

$$\begin{aligned} \mathbb{E} (R_t^\epsilon)^4 &= 3(c_2(\epsilon))^2 + c_4(\epsilon) \\ &= 3t^2 \sigma(\epsilon)^4 + t \int_{|x| \leq \epsilon} |x|^4 \nu(dx). \end{aligned}$$

Hence the first result of the proposition.

Set $n = [q/2]$. Since $0 \leq q/(2n) < 1$, using Jensen inequality, we get

$$\mathbb{E} |R_t^\epsilon|^q \leq \left(\mathbb{E} |R_t^\epsilon|^{2n} \right)^{\frac{q}{2n}}.$$

Thus, it suffices to prove that, for any $n \in \mathbb{N}$, there exists a constant $K_{n,t}$ such that

$$|\mathbb{E} (R_t^\epsilon)^n| \leq K_{n,t} \sigma_0(\epsilon)^n. \quad (2.3)$$

We will prove this inequality by induction. It is trivial for $n = 0, 1, 2$. Suppose that (2.3) holds for any $k < n$. We have, using (2.2)

$$\begin{aligned} |\mathbb{E}(R_t^\epsilon)^n| &= \left| \sum_{k=0}^{n-1} C_{n-1}^k c_{n-k}(\epsilon) \mathbb{E}(R_t^\epsilon)^k \right| \\ &\leq \max_{0 \leq k \leq n-1} (K_{k,t} C_{n-1}^k) \sum_{k=0}^{n-1} \sigma_0(\epsilon)^k |c_{n-k}(\epsilon)|. \end{aligned}$$

But

$$\begin{aligned} |c_{n-k}(\epsilon)| &\leq \int_{|x| \leq \epsilon} |x|^{n-k} \nu(dx) \\ &\leq \epsilon^{n-k-2} \int_{|x| \leq \epsilon} |x|^2 \nu(dx) \\ &\leq \sigma_0(\epsilon)^{n-k}. \end{aligned}$$

So

$$|\mathbb{E}(R_t^\epsilon)^n| \leq n \max_{0 \leq k \leq n-1} (K_{k,t} C_{n-1}^k) \sigma_0(\epsilon)^n.$$

This concludes the proof. \square

2.1. Estimates for smooth functions. Let X be a Lévy process and f a C -Lipschitz function where $C > 0$. Then

$$\begin{aligned} \mathbb{E}|f(X_t) - f(X_t^\epsilon)| &\leq C \mathbb{E}|X_t - X_t^\epsilon| \\ &= C \mathbb{E}|R_t^\epsilon| \\ &\leq C \sqrt{\mathbb{E}|R_t^\epsilon|^2}. \end{aligned}$$

Hence

$$\mathbb{E}|f(X_t) - f(X_t^\epsilon)| \leq C \sqrt{t} \sigma(\epsilon). \quad (2.4)$$

Note that we do not ask that $f(X_t)$ be integrable. If f is more regular, sharper estimates can be derived, as shown in the following proposition.

PROPOSITION 2.2. *Let X be an infinite activity Lévy process.*

1. *If $f \in C^1(\mathbb{R})$ and satisfies $\mathbb{E}|f'(X_t^\epsilon)| < \infty$, and if there exists $\beta > 1$ such that $\left(\sup_{\delta \in [0,1]} \mathbb{E} \left| f'(X_t^\delta + \theta R_t^\delta) - f'(X_t^\delta) \right|^\beta \right)^{\frac{1}{\beta}}$ is finite and integrable with respect to θ on $[0, 1]$, then*

$$\mathbb{E}(f(X_t) - f(X_t^\epsilon)) = o(\sigma_0(\epsilon)).$$

2. *If $f \in C^2(\mathbb{R})$ and satisfies $\mathbb{E}|f'(X_t^\epsilon)| + \mathbb{E}|f''(X_t^\epsilon)| < \infty$, and if there exists $\beta > 1$ such that $\left(\sup_{\delta \in [0,1]} \mathbb{E} \left| f''(X_t^\delta + \theta R_t^\delta) - f''(X_t^\delta) \right|^\beta \right)^{\frac{1}{\beta}}$ is finite and integrable with respect to θ on $[0, 1]$, then*

$$\mathbb{E}(f(X_t) - f(X_t^\epsilon)) = \frac{\sigma(\epsilon)^2 t}{2} \mathbb{E} f''(X_t^\epsilon) + o(\sigma_0(\epsilon)^2).$$

Note that, if f has bounded derivatives or f is the exponential function and $e^{\beta X_t}$ is integrable, where $\beta > 1$, the conditions in the above proposition are satisfied. Recall that the truncation of small jumps is used when $\nu(\mathbb{R}) = \infty$. In typical applications, we have $\liminf \sigma(\epsilon)/\epsilon > 0$, so that $o(\sigma_0(\epsilon)^2)$ is in fact $o(\sigma(\epsilon)^2)$.

Proof. Because $X_t - X_t^\epsilon \neq 0$ a.s. (Theorem 27.4 of [17]), we have

$$f(X_t) - f(X_t^\epsilon) = \int_0^1 f'(X_t^\epsilon + \theta R_t^\epsilon) R_t^\epsilon d\theta.$$

So

$$\begin{aligned} \mathbb{E}(f(X_t) - f(X_t^\epsilon)) &= \int_0^1 \mathbb{E} \left[\left(f'(X_t^\epsilon + \theta R_t^\epsilon) - f'(X_t^\epsilon) \right) R_t^\epsilon \right] d\theta \\ &\quad + \int_0^1 \mathbb{E} \left[f'(X_t^\epsilon) R_t^\epsilon \right] d\theta \\ &= \int_0^1 \mathbb{E} \left[\left(f'(X_t^\epsilon + \theta R_t^\epsilon) - f'(X_t^\epsilon) \right) R_t^\epsilon \right] d\theta, \end{aligned}$$

because R_t^ϵ and X_t^ϵ are independent and $\mathbb{E}R_t^\epsilon = 0$. Let $1 < \alpha < \beta$, by Hölder's inequality we have

$$\begin{aligned} \mathbb{E} \left| \left(f'(X_t^\epsilon + \theta R_t^\epsilon) - f'(X_t^\epsilon) \right) R_t^\epsilon \right| &\leq \left(\mathbb{E} \left| f'(X_t^\epsilon + \theta R_t^\epsilon) - f'(X_t^\epsilon) \right|^\alpha \right)^{\frac{1}{\alpha}} \\ &\quad \times \left(\mathbb{E} |R_t^\epsilon|^{\frac{\alpha}{\alpha-1}} \right)^{\frac{\alpha-1}{\alpha}}. \end{aligned}$$

By the assumption of uniform integrability

$$\lim_{\epsilon \rightarrow 0} \left(\mathbb{E} \left| f'(X_t^\epsilon + \theta R_t^\epsilon) - f'(X_t^\epsilon) \right|^\alpha \right)^{\frac{1}{\alpha}} = 0.$$

Hence using Proposition 2.1, we get

$$\mathbb{E}(f(X_t) - f(X_t^\epsilon)) = o(\sigma_0(\epsilon)).$$

We now prove the second part of the proposition. Using Taylor's formula we get

$$\begin{aligned} \mathbb{E}(f(X_t) - f(X_t^\epsilon)) &= \mathbb{E} \left[f'(X_t^\epsilon) (X_t - X_t^\epsilon) + \int_{X_t^\epsilon}^{X_t} f''(x) (X_t - x) dx \right] \\ &= \mathbb{E} \left[f'(X_t^\epsilon) R_t^\epsilon + \int_0^1 f''(X_t^\epsilon + \theta R_t^\epsilon) (1 - \theta) (R_t^\epsilon)^2 d\theta \right] \\ &= \mathbb{E} \left[\int_0^1 f''(X_t^\epsilon + \theta R_t^\epsilon) (1 - \theta) (R_t^\epsilon)^2 d\theta \right] \\ &= \mathbb{E} \left[\int_0^1 f''(X_t^\epsilon) (1 - \theta) (R_t^\epsilon)^2 d\theta \right] \\ &\quad + \mathbb{E} \left[\int_0^1 \left(f''(X_t^\epsilon + \theta R_t^\epsilon) - f''(X_t^\epsilon) \right) (1 - \theta) (R_t^\epsilon)^2 d\theta \right]. \end{aligned}$$

So

$$\begin{aligned} \mathbb{E}(f(X_t) - f(X_t^\epsilon)) &= \frac{\sigma(\epsilon)^2 t}{2} \mathbb{E} f''(X_t^\epsilon) \\ &\quad + \int_0^1 \mathbb{E} \left[\left(f''(X_t^\epsilon + \theta R_t^\epsilon) - f''(X_t^\epsilon) \right) (1 - \theta) (R_t^\epsilon)^2 \right] d\theta. \end{aligned}$$

Again, using Hölder's inequality, the assumption of uniform integrability and Proposition 2.1, we get

$$\mathbb{E}(f(X_t) - f(X_t^\epsilon)) = \frac{\sigma(\epsilon)^2 t}{2} \mathbb{E} f''(X_t^\epsilon) + o(\sigma_0(\epsilon)^2).$$

□

REMARK 2.3. Assume that X is an integrable infinite activity Lévy process and $t > 0$, $f \in C^1(\mathbb{R})$ and f' is Lipschitz. Then

$$\mathbb{E}(f(X_t) - f(X_t^\epsilon)) = O(\sigma(\epsilon)^2).$$

The proof is similar to the proof of Proposition 2.2. If f' is C -Lipschitz, and if we denote by f'' its a.e. derivative, we get $|f''| \leq C$. So we can prove that

$$\mathbb{E} \left[\left| f''(X_t^\epsilon + \theta R_t^\epsilon) - f''(X_t^\epsilon) \right| (R_t^\epsilon)^2 \right] \leq 2Ct\sigma(\epsilon)^2.$$

This concludes the proof.

We will consider now the case of the supremum process.

PROPOSITION 2.4. Let X be a Lévy process and f a K -Lipschitz function, then we have

$$\mathbb{E} |f(M_t) - f(M_t^\epsilon)| \leq 2K\sqrt{t}\sigma(\epsilon).$$

Proof. We have

$$\begin{aligned} \mathbb{E} \left| f \left(\sup_{0 \leq s \leq t} X_s \right) - f \left(\sup_{0 \leq s \leq t} X_s^\epsilon \right) \right| &\leq K \mathbb{E} \left| \sup_{0 \leq s \leq t} X_s - \sup_{0 \leq s \leq t} X_s^\epsilon \right| \\ &\leq K \mathbb{E} \sup_{0 \leq s \leq t} |R_s^\epsilon| \\ &\leq K \sqrt{\mathbb{E} \left(\sup_{0 \leq s \leq t} |R_s^\epsilon| \right)^2}. \end{aligned}$$

Note that R^ϵ is a càdlàg martingale. So, using Doob's inequality, we get

$$\begin{aligned} \mathbb{E} \left| f \left(\sup_{0 \leq s \leq t} X_s \right) - f \left(\sup_{0 \leq s \leq t} X_s^\epsilon \right) \right| &\leq 2K \sqrt{\mathbb{E} |R_t^\epsilon|^2} \\ &= 2K\sqrt{t}\sigma(\epsilon). \end{aligned}$$

□

REMARK 2.5. Suppose that X is a Lévy process and f a function from $\mathbb{R}^+ \times \mathbb{R}$ to \mathbb{R} , K -Lipschitz with respect to its second variable. Then

$$\left| \sup_{\tau \in \mathcal{T}_{[0,t]}} \mathbb{E} f(\tau, X_\tau) - \sup_{\tau \in \mathcal{T}_{[0,t]}} \mathbb{E} f(\tau, X_\tau^\epsilon) \right| \leq 2K\sqrt{t}\sigma(\epsilon),$$

where $\mathcal{T}_{[0,t]}$ denote the set of stopping times with values in $[0, t]$. The proof can be found in [9].

