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To cite this version:

Pierre Cardaliaguet, Anne Souquière. A differential game with a blind player. SIAM Journal on Control and Optimization, 2012, 50 (4), pp.2090–2116. hal-00520017

HAL Id: hal-00520017

https://hal.science/hal-00520017v1

Submitted on 22 Sep 2010

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A differential game with a blind player

Pierre Cardaliaguet*and Anne Souquière†‡

September 22, 2010

Abstract

We consider a zero sum differential game with lack of observation on one side. The initial state of the system is drawn at random according to some probability μ_0 on $\mathbb{R}^N$. Player I is informed of the initial position of state while player II knows only μ_0. Moreover Player I observes Player II's moves while Player II is blind and has no further information. We prove that in this game with a terminal payoff the value exists and is characterized as the unique viscosity solution of some Hamilton-Jacobi equation on a space of probability measures.

Keywords

Differential games - Asymmetric information - Hamilton-Jacobi equations - Viscosity solutions - Wasserstein space.

Introduction

We consider a two player zero sum differential game in $\mathbb{R}^N$ with finite horizon $T > 0$. Its dynamics is given by :

$$\begin{cases} x'(t) = f(x(t), u(t), v(t)) \ , \quad t \in [t_0, T], \ u(t) \in U, \ v(t) \in V \\ x(t_0) = x_0 \end{cases} \tag{1}$$

where Player I uses the measurable control $u \in \mathcal{U}(t_0) := L^1([t_0, T], U)$ and Player II the measurable control $v \in \mathcal{V}(t_0) := L^1([t_0, T], V)$. We denote by

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$(X_\cdot^{t_0,x_0,u,v})$ the solution of (1), which is unique under suitable assumptions on f stated below. In this zero sum game, Player I aims at minimizing a final cost $g(x(T))$, where $g : \mathbb{R}^N \to \mathbb{R}$, while player II aims at maximizing it. We introduce lack of observation in the following way:

- At time t_0, the initial state of the system, x_0, is drawn at random according to some probability measure μ_0 on $\mathbb{R}^N$;
- Player I is informed of x_0 while Player II is only informed of μ_0;
- During the game, Player II observes neither the state of the system and nor the control played by his or her opponent, while Player I has a full information on the control played so far by Player II (and therefore on the state of the system as well).

Our aim is to prove that the game has a value and to characterize this value as the unique viscosity solution of some Hamilton-Jacobi equation. Since the natural state of the system is the space of probability measures on $\mathbb{R}^N$, this Hamilton-Jacobi equation takes place in this space.

Games with asymmetric information were studied—mostly on examples—by several authors: see for instance Bernhard and Rapaport [2], Gal [8], Petrosjan [11] and Baras and James [1]. In this latter reference the authors introduce an underlying Hamilton-Jacobi equation in some infinite dimensional space, but, since the game is seen as a control problem with disturbance, the value function considered there differs considerably from ours.

Our game has actually much to do with a previous work of Cardaliaguet and Quincampoix [3] which analyses problems in which the only information that both players have on the initial position of the system is that it has been randomly choosen according to some probability known to both players. In [3], the players observe each other. This is a main difference with our problem, where the lack of observation of one player induces the use of completely asymmetric strategies. The introduction of a suitable notion of strategies to formalize this situation is one of the novelties of our paper. A dramatic consequence of the asymmetry of information is that the usual machinery of differential games (dynamic programming, which leads to the characterization of the value functions as the unique solution of some Hamilton-Jacobi equation) does not work. Indeed the lower value does not seem to satisfy any dynamic programming, because the uninformed player cannot actualize his or her strategy along the game since he or she sees nothing. However, and fortunately, it turns out that, in the game for

the upper value, the uninformed player, knowing the strategy of his or her oponent, can actualize his or her own strategy along the time. This leads to a dynamic programming for the upper value, which takes place in the space of probability measures on $\mathbb{R}^N$. From this we derive that the upper value satisfies a Hamilton-Jacobi equation in some suitable viscosity sense. The definition of the viscosity solution in this framework is inspired by, but slightly differs from, the one given in [3]. Other definitions of viscosity solution in the Wasserstein space have been used in the literature, in general for more singular dynamics (see for instance [6, 7, 9, 10]). Then the existence of a value (i.e., the fact that the upper value coincides with the lower one) relies on min-max arguments combined with PDE ones: we introduce an auxiliary game in which the uninformed player chooses a strategy by randomizing over a finite set of controls. Existence of a value for this game is obtained by min-max arguments. Then we show that this auxiliary game is close to the continuous one by using techniques from Crandall and Lions [5] on the stability of viscosity solutions for Hamilton-Jacobi equations in infinite dimension.

The paper is organized in the following way: in the first section, we define the strategies and state the assumptions on the game. In the second section, we prove that the upper and lower value functions are Lipschitz continuous. In the third section, we state the dynamic programming principle for the upper value function. In section 4, we show that if a function satisfies this dynamic programming principle, then it is the unique viscosity solution of some Hamilton-Jacobi equation. In section 5, we introduce some discrete approximation for the game. In the last section, we prove, using the discrete game, that the game has a value.

1 Definitions and assumptions

We first introduce some notations on the space of probability measures on $\mathbb{R}^N$. For a fixed closed subset K of $\mathbb{R}^N$ we denote by $\mathcal{W}(K)$ the set of Borel probability measures with support included in K and with finite second order moment. We set $\mathcal{W} = \mathcal{W}(\mathbb{R}^N)$. For any $\mu, \nu \in \mathcal{W}$, let $\Pi(\mu, \nu)$ be the set of probability measures on $\mathbb{R}^{2N}$ with first marginal μ and second marginal ν. Recall that the Wasserstein distance on $\mathcal{W}$ between μ and ν is defined as

$$\mathbf{d}^2(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^{2N}} |x - y|^2 d\pi(x, y) \ .$$

It is well-known that this infimum is in fact a minimum and we denote by $\Pi_{opt}(\mu, \nu)$ the set of minimizers in the above minimization problem. If μ

is a probability measure on a set X and $\varphi : X \to Y$, we denote by $\varphi\sharp\mu$ the pushforward image of μ by φ, defined by $\varphi\sharp\mu(A) = \mu(\varphi^{-1}(A))$ for any subset A of Y for which this definition makes sense.

Next we introduce notations and assumptions related to the game. The payoff only depends on the terminal state of the system. More precisely, if at time T the system is at some position $x(T)$, then the outcome of the game is $g(x(T))$, where $g : \mathbb{R}^N \to \mathbb{R}$ is a fixed Lipschitz continuous and bounded function. Assume that the initial state is choosen at random according to a probability measure $\mu_0 \in \mathcal{W}$ at the initial time t_0 and suppose for a while that the players use a pair of controls $(u, v) \in \mathcal{U}(t_0) \times \mathcal{V}(t_0)$ independent of the initial state. Then the outcome of the game is

$$\mathfrak{J}(t_0, \mu_0, u, v) = \int_{\mathbb{R}^N} g(X_T^{t_0,x,u,v}) d\mu_0(x) \ .$$

Throughout this paper we tacitely assume that the following conditions on the data are satisfied:

$$\begin{cases} i) & U \text{ and } V \text{ are compact subsets of some finite dimensional vector spaces,} \\ ii) & f \text{ is bounded, uniformly continuous on } \mathbb{R}^N \times U \times V, \\ & \quad \text{and uniformly Lipschitz continuous with respect to the } x \text{ variable,} \\ iii) & g \text{ is Lipschitz continuous and bounded.} \end{cases} \tag{2}$$

During the proofs, we denote by C a generic constant depending on N, f and g.

For any $0 \leq t_0 < t_1 \leq T$ we denote by $\mathcal{U}(t_0, t_1)$ the set of Lebesgue measurable maps $u : [t_0, t_1] \to U$. We abbreviate the notation into $\mathcal{U}(t_0)$ whenever $t_1 = T$. We endow $\mathcal{U}(t_0, t_1)$ with the L^1 distance

$$d_{\mathcal{U}(t_0,t_1)}(u_1, u_2) = \int_{t_0}^{t_1} |u_1(s) - u_2(s)| ds \qquad \forall u_1, u_2 \in \mathcal{U}(t_0, t_1) \ ,$$

and with the Borel $\sigma-$algebra associated with this distance. Recall that $\mathcal{U}(t_0, t_1)$ is then a Polish space (i.e., a complete separable metric space). We denote by $\Delta(\mathcal{U}(t_0, t_1))$ the set of Borel probability measures on $\mathcal{U}(t_0, t_1)$. This set is endowed with the weak-* topology, for which there is an associated distance defined as follows:

$$d_{\Delta(\mathcal{U}(t_0,t_1))}(P_1, P_2) = \sup \left\{ \int_{\mathcal{U}(t_0,t_1)} \varphi(u) d(P_1 - P_2)(u) \ , \right\} \ ,$$

where the supremum is taken over the set of Lipschitz continuous maps $\varphi : \mathcal{U}(t_0, t_1) \to [-1, 1]$ with a Lipschitz constant less than 1.

The sets $\mathcal{V}(t_0,t_1)$ and $\mathcal{V}(t_0)$ of Lebesgue measurable maps $v:[t_0,t_1]\to V$ and $v:[t_0,T]\to V$ are defined in a symmetric way and endowed with the L^1 distance and with the associate Borel $\sigma-$algebra. The set of Borel probability measures on $\mathcal{V}(t_0,t_1)$ is denoted by $\Delta(\mathcal{V}(t_0,t_1))$.

We say that a map $(x,v)\to P_x^v$ from $\mathbb{R}^N\times\mathcal{V}(t_0)$ into $\Delta(U(t_0))$ is measurable if, for any Borel subset A of $\mathcal{U}(t_0)$, the mapping $(x,v)\to P_x^v(A)$ is Borel measurable.

Definition 1.1. *A strategy for Player I for the initial time $t_0\in[0,T]$ is a measurable mapping $(x,v)\to P_x^v$ from $\mathbb{R}^N\times\mathcal{V}(t_0)$ into $\Delta(\mathcal{U}(t_0))$ which fulfills the following nonanticipativity condition: there is some delay $\tau>0$ such that, if two controls $v_1,v_2\in\mathcal{V}(t_0)$ coincide a.e. on $[t_0,t]$ for some $t\in[t_0,T]$ and if $R_{t_0,(t+\tau)\wedge T}$ denotes the restriction mapping from $\mathcal{U}(t_0)$ onto $\mathcal{U}(t_0,(t+\tau)\wedge T)$, then the measures $R_{t_0,(t+\tau)\wedge T}\sharp P_x^{v_1}$ and $R_{t_0,(t+\tau)\wedge T}\sharp P_x^{v_2}$ coincide (on $\mathcal{U}(t_0,(t+\tau)\wedge T)$) for any $x\in\mathbb{R}^N$.*

Note that strategies for Player I actually correspond to behavioral strategies in game theory, because Player I adapts his or her probability measure in function of the past behaviour of his or her oponent. The heuristic interpretation of a strategy P is that, if the state of the system is at the initial position x, then Player I answers (in a nonanticipative way) to a control $v\in\mathcal{V}(t_0)$ played by Player II a control $u\in\mathcal{U}(t_0)$ with probability $P_x^v(u)$.

We denote by $\Delta(A_x(t_0))$ the set of strategies for Player I, by $\Delta(\mathcal{A}_x^\tau(t_0))$ the set of such strategies which have a delay τ and by $\mathcal{A}_x^\tau(t_0)$ the subset of $\Delta(\mathcal{A}_x^\tau(t_0))$ consisting in *deterministic* strategies, i.e., strategies for which, for any $(x,v)\in\mathbb{R}^N\times\mathcal{V}(t_0)$, P_x^v is a Dirac mass. If $P\in\mathcal{A}_x^\tau(t_0)$, then there is a map $\alpha:\mathbb{R}^N\times\mathcal{V}(t_0)\to\mathcal{U}(t_0)$ such that $dP_x^v(u)=d\delta_{\alpha(x,v)}(u)$, and this map satisfies the nonanticipative property: for μ_0-a.e. $x\in\mathbb{R}^N$, if two controls $v_1,v_2\in\mathcal{V}(t_0)$ coincide a.e. in $[t_0,t]$ for some $t\in[t_0,T]$, then $\alpha(x,v_1)=\alpha(x,v_2)$ a.e. in $[t_0,(t+\tau)\wedge T]$. Generic elements of $\mathcal{A}_x^\tau(t_0)$ are systematically identified with the maps α.