The bound in Proposition 2.4 might not be optimal. This is what suggests the following result.

THEOREM 2.6. *Let X be an integrable infinite activity Lévy process, then*

$$0 \leq \mathbb{E}(M_t - M_t^\epsilon) = o(\sigma(\epsilon)).$$

Proof. Using Spitzer's identity (see Proposition 1 in Section 3 of [10] for details), we have

$$\begin{aligned} \mathbb{E}(M_t - M_t^\epsilon) &= \int_0^t \frac{\mathbb{E}X_s^+}{s} ds - \int_0^t \frac{\mathbb{E}(X_s^\epsilon)^+}{s} ds \\ &= \int_0^t \mathbb{E} \left(X_s^+ - (X_s^\epsilon)^+ \right) \frac{ds}{s}. \end{aligned}$$

Set $I_s^\epsilon = \mathbb{E} \left(X_s^+ - (X_s^\epsilon)^+ \right)$. So

$$\begin{aligned} I_s^\epsilon &= \mathbb{E} \left((X_s^\epsilon + R_s^\epsilon)^+ - (X_s^\epsilon)^+ \right) \\ &= \mathbb{E} X_s^\epsilon \left(\mathbf{1}_{X_s^\epsilon + R_s^\epsilon > 0} - \mathbf{1}_{X_s^\epsilon > 0} \right) + \mathbb{E} R_s^\epsilon \mathbf{1}_{X_s^\epsilon + R_s^\epsilon > 0} \\ &= \mathbb{E} X_s^\epsilon \left(\mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} - \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \right) + \mathbb{E} R_s^\epsilon \mathbf{1}_{X_s^\epsilon > -R_s^\epsilon} \\ &= \mathbb{E} X_s^\epsilon \left(\mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} - \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \right) + \mathbb{E} R_s^\epsilon \mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} \\ &\quad + \mathbb{E} R_s^\epsilon \mathbf{1}_{X_s^\epsilon > -R_s^\epsilon, X_s^\epsilon > 0} \\ &= \mathbb{E} X_s^\epsilon \left(\mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} - \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \right) + \mathbb{E} R_s^\epsilon \mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} \\ &\quad + \mathbb{E} R_s^\epsilon \left(\mathbf{1}_{X_s^\epsilon > 0} - \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \right). \end{aligned}$$

Besides R^ϵ and X^ϵ are independent and $\mathbb{E}R^\epsilon = 0$, thus

$$\begin{aligned} I_s^\epsilon &= \mathbb{E} X_s^\epsilon \left(\mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} - \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \right) + \mathbb{E} R_s^\epsilon \mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} \\ &\quad - \mathbb{E} R_s^\epsilon \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \\ &= \mathbb{E} (X_s^\epsilon + R_s^\epsilon) \mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} - \mathbb{E} (X_s^\epsilon + R_s^\epsilon) \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \\ &= \mathbb{E} (|R_s^\epsilon| - |X_s^\epsilon|) \mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} - \mathbb{E} (|X_s^\epsilon| - |R_s^\epsilon|) \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \\ &= \mathbb{E} (|R_s^\epsilon| - |X_s^\epsilon|) \left(\mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} + \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \right) \\ &= \mathbb{E} (|R_s^\epsilon| - |X_s^\epsilon|)^+ \left(\mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} + \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} \right). \end{aligned}$$

But we have

$$\mathbf{1}_{-R_s^\epsilon < X_s^\epsilon \leq 0} + \mathbf{1}_{0 < X_s^\epsilon \leq -R_s^\epsilon} = \mathbf{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbf{1}_{X_s^\epsilon = 0, R_s^\epsilon > 0}.$$

Hence

$$\begin{aligned} I_s^\epsilon &= \mathbb{E} (|R_s^\epsilon| - |X_s^\epsilon|)^+ \left(\mathbf{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbf{1}_{X_s^\epsilon = 0, R_s^\epsilon > 0} \right) \\ &= \mathbb{E} (|R_s^\epsilon| - |X_s^\epsilon|)^+ \mathbf{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{E} |R_s^\epsilon| \mathbf{1}_{X_s^\epsilon = 0, R_s^\epsilon > 0} \\ &= \mathbb{E} (|R_s^\epsilon| - |X_s^\epsilon|)^+ \mathbf{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{E} |R_s^\epsilon| \mathbf{1}_{R_s^\epsilon > 0} \mathbb{P}[X_s^\epsilon = 0] \\ &= \mathbb{E} (|R_s^\epsilon| - |X_s^\epsilon|)^+ \mathbf{1}_{R_s^\epsilon X_s^\epsilon < 0} + \mathbb{E} (R_s^\epsilon)^+ \mathbb{P}[X_s^\epsilon = 0]. \end{aligned}$$

But $I_s^\epsilon \geq 0$, therefore

$$\mathbb{E}(M_t - M_t^\epsilon) \geq 0.$$

We now prove that $\mathbb{E}(M_t - M_t^\epsilon) = o(\sigma(\epsilon))$. Using Cauchy-Schwarz inequality, we have

$$\begin{aligned} I_s^\epsilon &\leq \left(\mathbb{E}|R_s^\epsilon|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|}\right)^+\right)^2\right)^{\frac{1}{2}} + \left(\mathbb{E}|R_s^\epsilon|^2\right)^{\frac{1}{2}} \mathbb{P}[X_s^\epsilon = 0] \\ &\leq \sigma(\epsilon)\sqrt{s} \left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|}\right)^+\right)^2 + \mathbb{P}[X_s^\epsilon = 0]\right)^{\frac{1}{2}}. \end{aligned}$$

Note that $\nu(\mathbb{R}) = +\infty$, so $R_s^\epsilon \neq 0$ *a.s.* Then we get

$$\mathbb{E}(M_t - M_t^\epsilon) \leq \sigma(\epsilon) \int_0^t \left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|}\right)^+\right)^2 + \mathbb{P}[X_s^\epsilon = 0]\right)^{\frac{1}{2}} \frac{ds}{\sqrt{s}}.$$

Furthermore by dominated convergence, we have

$$\lim_{\epsilon \rightarrow 0} \mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|}\right)^+\right)^2 = 0.$$

Indeed $R_s^\epsilon \rightarrow 0$ *a.s.* and $X_s^\epsilon \rightarrow X_s$ *a.s.* with $X_s \neq 0$ *a.s.* We also have

$$\lim_{\epsilon \rightarrow 0} \mathbb{P}[X_s^\epsilon = 0] = \mathbb{P}[X_s = 0] = 0.$$

On the other hand

$$\left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|}\right)^+\right)^2 + \mathbb{P}[X_s^\epsilon = 0]\right)^{\frac{1}{2}} \leq 2.$$

Therefore by dominated convergence

$$\lim_{\epsilon \rightarrow 0} \int_0^t \left(\mathbb{E}\left(\left(1 - \frac{|X_s^\epsilon|}{|R_s^\epsilon|}\right)^+\right)^2 + \mathbb{P}[X_s^\epsilon = 0]\right)^{\frac{1}{2}} \frac{ds}{\sqrt{s}} = 0.$$

Hence

$$\mathbb{E}(M_t - M_t^\epsilon) = o(\sigma(\epsilon))$$

□

In financial applications, the function f in Proposition 2.4 is not always Lipschitz, as for call lookback option where the function is exponential. Hence the following proposition.

PROPOSITION 2.7. *Let X be a Lévy process, $p > 1$. We suppose that $\mathbb{E}e^{pM_t} < \infty$, then*

$$\mathbb{E}\left|e^{M_t} - e^{M_t^\epsilon}\right| \leq C_{p,t}\sigma_0(\epsilon),$$

where $C_{p,t}$ is a positive constant independent of ϵ .

LEMMA 2.8. Let $p > 0$. If $\mathbb{E}e^{pM_t} < \infty$, then

$$\sup_{0 \leq \delta \leq 1} \mathbb{E}e^{pM_t^\delta} < \infty$$

REMARK 2.9. For any $p > 0$, $\mathbb{E}e^{pM_t} < \infty$ if only if $\int_{x>1} e^{px}\nu(dx) < \infty$. Indeed, we have

$$\begin{aligned} X_t &= X_t^1 + R_t^1 \\ &= \gamma t + bB_t + \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s > 1\}} + \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s < -1\}} + R_t^1. \end{aligned}$$

Define

$$Z_t = \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s > 1\}}, \quad \forall t \geq 0.$$

The process Z is a Lévy process with Lévy measure ν^Z , which is the restriction of ν on $(1, +\infty)$. Thus, since $\sup_{0 \leq s \leq t} \left\{ p \sum_{0 \leq s \leq t} \Delta X_s \mathbf{1}_{\{\Delta X_s < -1\}} \right\} \leq 0$, we have

$$\mathbb{E}e^{pM_t} \leq \mathbb{E} \exp \left\{ p \sup_{0 \leq s \leq t} (\gamma s + bB_s + R_s^1) \right\} \mathbb{E}e^{p \sup_{0 \leq s \leq t} Z_s}.$$

So, by Theorem 25.3 and Theorem 25.18 of [17], if $\int_{x>1} e^{px}\nu(dx) < \infty$ then $\mathbb{E}e^{pM_t} < \infty$. On the other hand, we have $X_t \leq M_t$. Thus, the ‘‘only if’’ part, can be deduced from Theorem 25.3 of [17].

Proof. [Proof of Lemma 2.8] For $\delta \in (0, 1]$, define

$$\bar{R}^\delta = X^\delta - X^1.$$

The process \bar{R}^δ is the compensated sum of jumps belonging to $(\delta, 1]$. So

$$\begin{aligned} \mathbb{E}e^{pM_t^\delta} &\leq \mathbb{E}e^{p \sup_{0 \leq s \leq t} X_s^1 + p \sup_{0 \leq s \leq t} \bar{R}_s^\delta} \\ &\leq \mathbb{E}e^{p \sup_{0 \leq s \leq t} X_s^1} \mathbb{E}e^{p \sup_{0 \leq s \leq t} |\bar{R}_s^\delta|}. \end{aligned}$$

By hypothesis and Remark 2.9, $\mathbb{E}e^{p \sup_{0 \leq s \leq t} X_s^1} < \infty$. We need to bound $\mathbb{E}e^{p \sup_{0 \leq s \leq t} |\bar{R}_s^\delta|}$ independently of δ . We have

$$\begin{aligned} \mathbb{E}e^{p \sup_{0 \leq s \leq t} |\bar{R}_s^\delta|} &= \mathbb{E} \sum_{n=0}^{+\infty} \frac{(p \sup_{0 \leq s \leq t} |\bar{R}_s^\delta|)^n}{n!} \\ &= 1 + p \mathbb{E} \sup_{0 \leq s \leq t} |\bar{R}_s^\delta| + \sum_{n=2}^{+\infty} \frac{p^n}{n!} \mathbb{E} \left(\sup_{0 \leq s \leq t} |\bar{R}_s^\delta| \right)^n. \end{aligned}$$

By Doob’s inequality (\bar{R}^δ is a càdlàg martingale)

$$\begin{aligned} \mathbb{E}e^{p \sup_{0 \leq s \leq t} |\bar{R}_s^\delta|} &\leq 1 + p \sqrt{\mathbb{E} \left(\sup_{0 \leq s \leq t} |\bar{R}_s^\delta| \right)^2} + \sum_{n=2}^{+\infty} \frac{p^n}{n!} \left(\frac{n}{n-1} \right)^n \mathbb{E} |\bar{R}_t^\delta|^n \\ &\leq 1 + 2p \sqrt{\mathbb{E} |\bar{R}_t^\delta|^2} + \sum_{n=2}^{+\infty} \frac{p^n}{n!} 2^n \mathbb{E} |\bar{R}_t^\delta|^n \\ &= 2p \left(\sqrt{\mathbb{E} |\bar{R}_t^\delta|^2} - \mathbb{E} |\bar{R}_t^\delta| \right) + \sum_{n=0}^{+\infty} \frac{2^n p^n}{n!} \mathbb{E} |\bar{R}_t^\delta|^n. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{E}e^{p \sup_{0 \leq s \leq t} |\bar{R}_s^\delta|} &\leq 2p \left(\sqrt{\mathbb{E}|\bar{R}_t^\delta|^2} - \mathbb{E}|\bar{R}_t^\delta| \right) + \mathbb{E} \sum_{n=0}^{+\infty} \frac{2^n p^n}{n!} |\bar{R}_t^\delta|^n \\ &= 2p \left(\sqrt{\mathbb{E}|\bar{R}_t^\delta|^2} - \mathbb{E}|\bar{R}_t^\delta| \right) + \mathbb{E}e^{2p|\bar{R}_t^\delta|} \\ &\leq 2p \left(\sqrt{\mathbb{E}|\bar{R}_t^\delta|^2} - \mathbb{E}|\bar{R}_t^\delta| \right) + \mathbb{E}e^{2p\bar{R}_t^\delta} + \mathbb{E}e^{-2p\bar{R}_t^\delta}. \end{aligned}$$

But

$$\begin{aligned} \mathbb{E}|\bar{R}_t^\delta| &\leq \sqrt{\mathbb{E}|\bar{R}_t^\delta|^2} \\ &= \sqrt{t \int_{\delta < |x| \leq 1} x^2 \nu(dx)} \\ &\leq \sqrt{t \int_{|x| \leq 1} x^2 \nu(dx)}. \end{aligned}$$

On the other hand, we can prove that for any $\beta \in \mathbb{R}$, $(e^{\beta \bar{R}_t^\delta})_{0 \leq \delta \leq 1}$ is uniformly integrable. Indeed, we have

$$\mathbb{E}e^{\beta \bar{R}_t^\delta} = \exp \left\{ t \int_{\delta < |x| \leq 1} (e^{\beta x} - 1 - \beta x) \nu(dx) \right\}.$$

By Taylor-Lagrange Theorem, we have (for $|x| \leq 1$)

$$|e^{\beta x} - 1 - \beta x| = \frac{\beta^2 x^2}{2} e^{\beta \xi} \leq \frac{\beta^2 x^2}{2} e^{|\beta|},$$

where ξ is a real number between 0 and x . So

$$\mathbb{E}e^{\beta \bar{R}_t^\delta} \leq \exp \left\{ \frac{\beta^2 t}{2} e^{|\beta|} \int_{|x| \leq 1} x^2 \nu(dx) \right\}.$$

Hence

$$\sup_{0 \leq \delta \leq 1} \mathbb{E}e^{p \sup_{0 \leq s \leq t} |\bar{R}_s^\delta|} < \infty.$$

This concludes the proof. \square

Proof. [Proof of Proposition 2.7] By the mean value theorem, we have

$$e^{M_t} - e^{M_t^\epsilon} = (M_t - M_t^\epsilon) e^{\bar{M}_t^\epsilon},$$

where \bar{M}_t^ϵ is between M_t and M_t^ϵ . Let q be defined such that $\frac{1}{p} + \frac{1}{q} = 1$.

$$\begin{aligned} \mathbb{E} \left| e^{M_t} - e^{M_t^\epsilon} \right| &\leq \mathbb{E} |M_t - M_t^\epsilon| e^{\bar{M}_t^\epsilon} \\ &\leq \mathbb{E} \sup_{0 \leq s \leq t} |R_s^\epsilon| e^{\bar{M}_t^\epsilon} \\ &\leq \left(\mathbb{E} \left(\sup_{0 \leq s \leq t} |R_s^\epsilon| \right)^q \right)^{\frac{1}{q}} \left(\mathbb{E} e^{p \bar{M}_t^\epsilon} \right)^{\frac{1}{p}}. \end{aligned}$$

Hence, using Doob's inequality and then Proposition 2.1, we get

$$\begin{aligned} \mathbb{E} \left| e^{M_t} - e^{M_t^\epsilon} \right| &\leq \frac{q}{q-1} (\mathbb{E} |R_t^\epsilon|^q)^{\frac{1}{q}} \left(\mathbb{E} e^{p\bar{M}_t^\epsilon} \right)^{\frac{1}{p}} \\ &\leq C_{p,t} \sigma_0(\epsilon) \left(\mathbb{E} \left(e^{pM_t} + e^{pM_t^\epsilon} \right) \right)^{\frac{1}{p}}, \end{aligned}$$

where $C_{p,t}$ will denote a constant depending on p and t . We conclude the proof by Lemma 2.8. \square

2.2. Estimates for cumulative distribution functions. For cumulative distribution functions, bounds are expected to be bigger. However, in some cases we can get similar results as in Lipschitz case. In the first result below, we assume local boundedness of the probability density function of the Lévy process X and its supremum process M at a fixed time t . The regularity of the probability density function of a Lévy process is studied in [17, 3]. For the supremum process see [6, 9].

PROPOSITION 2.10. *Let X be a Lévy process.*

1. *If $b > 0$, then*

$$\sup_{x \in \mathbb{R}} |\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| \leq \frac{1}{\sqrt{2\pi b}} \sigma(\epsilon).$$

2. *If X_t has a locally bounded probability density function and $x \in \mathbb{R}$, then for any $q \in (0, 1)$,*

$$|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| \leq C_{x,t,q} \sigma_0(\epsilon)^{1-q}.$$

3. *If M_t has a locally bounded probability density function on $(0, +\infty)$ and $x > 0$, then for any $q \in (0, 1)$,*

$$|\mathbb{P}[M_t \geq x] - \mathbb{P}[M_t^\epsilon \geq x]| \leq C_{x,t,q} \sigma_0(\epsilon)^{1-q},$$

where $C_{x,t,q}$ means a positive constant depending on x , q and t .

LEMMA 2.11. *Let X and Y be two r.v.'s. We assume that X has a bounded density in a neighbourhood of $x \in \mathbb{R}$, and there exists $p \geq 1$ such that $\mathbb{E}|X - Y|^p$ is finite. Then there exists a constant $K_x > 0$, such that for any $\delta > 0$*

$$|\mathbb{P}[X \geq x] - \mathbb{P}[Y \geq x]| \leq K_x \delta + \frac{\mathbb{E}|X - Y|^p}{\delta^p}.$$

Proof. We have

$$|\mathbb{P}[X \geq x] - \mathbb{P}[Y \geq x]| = |\mathbb{P}[X \geq x, Y < x] - \mathbb{P}[X < x, Y \geq x]|.$$

We will study the above terms on the right of the equality.

$$\begin{aligned} \mathbb{P}[X \geq x, Y < x] &= \mathbb{P}[x \leq X < x + (X - Y)] \\ &= \mathbb{P}[x \leq X < x + (X - Y), |X - Y| \leq \delta] \\ &\quad + \mathbb{P}[x \leq X < x + (X - Y), |X - Y| > \delta] \\ &\leq \mathbb{P}[x \leq X < x + \delta] + \mathbb{P}[|X - Y| > \delta]. \end{aligned}$$

Suppose that X has a bounded probability density function in the interval $[x - \delta_0, x + \delta_0]$, $\delta_0 > 0$ fixed, and let

$$K_x = \max \left\{ \sup_{x - \delta_0 \leq t \leq x + \delta_0} f(t), \frac{1}{\delta_0} \right\}.$$

If $\delta < \delta_0$, then

$$\mathbb{P}[x \leq X < x + \delta] \leq \left(\sup_{x - \delta_0 \leq t \leq x + \delta_0} f(t) \right) \delta \leq K_x \delta.$$

If $\delta \geq \delta_0$, then

$$\mathbb{P}[x \leq X < x + \delta] \leq 1 = \frac{1}{\delta_0} \delta_0 \leq K_x \delta.$$

Hence, for any $\delta > 0$,

$$\mathbb{P}[x \leq X < x + \delta] \leq K_x \delta,$$

with K_x independent of δ . Thus, using Markov's inequality, we get

$$\mathbb{P}[X \geq x, Y < x] \leq K_x \delta + \frac{\mathbb{E}|X - Y|^p}{\delta^p}.$$

Similarly

$$\mathbb{P}[x - \delta \leq X < x] \leq K_x \delta.$$

So, it holds that

$$\mathbb{P}[X < x, Y \geq x] \leq K_x \delta + \frac{\mathbb{E}|X - Y|^p}{\delta^p}.$$

This concludes the proof. \square

Proof. [Proof of Proposition 2.10] We have

$$|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| = |\mathbb{P}[X_t \geq x, X_t^\epsilon < x] - \mathbb{P}[X_t < x, X_t^\epsilon \geq x]|. \quad (2.5)$$

But, in the case $b > 0$

$$\begin{aligned} \mathbb{P}[X_t \geq x, X_t^\epsilon < x] &= \mathbb{P}[x - (X_t - X_t^\epsilon) \leq X_t^\epsilon < x] \\ &= \mathbb{P}[x - R_t^\epsilon \leq bB_t + (X_t^\epsilon - bB_t) < x]. \end{aligned}$$

Note that the r.v.'s bB_t , $(X_t^\epsilon - bB_t)$ and R_t^ϵ are independent, and $\frac{1}{\sqrt{2\pi tb}}$ is a bound of the probability density function of bB_t . Since $P[x - s \leq bB_t < x] \leq \frac{|s|}{\sqrt{2\pi tb}}$, it thus follows that $P[x - R_t^\epsilon \leq bB_t < x] \leq \frac{1}{\sqrt{2\pi tb}} \mathbb{E}|R_t^\epsilon|$ and, in turn

$$P[x - R_t^\epsilon \leq bB_t + (X_t^\epsilon - bB_t) < x] \leq \frac{1}{\sqrt{2\pi tb}} \mathbb{E}|R_t^\epsilon|.$$

Therefore

$$\begin{aligned} \mathbb{P}[X_t \geq x, X_t^\epsilon < x] &\leq \frac{1}{\sqrt{2\pi tb}} \mathbb{E}|R_t^\epsilon| \\ &\leq \frac{1}{\sqrt{2\pi tb}} \sqrt{\mathbb{E}|R_t^\epsilon|^2} \\ &= \frac{1}{\sqrt{2\pi b}} \sigma(\epsilon). \end{aligned}$$

Similarly

$$\begin{aligned}\mathbb{P}[X_t < x, X_t^\epsilon \geq x] &= \mathbb{P}[x \leq X_t^\epsilon < x - (X_t - X_t^\epsilon)] \\ &= \mathbb{P}[x \leq bB_t + (X_t^\epsilon - bB_t) < x - R_t^\epsilon] \\ &\leq \frac{1}{\sqrt{2\pi b}}\sigma(\epsilon).\end{aligned}$$

Hence part 1 of the proposition follows from (2.5).