Since Player II observes neither the state nor his or her oponent behavior, the definition of his or her strategies is much simpler than for Player I:

Definition 1.2. *A strategy for Player II is a Borel probability measure Q on the set $\mathcal{V}(t_0)$.*

Recall that we denote by $\Delta(\mathcal{V}(t_0))$ the set of such strategies. Given $Q\in\Delta(\mathcal{V}(t_0))$ and $P\in\Delta(\mathcal{A}_x(t_0))$ we denote by $\mathfrak{J}(t_0,\mu_0,P,Q)$ the outcome of the two strategies P and Q:

$$\mathfrak{J}(t_0,\mu_0,P,Q)=\int_{\mathbb{R}^N\times\mathcal{U}(t_0)\times\mathcal{V}(t_0)}g\left(X_T^{t_0,x,u,v}\right)dP_x^v(u)dQ(v)d\mu_0(x)\ .$$

We are now ready to define the value functions. The *lower value* of the game is:

$$\begin{aligned}\mathbf{V}^-(t_0,\mu_0) &= \lim_{\tau\to 0^+}\sup_{Q\in\Delta(\mathcal{V}(t_0))}\inf_{P\in\Delta(\mathcal{A}_x^\tau(t_0))}\mathfrak{J}(t_0,\mu_0,P,Q)\\ &= \lim_{\tau\to 0^+}\mathbf{V}_\tau^-(t_0,\mu_0)\end{aligned}$$

where we have set

$$\begin{aligned}\mathbf{V}_\tau^-(t_0,\mu_0) &= \sup_{Q\in\Delta(\mathcal{V}(t_0))}\inf_{P\in\Delta(\mathcal{A}_x^\tau(t_0))}\mathfrak{J}(t_0,\mu_0,P,Q)\\ &= \sup_{Q\in\Delta(\mathcal{V}(t_0))}\inf_{\alpha\in\mathcal{A}_x^\tau(t_0)}\mathfrak{J}(t_0,\mu_0,\alpha,Q)\end{aligned} \tag{3}$$

The *upper value* of the game is defined in a symmetrical way:

$$\begin{aligned}\mathbf{V}^+(t_0,\mu_0) &= \lim_{\tau\to 0}\inf_{P\in\Delta(\mathcal{A}_x^\tau(t_0))}\sup_{Q\in\Delta(\mathcal{V}(t_0))}\mathfrak{J}(t_0,\mu_0,P,Q)\\ &= \inf_{P\in\Delta(\mathcal{A}_x(t_0,\mu_0))}\sup_{v\in\mathcal{V}(t_0)}\mathfrak{J}(t_0,\mu_0,P,v)\end{aligned}$$

2 Regularity of the value functions

We begin by proving the Lipschitz continuity of the upper and lower value functions, which is important for the characterization of the value as a viscosity solution of some Hamilton-Jacobi equation.

Proposition 2.1 (Regularity of the value functions)**.** *The value functions* $\mathbf{V}^+$ *and* $\mathbf{V}^-$ *are Lipschitz continuous on* $[0,T]\times\mathcal{W}$.

Proof. We start with the Lipschitz continuity of $\mathbf{V}^+$ with respect to the μ variable. Let $t_0\in[0,T]$, $\mu,\nu\in\mathcal{W}$ and choose $\gamma\in\Pi_{opt}(\nu,\mu)$ some optimal transport plan between μ and ν. Let us recall that γ admits a desintegration of the form $d\gamma(x,y)=d\gamma_x(y)d\nu(x)$ where the map $x\to\gamma_x$ is measurable, i.e., such that the map $x\to\gamma_x(A)$ is Borel measurable for any Borel set $A\subset\mathbb{R}^N$. Let $P\in\Delta(\mathcal{A}_x(t_0))$ be an ϵ-optimal strategy for $\mathbf{V}^+(t_0,\mu)$, i.e., P satisfies

$$\sup_{v\in\mathcal{V}(t_0)}\mathfrak{J}(t_0,\mu,P,v)\le\mathbf{V}^+(t_0,\mu)+\epsilon\ .$$

We define the strategy $\tilde{P}\in\Delta(\mathcal{A}_x(t_0))$ by

$$\int_{\mathcal{U}(t_0)}\varphi(u)d\tilde{P}_x^v(u)=\int_{\mathbb{R}^N\times\mathcal{U}(t_0)}\varphi(u)dP_y^v(u)d\gamma_x(y)$$

for any $(x,v) \in \mathbb{R}^N \times \mathcal{V}(t_0)$ and for any nonnegative Borel measurable map $\varphi : \mathcal{U}(t_0) \to \mathbb{R}$. Let now $v \in \mathcal{V}(t_0)$ and let us estimate $\mathfrak{J}(t_0, \nu, \tilde{P}, v)$: we have

$$\begin{aligned}
\mathfrak{J}(t_0, \nu, \tilde{P}, v) &= \int_{\mathbb{R}^{2N} \times \mathcal{U}(t_0)} g\left(X_T^{t_0,x,u,v}\right) dP_y^v(u) d\gamma_x(y) d\nu(x) \\
&\leq \int_{\mathbb{R}^{2N} \times \mathcal{U}(t_0)} \left[g\left(X_T^{t_0,y,u,v}\right) + C|x-y|\right] dP_y^v(u) d\gamma(x,y) \\
&\leq \int_{\mathbb{R}^{N} \times \mathcal{U}(t_0)} g\left(X_T^{t_0,y,u,v}\right) dP_y^v(u) d\mu(y) + C \int_{\mathbb{R}^N \times \mathbb{R}^N} |x-y| d\gamma(x,y) \\
&\leq \mathbf{V}^+(t_0, \mu) + \epsilon + C\mathbf{d}(\mu, \nu)
\end{aligned}$$

Therefore

$$\mathbf{V}^+(t_0, \nu) \leq \sup_{v \in \mathcal{V}(t_0)} \mathfrak{J}(t_0, \nu, \tilde{P}, v) \leq \mathbf{V}^+(t_0, \mu) + \epsilon + C\mathbf{d}(\mu, \nu) \ .$$

This proves the Lipschitz continuity of $\mathbf{V}^+$ with respect to second variable, uniformly with respect to the time variable.

We now prove that $\mathbf{V}^+$ is Lipschitz continuous with respect to the time variable. Fix $t_0 < t_1 \leq T$, $\mu \in \mathcal{W}$ and $v_0 \in \mathcal{V}(t_0)$. We choose some ϵ-optimal strategy P for Player I in $\mathbf{V}^+(t_0, \mu)$ and define the strategy $\tilde{P} \in \Delta(\mathcal{A}_x(t_1))$ by:

$$\int_{\mathcal{U}(t_1)} \varphi(u_1) d\tilde{P}_x^v(u_1) = \int_{\mathcal{U}(t_0)} \varphi(u_{|_{[t_1,T]}}) dP_x^{(v_0,v)}(u)$$

where (v_0, v) denotes the concatenation of the controls v_0 and v, for any $(x,v) \in \mathbb{R}^N \times \mathcal{V}(t_1)$ and any nonnegative Borel measurable map $\varphi : \mathcal{U}(t_1) \to \mathbb{R}$. Then, for any $v \in \mathcal{V}(t_1)$, we have

$$\begin{aligned}
\mathfrak{J}(t_1, \mu, \tilde{P}, v) &= \int_{\mathbb{R}^N \times \mathcal{U}(t_0)} g\left(X_T^{t_1,x,u_{|_{[t_1,T]}},v}\right) dP_x^{(v_0,v)}(u) d\mu(x) \\
&\leq \int_{\mathbb{R}^N \times \mathcal{U}(t_0)} \left[g\left(X_T^{t_0,x,u,(v_0,v)}\right) + C(t_1 - t_0)\right] dP_x^{(v_0,v)}(u) d\mu(x) \\
&\leq \mathbf{V}^+(t_0, \mu) + \epsilon + C(t_1 - t_0)
\end{aligned}$$

Therefore we get:

$$\mathbf{V}^+(t_1, \mu) - \mathbf{V}^+(t_0, \mu) \leq \epsilon + C(t_1 - t_0) \ .$$

For the reverse inequality, let $P \in \Delta(\mathcal{A}_x(t_1))$ be some ϵ-optimal strategy for player I in $\mathbf{V}^+(t_1, \mu)$. We fix $u_0 \in \mathcal{U}(t_0)$ and define the strategy $\tilde{P} \in \Delta(\mathcal{A}_x(t_0))$ by

$$\int_{\mathcal{U}(t_0)} \varphi(u) d\tilde{P}_x^v(u) = \int_{\mathcal{U}(t_1)} \varphi((u_0, u_1)) dP_x^{v_{|_{[t_1,T]}}}(u_1)$$

for any $(x,v) \in \mathbb{R}^N \times \mathcal{V}(t_0)$ and for any nonnegative Borel measurable map $\varphi : \mathcal{U}(t_0) \to \mathbb{R}$. Then, for any $v \in \mathcal{V}(t_0)$, we have

$$\begin{aligned}
\mathfrak{J}(t_0,\mu,\tilde{P},v) &= \int_{\mathbb{R}^N \times \mathcal{U}(t_1)} g\left(X_T^{t_0,x,(u_0,u_1),v}\right) dP_x^{v_{|[t_1,T]}}(u_1) d\mu(x) \\
&\leq \int_{\mathbb{R}^N \times \mathcal{U}(t_1)} \left[g\left(X_T^{t_1,x,u_1,v_{|[t_1,T]}}\right) + C(t_1-t_0) \right] dP_x^{v_{|[t_1,T]}}(u_1) d\mu(x) \\
&\leq \mathbf{V}^+(t_1,\mu) + \epsilon + C(t_1-t_0)
\end{aligned}$$

Hence:

$$\mathbf{V}^+(t_0,\mu) - \mathbf{V}^+(t_1,\mu) \leq \epsilon + C(t_1-t_0) \ .$$

which shows that $\mathbf{V}^+$ is Lipschitz continuous with respect to the time variable, uniformly with respect to the μ variable, since ϵ is arbitrary.

The proof of the Lipschitz continuity for $\mathbf{V}^-$ goes along the same lines, so we omit it. □

3 Dynamic programming for the upper value function

We prove in this section that $\mathbf{V}^+$ satisfies some dynamic programming principle. We have to define how Player II's information evolves in time. In the game $\mathbf{V}^+(t_0,\mu_0)$, Player II knows the initial distribution of the state variable as well as his or her opponent's strategy P. If he or she plays the control $v \in \mathcal{V}(t_0)$, his or her information on the state of the system at time $t_1 \in (t_0,T]$ is the probability measure $\mu_{t_1}^{t_0,\mu_0,P,v}$ defined by:

$$\forall \varphi \in \mathcal{C}_b(\mathbb{R}^N,\mathbb{R}),\ \int_{\mathbb{R}^N} \varphi(x) d\mu_{t_1}^{t_0,\mu_0,P,v}(x) = \int_{\mathbb{R}^N \times \mathcal{U}(t_0)} \varphi(X_{t_1}^{t_0,x,u,v}) dP_x^v(u) d\mu_0(x) \ .$$

Note that $\mu_{t_1}^{t_0,\mu_0,P,v}$ belongs to $\mathcal{W}$.

Proposition 3.1 (Dynamic programming principle for $\mathbf{V}^+$)**.** *For any* (t_0,t_1,μ_0) *such that* $t_1 \in (t_0,T]$*, we have:*

$$\mathbf{V}^+(t_0,\mu_0) = \inf_{P \in \Delta(\mathcal{A}_x(t_0))} \sup_{v \in \mathcal{V}(t_0)} \mathbf{V}^+(t_1,\mu_{t_1}^{t_0,\mu_0,P,v}) \ .$$

Proof. We denote by $W(t_0,t_1,\mu_0)$ the right-hand side of the previous equality. Arguing as for Proposition 2.1, one can show that W is Lipschitz continuous with respect to the measure variable.

Let us now show that we can assume in addition to (2) that f has a uniformly bounded $\mathcal{C}^2$ norm with respect to the x variable. Indeed, from our assumptions on f, if we mollify f with respect to the x variable, we obtain a sequence of uniformly continuous functions $f_n : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$, with a modulus of continuity independent of n, uniformly (with respect to n) Lipschitz continuous in space and which converge uniformly to f on $\mathbb{R}^N \times U \times V$. We easily check that the upper value function $\mathbf{V}_n^+$ for f_n corresponding to f_n converges to $\mathbf{V}^+$ and that the W_n converge to W uniformly on $[0,T] \times \mathcal{W}$. We also note that the transported measures $\mu_{t_1}^{n,t_0,\mu_0,P,v}$ for f_n converges in $\mathcal{W}$ to $\mu_{t_1}^{t_0,\mu_0,P,v}$ uniformly with respect to P and v. So, if Lemma 3.1 holds for the f_n, it also holds for f. Therefore we can assume, from now on, that f has a uniformly bounded $\mathcal{C}^2$ norm with respect to the x variable.

Let us first prove that $\mathbf{V}^+(t_0,\mu_0) \leq W(t_0,t_1,\mu_0)$ under the additional assumptions that $\mu_0 \in \mathcal{W}(K)$, where K is some compact in $\mathbb{R}^N$. This extra assumption is removed later. The first step consists in regularizing μ_0. Let $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be a smooth mollifier: $\rho \geq 0$ is even, has a support in the unit ball and satisfies $\int_{\mathbb{R}^N} \rho(x)dx = 1$. Let $\rho_\epsilon(x) = \epsilon^N \rho(\frac{x}{\epsilon})$, $f_\epsilon = \rho_\epsilon * \mu_0$ and $\mu_\epsilon = f_\epsilon dx$. By standard arguments, we have that $\mathbf{d}(\mu_0,\mu_\epsilon) \leq \epsilon$.