Consider now the second part of the proposition. By Lemma 2.11, there exists a constant $K_{x,t} > 0$ such that for any $p > 1$, we have

$$\begin{aligned}|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| &\leq K_{x,t}\delta + \frac{\mathbb{E}|X_t - X_t^\epsilon|^p}{\delta^p} \\ &= K_{x,t}\delta + \frac{\mathbb{E}|R_t^\epsilon|^p}{\delta^p}.\end{aligned}$$

But by Proposition 2.1, there exists a constant $K_{p,t} > 0$ such that

$$\mathbb{E}|R_t^\epsilon|^p \leq K_{p,t}\sigma_0(\epsilon)^p.$$

So

$$\begin{aligned}|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| &\leq K_{x,t}\delta + K_{p,t}\frac{\sigma_0(\epsilon)^p}{\delta^p} \\ &\leq \max(K_{x,t}, K_{p,t})\left(\delta + \frac{\sigma_0(\epsilon)^p}{\delta^p}\right).\end{aligned}$$

Thus by choosing $\delta = \sigma_0(\epsilon)^{\frac{p}{p+1}}$, we get

$$|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| \leq 2\max(K_{x,t}, K_{p,t})\sigma_0(\epsilon)^{\frac{p}{p+1}}.$$

Therefore for any $q \in (0, 1)$, we have

$$|\mathbb{P}[X_t \geq x] - \mathbb{P}[X_t^\epsilon \geq x]| \leq C_{x,t,q}\sigma_0(\epsilon)^{1-q}.$$

For the third part of the proposition, set

$$I = |\mathbb{P}[M_t \geq x] - \mathbb{P}[M_t^\epsilon \geq x]|.$$

By Lemma 2.11, there exists a constant $K'_{x,t} > 0$ such that

$$I \leq K'_{x,t}\delta + \frac{\mathbb{E}|M_t - M_t^\epsilon|^p}{\delta^p}.$$

On the other hand

$$\begin{aligned}\mathbb{E}|M_t - M_t^\epsilon|^p &\leq \mathbb{E}\left(\sup_{0 \leq s \leq t} |X_s - X_s^\epsilon|\right)^p \\ &= \mathbb{E}\left(\sup_{0 \leq s \leq t} |R_s^\epsilon|\right)^p.\end{aligned}$$

By Doob's inequality, we have

$$\begin{aligned}\mathbb{E}|M_t - M_t^\epsilon|^p &\leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|R_t^\epsilon|^p \\ &\leq K_{p,t}\left(\frac{p}{p-1}\right)^p \sigma_0(\epsilon)^p.\end{aligned}$$

Part 3 of the proposition then follows by choosing $\delta = \sigma_0(\epsilon)^{\frac{p}{p+1}}$. \square

3. Approximation of the compensated sum of small jumps by a Brownian motion. In this section we will replace R^ϵ by a Brownian motion. This method gives better results, subject to a convergence assumption. In fact, Asmussen and Rosinski proved ([1], Theorem 2.1) that, if X is a Lévy process, then the process $\sigma(\epsilon)^{-1}R^\epsilon$ converges in distribution to a standard Brownian motion, when $\epsilon \rightarrow 0$, if only if for any $k > 0$

$$\lim_{\epsilon \rightarrow 0} \frac{\sigma(k\sigma(\epsilon) \wedge \epsilon)}{\sigma(\epsilon)} = 1. \quad (3.1)$$

Condition (3.1) is implied by the condition

$$\lim_{\epsilon \rightarrow 0} \frac{\sigma(\epsilon)}{\epsilon} = +\infty. \quad (3.2)$$

The conditions (3.1) and (3.2) are equivalent, if ν does not have atoms in some neighbourhood of zero ([1], Proposition 2.1).

Note that, if the Asmussen-Rosinski's condition is not satisfied we can prove that the bounds obtained in Section 2 are valid for the Brownian approximation, using the same argument as in Section 2.

3.1. Estimates for smooth functions. The errors resulting from Brownian approximation have not been much studied in the literature, at least theoretically. There are some results which we can find in [7, 8].

PROPOSITION 3.1. *Let X be an infinite activity Lévy process and $t > 0$,*

1. *If $f \in C^1(\mathbb{R})$ and satisfies $\mathbb{E}|f'(X_t^\epsilon)| < \infty$, and if there exists $\beta > 1$ such*

that $\left(\sup_{\epsilon \in [0,1]} \mathbb{E} \left| f'(X_t^\epsilon + \theta\sigma(\epsilon)\hat{W}_t) - f'(X_t^\epsilon) \right|^\beta \right)^{\frac{1}{\beta}}$ and

$\left(\sup_{\epsilon \in [0,1]} \mathbb{E} \left| f'(X_t^\epsilon + \theta R_t^\epsilon) - f'(X_t^\epsilon) \right|^\beta \right)^{\frac{1}{\beta}}$ are finite and integrable with respect to θ on $[0, 1]$, then

$$\mathbb{E} \left(f(X_t) - f(\hat{X}_t^\epsilon) \right) = o(\sigma_0(\epsilon)).$$

2. *If $f \in C^2(\mathbb{R})$ and satisfies $\mathbb{E}|f'(X_t^\epsilon)| + \mathbb{E}|f''(X_t^\epsilon)| < \infty$, and if there exists*

$\beta > 1$ such that $\left(\sup_{\epsilon \in [0,1]} \mathbb{E} \left| f''(X_t^\epsilon + \theta\sigma(\epsilon)\hat{W}_t) - f''(X_t^\epsilon) \right|^\beta \right)^{\frac{1}{\beta}}$ and

$\left(\sup_{\epsilon \in [0,1]} \mathbb{E} \left| f''(X_t^\epsilon + \theta R_t^\epsilon) - f''(X_t^\epsilon) \right|^\beta \right)^{\frac{1}{\beta}}$ are finite and integrable with respect to θ on $[0, 1]$, then

$$\mathbb{E} \left(f(X_t) - f(\hat{X}_t^\epsilon) \right) = o(\sigma_0(\epsilon)^2).$$

Examples of functions satisfying the above conditions are noted after Proposition 2.2.

Proof. By Proposition 2.2, we have

$$\mathbb{E} \left(f(X_t) - f(X_t^\epsilon) \right) = \frac{\sigma(\epsilon)^2 t}{2} \mathbb{E} f''(X_t^\epsilon) + o(\sigma_0(\epsilon)^2).$$

On the other hand, using the same reasoning as the proof of Proposition 2.2 (we will replace R^ϵ by $\sigma(\epsilon)\hat{W}$) we get

$$\mathbb{E} \left(f \left(X_t^\epsilon + \sigma(\epsilon)\hat{W}_t \right) - f(X_t^\epsilon) \right) = \frac{\sigma(\epsilon)^2 t}{2} \mathbb{E} f''(X_t^\epsilon) + o(\sigma_0(\epsilon)^2).$$

Hence

$$\mathbb{E} \left(f(X_t) - f(\hat{X}_t^\epsilon) \right) = o(\sigma_0(\epsilon)^2).$$

□

The combination of Proposition 6.2 of [7] and the Spitzer's identity for Lévy processes (Proposition 1 of [10]) leads to the following result.

PROPOSITION 3.2. *Let X be an integrable infinite activity Lévy process, then*

$$\left| \mathbb{E}M_t - \mathbb{E}\hat{M}_t^\epsilon \right| \leq 4 \max \left(1 + \sqrt{\frac{2}{\pi}}, A \right) \sigma(\epsilon) \rho(\epsilon) \left(1 + \log \left(\frac{\sqrt{t}}{2\rho(\epsilon)} \right) \right),$$

where A is a positive constant < 16.5 . Compared to the estimate of Theorem 2.6, we have gained a factor of about $\rho(\epsilon)$.

Proof. Using Spitzer's identity (see Proposition 1 of [10] for details), we have

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} X_s &= \int_0^t \frac{\mathbb{E}X_s^+}{s} ds \\ \mathbb{E} \sup_{0 \leq s \leq t} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right) &= \int_0^t \frac{\mathbb{E} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right)^+}{s} ds. \end{aligned}$$

Set

$$I^\epsilon = \left| \mathbb{E} \sup_{0 \leq s \leq t} X_s - \mathbb{E} \sup_{0 \leq s \leq t} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right) \right|.$$

Let $\delta > 0$, then

$$\begin{aligned} I^\epsilon &= \left| \int_0^t \frac{\mathbb{E}X_s^+ - \mathbb{E} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right)^+}{s} ds \right| \\ &\leq \int_0^\delta \left| \mathbb{E}X_s^+ - \mathbb{E} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right)^+ \right| \frac{ds}{s} + \int_\delta^t \left| \mathbb{E}X_s^+ - \mathbb{E} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right)^+ \right| \frac{ds}{s}. \end{aligned}$$

We will call I_1^ϵ (resp. I_2^ϵ) the first (resp. the second) term of the last expression. Note that the function $x \rightarrow x^+$ is 1-Lipschitz, So by Proposition 6.2 of [7], we have

$$\left| \mathbb{E} \left(X_s^\epsilon + R_s^\epsilon \right)^+ - \mathbb{E} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right)^+ \right| \leq A\sigma(\epsilon)\rho(\epsilon),$$

where $A < 16.5$. So

$$I_2^\epsilon \leq A\sigma(\epsilon)\rho(\epsilon) \int_\delta^t \frac{ds}{s} = A\sigma(\epsilon)\rho(\epsilon) \log \left(\frac{t}{\delta} \right).$$

On the other hand

$$\left| \mathbb{E}X_s^+ - \mathbb{E} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right)^+ \right| \leq \mathbb{E} \left| R_s^\epsilon - \sigma(\epsilon) \hat{W}_s \right| \leq \left(1 + \sqrt{\frac{2}{\pi}} \right) \sqrt{s} \sigma(\epsilon).$$

Hence

$$I_1^\epsilon \leq \left(1 + \sqrt{\frac{2}{\pi}} \right) \sigma(\epsilon) \int_0^\delta \frac{ds}{\sqrt{s}} = A_1 \sigma(\epsilon) \sqrt{\delta},$$

where $A_1 = 2 \left(1 + \sqrt{\frac{2}{\pi}}\right)$. Finally

$$I^\epsilon \leq \max(A, A_1) \sigma(\epsilon) \left(\sqrt{\delta} + \rho(\epsilon) \log \left(\frac{t}{\delta} \right) \right).$$

The right term of the above inequality is minimal for $\delta = 4\rho(\epsilon)^2$. So

$$I^\epsilon \leq 2 \max(A, A_1) \sigma(\epsilon) \rho(\epsilon) \left(1 + \log \left(\frac{\sqrt{t}}{2\rho(\epsilon)} \right) \right).$$

This concludes the proposition. \square

3.2. Estimates by Skorokhod embedding. We will use a powerful tool to prove the results of this section. This is the Skorokhod embedding Theorem. We will begin by defining some useful notations.