Lemma 3.2. *For any strategy $P \in \Delta(\mathcal{A}_x(t_0))$, there exists a strategy $P_\epsilon \in \Delta(\mathcal{A}_x(t_0))$, with the same delay as P, a compact set $K_1 \subset \mathbb{R}^N$ and a constant C such that, for any $v \in \mathcal{V}(t_0)$ and any $t \in [t_0,T]$,*

1. $\mathbf{d}\left(\mu_t^{t_0,\mu_0,P,v}, \mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}\right) \leq C\epsilon$,
2. $\mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}$ *has a support in K_1 and a density $f_t^{v,\epsilon}$ bounded in $\mathcal{C}^1(K_1)$ by C,*

Proof. Let $P_\epsilon \in \mathcal{A}_x(t_0)$ be defined by: if $f_\epsilon(x) > 0$, then we set

$$\int_{\mathcal{U}(t_0)} \varphi(u) dP_{\epsilon,x}^v(u) = \frac{1}{f_\epsilon(x)} \int_{\mathbb{R}^N \times \mathcal{U}(t_0)} \varphi(u)\rho_\epsilon(x-y) dP_y^v(u) d\mu_0(y)$$

for any $(x,v) \in \mathbb{R}^N \times \mathcal{V}(t_0)$ and for any nonnegative Borel measurable map $\varphi : \mathcal{U}(t_0) \to \mathbb{R}$. If $f_\epsilon(x) = 0$, we just set $dP_{\epsilon,x}^v(u) = dP_x^v(u)$.

Since μ_ϵ has bounded support and the dynamics is bounded, there is a compact set K_1 such that $\mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}$ has a support contained in K_1 for any $v \in \mathcal{V}(t_0)$ and any $t \in [t_0,T]$.

We now compare $\mu_t^{t_0,\mu_0,P,v}$ to $\mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}$ for any $v \in \mathcal{V}(t_0)$ and any $t \in [t_0,T]$: we have

$$\mathbf{d}^2(\mu_t^{t_0,\mu_0,P,v}, \mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}) \leq \int_{\mathbb{R}^{2N}\times\mathcal{U}(t_0)} \left|X_t^{t_0,x,u,v} - X_t^{t_0,y,u,v}\right|^2 \rho_\epsilon(y-x) dP_x^v(u) d\mu_0(x) dy$$

because the probability measure γ on $\mathbb{R}^{2N}$ defined by

$$\int_{\mathbb{R}^{2N}} \varphi(x,y) d\gamma(x,y) = \int_{\mathbb{R}^{2N}\times\mathcal{U}(t_0)} \varphi(X_t^{t_0,x,u,v}, X_t^{t_0,y,u,v)}) \rho_\epsilon(y-x) dP_x^v(u) d\mu_0(x) dy$$

satisfies $\gamma \in \Pi(\mu_t^{t_0,\mu_0,P,v}, \mu_t^{t_0,\mu_\epsilon,P_\epsilon,v})$. Since $\left|X_t^{t_0,x,u,v} - X_t^{t_0,y,u,v}\right| \leq C|x-y|$, we get

$$\begin{aligned}
&\mathbf{d}^2(\mu_t^{t_0,\mu_0,P,v}, \mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}) \\
&\quad \leq C \int_{\mathbb{R}^{2N}\times\mathcal{U}(t_0)} |x-y|^2 \rho_\epsilon(y-x) dP_x^v(u) d\mu_0(x) dy \\
&\quad \leq C\epsilon^2 \int_{\mathbb{R}^{2N}\times\mathcal{U}(t_0)} \rho_\epsilon(y-x) dP_x^v(u) d\mu_0(x) dy \ \leq \ C\epsilon^2 \ .
\end{aligned}$$

Therefore for all $v \in \mathcal{V}(t_0)$ and any $t \in [t_0,T]$:

$$\mathbf{d}(\mu_t^{t_0,\mu_0,P,v}, \mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}) \leq C\epsilon \ .$$

We now check that the measure $\mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}$ is absolutely continuous and has a density bounded in $\mathcal{C}^1(K_1)$ uniformly with respect to v and t. We first note that for fixed $(t,u,v) \in [t_0,T]\times\mathcal{U}(t_0)\times\mathcal{V}(t_0)$ the map $\mathcal{T}(t,u,v) : x \mapsto X_t^{t_0,x,u,v}$ is of class $\mathcal{C}^2$ with a $\mathcal{C}^2$ inverse because the dynamics f is of class $\mathcal{C}^2$ with respect to the x variable. We denote by $\mathcal{T}(t,u,v)^{-1}$ this inverse. We have, for all $\varphi \in C_b^0(\mathbb{R}^N,\mathbb{R})$:

$$\begin{aligned}
&\int_{\mathbb{R}^N} \varphi(x) d\mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}(x) \\
&\quad = \int_{\mathbb{R}^{2N}\times\mathcal{U}(t_0)} \varphi(X_t^{t_0,x,u,v}) \rho_\epsilon(x-y) dP_y^v(u) d\mu_0(y) dx \\
&\quad = \int_{\mathbb{R}^{2N}\times\mathcal{U}(t_0)} \varphi(z) \rho_\epsilon(\mathcal{T}(t,u,v)^{-1}(z)-y) |\det J_{\mathcal{T}(t,u,v)^{-1}}(z)| dP_y^v(u) d\mu_0(y) \ dz
\end{aligned}$$

Therefore $\mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}$ is absolutely continuous with a density given by

$$f_t^{v,\epsilon}(z) = \int_{\mathbb{R}^N\times\mathcal{U}(t_0)} \rho_\epsilon(\mathcal{T}(t,u,v)^{-1}(z)-y) |\det J_{\mathcal{T}(t,u,v)^{-1}}(z)| dP_y^v(u) d\mu_0(y) \ .$$

Note that $f_t^{v,\epsilon}$ is bounded in $\mathcal{C}^1$, uniformly with respect to v and t, thanks to our assumptions on the dynamics f. $\square$

We now proceed in the proof of inequality $\mathbf{V}^+(t_0,\mu_0) \leq W(t_0,t_1,\mu_0)$ under the additional assumption that $\mu_0 \in \mathcal{W}(K)$. Let $P_0 \in \Delta(\mathcal{A}_x(t_0))$ be an $\epsilon-$optimal strategy for $W(t_0,t_1,\mu_0)$ and $P_\epsilon \in \Delta(\mathcal{A}_x(t_0))$ be the strategy associated to P_0 as in Lemma 3.2.

We first note that P_ϵ is $C\epsilon-$optimal for $W(t_0,t_1,\mu_\epsilon)$: indeed we have

$$\begin{aligned}\sup_{v\in\mathcal{V}(t_0)} \mathbf{V}^+(t_1,\mu_{t_1}^{t_0,\mu_\epsilon,P_\epsilon,v}) &\leq \sup_{v\in\mathcal{V}(t_0)} \mathbf{V}^+(t_1,\mu_{t_1}^{t_0,\mu_0,P_0,v}) + C\mathbf{d}(\mu_{t_1}^{t_0,\mu_\epsilon,P_\epsilon,v},\mu_{t_1}^{t_0,\mu_0,P_0,v}) \\ &\leq W(t_0,t_1,\mu_0)+C\epsilon \;\leq\; W(t_0,t_1,\mu_\epsilon)+C\epsilon\,.\end{aligned}$$

For $v \in \mathcal{V}(t_0)$ and $t \in [t_0,T]$, let $f_t^{v,\epsilon}$ be the density of the measure $\mu_t^{t_0,\mu_\epsilon,P_\epsilon,v}$. We denote by $\mathbf{F}$ the closure in $L^1(\mathbb{R}^N)$ of the set $\{f_t^{v,\epsilon}\ ,\ v \in \mathcal{V}(t_0),\ t\in[t_0,T]\}$. Since, from Lemma 3.2, the elements of $\mathbf{F}$ have a support contained in a fixed compact set K_1 and are uniformly bounded in $\mathcal{C}^1$, $\mathbf{F}$ is a compact subset of $L^1(\mathbb{R}^N)$. Therefore, for any fixed $\eta>0$, we can find a partition $(O_i)_{i=1,\dots,n}$ of $\mathbf{F}$ into Borel (for the L^1-topology) subsets with a diameter in L^1 less than η. Let $f_i \in O_i$, $\mu_i = f_i dx$ and $P_i \in \Delta(\mathcal{A}_x(t_1))$ be an $(\epsilon/6)-$optimal strategy for $\mathbf{V}^+(t_1,\mu_i)$. Let us check that, if η is small enough, then the strategy P_i is still $\epsilon/2-$optimal for $\mathbf{V}^+(t_1,\mu)$ for any measure $\mu \in \mathbf{F}$ such that $\|h_\mu - f_i\|_1 \leq \eta$, where h_μ is the density of μ. Indeed, for all $v \in \mathcal{V}(t_1)$, we have

$$\begin{aligned}&|\mathfrak{J}(t_1,\mu,P_i,v)-\mathfrak{J}(t_1,\mu_i,P_i,v)| \\ &\quad\leq \int_{\mathbb{R}^N\times\mathcal{U}(t_0)} \left|g(X_T^{t_1,x,u,v})(f_i(x)-h_\mu(x))\right| dP_{i,x}^v(u)dx \\ &\quad\leq \|g\|_\infty \|h_\mu - f_i\|_{L^1} \leq \eta\|g\|_\infty \;\leq\; \epsilon/6\,.\end{aligned}$$

So

$$\begin{aligned}\sup_{v\in\mathcal{V}(t_1)} \mathfrak{J}(t_1,\mu,P_i,v) &\leq \sup_{v\in\mathcal{V}(t_0)} \mathfrak{J}(t_1,\mu_i,P_i,v)+\epsilon/6 \\ &\leq \mathbf{V}^+(t_1,\mu_i)+\epsilon/3 \;\leq\; \mathbf{V}^+(t_1,\mu)+\epsilon/2\,.\end{aligned}$$

Let τ be a common delay for P_ϵ and for all the P_i $(i=1,\dots,I)$. For $v\in\mathcal{V}(t_0)$, we set $\mu_1^v = \mu_{t_1-\tau}^{t_0,\mu_\epsilon,P_\epsilon,v}$. Since $\mathcal{U}(t_0)=\mathcal{U}(t_0,t_1)\times\mathcal{U}(t_1)$, we can write any $u\in\mathcal{U}(t_0)$ as $u=(u_1,u_2)$ where $u_1\in\mathcal{U}(t_0,t_1)$ and $u_2\in\mathcal{U}(t_1)$. We define the strategy $P\in\Delta(\mathcal{A}(t_0))$ by

$$\int_{\mathcal{U}(t_0)} \varphi(u)dP_x^v(u) = \sum_{i=1}^I \mathbf{1}_{\mu_1^v\in O_i} \int_{\mathcal{U}(t_0,t_1)\times\mathcal{U}(t_1)} \varphi((u_1,u_2))dP_{i,X_{t_1-\tau}^{t_0,x,u_1,v}}^{v_{|[t_1,T]}}(u_2)dP_{\epsilon,x}^v(u_1)$$

(where, with a slight abuse of notation, $P^v_{\epsilon,x}$ still denotes the natural restriction of the measure $P^v_{\epsilon,x}$ to $\mathcal{U}(t_0,t_1)$) for any $(x,v)\in \mathbb{R}^N\times\mathcal{V}(t_0)$ and any nonnegative Borel measurable map $\varphi:\mathcal{U}(t_0)\to\mathbb{R}$. Then

$$
\begin{aligned}
&\mathfrak{J}(t_0,\mu_\epsilon,P,v)\\
&=\sum_{i=1}^I \mathbf{1}_{\mu_1^v\in O_i}\int_{\mathbb{R}^N\times\mathcal{U}(t_0,t_1)\times\mathcal{U}(t_1)} g\left(X_T^{t_1-\tau,X^{t_0,x,u_1,v}_{t_1-\tau},(u_1|_{[t_1-\tau,t_1]},u_2),v|_{[t_1-\tau,T]}}\right) dP^{v|_{[t_1,T]}}_{i,X^{t_0,x,u_1,v}_{t_1-\tau}}(u_2)dP^v_{\epsilon,x}(u_1)\\
&\le\sum_{i=1}^I \mathbf{1}_{\mu_1^v\in O_i}\int_{\mathbb{R}^N\times\mathcal{U}(t_0,t_1)\times\mathcal{U}(t_1)}\left[g\left(X_T^{t_1,X^{t_0,x,u_1,v}_{t_1-\tau},u_2,v|_{[t_1,T]}}\right)+C\tau\right] dP^{v|_{[t_1,T]}}_{i,X^{t_0,x,u_1,v}_{t_1-\tau}}(u_2)dP^v_{\epsilon,x}(u_1)d\mu_\epsilon(x)\\
&\le\sum_{i=1}^I \mathbf{1}_{\mu_1^v\in O_i}\int_{\mathbb{R}^N\times\mathcal{U}(t_1)} g\left(X_T^{t_1,y,u_2,v|_{[t_1,T]}}\right) dP^{v|_{[t_1,T]}}_{i,y}(u_2)d\mu_1^v(y)+C\tau\\
&=\sum_{i=1}^I \mathbf{1}_{\mu_1^v\in O_i}\mathfrak{J}(t_1,\mu_1^v,P_i,v|_{[t_1,T]})+C\tau
\end{aligned}
$$

Note that, if $\mu_1^v\in O_i$, then $\|f^{v,\epsilon}_{t_1-\tau}-f_i\|_{L^1}\le\eta$, so that, by the choice of η, we get

$$\mathfrak{J}(t_1,\mu_1^v,P_i,v|_{[t_1,T]})\le\mathbf{V}^+(t_1,\mu_1^v)+\epsilon/2\ .$$

Therefore, recalling the definition of μ_1^v and noticing that $\mathbf{d}(\mu_1^v,\mu_{t_1}^{t_0,\mu_\epsilon,P_\epsilon,v})\le C\tau$, we get

$$\mathfrak{J}(t_0,\mu_\epsilon,P,v)\le\mathbf{V}^+(t_1,\mu_1^v)+\epsilon+C\tau\le\mathbf{V}^+(t_1,\mu_{t_1}^{t_0,\mu_\epsilon,P_\epsilon,v})+\epsilon+C\tau\ .$$

Now, since P_ϵ is $C\epsilon-$optimal for $W(t_0,t_1,\mu_\epsilon)$ we obtain

$$\mathbf{V}^+(t_0,\mu_\epsilon)\le\sup_{v\in\mathcal{V}(t_0)}\mathfrak{J}(t_0,\mu_\epsilon,P,v)\le W(t_0,t_1,\mu_\epsilon)+C(\epsilon+\tau)\ .$$

Using again the fact that $\mathbf{V}^+$ and W are Lipschitz continuous we have, as ϵ and τ are is arbitrary:

$$\mathbf{V}^+(t_0,\mu_0)\le W(t_0,t_1,\mu_0)\ .$$

Now we have to prove that the result still holds for measures with unbounded support. Let $\mu_0\in\mathcal{W}$. For all $\epsilon>0$, there exists some closed ball K_ϵ centered at 0 such that

$$\int_{\mathbb{R}^N\setminus K_\epsilon}|x|^2d\mu_0(x)\le\epsilon^2\qquad\text{and}\qquad\mu_0(\mathbb{R}^N\setminus K_\epsilon)\le\epsilon^2\ .$$

Let $T:\mathbb{R}^N\to\mathbb{R}^N$ be such that $T(x)=x$ for all $x\in K_\epsilon$ and $T(x)=0$ for all $x\notin K_\epsilon$ and let us set $\mu_\epsilon=T\sharp\mu_0$. Then $\mu_\epsilon\in\mathcal{W}(K_\epsilon)$ and $\mathbf{d}(\mu_0,\mu_\epsilon)\le\epsilon$.

Let us now check that for all $(P,v) \in \Delta(\mathcal{A}_x(t_0)) \times \mathcal{V}(t_0)$, $\mu_1 := \mu_{t_1}^{t_0,\mu_0,P,v}$ is close to $\mu_1^\epsilon := \mu_{t_1}^{t_0,\mu_\epsilon,P,v}$. Indeed we have:

$$\begin{aligned}
\mathbf{d}^2(\mu_1,\mu_1^\epsilon) &\leq \int_{\mathbb{R}^N} \left| \int_{\mathcal{U}(t_0)} X_{t_1}^{t_0,x,u,v} dP_x^v(u) - \int_{\mathcal{U}(t_0)} X_{t_1}^{t_0,T(x),u,v} dP_{T(x)}^v(u) \right|^2 d\mu_0(x) \\
&\leq \int_{\mathbb{R}^N \setminus K_\epsilon} \left| \int_{\mathcal{U}(t_0)} X_{t_1}^{t_0,x,u,v} dP_x^v(u) - \int_{\mathcal{U}(t_0)} X_{t_1}^{t_0,0,u,v} dP_0^v(u) \right|^2 d\mu_0(x) \\
&\leq \int_{\mathbb{R}^N \setminus K_\epsilon} 2\left[|x|^2 + 4(t_1-t_0)^2 \|f\|_\infty^2\right] d\mu_0(x) \;\leq\; C\epsilon^2
\end{aligned}$$

Then the Lipschitz continuity of the upper value leads to:

$$\begin{aligned}
\mathbf{V}^+(t_0,\mu_0) &\leq \mathbf{V}^+(t_0,\mu_\epsilon) + C\epsilon \\
&\leq \inf_{P\in\Delta(\mathcal{A}_x(t_0))} \sup_{v\in\mathcal{V}(t_0)} \mathbf{V}^+(t_1,\mu_{t_1}^{t_0,\mu_\epsilon,P,v}) + C\epsilon \\
&\leq \inf_{P\in\Delta(\mathcal{A}_x(t_0))} \sup_{v\in\mathcal{V}(t_0)} \mathbf{V}^+(t_1,\mu_{t_1}^{t_0,\mu_0,P,v}) + C\epsilon\,.
\end{aligned}$$

Hence $\mathbf{V}^+(t_0,\mu_0) \leq W(t_0,t_1,\mu_0)$ as ϵ is arbitrary.

We now prove that

$$\mathbf{V}^+(t_0,\mu_0) \geq W(t_0,t_1,\mu_0)\,. \tag{4}$$

Let P be an ϵ-optimal strategy for player I for $\mathbf{V}^+(t_0,\mu_0)$. Let us fix $v_0 \in \mathcal{V}(t_0)$ and set $\mu_1 = \mu_{t_1}^{t_0,\mu_0,P,v_0}$. For all $v \in \mathcal{V}(t_1)$, we define the measure $\tilde{P}^v$ on $\mathbb{R}^N \times \mathcal{U}(t_1)$ by

$$\int_{\mathbb{R}^N\times\mathcal{U}(t_1)} \varphi(x,u_2) d\tilde{P}^v(x,u_2) = \int_{\mathbb{R}^N\times\mathcal{U}(t_0)} \varphi(X_{t_1}^{t_0,x,u,v_0}, u_{|_{[t_1,T]}}) dP_x^{(v_0|_{[t_0,t_1]},v)}(u) d\mu_0(x)$$

for any $v \in \mathcal{V}(t_1)$ and any nonnegative Borel measurable function $\varphi : \mathbb{R}^N \times \mathcal{U}(t_1) \to \mathbb{R}$. We note that the first marginal of $\tilde{P}^v$ is $\mu_{t_1}^{t_0,\mu_0,P,v_0}$. Since $\mathbb{R}^N \times \mathcal{U}(t_1)$ is a Polish space, we can desintegrate $\tilde{P}^v$ with respect to $\mu_{t_1}^{t_0,\mu_0,P,v_0}$: $d\tilde{P}^v(x,u) = d\tilde{P}_x^v(u) d\mu_{t_1}^{t_0,\mu_0,P,v_0}(x)$, where the mapping $(x,v) \to$

$\tilde{P}_x^v$ is measurable. Then $\tilde{P}$ belongs to $\Delta(\mathcal{A}_x(t_1))$ and we have:

$$\begin{aligned}
\mathbf{V}^+(t_1, \mu_{t_1}^{t_0,\mu_0,P,v_0}) &\leq \sup_{v\in\mathcal{V}(t_1)} \int_{\mathbb{R}^N\times\mathcal{U}(t_1)} g(X_T^{t_1,x,u_2,v}) d\tilde{P}_x^v(u_2) d\mu_{t_1}^{t_0,\mu_0,P,v_0}(x) \\
&\leq \sup_{v\in\mathcal{V}(t_1)} \int_{\mathbb{R}^N\times\mathcal{U}(t_0,t_1)\times\mathcal{U}(t_1)} g\left(X_T^{t_1,X_{t_1}^{t_0,x,u_1,v_0},u_2,v}\right) dP_x^{(v_0|_{[t_0,t_1]},v)}((u_1,u_2)) d\mu_0(x) \\
&\leq \sup_{v\in\mathcal{V}(t_1)} \int_{\mathbb{R}^N\times\mathcal{U}(t_0)} g\left(X_T^{t_0,x,u,(v_0|_{[t_0,t_1]},v)}\right) dP_x^{(v_0|_{[t_0,t_1]},v)}(u) d\mu_0(x) \\
&\leq \mathbf{V}^+(t_0,\mu_0) + \epsilon
\end{aligned}$$

Hence inequality (4) holds since v_0 and ϵ are arbitrary. □

4 Characterization of the upper value function

We prove in this section that if a function satisfies the previous dynamic programming principle, then it is the unique viscosity solution of some Hamilton-Jacobi equation.

We consider the Hamiltonian H, defined for any $\mu \in \mathcal{W}$ and for any $\mathbf{p} \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$, by

$$H(\mu,\mathbf{p}) = \sup_{\mathbf{v}\in\Delta(V)} \int_{\mathbb{R}^N} \inf_{\mathbf{u}\in\Delta(U)} \int_{U\times V} \langle f(x,u,v), \mathbf{p}(x)\rangle d\mathbf{u}(u) d\mathbf{v}(v) d\mu(x) \ , \quad (5)$$

where $\Delta(U)$ and $\Delta(V)$ denote the sets of Borel probability measures on the compact sets U and V respectively. Let $\mathbf{V} : [0,T]\times\mathcal{W} \to \mathbb{R}$ be a Lipschitz continuous map. We say $\mathbf{V}$ is a subsolution to

$$\mathbf{V}_t + H(\mu, D_\mu \mathbf{V}) = 0 \qquad \text{in } [0,T)\times\mathcal{W} \quad (6)$$

if, for any test function $\varphi(t,\mu)$ of the form

$$\varphi(t,\mu) = \frac{\alpha}{2}\mathbf{d}^2(\bar{\mu},\mu) + \eta\mathbf{d}(\bar{\nu},\mu) + \psi(t)$$

(where $\psi : \mathbb{R} \to \mathbb{R}$ is smooth, $\alpha, \eta > 0$ and $\bar{\mu}, \bar{\nu} \in \mathcal{W}$) such that $\mathbf{V} - \varphi$ has a local maximum at $(\bar{\nu}, \bar{t}) \in [0,T)\times\mathcal{W}$ and for any optimal transport plan $\bar{\pi} \in \Pi_{opt}(\bar{\mu},\bar{\nu})$, one has

$$\psi'(\bar{t}) + H(\bar{\nu}, -\alpha\mathbf{p}) \geq -\|f\|_\infty \eta$$

where $\mathbf{p}$ is the unique element of $L^2_{\bar\nu}(\mathbb{R}^N, \mathbb{R}^N)$ associated to $\bar\pi$ such that

$$\int_{\mathbb{R}^N} \langle \xi(y), x-y\rangle d\bar\pi(x,y) = \int_{\mathbb{R}^N} \langle \xi(y), \mathbf{p}(y)\rangle d\bar\nu(y) \qquad \forall \xi \in L^2_{\bar\nu}(\mathbb{R}^N, \mathbb{R}^N) \tag{7}$$

(see [3]). In the same way, we say $\mathbf{V}$ is a supersolution to (6) if, for any test function $\varphi(t,\mu)$ of the form

$$\varphi(t,\mu) = -\frac{\alpha}{2}\mathbf{d}^2(\bar\mu,\mu) - \eta\mathbf{d}(\bar\nu,\mu) + \psi(t)$$

(where $\psi : \mathbb{R} \to \mathbb{R}$ is smooth, $\alpha, \eta > 0$ and $\bar\mu, \bar\nu \in \mathcal{W}$) such that $\mathbf{V} - \varphi$ has a local minimum at $(\bar\nu, \bar t) \in [0,T) \times \mathcal{W}$, one has

$$\psi'(\bar t) + H(\bar\nu, \alpha\mathbf{p}) \le \|f\|_\infty \eta \ .$$

Proposition 4.1 (Comparison principle)**.** *Let w_1 be a Lipschitz continuous subsolution of* (6) *and w_2 be a Lipschitz continuous supersolution such that $w_1(T,\mu) \le w_2(T,\mu)$ for any $\mu \in \mathcal{W}$. Then $w_1 \le w_2$ in $[0,T] \times \mathcal{W}$.*

In particular, given a Lipschitz continuous terminal condition $\tilde g : \mathcal{W} \to \mathbb{R}$, the Hamilton-Jacobi equation (6) has at most one Lipschitz continuous solution $\mathbf{V}$ which satisfies $\mathbf{V}(T,\mu) = \tilde g(\mu)$ for any $\mu \in \mathcal{W}$.

Note that other definitions of viscosity solution in the space $\mathcal{W}$ have been introducted recently: see for instance [6, 7, 9, 10]. Our definition is closely related to the one of [3], which seems more appropriate for the kind of problem we have to handle.