DEFINITION 3.3. *We define*

$$\beta(\epsilon) = \frac{\int_{|x| \leq \epsilon} x^4 \nu(dx)}{(\sigma_0(\epsilon))^4}, \quad \beta_1^t(\epsilon) = \beta(\epsilon)^{\frac{1}{6}} \left(\sqrt{\log \left(\frac{t}{\beta(\epsilon)^{\frac{1}{3}}} + 3 \right)} + 1 \right),$$

$$\beta_2^t(\epsilon) = \beta(\epsilon)^{\frac{1}{4}} \left(\log \left(\frac{t}{\beta(\epsilon)^{\frac{1}{4}}} + 3 \right) + 1 \right), \quad \beta_{p,\theta}(\epsilon) = \beta(\epsilon)^{\frac{p\theta}{p+4\theta}} \left(\log \left(\frac{1}{\beta(\epsilon)^{\frac{p\theta}{p+4\theta}}} \right) \right)^p.$$

REMARK 3.4. *Note that under the condition (3.2), we have*

$$\lim_{\epsilon \rightarrow 0} \beta(\epsilon) = 0. \quad (3.3)$$

The proof of Proposition 3.2 cannot be extended to the Lipschitz functions, because the reformulation of the spitzer identity for Lévy processes cannot be applied in that case. We have to use another method.

THEOREM 3.5. *Let X be an integrable infinite activity Lévy process, and f be a Lipschitz function, then*

$$\left| \mathbb{E}f(M_t) - \mathbb{E}f(\hat{M}_t^\epsilon) \right| \leq C \sigma_0(\epsilon) \beta_1^t(\epsilon),$$

where C is a positive constant independent of ϵ .

Compared to the estimate of Theorem 2.6, we have gained the factor $\beta_1^t(\epsilon)$.

Proof. Set

$$I_f^\epsilon = \left| \mathbb{E} \left(f \left(\sup_{0 \leq s \leq t} X_s \right) - f \left(\sup_{0 \leq s \leq t} \left(X_s^\epsilon + \sigma(\epsilon) \hat{W}_s \right) \right) \right) \right|$$

$$I_f^\epsilon(n) = \left| \mathbb{E} \left(f \left(\sup_{0 \leq k \leq n} X_{\frac{kt}{n}} \right) - f \left(\sup_{0 \leq k \leq n} \left(X_{\frac{kt}{n}}^\epsilon + \sigma(\epsilon) \hat{W}_{\frac{kt}{n}} \right) \right) \right) \right|.$$

Because f is, say, K -Lipschitz, we can show that

$$I_f^\epsilon(n) \leq K \mathbb{E} \left(\sup_{0 \leq s \leq t} |R_s^\epsilon| + \sigma(\epsilon) \sup_{0 \leq s \leq t} |\hat{W}_s| \right).$$

As the right hand side expression is integrable, by dominated convergence we can deduce that $\lim_{n \rightarrow +\infty} I_f^\epsilon(n) = I_f^\epsilon$. So we will consider $I_f^\epsilon(n)$. For $k \in \{1, \dots, n\}$, we have

$$R_{\frac{kt}{n}}^\epsilon = \frac{1}{\sqrt{n}} \sum_{j=1}^k V_j^n,$$

where $V_j^n = \sqrt{n} \left(R_{\frac{jt}{n}}^\epsilon - R_{(j-1)\frac{t}{n}}^\epsilon \right)$. The r.v.'s $(V_j^n)_{j \in \{1, \dots, n\}}$ are i.i.d. and have the same distribution as $\sqrt{n} R_{\frac{t}{n}}^\epsilon$. But $\mathbb{E}V_1^n = 0$, and $\text{var}(V_1^n) = \sigma(\epsilon)^2 t$, by Theorem 1 of [19] (see pp. 163) there exists positive i.i.d. r.v.'s, $(\tau_j)_{j \in \{1, \dots, n\}}$, and a standard Brownian motion, \hat{B} , such that $\left(\sum_{j=1}^k V_j^n, k \in \{1, \dots, n\} \right)$ and $\left(\hat{B}_{\tau_1 + \dots + \tau_k}, k \in \{1, \dots, n\} \right)$ have the same joint distribution. Thus $\left(R_{\frac{kt}{n}}^\epsilon, k \in \{1, \dots, n\} \right)$ and $\left(\hat{B}_{\frac{\tau_1 + \dots + \tau_k}{n}}, k \in \{1, \dots, n\} \right)$ have the same joint distribution. Furthermore $\mathbb{E}\tau_1 = \text{var}(V_1)$ and for any $q \geq 1$, there exists a positive constant L_q , such that

$$\mathbb{E}\tau_1^q \leq L_q \mathbb{E}V_1^{2q}. \quad (3.4)$$

We also have $\left(\sigma(\epsilon) \hat{W}_{\frac{kt}{n}}, k \in \{1, \dots, n\} \right) =^d \left(\hat{B}_{\frac{\sigma(\epsilon)^2 kt}{n}}, k \in \{1, \dots, n\} \right)$. Set

$$T_k = \frac{\tau_1 + \dots + \tau_k}{n}, \quad T_k^\epsilon = \frac{\sigma(\epsilon)^2 kt}{n}.$$

Thus

$$I_f^\epsilon(n) \leq K \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right|.$$

The following theorem concludes the proof. \square

THEOREM 3.6.

1. We have

$$\limsup_{n \rightarrow +\infty} \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right| \leq C \sigma_0(\epsilon) \beta_1^t(\epsilon),$$

2. and

$$\limsup_{n \rightarrow +\infty} \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right|^2 \leq C \sigma_0(\epsilon)^2 \beta_2^t(\epsilon).$$

3. For any real $p \geq 1$ and for any real $\theta \in (0, 1)$, we also have

$$\limsup_{n \rightarrow +\infty} \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right|^p \leq C_{p,\theta} \sigma_0(\epsilon)^p \beta_{p,\theta}^t(\epsilon),$$

where C and $C_{p,\theta}$ are constants independent of ϵ .

This theorem is the main result of this section. The following result will be also useful for the sequel.

LEMMA 3.7. Let $\delta > 0$, then we have

$$\limsup_{n \rightarrow +\infty} \mathbb{P} \left[\sup_{0 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] \leq \frac{4L_2 t \sigma_0(\epsilon)^4 \beta(\epsilon)}{\delta^2}.$$

Proof. By Markov's inequality, we have

$$\mathbb{P} \left[\sup_{0 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] \leq \frac{\mathbb{E} \sup_{1 \leq k \leq n} |T_k - T_k^\epsilon|^2}{\delta^2}.$$

On the other hand $(T_k - T_k^\epsilon)_{1 \leq k \leq n}$ is a martingale, so using Doob's inequality, we get

$$\begin{aligned} \mathbb{P} \left[\sup_{0 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] &\leq \frac{4\mathbb{E} |T_n - T_n^\epsilon|^2}{\delta^2} \\ &\leq \frac{4 \operatorname{var}(\tau_1)}{\delta^2 n} \\ &\leq \frac{4 \mathbb{E}\tau_1^2}{\delta^2 n} \\ &\leq \frac{4L_2\mathbb{E}V_1^4}{n\delta^2}, \quad \text{by (3.4)} \\ &= \frac{4L_2n\mathbb{E}\left(R_{\frac{t}{n}}^\epsilon\right)^4}{\delta^2}. \end{aligned}$$

But by Proposition 2.1, we have $\mathbb{E} \left| R_{\frac{t}{n}}^\epsilon \right|^4 = \frac{t}{n} \int_{|x| \leq \epsilon} x^4 \nu(dx) + 3 \left(\frac{t}{n} \sigma(\epsilon)^2 \right)^2$. So

$$\lim_{n \rightarrow +\infty} n\mathbb{E} \left(R_{\frac{t}{n}}^\epsilon \right)^4 = t \int_{|x| \leq \epsilon} |x|^4 \nu(dx) = \sigma_0(\epsilon)^4 \beta(\epsilon)t.$$

Thus

$$\limsup_{n \rightarrow +\infty} \mathbb{P} \left[\sup_{0 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] \leq \frac{4L_2t\sigma_0(\epsilon)^4\beta(\epsilon)}{\delta^2}.$$

□

Proof. [Proof of Theorem 3.6] For $\delta > 0$, we have

$$\begin{aligned} \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right| &\leq \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right| \mathbf{1}_{\{\sup_{1 \leq k \leq n} |T_k - T_k^\epsilon| \leq \delta\}} \\ &\quad + \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right| \mathbf{1}_{\{\sup_{1 \leq k \leq n} |T_k - T_k^\epsilon| > \delta\}}. \end{aligned}$$

Set

$$\begin{aligned} I_1 &= \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right| \mathbf{1}_{\{\sup_{1 \leq k \leq n} |T_k - T_k^\epsilon| \leq \delta\}} \\ I_2 &= \mathbb{E} \sup_{1 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right| \mathbf{1}_{\{\sup_{1 \leq k \leq n} |T_k - T_k^\epsilon| > \delta\}}. \end{aligned}$$

On $\{\sup_{1 \leq k \leq n} |T_k - T_k^\epsilon| \leq \delta\}$, set, for k fixed

$$\begin{aligned} s_1 &= T_k^\epsilon \wedge T_k \\ s_2 &= T_k^\epsilon \vee T_k. \end{aligned}$$

We have $s_1 \leq s_2 \leq s_1 + \delta$. Let j be such that $j\delta \leq s_1 < (j+1)\delta$, we have $s_1 \leq s_2 \leq (j+2)\delta$. If $j\delta \leq s_1 \leq s_2 \leq (j+1)\delta$, we have

$$\begin{aligned} \left| \hat{B}_{s_1} - \hat{B}_{s_2} \right| &\leq \left| \hat{B}_{s_1} - \hat{B}_{j\delta} \right| + \left| \hat{B}_{j\delta} - \hat{B}_{s_2} \right| \\ &\leq 2 \sup_{0 \leq j \leq \lceil \frac{\sigma(\epsilon)^2 t}{\delta} \rceil + 1} \sup_{j\delta \leq u \leq (j+1)\delta} \left| \hat{B}_u - \hat{B}_{j\delta} \right|. \end{aligned}$$

If $j\delta \leq s_1 \leq (j+1)\delta \leq s_2 \leq (j+2)\delta$, we have

$$\begin{aligned} \left| \hat{B}_{s_1} - \hat{B}_{s_2} \right| &\leq \left| \hat{B}_{s_1} - \hat{B}_{j\delta} \right| + \left| \hat{B}_{j\delta} - \hat{B}_{(j+1)\delta} \right| + \left| \hat{B}_{(j+1)\delta} - \hat{B}_{s_2} \right| \\ &\leq 3 \sup_{0 \leq j \leq \lceil \frac{\sigma(\epsilon)^2 t}{\delta} \rceil + 2} \sup_{j\delta \leq u \leq (j+1)\delta} \left| \hat{B}_u - \hat{B}_{j\delta} \right|. \end{aligned}$$

Hence

$$\begin{aligned} I_1 &\leq 3\mathbb{E} \sup_{0 \leq j \leq \lceil \frac{\sigma(\epsilon)^2 t}{\delta} \rceil + 2} \sup_{j\delta \leq u \leq (j+1)\delta} \left| \hat{B}_u - \hat{B}_{j\delta} \right| \\ &= 3\mathbb{E} \sup_{1 \leq j \leq \lceil \frac{\sigma(\epsilon)^2 t}{\delta} \rceil + 3} \sup_{(j-1)\delta \leq u \leq j\delta} \left| \hat{B}_u - \hat{B}_{j\delta} \right|. \end{aligned}$$

The r.v.'s $\left(\sup_{(j-1)\delta \leq u \leq j\delta} \left| \hat{B}_u - \hat{B}_{j\delta} \right| \right)_{1 \leq j \leq \lceil \frac{\sigma(\epsilon)^2 t}{\delta} \rceil + 3}$ are i.i.d. with the same distribution as $\sup_{0 \leq u \leq \delta} \left| \hat{B}_u \right| \sim \sqrt{\delta} \sup_{0 \leq u \leq 1} \left| \hat{B}_u \right|$. Then

$$I_1 \leq 3\sqrt{\delta} \mathbb{E} \sup_{1 \leq j \leq \lceil \frac{\sigma(\epsilon)^2 t}{\delta} \rceil + 3} V_j,$$

where $(V_j)_{1 \leq j \leq \lceil \frac{\sigma(\epsilon)^2 t}{\delta} \rceil + 3}$ are i.i.d. r.v.'s with the same distribution as