Proof. The proof borrows its main arguments from [4], and follows closely [3]. We denote by K the common Lipschitz constant of w_1, w_2 and f. Without loss of generality we can assume that

$$\inf_{\mu\in\mathcal{W}} w_2(T,\mu) - w_1(T,\mu) = 0 \ . \tag{8}$$

Our aim is to prove that

$$\inf_{(t,\mu)\in[0,T]\times\mathcal{W}} w_2(t,\mu) - w_1(t,\mu) = 0 \ .$$

Assume on the contrary that

$$\inf_{(t,\mu)\in[0,T]\times\mathcal{W}} w_2(t,\mu) - w_1(t,\mu) = -\xi < 0 \ .$$

Fix (t_0, μ_0) such that $(w_2 - w_1)(t_0, \mu_0) < -\xi/2$. Denote by

$$\varphi_{\epsilon\eta}(s, \mu, t, \nu) = w_2(t, \nu) - w_1(s, \mu) + \frac{1}{\epsilon}\mathbf{d}^2(\mu, \nu) + \frac{1}{\epsilon}(t - s)^2 - \eta s \ .$$

The function $\varphi_{\epsilon\eta}$ is continuous and bounded from below. Using some modified version of Ekeland's variational Lemma (Lemma 6.4 below), we have that, for all $\delta > 0$, there is $(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu})$ such that for all (s, μ, t, ν):

$$\begin{array}{l} \varphi_{\epsilon\eta}(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \varphi_{\epsilon\eta}(t_0, \mu_0, t_0, \mu_0) \\ \varphi_{\epsilon\eta}(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \varphi_{\epsilon\eta}(s, \mu, t, \nu) + \delta[\mathbf{d}(\mu, \bar{\mu}) + \mathbf{d}(\nu, \bar{\nu})] \end{array} \tag{9}$$

Let us fix $\pi \in \Pi_{opt}(\bar{\nu}, \bar{\mu})$. We first give a bound on the distance between $(\bar{s}, \bar{\mu})$ and $(\bar{t}, \bar{\nu})$. Since

$$\varphi_{\epsilon\eta}(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \varphi_{\epsilon\eta}(\bar{s}, \bar{\mu}, \bar{s}, \bar{\mu}) + \delta\mathbf{d}(\bar{\mu}, \bar{\nu}) \ ,$$

we have

$$\begin{aligned} & w_2(\bar{s}, \bar{\mu}) - w_1(\bar{s}, \bar{\mu}) - \eta\bar{s} + \delta\mathbf{d}(\bar{\mu}, \bar{\nu}) \\ & \quad \geq w_2(\bar{t}, \bar{\nu}) - w_1(\bar{s}, \bar{\mu}) + \frac{1}{\epsilon}\mathbf{d}^2(\bar{\mu}, \bar{\nu}) + \frac{1}{\epsilon}(\bar{t} - \bar{s})^2 - \eta\bar{s} \\ & \quad \geq w_2(\bar{s}, \bar{\mu}) - K|\bar{s} - \bar{t}| - Kd(\bar{\nu}, \bar{\mu}) - w_1(\bar{s}, \bar{\mu}) + \frac{1}{\epsilon}\mathbf{d}^2(\bar{\mu}, \bar{\nu}) + \frac{1}{\epsilon}(\bar{t} - \bar{s})^2 - \eta\bar{s} \ , \end{aligned}$$

that reduces to

$$\mathbf{d}(\bar{\nu}, \bar{\mu}) + |\bar{t} - \bar{s}| \leq 2\epsilon(K + \delta) \ . \tag{10}$$

We now seek some contradiction assuming that $\bar{t}, \bar{s} \neq T$. We first use the fact that

$$\varphi_{\epsilon\eta}(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \varphi_{\epsilon\eta}(s, \mu, \bar{t}, \bar{\nu}) + \delta\mathbf{d}(\mu, \bar{\mu}) \ ,$$

namely

$$\begin{aligned} & w_2(\bar{t}, \bar{\nu}) - w_1(\bar{s}, \bar{\mu}) + \frac{1}{\epsilon}\mathbf{d}^2(\bar{\mu}, \bar{\nu}) + \frac{1}{\epsilon}(\bar{t} - \bar{s})^2 - \eta\bar{s} \\ & \quad \leq w_2(\bar{t}, \bar{\nu}) - w_1(s, \mu) + \frac{1}{\epsilon}\mathbf{d}^2(\mu, \bar{\nu}) + \frac{1}{\epsilon}(\bar{t} - s)^2 - \eta s + \delta\mathbf{d}(\mu, \bar{\mu}) \ , \end{aligned}$$

leading to

$$w_1(s, \mu) - \frac{1}{\epsilon}\mathbf{d}^2(\mu, \bar{\nu}) - \delta\mathbf{d}(\mu, \bar{\mu}) - \frac{1}{\epsilon}(\bar{t} - s)^2 + \eta s \leq w_1(\bar{s}, \bar{\mu}) - \frac{1}{\epsilon}\mathbf{d}^2(\bar{\mu}, \bar{\nu}) - \frac{1}{\epsilon}(\bar{t} - \bar{s})^2 + \eta\bar{s} \ .$$

If we set $\varphi(s, \mu) = \frac{1}{\epsilon}\mathbf{d}^2(\mu, \bar{\nu}) + \delta\mathbf{d}(\mu, \bar{\mu}) + \frac{1}{\epsilon}(\bar{t} - s)^2 - \eta s$, then the function $w_1 - \varphi$ has a maximum at $(\bar{s}, \bar{\mu})$. The function w_1 being a subsolution, we get by definition:

$$-\eta - \frac{2}{\epsilon}(\bar{t} - \bar{s}) + H(\bar{\mu}, -\frac{2}{\epsilon}\mathbf{p}) \geq -\delta\|f\|_\infty \tag{11}$$

where $\mathbf{p}$ is defined by:

$$\int_{\mathbb{R}^{2N}} \langle \xi(y), x-y\rangle d\pi(x,y) = \int_{\mathbb{R}^N} \langle \xi(y), \mathbf{p}(y)\rangle d\bar{\mu}(y) \qquad \forall \xi \in L^2_{\bar{\mu}}(\mathbb{R}^N, \mathbb{R}^N)\ .$$

The same argument applied to

$$\varphi_{\epsilon\eta}(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \varphi_{\epsilon\eta}(\bar{s}, \bar{\mu}, t, \nu) + \delta \mathbf{d}(\nu, \bar{\nu})$$

leads to

$$-\frac{2}{\epsilon}(\bar{t} - \bar{s}) + H(\bar{\nu}, \frac{2}{\epsilon}\mathbf{q}) \leq \delta \|f\|_\infty \tag{12}$$

where $\mathbf{q}$ satisfies

$$\int_{\mathbb{R}^{2N}} \langle \xi(y), x-y\rangle d\bar{\pi}(x,y) = \int_{\mathbb{R}^N} \langle \xi(x), \mathbf{q}(x)\rangle d\bar{\nu}(x) \qquad \forall \xi \in L^2_{\bar{\nu}}(\mathbb{R}^N, \mathbb{R}^N)\ ,$$

$\bar{\pi}$ being defined by

$$\int_{\mathbb{R}^{2N}} \varphi(x,y) d\bar{\pi}(x,y) = \int_{\mathbb{R}^N} \varphi(y,x) d\pi(x,y)\ \forall \varphi \in L^2_{\pi}(\mathbb{R}^{2N}, \mathbb{R}^{2N})\ .$$

Note that

$$\int_{\mathbb{R}^{2N}} \langle \xi(x), x-y\rangle d\pi(x,y) = \int_{\mathbb{R}^N} \langle \xi(x), -\mathbf{q}(x)\rangle d\bar{\nu}(y) \qquad \forall \xi \in L^2_{\bar{\nu}}(\mathbb{R}^N, \mathbb{R}^N)\ .$$

Combining (11) and (12) we get

$$\eta + H(\bar{\nu}, \frac{2}{\epsilon}\mathbf{q}) - H(\bar{\mu}, -\frac{2}{\epsilon}\mathbf{p}) \leq 2\delta \|f\|_\infty\ . \tag{13}$$

Let us now recall some continuity property of the Hamiltonian H defined by (5):

Lemma 4.2. *Let $(\bar{\nu}, \bar{\mu}) \in \mathcal{W}^2$ and $(\mathbf{p}, \mathbf{q}) \in L^2_{\bar{\mu}}(\mathbb{R}^N, \mathbb{R}^N) \times L^2_{\bar{\nu}}(\mathbb{R}^N, \mathbb{R}^N)$ be such that, for some $\pi \in \Pi_{opt}(\bar{\nu}, \bar{\mu})$,*

$$\int_{\mathbb{R}^{2N}} \langle \xi(y), x-y\rangle d\pi(x,y) = \int_{\mathbb{R}^N} \langle \xi(y), \mathbf{p}(y)\rangle d\bar{\mu}(y) \qquad \forall \xi \in L^2_{\bar{\mu}}(\mathbb{R}^N, \mathbb{R}^N)$$

and

$$\int_{\mathbb{R}^{2N}} \langle \xi(x), x-y\rangle d\pi(x,y) = \int_{\mathbb{R}^N} \langle \xi(x), -\mathbf{q}(x)\rangle d\bar{\nu}(x) \qquad \forall \xi \in L^2_{\bar{\nu}}(\mathbb{R}^N, \mathbb{R}^N)\ .$$

Then we have:

$$|H(\bar{\mu}, \mathbf{p}) - H(\bar{\nu}, -\mathbf{q})| \leq K \mathbf{d}^2(\bar{\nu}, \bar{\mu})$$

where K stands for the Lipschitz constant of the dynamics.

Proof. The proof is the same as in [3], Lemma 6. □

Therefore, we have:

$$|H(\bar{\mu}, -\frac{2}{\epsilon}\mathbf{p}) - H(\bar{\nu}, \frac{2}{\epsilon}\mathbf{q}_y)| \leq \frac{2K}{\epsilon}\mathbf{d}^2(\bar{\nu}, \bar{\mu}) \ .$$

Thus using the previous inequality and estimate (10) in (13), we get

$$\eta \leq 2\delta\|f\|_\infty + 8K\epsilon(K+\delta)^2$$

leading to a contradiction for ϵ, δ sufficiently small.

This implies that we have $\bar{t} = T$ or $\bar{s} = T$. Assume for example that $\bar{s} = T$. We have

$$\varphi_{\epsilon\eta}(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \leq \varphi_{\epsilon\eta}(t_0, \mu_0, t_0, \mu_0) \leq -\xi/2 \ .$$

Therefore using (8) and (9) we obtain:

$$\begin{aligned}
-\xi/2 &\geq w_2(\bar{t}, \bar{\nu}) - w_1(T, \bar{\mu}) + \frac{1}{\epsilon}\mathbf{d}^2(\bar{\mu}, \bar{\nu}) + \frac{1}{\epsilon}(T-\bar{t})^2 - \eta T \\
&\geq -K|T-\bar{t}| - K\mathbf{d}(\bar{\nu}, \bar{\mu}) + \frac{1}{\epsilon}\mathbf{d}^2(\bar{\mu}, \bar{\nu}) + \frac{1}{\epsilon}(T-\bar{t})^2 - \eta T \\
&\geq -K[|T-\bar{t}| + \mathbf{d}(\bar{\mu}, \bar{\nu})] + \frac{1}{2\epsilon}[|T-\bar{t}| + \mathbf{d}(\bar{\mu}, \bar{\nu})]^2 - \eta T
\end{aligned}$$

Using (10), we finally get:

$$\xi/2 \leq 2\epsilon(K+\delta)(2K+\delta) + \eta T$$

which is impossible for ϵ and η small enough. □

Proposition 4.3. *The upper value function* $\mathbf{V}^+$ *is the unique Lipschitz continuous viscosity solution of the Hamilton-Jacobi equation* (6) *satisfying the terminal condition:*

$$\mathbf{V}^+(T, \mu) = \int_{\mathbb{R}^N} g(x) d\mu(x) \tag{14}$$

Proof. We only prove that $\mathbf{V}^+$ is some solution, uniqueness being an obvious consequence of Proposition 4.1. Let us recall that $\mathbf{V}^+$ satisfies the dynamic programming principle

$$\mathbf{V}^+(t_0, \bar{\nu}) = \inf_{P \in \Delta(\mathcal{A}_x(t_0))} \sup_{v \in \mathcal{V}(t_0)} \mathbf{V}^+(t_0 + h, \mu_{t_0+h}^{t_0, \bar{\nu}, P, v}) \tag{15}$$

where $\mu_{t_0+h}^{t_0,\bar{\nu},P,v}$ is the measure defined by

$$\forall\varphi \in \mathcal{C}_b(\mathbb{R}^N,\mathbb{R}),\ \int_{\mathbb{R}^N} \varphi(x)d\mu_{t_0+h}^{t_0,\bar{\nu},P,v}(x) = \int_{\mathbb{R}^N\times\mathcal{U}(t_0)} \varphi(X_{t_0+h}^{t_0,x,u,v})dP^v(x,u)\ .$$

Let us show that $\mathbf{V}^+$ is a subsolution. Let $\varphi = \varphi(t,\mu)$ be a test function of the form

$$\varphi(t,\mu) = \frac{\alpha}{2}\mathbf{d}^2(\bar{\mu},\mu) + \eta\mathbf{d}(\bar{\nu},\mu) + \psi(t)$$

(where $\psi : \mathbb{R} \to \mathbb{R}$ is smooth, $\alpha,\eta > 0$ and $\bar{\mu},\bar{\nu} \in \mathcal{W}$), such that $\mathbf{V}^+ - \varphi$ has a local maximum at $(\bar{\nu},t_0)$. Without loss of generality we assume that $\varphi(t_0,\bar{\nu}) = \mathbf{V}^+(t_0,\bar{\nu})$. Then $\mathbf{V}^+ \le \varphi$. We fix an optimal plan $\bar{\pi} \in \Pi_{opt}(\bar{\mu},\bar{\nu})$. From (15), we get

$$0 \le \inf_{P\in\Delta(\mathcal{A}_x(t_0))}\ \sup_{v\in\mathcal{V}(t_0)} \left[\varphi(t_0+h,\mu_{t_0+h}^{t_0,\bar{\nu},P,v}) - \varphi(t_0,\bar{\nu})\right]\ . \tag{16}$$

Setting for simplicity $\nu(t_0+h) = \mu_{t_0+h}^{t_0,\bar{\nu},P,v}$ and recalling the definition of φ we have

$$\varphi(t_0+h,\nu(t_0+h))-\varphi(t_0,\bar{\nu}) = \frac{\alpha}{2}\left[\mathbf{d}^2(\bar{\mu},\nu(t_0+h)) - \mathbf{d}^2(\bar{\mu},\bar{\nu})\right]+\eta\mathbf{d}(\bar{\nu},\nu(t_0+h))+\psi(t_0+h)-\psi(t_0) \tag{17}$$

where

$$\mathbf{d}(\bar{\nu},\nu(t_0+h)) \le \left[\int_{\mathbb{R}^N\times\mathcal{U}(t_0)} \left|y - X_{t_0+h}^{t_0,y,u,v}\right|^2 dP_y^v(u)d\bar{\nu}(y)\right]^{\frac{1}{2}} \le \|f\|_\infty h\ .$$

Recalling the definition of $\mathbf{p}$ in (7), we also have

$$\begin{aligned}
&\mathbf{d}^2(\bar{\mu},\nu(t_0+h))\\
&\quad\le \int_{\mathbb{R}^{2N}\times\mathcal{U}(t_0)} \left|x - X_{t_0+h}^{t_0,y,u,v}\right|^2 dP_y^v(u)d\bar{\pi}(x,y)\\
&\quad\le \mathbf{d}^2(\bar{\mu},\bar{\nu}) - 2\int_{\mathbb{R}^{2N}} \langle x-y, \left[\int_{\mathcal{U}(t_0)}\int_{t_0}^{t_0+h} f(X_s^{t_0,y,u,v},u(s),v(s))ds\ dP_y^v(u)\right]\rangle d\bar{\pi}(x,y) + Ch^2\\
&\quad\le \mathbf{d}^2(\bar{\mu},\bar{\nu}) - 2\int_{\mathbb{R}^N} \langle \mathbf{p}(y), \left[\int_{\mathcal{U}(t_0)}\int_{t_0}^{t_0+h} f(y,u(s),v(s))ds\ dP_y^v(u)\right]\rangle d\bar{\nu}(y) + Ch^2
\end{aligned}$$

Let (U_n) be an increasing family of finite subsets of U such that $\bigcup_n U_n$ is dense in U and $\mathbf{U}^n$ be the set of Borel measurable maps $x \to \mathbf{u}_x$ from $\mathbb{R}^N$ into $\Delta(U_n)$. The main point in this discretization is that $\Delta(U_n)$ is a

compact subset of some finite dimensional space. Therefore $\mathbf{U}^n$, endowed with the weak topology of $L^2_{\bar\nu}$, is convex and compact. Since U_n can be viewed as a subset of $\mathcal{U}(t_0)$, one can associate with a map $\mathbf{u} \in \mathbf{U}^n$ a strategy $P_{\mathbf{u}} \in \Delta(\mathcal{A}_x(t_0))$ defined by the equality

$$\int_{\mathbb{R}^N \times \mathcal{U}(t_0)} \varphi(x,u) dP^v_{\mathbf{u}} = \int_{\mathbb{R}^N \times U_n} \varphi(x,u) d\mathbf{u}_x(u) d\bar\nu(x)$$

for any $v \in \mathcal{V}(t_0)$ and any nonnegative Borel measurable map $\varphi : \mathbb{R}^N \times \mathcal{U}(t_0) \to \mathbb{R}$. Recalling (16) and (17), we get

$$\begin{aligned} 0 \le\ & \psi'(t_0) + \|f\|_\infty \eta + Ch \\ & + \alpha \inf_{\mathbf{u}\in\mathbf{U}^n} \sup_{Q\in\Delta(\mathcal{V}(t_0))} \int_{\mathbb{R}^N} \langle -\mathbf{p}(y), \int_{U_n\times\mathcal{V}(t_0)} \frac{1}{h}\int_{t_0}^{t_0+h} f(y,u,v(s))ds\ d\mathbf{u}_y(u) dQ(v)\rangle d\bar\nu(y)\ . \end{aligned}$$

By Sion's min-max Theorem we get

$$\begin{aligned} & \inf_{\mathbf{u}\in\mathbf{U}^n} \sup_{Q\in\Delta(\mathcal{V}(t_0))} \int_{\mathbb{R}^N} \langle -\mathbf{p}(y), \int_{U_n\times\mathcal{V}(t_0)} \frac{1}{h}\int_{t_0}^{t_0+h} f(y,u,v(s))ds\ d\mathbf{u}_y(u) dQ(v)\rangle d\bar\nu(y) \\ & = \sup_{Q\in\Delta(\mathcal{V}(t_0))} \inf_{\mathbf{u}\in\mathbf{U}^n} \int_{\mathbb{R}^N} \langle -\mathbf{p}(y), \int_{U_n\times\mathcal{V}(t_0)} \frac{1}{h}\int_{t_0}^{t_0+h} f(y,u,v(s))ds\ d\mathbf{u}_y(u) dQ(v)\rangle d\bar\nu(y) \\ & = \sup_{Q\in\Delta(\mathcal{V}(t_0))} \int_{\mathbb{R}^N} \inf_{\mathbf{u}\in\Delta(U_n)} \langle -\mathbf{p}(y), \int_{U_n\times\mathcal{V}(t_0)} \frac{1}{h}\int_{t_0}^{t_0+h} f(y,u,v(s))ds\ d\mathbf{u}(u) dQ(v)\rangle d\bar\nu(y) \\ & \le \sup_{Q\in\Delta(\mathcal{V}(t_0))} \int_{\mathbb{R}^N} \inf_{\mathbf{u}\in\Delta(U_n)} \text{ess-sup}_{s\in[t_0,t_0+h]} \langle -\mathbf{p}(y), \int_{U_n\times\mathcal{V}(t_0)} f(y,u,v(s))\ d\mathbf{u}(u) dQ(v)\rangle d\bar\nu(y) \\ & \le \sup_{\mathbf{v}\in\Delta(V)} \int_{\mathbb{R}^N} \inf_{\mathbf{u}\in\Delta(U_n)} \langle -\mathbf{p}(y), \int_{U_n\times V} f(y,u,v)\ d\mathbf{u}(u) d\mathbf{v}(v)\rangle d\bar\nu(y) \end{aligned}$$

So

$$0 \ \le\ \psi'(t_0) + \alpha \sup_{\mathbf{v}\in\Delta(V)} \int_{\mathbb{R}^N} \inf_{\mathbf{u}\in\Delta(U_n)} \langle -\mathbf{p}(y), \int_{U_n\times V} f(y,u,v)\ d\mathbf{u}(u) d\mathbf{v}(v)\rangle d\bar\nu(y) + \|f\|_\infty \eta + Ch$$

Letting $h \to 0$ and $n \to +\infty$ gives the desired inequality since $\bigcup_n \Delta(U_n)$ is dense in $\Delta(U)$.

We now check that $\mathbf{V}^+$ is a supersolution. Let $\varphi = \varphi(t,\mu)$ be a test function of the form

$$\varphi(t,\mu) = -\frac{\alpha}{2}\mathbf{d}^2(\bar\mu,\mu) - \eta \mathbf{d}(\bar\nu,\mu) + \psi(t)$$

(where $\psi : \mathbb{R} \to \mathbb{R}$ is smooth, $\alpha, \eta > 0$ and $\bar\mu, \bar\nu \in \mathcal{W}$), such that $\mathbf{V}^+ - \varphi$ has a local minimum at $(\bar\nu, t_0)$. We again assume that $\varphi(t_0,\bar\nu) = \mathbf{V}^+(t_0,\bar\nu)$,

so that $\mathbf{V}^+ \geq \varphi$. Let us apply the dynamic programming at time t_0 and for $\bar{\nu}$. We get

$$0 \geq \inf_{P\in\Delta(\mathcal{A}_x(t_0))} \sup_{v\in\mathcal{V}(t_0)} \left[\varphi(t_0+h, \mu_{t_0+h}^{t_0,\bar{\nu},P,v}) - \varphi(t_0,\bar{\nu})\right]$$

Setting as before $\bar{\nu}(t_0+h) = \mu_{t_0+h}^{t_0,\bar{\nu},P,v}$, we have, for any $P \in \Delta(\mathcal{A}_x(t_0))$ and $v \in \mathcal{V}(t_0)$,

$$\mathbf{d}^2(\bar{\mu},\nu(t_0+h)) \;\leq\; \mathbf{d}^2(\bar{\mu},\bar{\nu}) - 2\int_{\mathbb{R}^N} \langle \mathbf{p}(y), \left[\int_{\mathcal{U}(t_0)} \int_{t_0}^{t_0+h} f(y,u(s),v(s))ds \; dP_y^v(u)\right]\rangle d\bar{\nu}(y) - Ch^2$$

so that

$$\begin{aligned} 0 \;&\geq\; \psi'(t_0) - \|f\|_\infty \eta - Ch \\ &+\alpha \inf_{P\in\Delta(\mathcal{A}_x(t_0))} \sup_{Q\in\Delta(\mathcal{V}(t_0))} \int_{\mathbb{R}^N} \langle \mathbf{p}(y), \left[\int_{\mathcal{U}(t_0)\times\mathcal{V}(t_0)} \frac{1}{h}\int_{t_0}^{t_0+h} f(y,u(s),v(s))ds \; dP_y^v(u) dQ(v)\right]\rangle d\bar{\nu}(y) \end{aligned} \tag{18}$$

Let $P_h \in \Delta(\mathcal{A}_x(t_0))$ be $h-$optimal in the above expression. We denote by τ_h its delay and set $n_h = h/\tau_h$. Reducing τ_h if necessary, we can suppose that n_h is a positive integer. Let us set $t_k = t_0 + k\tau_h$ for $k = 0,\ldots,n_h$. Let us now fix $\mathbf{v} \in \Delta(V)$. With $\mathbf{v}$ we associate the strategy $Q_{h,\mathbf{v}}$ consisting in choosing randomly, on each time interval $[t_k, t_{k+1}]$ (where $k = 0,\ldots,n_h-1$) a time independant control v according to the probability $\mathbf{v}$. We now claim that

$$\begin{aligned} \int_{\mathcal{U}(t_0)\times\mathcal{V}(t_0)} \langle \mathbf{p}(y), \int_{t_0}^{t_0+h} f(y,u(s),v(s))ds\rangle \; dP_{h,y}^v(u) dQ_{h,\mathbf{v}}(v) \\ \geq h \inf_{\mathbf{u}\in\Delta(U)} \int_{U\times V} \langle \mathbf{p}(y), f(y,u,v)\rangle \; d\mathbf{u}(u) d\mathbf{v}(v) \end{aligned} \tag{19}$$

for $\bar{\nu}-$a.e. y. For this it is enough to show by backward induction on $k \in \{0,\ldots,n_h\}$ that

$$\begin{aligned} &\int_{\mathcal{U}(t_0)\times\mathcal{V}(t_0)} \langle \mathbf{p}(y), \int_{t_0}^{t_0+h} f(y,u(s),v(s))ds\rangle \; dP_{h,y}^v(u) dQ_{h,\mathbf{v}}(v) \\ &\quad\geq \int_{\mathcal{U}(t_0,t_k)\times\mathcal{V}(t_0,t_k)} \langle \mathbf{p}(y), \int_{t_0}^{t_k} f(y,u(s),v(s))ds\rangle \; dP_{h,y}^{k,v}(u) dQ_{h,\mathbf{v}}^k(v) \\ &\qquad +(n_h-k)\tau_h \inf_{\mathbf{u}\in\Delta(U)} \int_{U\times V} \langle \mathbf{p}(y), f(y,u,v)\rangle \; d\mathbf{u}(u) d\mathbf{v}(v) \end{aligned} \tag{20}$$

where P_h^k and $Q_{h,\mathbf{v}}^k$ are defined as the restriction of the strategies P_h and $Q_{h,\mathbf{v}}$ to the time interval $[t_0, t_0 + k\tau_h]$. Note that the above inequality is obvious for $k = n_h$. Let us assume that it holds for $k+1$ and prove that it still hold for k. We use the decomposition

$$\mathcal{V}(t_0, t_{k+1}) = \mathcal{V}(t_0, t_k) \times \mathcal{V}(t_k, t_{k+1})$$

and write $v = (v_1, v_2)$ for any $v \in \mathcal{V}(t_0, t_{k+1})$, where $v_1 \in \mathcal{V}(t_0, t_k)$ and $v_2 \in \mathcal{V}(t_k, t_{k+1})$. By definition of $Q_{h,\mathbf{v}}^{k+1}$, we have

$$\int_{\mathcal{V}(t_0,t_{k+1})} dQ_{h,\mathbf{v}}^{k+1}(v) = \int_{\mathcal{V}(t_0,t_k)\times V} dQ_{h,\mathbf{v}}^{k}(v_1) d\mathbf{v}(v_2)$$