$\sup_{0 \leq u \leq 1} \left| \hat{B}_u \right|$. On the other hand, we know that if $(V_j)_{1 \leq j \leq m}$ are i.i.d. r.v.'s satisfying $\mathbb{E} e^{\alpha V_1^2} < \infty$ where α is a positive real, then

$$\mathbb{E} \sup_{1 \leq j \leq m} V_j \leq g \left(m \mathbb{E} e^{\alpha V_1^2} \right),$$

where $g : x \in [1, +\infty) \rightarrow \sqrt{\frac{1}{\alpha} \log(x)}$. Indeed, since g is concave, we have

$$\begin{aligned} \mathbb{E} \sup_{1 \leq j \leq m} V_j &= \mathbb{E} \sup_{1 \leq j \leq m} g \left(e^{\alpha V_j^2} \right) \\ &= \mathbb{E} g \left(\sup_{1 \leq j \leq m} e^{\alpha V_j^2} \right), \text{ because } g \text{ is non-decreasing} \\ &\leq g \left(\mathbb{E} \sup_{1 \leq j \leq m} e^{\alpha V_j^2} \right), \text{ by Jensen's inequality} \\ &\leq g \left(\mathbb{E} \sum_{j=1}^m e^{\alpha V_j^2} \right), \text{ because } g \text{ is non-decreasing} \\ &\leq g \left(m \mathbb{E} e^{\alpha V_1^2} \right). \end{aligned}$$

In our case $V_1 = \sup_{0 \leq u \leq 1} |\hat{B}_u|$. So

$$V_1 \leq \sup_{0 \leq u \leq 1} \hat{B}_u + \sup_{0 \leq u \leq 1} (-\hat{B}_u).$$

Thus

$$\begin{aligned} e^{\alpha V_1^2} &\leq e^{2\alpha((\sup_{0 \leq u \leq 1} \hat{B}_u)^2 + (\sup_{0 \leq u \leq 1} (-\hat{B}_u))^2)} \\ &\leq \left(\mathbb{E} e^{4\alpha(\sup_{0 \leq u \leq 1} \hat{B}_u)^2} \right)^{\frac{1}{2}} \left(\mathbb{E} e^{4\alpha(\sup_{0 \leq u \leq 1} (-\hat{B}_u))^2} \right)^{\frac{1}{2}} \\ &\leq \mathbb{E} e^{4\alpha(\sup_{0 \leq u \leq 1} \hat{B}_u)^2}. \end{aligned}$$

But the probability density function of $\sup_{0 \leq u \leq 1} \hat{B}_u$ is $2\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \mathbb{1}_{x \geq 0}$, so an easy computation give that for any real $\beta < \frac{1}{2}$

$$\mathbb{E} e^{\beta(\sup_{0 \leq u \leq 1} \hat{B}_u)^2} = (1 - 2\beta)^{-\frac{1}{2}}. \quad (3.5)$$

Thus choosing $\alpha < \frac{1}{8}$, we get

$$\mathbb{E} e^{\alpha V_1^2} \leq (1 - 8\alpha)^{-\frac{1}{2}}.$$

Hence for $\alpha \in (0, \frac{1}{8})$

$$\begin{aligned} I_1 &\leq 3\sqrt{\delta} \sqrt{\frac{1}{\alpha} \log \left(\left(\left\lceil \frac{\sigma(\epsilon)^2 t}{\delta} \right\rceil + 3 \right) \mathbb{E} e^{\alpha V_1^2} \right)} \\ &\leq 3\sqrt{\frac{1}{\alpha} \left(1 + \frac{\log(\mathbb{E} e^{\alpha V_1^2})}{\log(3)} \right)} \sqrt{\delta \log \left(\left\lceil \frac{\sigma(\epsilon)^2 t}{\delta} \right\rceil + 3 \right)} \\ &\leq 3\sqrt{\frac{1}{\alpha} \left(1 - \frac{\log(1 - 8\alpha)}{2 \log(3)} \right)} \sqrt{\delta \log \left(\left\lceil \frac{\sigma(\epsilon)^2 t}{\delta} \right\rceil + 3 \right)}. \end{aligned}$$

So

$$\begin{aligned} I_1 &\leq C_\alpha \sqrt{\delta \log \left(\left\lceil \frac{\sigma(\epsilon)^2 t}{\delta} \right\rceil + 3 \right)} \\ &\leq C_\alpha \sqrt{\delta \log \left(\frac{\sigma(\epsilon)^2 t}{\delta} + 3 \right)}. \end{aligned}$$

Notice that $C_\alpha = 3\sqrt{\frac{1}{\alpha} \left(1 - \frac{\log(1 - 8\alpha)}{2 \log(3)} \right)}$ and the first inequality come from the fact that for any $x, y \in \mathbb{R}^+$ we have $\log(x + 3) + y \leq \left(1 + \frac{y}{\log(3)} \right) \log(x + 3)$. Consider now I_2 . We have

$$\begin{aligned} I_2 &\leq \left(\mathbb{E} \left(\sup_{1 \leq k \leq n} |\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}| \right)^2 \right)^{\frac{1}{2}} \left(\mathbb{P} \left[\sup_{1 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] \right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E} \left(\sup_{1 \leq k \leq n} |\hat{B}_{T_k}| + \sup_{1 \leq k \leq n} |\hat{B}_{T_k^\epsilon}| \right)^2 \right)^{\frac{1}{2}} \left(\mathbb{P} \left[\sup_{1 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] \right)^{\frac{1}{2}}. \end{aligned}$$

So

$$I_2 \leq \left(\left(\mathbb{E} \sup_{0 \leq s \leq t} |R_s^\epsilon|^2 \right)^{\frac{1}{2}} + \left(\mathbb{E} \sup_{0 \leq s \leq \sigma(\epsilon)^2 t} |\hat{B}_s|^2 \right)^{\frac{1}{2}} \right) \left(\mathbb{P} \left[\sup_{0 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] \right)^{\frac{1}{2}}.$$

Using Doob's inequality, we get

$$I_2 \leq 2 \left(\left(\mathbb{E} |R_t^\epsilon|^2 \right)^{\frac{1}{2}} + \left(\mathbb{E} |\hat{B}_{\sigma(\epsilon)^2 t}|^2 \right)^{\frac{1}{2}} \right) \left(\mathbb{P} \left[\sup_{0 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] \right)^{\frac{1}{2}}.$$

Hence

$$I_2 \leq 4\sqrt{t}\sigma(\epsilon) \left(\mathbb{P} \left[\sup_{1 \leq k \leq n} |T_k - T_k^\epsilon| > \delta \right] \right)^{\frac{1}{2}}.$$

So, by Lemma 3.7, we have

$$\limsup_{n \rightarrow +\infty} I_2 \leq 4\sqrt{t}\sigma(\epsilon) \left(\frac{4L_2 t \sigma_0(\epsilon)^4 \beta(\epsilon)}{\delta^2} \right)^{\frac{1}{2}}.$$

Hence

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \mathbb{E} \sup_{1 \leq k \leq n} |\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}| &\leq C_\alpha \sqrt{\delta \log \left(\frac{\sigma(\epsilon)^2 t}{\delta} + 3 \right)} + \frac{8t}{\delta} \sigma(\epsilon) \sigma_0(\epsilon)^2 \sqrt{\beta(\epsilon)} L_2 \\ &\leq C_{\alpha,t} \left(\sqrt{\delta \log \left(\frac{\sigma_0(\epsilon)^2 t}{\delta} + 3 \right)} + \frac{1}{\delta} \sigma_0(\epsilon)^3 \sqrt{\beta(\epsilon)} \right), \end{aligned}$$

where $C_{\alpha,t} = \max(C_\alpha, 8\sqrt{L_2 t})$. Choosing $\delta = \sigma_0(\epsilon)^2 \beta(\epsilon)^{\frac{1}{3}}$, we get

$$\limsup_{n \rightarrow +\infty} \mathbb{E} \sup_{1 \leq k \leq n} |\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}| \leq C_{\alpha,t} \sigma_0(\epsilon) \beta(\epsilon)^{\frac{1}{6}} \left(\sqrt{\log \left(\frac{t}{\beta(\epsilon)^{\frac{1}{3}}} + 3 \right)} + 1 \right).$$

For the second part of the theorem we will use the following definition for g :

$$g(x) = \frac{1}{\alpha} \log(x), \quad x \in [1, +\infty).$$

And for the third part of the theorem, the function g will be defined as follows

$$g(x) = \left(\frac{1}{\alpha} \log(x) \right)^p.$$

The proofs of the second and the third part of the theorem can be found in [9]. \square

With the method used above, we can derived results for functions depending on X at a given time. To prove the following propositions, we will use the same notations as in the proof of Theorem 3.5.

PROPOSITION 3.8. *Let X be an infinite activity Lévy process, and f be a Lipschitz function. Then*

$$\left| \mathbb{E} f(X_t) - \mathbb{E} f(\hat{X}_t^\epsilon) \right| \leq C \beta_1^t(\epsilon) \sigma_0(\epsilon),$$

where C is a positive constant.

Proof. We have

$$R_t^\epsilon = {}^d \hat{B}_{T_n}, \quad \sigma(\epsilon)\hat{W}_t = {}^d \hat{B}_{T_n^\epsilon}.$$

So, if f is K -Lipschitz, we have

$$\begin{aligned} \left| \mathbb{E}f(X_t) - \mathbb{E}f\left(X_t^\epsilon + \sigma(\epsilon)\hat{W}_t\right) \right| &= \left| \mathbb{E}f\left(X_t^\epsilon + \hat{B}_{T_n}\right) - \mathbb{E}f\left(X_t^\epsilon + \hat{B}_{T_n^\epsilon}\right) \right| \\ &\leq K\mathbb{E}\left|\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right|. \end{aligned}$$

We conclude with Theorem 3.6. \square

If we replace condition f' bounded by f' Lipschitz, then we get the following result.

PROPOSITION 3.9. *Let X be a Lévy process, f be a continuously differentiable function satisfying f' is Lipschitz. Then*

$$\left| \mathbb{E}f(X_t) - \mathbb{E}\left(2f(X_t^\epsilon) - f(\hat{X}_t^\epsilon)\right) \right| \leq C\beta_2^t(\epsilon)\sigma_0(\epsilon)^2,$$

where C is a positive constant.

Proof. We have

$$R_t^\epsilon = {}^d \hat{B}_{T_n}, \quad \sigma(\epsilon)\hat{W}_t = {}^d \hat{B}_{T_n^\epsilon}.$$

We have shown in the proof of proposition 2.2 that

$$\mathbb{E}f(X_t) - \mathbb{E}f(X_t^\epsilon) = \int_0^1 \mathbb{E}\left[f'(X_t^\epsilon + \theta R_t^\epsilon) R_t^\epsilon\right] d\theta.$$

Because R^ϵ is independent of X^ϵ and \hat{W} , we have

$$\mathbb{E}f(X_t) - \mathbb{E}f(X_t^\epsilon) = \int_0^1 \mathbb{E}\left[\left(f'(X_t^\epsilon + \theta R_t^\epsilon) - f'(X_t^\epsilon + \theta\sigma(\epsilon)\hat{W}_t)\right) R_t^\epsilon\right] d\theta.$$

By the same way, we will get

$$\mathbb{E}f(\hat{X}_t^\epsilon) - \mathbb{E}f(X_t^\epsilon) = \int_0^1 \mathbb{E}\left[\left(f'(X_t^\epsilon + \theta\sigma(\epsilon)\hat{W}_t) - f'(X_t^\epsilon + \theta R_t^\epsilon)\right) \sigma(\epsilon)\hat{W}_t\right] d\theta.$$

Set $I^\epsilon = \mathbb{E}f(X_t) - \mathbb{E}\left(2f(X_t^\epsilon) - f(\hat{X}_t^\epsilon)\right)$. Thus

$$\begin{aligned} I^\epsilon &= \mathbb{E}\left(f(X_t) - f(X_t^\epsilon)\right) + \mathbb{E}\left(f(\hat{X}_t^\epsilon) - f(X_t^\epsilon)\right) \\ &= \int_0^1 \mathbb{E}\left[\left(f'(X_t^\epsilon + \theta R_t^\epsilon) - f'(X_t^\epsilon + \theta\sigma(\epsilon)\hat{W}_t)\right) \left(R_t^\epsilon - \sigma(\epsilon)\hat{W}_t\right)\right] d\theta \\ &= \int_0^1 \mathbb{E}\left[\left(f'(X_t^\epsilon + \theta\hat{B}_{T_n}) - f'(X_t^\epsilon + \theta\hat{B}_{T_n^\epsilon})\right) \left(\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right)\right] d\theta. \end{aligned}$$

Therefore, if f is K -Lipschitz, we get

$$\begin{aligned} \left| \mathbb{E}f(X_t) - \mathbb{E}\left(2f(X_t^\epsilon) - f(\hat{X}_t^\epsilon)\right) \right| &\leq \int_0^1 K\theta\mathbb{E}\left(\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right)^2 d\theta \\ &= \frac{K}{2}\mathbb{E}\left(\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right)^2. \end{aligned}$$

We conclude with Theorem 3.6. \square

For exotic payoffs, we have the following results.

PROPOSITION 3.10. *Let X be an infinite activity Lévy process, and f be a function from $\mathbb{R}^+ \times \mathbb{R}$ in \mathbb{R} K -Lipschitz with respect to its second variable. We assume that $\frac{\epsilon}{\sigma(\epsilon)}$ is bounded. Then*

$$\left| \sup_{\tau \in \mathcal{T}_{[0,t]}} \mathbb{E}f(\tau, X_\tau) - \sup_{\tau \in \mathcal{T}_{[0,t]}} \mathbb{E}f\left(\tau, \hat{X}_\tau^\epsilon\right) \right| \leq C\sigma_0(\epsilon)\beta_1^t(\epsilon),$$

where $\mathcal{T}_{[0,t]}$ is the set of stopping times with values in $[0, t]$, and C is a positive constant independent of ϵ .

Proof. Set $t_k = \frac{kt}{n}$, and

$$\begin{aligned} p(t) &= \sup_{\tau \in \mathcal{T}_{[0,t]}} \mathbb{E}f(\tau, X_\tau), \quad p^\epsilon(t) = \sup_{\tau \in \mathcal{T}_{[0,t]}} \mathbb{E}f\left(\tau, \hat{X}_\tau^\epsilon\right) \\ p_n(t) &= \sup_{\tau \in \mathcal{T}_{[0,t]}^n} \mathbb{E}f(\tau, X_\tau), \quad p_n^\epsilon(t) = \sup_{\tau \in \mathcal{T}_{[0,t]}^n} \mathbb{E}f\left(\tau, \hat{X}_\tau^\epsilon\right), \end{aligned}$$

where $\mathcal{T}_{[0,t]}^n$ is the set of stopping times with values in $\{t_k, k = 1, \dots, n\}$. We know that

$$\lim_{n \rightarrow +\infty} p_n(t) = p(t), \quad \lim_{n \rightarrow +\infty} p_n^\epsilon(t) = p^\epsilon(t).$$

Let $\theta \in \mathcal{T}_{[0,t]}^n$, we have

$$\begin{aligned} \mathbb{E}f(\theta, X_\theta) &= \mathbb{E}f(\theta, X_\theta^\epsilon) + \mathbb{E}\left(f(\theta, X_\theta) - f\left(\theta, \hat{X}_\theta^\epsilon\right)\right) \\ &\leq p_n^\epsilon(t) + \mathbb{E} \sup_{1 \leq k \leq n} \left(f(t_k, X_{t_k}) - f\left(t_k, \hat{X}_{t_k}^\epsilon\right)\right) \\ &= p_n^\epsilon(t) + \mathbb{E} \sup_{1 \leq k \leq n} \left(f\left(t_k, X_{t_k}^\epsilon + R_{t_k}^\epsilon\right) - f\left(t_k, X_{t_k}^\epsilon + \sigma(\epsilon)\hat{W}_{t_k}\right)\right) \\ &= p_n^\epsilon(t) + \mathbb{E} \sup_{1 \leq k \leq n} \left(f\left(t_k, X_{t_k}^\epsilon + B_{T_k}\right) - f\left(t_k, X_{t_k}^\epsilon + B_{T_k}^\epsilon\right)\right) \\ &\leq p_n^\epsilon(t) + K\mathbb{E} \sup_{1 \leq k \leq n} \left|B_{T_k} - B_{T_k}^\epsilon\right|. \end{aligned}$$

So

$$p_n(t) \leq p_n^\epsilon(t) + K\mathbb{E} \sup_{1 \leq k \leq n} \left|B_{T_k} - B_{T_k}^\epsilon\right|.$$

By the same reasoning, we get

$$p_n^\epsilon(t) \leq p_n(t) + K\mathbb{E} \sup_{1 \leq k \leq n} \left|B_{T_k} - B_{T_k}^\epsilon\right|.$$

Therefore

$$|p_n(t) - p_n^\epsilon(t)| \leq K\mathbb{E} \sup_{1 \leq k \leq n} \left|B_{T_k} - B_{T_k}^\epsilon\right|.$$

We conclude with Theorem 3.6. \square

PROPOSITION 3.11. *Let X be a Lévy process and $p > 1$. We assume that $\mathbb{E}e^{pM_t} < \infty$, then for any $x \in \mathbb{R}$ and for any $\theta \in (0, 1)$*

$$\left| \mathbb{E} \left(e^{M_t} - x \right)^+ - \mathbb{E} \left(e^{\hat{M}_t^\epsilon} - x \right)^+ \right| \leq C_{p,\theta} \sigma_0(\epsilon) \left(\beta_{\frac{p}{p-1}, \theta}(\epsilon) \right)^{1-\frac{1}{p}},$$

where $C_{p,\theta}$ is a positive constant independent of ϵ .

Proof. Define

$$M_t^n = \sup_{0 \leq k \leq n} \left(X_{k\frac{t}{n}}^\epsilon + R_{k\frac{t}{n}}^\epsilon \right), \quad \hat{M}_t^{\epsilon,n} = \sup_{0 \leq k \leq n} \left(X_{k\frac{t}{n}}^\epsilon + \sigma(\epsilon) \hat{W}_{k\frac{t}{n}}^\epsilon \right).$$

We know that $\lim_{n \rightarrow +\infty} M_t^n = M_t$ a.s. and $\lim_{n \rightarrow +\infty} \hat{M}_t^{\epsilon,n} = \hat{M}_t^\epsilon$ a.s.. Set

$$U_t^n = \sup_{0 \leq k \leq n} \left(X_{k\frac{t}{n}}^\epsilon + \hat{B}_{T_k} \right), \quad \hat{U}_t^{\epsilon,n} = \sup_{0 \leq k \leq n} \left(X_{k\frac{t}{n}}^\epsilon + \hat{B}_{T_k^\epsilon} \right).$$

So $M_t^n =^d U_t^n$ and $\hat{M}_t^{\epsilon,n} =^d \hat{U}_t^{\epsilon,n}$. By mean value theorem, we have

$$e^{U_t^n} - e^{\hat{U}_t^{\epsilon,n}} = \left(U_t^n - \hat{U}_t^{\epsilon,n} \right) e^{\bar{U}_t^{\epsilon,n}},$$

where $\bar{U}_t^{\epsilon,n}$ is between U_t^n and $\hat{U}_t^{\epsilon,n}$. Set

$$I_n^\epsilon = \left| \mathbb{E} \left(e^{U_t^n} - x \right)^+ - \mathbb{E} \left(e^{\hat{U}_t^{\epsilon,n}} - x \right)^+ \right|.$$

Thus

$$\begin{aligned} I_n^\epsilon &\leq \mathbb{E} \left| e^{U_t^n} - e^{\hat{U}_t^{\epsilon,n}} \right| \\ &\leq \mathbb{E} \left| U_t^n - \hat{U}_t^{\epsilon,n} \right| e^{\bar{U}_t^{\epsilon,n}} \\ &\leq \mathbb{E} \sup_{0 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right| e^{\bar{U}_t^{\epsilon,n}} \\ &\leq \left(\mathbb{E} \sup_{0 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \left(\mathbb{E} e^{p\bar{U}_t^{\epsilon,n}} \right)^{\frac{1}{p}} \\ &\leq \left(\mathbb{E} \sup_{0 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \left(\mathbb{E} \left(e^{pM_t^n} + e^{p\hat{M}_t^{\epsilon,n}} \right) \right)^{\frac{1}{p}} \\ &\leq \left(\mathbb{E} \sup_{0 \leq k \leq n} \left| \hat{B}_{T_k} - \hat{B}_{T_k^\epsilon} \right|^{\frac{p}{p-1}} \right)^{1-\frac{1}{p}} \left(\mathbb{E} \left(e^{pM_t} + e^{p\hat{M}_t^\epsilon} \right) \right)^{\frac{1}{p}}. \end{aligned}$$

But

$$\begin{aligned} \mathbb{E} \left(e^{pM_t} + e^{p\hat{M}_t^\epsilon} \right) &\leq \mathbb{E} \left(e^{pM_t} + e^{p\sigma(\epsilon) \sup_{0 \leq s \leq t} \hat{W}_s} e^{pM_t^\epsilon} \right) \\ &\leq \mathbb{E} e^{pM_t} + 2e^{\frac{p^2}{2}\sigma(\epsilon)^2 t} \mathbb{E} e^{pM_t^\epsilon} \\ &\leq 2e^{\frac{p^2}{2}\sigma(\epsilon)^2 t} \mathbb{E} \left(e^{pM_t} + e^{pM_t^\epsilon} \right). \end{aligned}$$

So using dominated convergence, Theorem 3.6 and Lemma 2.8, we get

$$\begin{aligned} \left| \mathbb{E} \left(e^{M_t} - x \right)^+ - \mathbb{E} \left(e^{M_t^\epsilon} - x \right)^+ \right| &= \lim_{n \rightarrow +\infty} \left| \mathbb{E} \left(e^{M_t^n} - x \right)^+ - \mathbb{E} \left(e^{\hat{M}_t^{\epsilon, n}} - x \right)^+ \right| \\ &= \limsup_{n \rightarrow +\infty} \left| \mathbb{E} \left(e^{U_t^n} - x \right)^+ - \mathbb{E} \left(e^{\hat{U}_t^{\epsilon, n}} - x \right)^+ \right| \\ &\leq C_{p, \theta} \sigma_0(\epsilon) \left(\beta_{\frac{p}{p-1}, \theta}(\epsilon) \right)^{1-\frac{1}{p}}. \end{aligned}$$

□

3.3. Estimates for cumulative distribution functions. The bounds obtained in this section are better than those obtained by truncation, provided that condition (3.2) is satisfied.