Therefore, since P has a delay τ_h we get, omitting the arguments of $\mathbf{p}$ and f for simplicity

$$\begin{aligned}
&\int_{\mathcal{U}(t_0,t_{k+1})\times\mathcal{V}(t_0,t_{k+1})} \langle \mathbf{p}, \int_{t_0}^{t_{k+1}} f ds\rangle \; dP_{h,y}^{k+1,v}(u) dQ_{h,\mathbf{v}}^{k+1}(v) \\
&\quad = \int_{\mathcal{U}(t_0,t_{k+1})\times\mathcal{V}(t_0,t_k)\times V} \langle \mathbf{p}, \int_{t_0}^{t_{k+1}} f ds\rangle \; d\mathbf{v}(v_2) dP_{h,y}^{k+1,v_1}(u) dQ_{h,\mathbf{v}}^{k}(v_1) \\
&\quad \geq \int_{\mathcal{U}(t_0,t_k)\times\mathcal{V}(t_0,t_k)} \langle \mathbf{p}, \int_{t_0}^{t_k} f ds\rangle dP_{h,y}^{k,v_1}(u) dQ_{h,\mathbf{v}}^{k}(v_1) \\
&\qquad + \tau_h \inf_{\mathbf{u}\in\Delta(U)} \int_{U\times V} \langle \mathbf{p}, f ds\rangle d\mathbf{u}(u) d\mathbf{v}(v)
\end{aligned}$$

because

$$\begin{aligned}
&\int_{\mathcal{U}(t_0,t_{k+1})\times\mathcal{V}(t_0,t_k)\times V} \langle \mathbf{p}, \int_{t_k}^{t_{k+1}} f(y, u(t), v_2) ds\rangle \; d\mathbf{v}(v_2) dP_{h,y}^{k+1,v_1}(u) dQ_{h,\mathbf{v}}^{k}(v_1) \\
&\quad \geq \inf_{P\in\Delta(\mathcal{U}(t_k,t_{k+1}))} \int_{\mathcal{U}(t_k,t_{k+1})\times V} \langle \mathbf{p}, \int_{t_k}^{t_{k+1}} f(y, u(t), v_2) ds\rangle \; d\mathbf{v}(v_2) dP(u) \\
&\quad = \tau_h \inf_{\mathbf{u}\in\Delta(U)} \int_{U\times V} \langle \mathbf{p}, \int_{t_k}^{t_{k+1}} f(y, u, v_2) ds\rangle \; d\mathbf{u}(u) d\mathbf{v}(v_2)
\end{aligned}$$

where $\Delta(\mathcal{U}(t_k, t_{k+1}))$ stands for the Borel probability measures on the set $\mathcal{U}(t_k, t_{k+1})$. This gives (20) by induction.

Combining (18) with (19) we get

$$0 \;\geq\; \psi'(t_0) - \|f\|_\infty \eta - Ch + \alpha \int_{\mathbb{R}^N} \inf_{\mathbf{u}\in\Delta(U)} \int_{U\times V} \langle \mathbf{p}(y), f(y, u, v)\rangle \; d\mathbf{u}(u) d\mathbf{v}(v) d\bar{\nu}(y) \;,$$

and we obtain the desired inequality by letting $h \to 0$, since $\mathbf{v}$ is arbitrary. □

5 The discretized game

In order to prove that the game has a value, we have to introduce some auxiliary discretized game for which the existence of the value can be obtained by classical min-max arguments.

5.1 Discrete strategies of Player II

In this new game, the actions of Player II are random controls defined on a suitable finite set. More precisely, let us fix an integer $n \geq 1$. Let $\tau_n = \frac{T}{n}$ be the time step and $t_i^n = i\tau_n$ (for $i = 0, \ldots, n$) be a grid on $[0, T]$. We consider an increasing family (V_n) of finite subsets of V such that, for any $n \geq 1$ and any $v \in V$, there is some $v_n \in V_n$ with $|v - v_n| \leq 1/n$. For each $n \geq 1$ and $t_0 \in [0, T]$, we denote by $\mathcal{V}_n(t_0)$ the finite subset of $\mathcal{V}(t_0)$ consisting in step functions with constant value on each interval $[t_i^n, t_{i+1}^n)$ and taking values in V_n. Let $\Delta(\mathcal{V}_n(t_0))$ be the set of all probability measures over $\mathcal{V}_n(t_0)$. If we denote by N_n the cardinal of $\mathcal{V}_n(t_0)$, then the set $\Delta(\mathcal{V}_n(t_0))$ can be identified with the simplex of $\mathbb{R}^{N_n}$ because each element of $\Delta(\mathcal{V}_n(t_0))$ can be written as $\tilde{v} = \sum_{i=1}^{N_n} p_i v_i$ for some $p \in (\mathbb{R}^+)^{N_n}$ such that $p \cdot \vec{1} = 1$. Then we set $d^\epsilon(\tilde{v}, \tilde{v}') = d^\epsilon(\sum_{i=1}^{N_n} p_i v_i, \sum_{i=1}^{N_n} p_i' v_i) = \|p - p'\|_1$. The set $\Delta(\mathcal{V}_n(t_0))$ can therefore be viewed as a compact, convex subset of $\mathbb{R}^{N_n}$.

5.2 Discretized game

The discretized game is the game where Player I plays some strategy $P \in \Delta(\mathcal{A}_x^{\tau_n}(t_0))$ and Player II plays some random control $\tilde{v} \in \Delta(\mathcal{V}_n(t_0))$. Our aim is to use Sion's Theorem in order to prove that the discretized game has a value. For this we see $\Delta(\mathcal{A}_x(t_0))$ as a convex subset of the vector space of the set of maps from $\mathbb{R}^N \times \mathcal{V}(t_0)$ into the set of Borel signed measures on $\mathcal{U}(t_0)$. We can endowed $\Delta(\mathcal{A}_x(t_0))$ with the distance

$$d(P, \tilde{P}) = \sup_{(x,v) \in \mathbb{R}^N \times \mathcal{V}(t_0)} d_{\Delta(\mathcal{U}(t_0))}(P_x^v, \tilde{P}_x^v)$$

(recall that the distance $d_{\Delta(\mathcal{U}(t_0))}$ on $\Delta(\mathcal{U}(t_0))$ is defined in section 1). For any fixed $(t_0, \mu_0) \in [0, T] \times \mathcal{W}$, we note that the map $(P, Q) \to \mathfrak{J}(t_0, \mu_0, P, Q)$ is linear with respect to P and to Q and continuous with respect to both variables on $\Delta(\mathcal{A}_x^{\tau_n}(t_0))$ and $\Delta(\mathcal{V}_n(t_0))$. Indeed the continuity with respect to Q is obvious since $\Delta(\mathcal{V}_n(t_0))$ is finite dimensional. The continuity with respect to P also holds because, since the map $u \to g(X_T^{t_0,x,u,v})$ is continuous on $\mathcal{U}(t_0)$ for any (x, v), the map $p \to \int_{\mathcal{U}(t_0)} g(X_T^{t_0,x,u,v}) dp(u)$ is

continuous on $\Delta(\mathcal{U}(t_0))$ for any (x,v). The continuity of the map $P \to \int_{\mathbb{R}^N \times \mathcal{U}(t_0)\times\mathcal{V}(t_0)} g(X_T^{t_0,x,u,v}) dP_x^v(u) dQ(v) d\mu_0(x)$ on $\Delta(\mathcal{A}_x(t_0))$ then follows from Lebesgue dominate convergence Theorem since g is bounded.

We can now use Sion's minmax Theorem to get:

Lemma 5.1. *For all $n \in \mathbb{N}^*$, the discretized game on $\Delta(\mathcal{A}_x^{\tau_n}(t_0)) \times \Delta(\mathcal{V}_n(t_0))$ has a value, denoted by $\mathbf{V}_n(t_0,\mu_0)$:*

$$\begin{aligned}
\mathbf{V}_n(t_0,\mu_0) &= \inf_{P\in\Delta(\mathcal{A}_x^{\tau_n}(t_0))} \sup_{Q\in\Delta(\mathcal{V}_n(t_0))} \int_{\mathbb{R}^N\times\mathcal{U}(t_0)\times\mathcal{V}_n(t_0)} g\left(X_T^{t_0,x,u,v}\right) dP_x^v(u) dQ(v) d\mu_0(x) \\
&= \inf_{P\in\Delta(\mathcal{A}_x^{\tau_n}(t_0))} \sup_{v\in\mathcal{V}_n(t_0)} \int_{\mathbb{R}^N\times\mathcal{U}(t_0)} g\left(X_T^{t_0,x,u,v}\right) dP_x^v(u) d\mu_0(x) \\
&= \sup_{Q\in\Delta(\mathcal{V}_n(t_0))} \inf_{P\in\Delta(\mathcal{A}_x^{\tau_n}(t_0))} \int_{\mathbb{R}^N\times\mathcal{U}(t_0)\times\mathcal{V}_n(t_0)} g\left(X_T^{t_0,x,u,v}\right) dP_x^v(u) dQ(v) d\mu_0(x) \\
&= \sup_{Q\in\Delta(\mathcal{V}_n(t_0))} \inf_{\alpha\in\mathcal{A}_x^{\tau_n}(t_0)} \int_{\mathbb{R}^N\times\mathcal{V}_n(t_0)} g\left(X_T^{t_0,x,\alpha(x,v),v}\right) dQ(v) d\mu_0(x)
\end{aligned}$$

Moreover $\mathbf{V}_n$ is Lipschitz continuous in both variables uniformly with respect to n.

Proof. The Lipschitz continuity of $\mathbf{V}_n$ can be established as in Proposition 2.1. □

Recalling the definition of $\mathbf{V}_\tau$ in (3) one easily gets:

$$\mathbf{V}_n(t_0,\mu_0) \le \mathbf{V}_{\tau_n}^-(t_0,\mu_0)\ .$$

It remains to check that $\liminf_{n\to\infty} \mathbf{V}_n(t_0,\mu_0) \ge \mathbf{V}^+(t_0,\mu_0)$. This is the aim of the next section. For this we need two preliminary lemmas:

Lemma 5.2. *The value function $\mathbf{V}_n$ satisfies the dynamic programming principle:*

$$\mathbf{V}_n(t_k^n,\mu) = \inf_{P\in\Delta(\mathcal{A}_x^{\tau_n}(t_k^n))} \sup_{v\in\mathcal{V}_n(t_k^n)} \mathbf{V}_n(t_{k+1}^n, \mu_{t_{k+1}^n}^{t_k^n,\mu,P,v})\ .$$

Proof. The proof is closely related to that of Proposition 3.1, so we only explain the main differences. Let us denote by $W(t_k^n,\mu)$ the right-hand side of the above equality. One can check, as in the proof of Proposition 2.1, that W is Lipschitz continuous with respect to μ. Inequality $\mathbf{V}_n \ge W$ can be established as in Proposition 3.1.

Let us now prove the inequality $\mathbf{V}_n \leq W$. Let $\epsilon > 0$ and P_0 be $\epsilon-$optimal for $W(t_k^n, \mu)$. Let us fix $\delta > 0$ small and $v \in \mathcal{V}_n(t_k^n)$. We set $s_0 = t_k^N$ and $s_1 = t_{k+1}^N$. At time $s_1 + \delta$, Player I knows which constant control $v_i \in V_n$ Player II has been playing on the time interval $[s_0, s_1]$, so he or she knows the measure $\nu_i := \mu_{s_1}^{s_0,\mu,P_0,v_i}$. Let P_i be $\epsilon-$optimal for $\mathbf{V}_n(s_1, \nu_i)$. We restrict the strategy P_i to the time interval $[s_1 + \delta, T]$ by setting

$$\int_{\mathcal{U}(s_1+\delta)} \varphi(u_2) d\tilde{P}_{i,x}^v(u_2) = \int_{\mathcal{U}(s_1)} \varphi(u_{|_{[s_1+\delta,T]}})) dP_{i,x}^{v_{|_{[s_1,T]}}}(u)$$

for any $(x, v) \in \mathbb{R}^N \times \mathcal{V}(s_0)$ and any nonnegative Borel measurable map $\varphi : \mathcal{U}(s_1 + \delta) \to \mathbb{R}$. We finally define the strategy $P \in \Delta(\mathcal{A}_x(s_0))$ by using the identification $\mathcal{U}(s_0) = \mathcal{U}(s_0, s_1 + \delta) \times \mathcal{U}(s_1 + \delta)$:

$$dP_x^v((u_1, u_2)) = dP_{0,x}^v(u_1) \left(\sum_{v_i \in V_n} \mathbf{1}_{\{v_{|_{[s_0,s_1]}} = v_i\}} d\tilde{P}_{i,X_{s_1}^{s_0,x,P_0,v_i}}^v(u_2) \right) .$$