PROPOSITION 3.12. *Let X be an infinite activity Lévy process. Set*

$$\tilde{\beta}_{\rho, \theta}(\epsilon) = \beta_{\frac{1}{\rho}-1, \theta}(\epsilon)$$

1. *If $b > 0$, then*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P} [X_t \geq x] - \mathbb{P} [\hat{X}_t^\epsilon \geq x] \right| \leq C \sigma_0(\epsilon) \beta_1^t(\epsilon).$$

2. *If X_t has a locally bounded probability density function and $x \in \mathbb{R}$, then for any pair of reals $\theta, \rho \in (0, 1)$*

$$\left| \mathbb{P} [X_t \geq x] - \mathbb{P} [\hat{X}_t^\epsilon \geq x] \right| \leq C_{x, t, \rho, \theta} \sigma_0(\epsilon)^{1-\rho} \left(\tilde{\beta}_{\rho, \theta}(\epsilon) \right)^\rho.$$

3. *If M_t has a locally bounded probability density function on $(0, +\infty)$ and $x > 0$, then for any pair of reals $\theta, \rho \in (0, 1)$*

$$\left| \mathbb{P} [M_t \geq x] - \mathbb{P} [\hat{M}_t^\epsilon \geq x] \right| \leq C_{x, t, \rho, \theta} \sigma_0(\epsilon)^{1-\rho} \left(\tilde{\beta}_{\rho, \theta}(\epsilon) \right)^\rho.$$

The positive constants C and $C_{x, t, \rho, \theta}$ independent of ϵ .

Proof. We will use the same notations as in the proofs of Theorem 3.5 and Proposition 3.11. Recall that $R_t^\epsilon = {}^d \hat{B}_{T_n}^\epsilon$ and $\sigma(\epsilon) \hat{W}_t = {}^d \hat{B}_{T_n}^\epsilon$. Set

$$Y_t = X_t^\epsilon + \hat{B}_{T_n}, \quad \hat{Y}_t^\epsilon = X_t^\epsilon + \hat{B}_{T_n}^\epsilon.$$

Thus

$$\begin{aligned} \left| \mathbb{P} [X_t \geq x] - \mathbb{P} [\hat{X}_t^\epsilon \geq x] \right| &= \left| \mathbb{P} [Y_t \geq x] - \mathbb{P} [\hat{Y}_t^\epsilon \geq x] \right| \\ &= \left| \mathbb{P} [Y_t \geq x, \hat{Y}_t^\epsilon < x] - \mathbb{P} [Y_t < x, \hat{Y}_t^\epsilon \geq x] \right|. \end{aligned}$$

But, in the case $b > 0$

$$\begin{aligned} \mathbb{P} [Y_t \geq x, \hat{Y}_t^\epsilon < x] &= \mathbb{P} \left[x - \left(Y_t - \hat{Y}_t^\epsilon \right) \leq \hat{Y}_t^\epsilon < x \right] \\ &= \mathbb{P} \left[x - \left(\hat{B}_{T_n} - \hat{B}_{T_n}^\epsilon \right) \leq b B_t + \left(\hat{Y}_t^\epsilon - b B_t \right) < x \right]. \end{aligned}$$

By construction, bB_t is independent of $(\hat{Y}_t^\epsilon - bB_t)$ and of $(\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon})$. We know that $\frac{1}{b\sqrt{2\pi t}}$ is an upper bound of the probability density function of bB_t , so

$$\mathbb{P}\left[Y_t \geq x, \hat{Y}_t^\epsilon < x\right] \leq \frac{1}{b\sqrt{2\pi t}} \mathbb{E}\left|\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right|.$$

We get the first part of the proposition by using Theorem 3.6. Consider now the second part of the proposition. By Lemma 2.11, we know that there exists $K_{x,t} > 0$ such that

$$\begin{aligned} \left|\mathbb{P}[Y_t \geq Y] - \mathbb{P}[\hat{Y}_t^\epsilon \geq x]\right| &\leq K_{x,t}\delta + \frac{\mathbb{E}\left|Y_t - \hat{Y}_t^\epsilon\right|^p}{\delta^p} \\ &= K_{x,t}\delta + \frac{\mathbb{E}\left|\hat{B}_{T_n} - \hat{B}_{T_n^\epsilon}\right|^p}{\delta^p}. \end{aligned}$$

By Theorem 3.6, we know that there exists $C_{t,p,\theta} > 0$ such that

$$\begin{aligned} \left|\mathbb{P}[Y_t \geq Y] - \mathbb{P}[\hat{Y}_t^\epsilon \geq x]\right| &\leq K_{x,t}\delta + C_{t,p,\theta} \frac{\sigma_0(\epsilon)^p \beta_{p,\theta}(\epsilon)}{\delta^p} \\ &\leq \max(K_{x,t}, C_{t,p,\theta}) \left(\delta + \frac{\sigma_0(\epsilon)^p \beta_{p,\theta}(\epsilon)}{\delta^p}\right) \\ &\leq 2 \max(K_{x,t}, C_{t,p,\theta}) \sigma_0(\epsilon)^{1-\frac{1}{p+1}} \beta_{p,\theta}(\epsilon)^{\frac{1}{p+1}}. \end{aligned}$$

The last inequality is obtained by choosing $\delta = \sigma_0(\epsilon)^{1-\frac{1}{p+1}} \beta_{p,\theta}(\epsilon)^{\frac{1}{p+1}}$. We set

$$\begin{aligned} I(\epsilon) &= \left|\mathbb{P}[M_t \geq x] - \mathbb{P}[\hat{M}_t^\epsilon \geq x]\right| \\ I^n(\epsilon) &= \left|\mathbb{P}[M_t^n \geq x] - \mathbb{P}[\hat{M}_t^{\epsilon,n} \geq x]\right|, \end{aligned}$$

Note that $\lim_{\epsilon \rightarrow 0} I^n(\epsilon) = I(\epsilon)$. So (see the proof of Proposition 3.11)

$$I^n(\epsilon) = \left|\mathbb{P}[U_t^n \geq x] - \mathbb{P}[\hat{U}_t^{\epsilon,n} \geq x]\right|.$$

Using the proof of Lemma 2.11, we get

$$\begin{aligned} I^n(\epsilon) &\leq \mathbb{P}[x \leq U_t^n < x + \delta] + \frac{\mathbb{E}\left|U_t^n - \hat{U}_t^{\epsilon,n}\right|^p}{\delta^p} \\ &\leq \mathbb{P}[x \leq U_t^n < x + \delta] + \frac{\mathbb{E} \sup_{1 \leq k \leq n} \left|\hat{B}_{T_k} - \hat{B}_{T_k^\epsilon}\right|^p}{\delta^p}. \end{aligned}$$

But

$$\begin{aligned} \lim_{n \rightarrow +\infty} \mathbb{P}[x \leq U_t^n < x + \delta] &= \lim_{n \rightarrow +\infty} \mathbb{P}[x \leq M_t^n < x + \delta] \\ &= \mathbb{P}[x \leq M_t < x + \delta] \\ &\leq C_{x,t}\delta, \end{aligned}$$

where $C_{x,t}$ is a positive constant given by Lemma 2.11. Thus by Theorem 3.6

$$\begin{aligned} I(\epsilon) &\leq C_{x,t}\delta + \frac{C_{t,p,\theta}}{\delta^p} \sigma_0(\epsilon)^p \beta_{p,\theta}(\epsilon) \\ &\leq 2 \max(C_{x,t}, C_{t,p,\theta}) \sigma_0(\epsilon)^{1-\frac{1}{p+1}} \beta_{p,\theta}(\epsilon)^{\frac{1}{p+1}}. \end{aligned}$$

The last inequality is obtained by choosing $\delta = \sigma_0(\epsilon)^{1-\frac{1}{p+1}} \beta_{p,\theta}(\epsilon)^{\frac{1}{p+1}}$. \square

REFERENCES

- [1] Asmussen, S & Rosinski, J 2001, ‘Approximations of small jumps of Lévy processes with a view towards simulation’, *J. Appl. Probab.*, vol. 38, pp. 482-493.
- [2] Barndorff-Nielsen, OE 1997, ‘Normal inverse Gaussian distributions and stochastic volatility modelling’, *Scand. J. Statistics*, vol. 24, pp. 1-13.
- [3] Bertoin, J 1996, *Lévy Processes*, Cambridge University Press.
- [4] Broadie, M, & Yamamoto, Y 2005, ‘A double-exponential fast Gauss transform algorithm for pricing discrete path-dependent options’, *Operations Research*, vol. 53, pp. 764-779.
- [5] Carr, P. P., Geman, H, Madan DB & Yor, M 2002, ‘The fine structure of asset returns: An empirical investigation’, *Journal of Business*, vol. 75, pp. 305-332.
- [6] Chaumont, L 2011, ‘On the law of the supremum of Lévy processes’, to appear in *Ann. Probab.*
- [7] Cont, R & Tankov, P 2004, *Financial modelling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series, Boca Raton.
- [8] Cont, R & Voltchkova, E 2005, ‘Integro-differential equations for option prices in exponential Lévy models’, *Finance Stochast.*, vol. 9, no. 3, pp. 299-325.
- [9] Dia, EHA 2010, ‘Options exotiques dans les modèles exponentiels de Lévy’, Doctoral thesis, Université Paris-Est, <http://tel.archives-ouvertes.fr/tel-00520583/en/>.
- [10] Dia, EHA & Lamberton, D 2011, ‘Connecting discrete and continuous lookback or hindsight options in exponential Lévy models’, to appear in *Advances in Applied Probability*.
- [11] Eberlein, E 2001, ‘Application of generalized hyperbolic Lévy motions to finance’, in *Lévy Processes: Theory and Applications*, eds OE Barndorff-Nielsen, T Mikosch & S Resnick, Birkhäuser Verlag, pp. 319-337.
- [12] Feng, L & Linetsky, V 2009, ‘Computing Exponential Moments of the Discrete Maximum of a Lévy process and Lookback Options’, *Finance and Stochastics*, vol. 13, no. 4, pp. 501-529.
- [13] Feng, L & Linetsky, V 2008, ‘Pricing Discretely Monitored Barrier Options and Defaultable Bonds in Lévy Process Models: a Fast Hilbert Transform Approach’, *Mathematical Finance*, vol. 18, no. 3, pp. 337-384.
- [14] Morris, CN 1982. ‘Natural exponential families with quadratic variance functions’, *Ann. Statist.*, vol 10, pp. 65-80.
- [15] Kou, SG & Petrella, G 2004, ‘Numerical Pricing of Discrete Barrier and Lookback Options via Laplace Transforms’ *Journal of Computational Finance*, vol. 8, pp. 1-37.
- [16] Rydberg, TH 1997, ‘The normal inverse gaussian lévy process: simulation and approximation’, *Stochastic Models*, vol. 13, no. 4, pp. 887-910.
- [17] Sato, K 1999, *Lévy processes and infinitely divisible distributions*, Cambridge university press.
- [18] Signahl, M 2003, ‘On error rates in Normal approximations and simulation schemes for Lévy processes’, *Stoch. Models.*, vol. 19, pp. 287-298.
- [19] Skorokhod, AV 1965, *Studies in the Theory of Random Processes*, Reading, Mass., Addison-Wesley – cop, 1965.