This means that Player I plays the strategy P_0 on the time interval $[s_0, s_1+\delta]$, and then switches at time $s_1 + \delta$ to the strategy $\tilde{P}_i$ evaluated at the point $X_{s_1}^{s_0,x,P_0,v_i}$ if the control played by Player II on $[s_0, s_1]$ has been v_i. It is then a routine computation to show that the strategy P satisfies

$$\mathfrak{J}(s_0, \mu_\epsilon, P, v) \leq W(s_0, \mu) + C\epsilon + O(1) ,$$

where $O(1) \to 0$ as $\delta \to 0$, uniformly with respect to v and we conclude as for Proposition 3.1 that $\mathbf{V}_n \leq W$. □

Lemma 5.3. *The value $\mathbf{V}_n$ of the discretized game satisfies for all test function*

$$\varphi(t, \mu) = -\frac{\alpha}{2}\mathbf{d}^2(\bar{\mu}, \mu) - \eta\mathbf{d}(\bar{\nu}, \mu) + \psi(t)$$

(where $\psi : \mathbb{R} \to \mathbb{R}$ is smooth, $\alpha, \eta > 0$ and $\bar{\mu}, \bar{\nu} \in \mathcal{W}$) such that $\mathbf{V}_n - \varphi$ has a global minimum at $(t_k^n, \bar{\nu})$ and for any optimal plan $\pi \in \Pi_{opt}(\bar{\mu}, \bar{\nu})$:

$$0 \geq \psi'(t_k^n) + H_n(\bar{\nu}, \alpha\mathbf{p}) - \|f\|_\infty \eta - O\left(\frac{1}{n}\right) , \tag{21}$$

where $\mathbf{p}$ is defined by

$$\int_{\mathbb{R}^{2N}} \langle \xi(y), x - y\rangle d\pi(x, y) = \int_{\mathbb{R}^N} \langle \xi(y), \mathbf{p}(y)\rangle d\bar{\nu}(y) \qquad \forall \xi \in L^2_{\bar{\nu}}(\mathbb{R}^N, \mathbb{R}^N)$$

and where the Hamiltonian H_n is given by:

$$H_n(\mu, \mathbf{p}) = \max_{\mathbf{v}\in\Delta(V^n)} \int_{\mathbb{R}^N} \min_{\mathbf{u}\in\Delta(U)} \int_{U\times V} \langle \mathbf{p}, f(y, u, v)\rangle d\mathbf{u}(u) d\mathbf{v}(v) \; d\mu(y). \tag{22}$$

Proof. We assume, without loss of generality, that $\mathbf{V}_n(t_k^n, \bar{\nu}) = \varphi(t_k^n, \bar{\nu})$. Applying the dynamic programming principle of Lemma 5.1 we have

$$0 \geq \inf_{P \in \mathcal{A}_x^{\tau_n}(t_k^n)} \sup_{Q \in \Delta(\mathcal{V}_n(t_k^n))} [\varphi(t_k^n + \tau_n, \nu(t_k^n + \tau_n)) - \varphi(t_k^n, \bar{\nu})]$$

where we have set

$$\nu(t_k^n + \tau_n) = \mu_{t_k^n + \tau_n}^{t_k^n, \bar{\nu}, P, Q} .$$

Using the special form of φ, we get:

$$0 \geq \inf_{P \in \mathcal{A}_x^{\tau_n}(t_k^n)} \sup_{Q \in \Delta(\mathcal{V}_n(t_k^n))} \left[\psi(t_k^n + \tau_n) - \psi(t_k^n) + \frac{\alpha}{2}[\mathbf{d}^2(\bar{\mu}, \bar{\nu}) - \mathbf{d}^2(\bar{\mu}, \nu(t_k^n + \tau_n))] - \eta \mathbf{d}(\bar{\nu}, \nu(t_k^n + \tau_n)\right]$$

Arguing as in Section 4 we have:

$$\begin{aligned} & 0 \geq \tau_n \psi'(t_k^n) - O(\tau_n^2) \\ & + \inf_{P \in \mathcal{A}_x^{\tau_n}(t_k^n)} \sup_{Q \in \Delta(\mathcal{V}_n(t_k^n))} \left[\alpha \int_{\mathbb{R}^N \times \mathcal{U}(t_k^N) \times \mathcal{V}(t_k^N)} \langle \mathbf{p}(y), \int_{t_k^N}^{t_{k+1}^N} f(y, u(s), v(s)) ds \rangle dP_y^v(u) dQ(v) d\bar{\nu}(y) - \eta \|f\| \right. \end{aligned}$$

We now note that the restriction of the strategy Q to $[t_k^n, t_k^n + \tau]$ just consists in playing a constant control at random according to some probability measure $\mathbf{v} \in \Delta(V_n)$. Moreover the strategy P, having for delay τ_n, does not depend on v on this time interval and therefore amounts to playing at random a control u with probability $P_x(u)$ independent of v. Denoting by $\Delta(\mathcal{U}(t_k^n)))$ the set of probability measures on $\mathcal{U}(t_k^n)$ and using the min-max Theorem, we have

$$\begin{aligned} 0 &\geq \tau_n \psi'(t_k^n) - \eta \|f\|_\infty \tau_n - O(\tau_n^2) \\ &\quad + \alpha \max_{\mathbf{v} \in \Delta(V_n)} \int_{\mathbb{R}^N} \inf_{P \in \Delta(\mathcal{U}(t_k^n))} \int_{\mathcal{U}(t_k^n) \times V} \langle \mathbf{p}(y), \int_{t_k^N}^{t_{k+1}^N} f(y, u(s), v) ds \rangle dP(u) d\mathbf{v}(v) \ d\bar{\nu}(y) \\ &\geq \tau_n \psi'(t_k^n) - \eta \|f\|_\infty \tau_n - O(\tau_n^2) \\ &\quad + \alpha \tau_n \max_{\mathbf{v} \in \Delta(V_n)} \int_{\mathbb{R}^N} \min_{\mathbf{u} \in \Delta(U)} \int_{U \times V} \langle \mathbf{p}(y), f(y, u, v) \rangle d\mathbf{u}(u) d\mathbf{v}(v) \ d\bar{\nu}(y) . \end{aligned}$$

□

6 Existence and Characterization of the value

Theorem 6.1. *The game has a value, namely:*

$$\mathbf{V}^+ = \mathbf{V}^-$$

characterized as the unique viscosity solution of the Hamilton-Jacobi equation (6).

Proof. We have already noticed that $\mathbf{V}_n(t_0,\mu_0) \le \mathbf{V}^-_{\tau_n}(t_0,\mu_0)$. It remains to check that $\liminf_{n\to\infty} \mathbf{V}_n(t_0,\mu_0) \ge \mathbf{V}^+(t_0,\mu_0)$. The main idea is to use the stability of viscosity solutions as in [5]. From Proposition 4.3 we know that $\mathbf{V}^+$ is a solution of the Hamilton-Jacobi equation (6), while Lemma 5.3 states that $\mathbf{V}_n$ is "almost a subsolution" of that equation. Moreover, the functions $\mathbf{V}_n$ and $\mathbf{V}^+$ are bounded and Lipschitz continuous with the same Lipschitz constant denoted by K and we have $\mathbf{V}^+(T,\mu) = \mathbf{V}_n(T,\mu)$ for all $\mu \in \mathcal{W}$.

Let us introduce the functions $\mathbf{U}_n(t,\mu) = e^t\mathbf{V}_n(t,\mu)$ and $\mathbf{U}^+(t,\mu) = e^t\mathbf{V}^+(t,\mu)$. The function $\mathbf{U}^+$ is a viscosity solution of

$$-\mathbf{U} + \mathbf{U}_t + H(\mu, D_\mu \mathbf{U}) = 0 \tag{23}$$

with final condition $\mathbf{U}^+(T,\mu) = e^T \int_{\mathbb{R}^N} g(x)d\mu(x)$. For $(s,\mu),(t,\nu) \in [0,T]\times \mathcal{W}$, with $s \in \{t_i^n,\ i = 0,\dots,n\}$, we set

$$\varphi^n_\epsilon(s,\mu,t,\nu) = \mathbf{U}^+(t,\nu) - \mathbf{U}_n(s,\mu) - \frac{1}{\epsilon}\mathbf{d}^2(\mu,\nu) - \frac{1}{\epsilon}(t-s)^2 \ .$$

The function φ^n_ϵ is continuous and bounded from above. From Lemma 6.4, for all $\delta > 0$, there exists $(\bar s,\bar\mu,\bar t,\bar\nu)$

$$\varphi^n_\epsilon(\bar s,\bar\mu,\bar t,\bar\nu) \ge \varphi^n_\epsilon(s,\mu,t,\nu) - \delta[\mathbf{d}(\mu,\bar\mu) + \mathbf{d}(\nu,\bar\nu)] \qquad \forall (s,\mu,t,\nu) \tag{24}$$

and

$$\varphi^n_\epsilon(\bar s,\bar\mu,\bar t,\bar\nu) \ge \sup_{(s,\mu,t,\nu)} \varphi^n_\epsilon(s,\mu,t,\nu) - \delta \ .$$

This auxiliary function gives a bound on $\sup_{(t,\mu)}[\mathbf{V}^+(t,\mu) - \mathbf{V}_n(t,\mu)]$:

$$\sup_{(t,\mu)}[\mathbf{V}^+(t,\mu) - \mathbf{V}_n(t,\mu)] \le \sup_{(t,\mu)}[\mathbf{U}^+(t,\mu) - \mathbf{U}_n(t,\mu)] \le \varphi^n_\epsilon(\bar s,\bar\mu,\bar t,\bar\nu) + \delta \ . \tag{25}$$

Now we use the fact that

$$\varphi^n_\epsilon(\bar s,\bar\mu,\bar t,\bar\nu) \ge \varphi^n_\epsilon(\bar s,\bar\mu,\bar s,\bar\mu) - \delta\mathbf{d}(\bar\mu,\bar\nu) \ ,$$

namely

$$\begin{aligned}
&\mathbf{U}^+(\bar s,\bar\mu) - \mathbf{U}_n(\bar s,\bar\mu) - \delta\mathbf{d}(\bar\mu,\bar\nu)\\
&\le \mathbf{U}^+(\bar t,\bar\nu) - \mathbf{U}_n(\bar s,\bar\mu) - \frac{1}{\epsilon}\mathbf{d}^2(\bar\mu,\bar\nu) - \frac{1}{\epsilon}(\bar t-\bar s)^2\\
&\le \mathbf{U}^+(\bar s,\bar\mu) + K|\bar s-\bar t| + K\mathbf{d}(\bar\nu,\bar\mu) - \mathbf{U}_n(\bar s,\bar\mu) - \frac{1}{\epsilon}\mathbf{d}^2(\bar\mu,\bar\nu) - \frac{1}{\epsilon}(\bar t-\bar s)^2 \ ,
\end{aligned}$$

to get the following bound on the distance between $(\bar{t},\bar{\nu})$ and $(\bar{s},\bar{\mu})$:

$$\mathbf{d}(\bar{\nu},\bar{\mu}) + |\bar{t}-\bar{s}| \leq 2\epsilon(K+\delta) \ . \tag{26}$$

Let us first assume that $\bar{s} = T$ (the case $\bar{t} = T$ could be treated similarly). Then we have:

$$\begin{aligned}\varphi^n_\epsilon(\bar{s},\bar{\mu},\bar{t},\bar{\nu}) &\leq \mathbf{U}^+(T,\bar{\nu}) - \mathbf{U}_n(T,\bar{\mu}) + K|T-\bar{t}| \\ &\leq e^T \int_{\mathbb{R}^N} g(x)d\bar{\nu}(x) - e^T \int_{\mathbb{R}^N} g(x)d\bar{\mu}(x) + K|T-\bar{t}| \\ &\leq \mathrm{Lip}(g)e^T\mathbf{d}(\bar{\nu},\bar{\mu}) + K|T-\bar{t}| \ \leq \ C\epsilon\end{aligned}$$

Thus using (25) we get:

$$\sup_{(t,\mu)}[\mathbf{V}^+(t,\mu) - \mathbf{V}_n(t,\mu)] \leq C\epsilon + \delta \ ,$$

so that, passing to the limit as $\epsilon,\delta \to 0$, we obtain:

$$\sup_{(t,\mu)}[\mathbf{V}^+(t,\mu) - \mathbf{V}_n(t,\mu)] \leq 0 \ .$$

We now assume that $\bar{t} \neq T$ and $\bar{s} \neq T$. Let us fix some optimal transport plan $\pi \in \Pi_{opt}(\bar{\nu},\bar{\mu})$. We first use the fact that

$$\varphi_\epsilon(\bar{s},\bar{\mu},\bar{t},\bar{\nu}) \geq \varphi_\epsilon(s,\mu,\bar{t},\bar{\nu}) - \delta\mathbf{d}(\mu,\bar{\mu}) \ ,$$

namely

$$\begin{aligned}&\mathbf{U}^+(\bar{t},\bar{\nu}) - \mathbf{U}_n(\bar{s},\bar{\mu}) - \frac{1}{\epsilon}\mathbf{d}^2(\bar{\mu},\bar{\nu}) - \frac{1}{\epsilon}(\bar{t}-\bar{s})^2 \\ &\quad \geq \mathbf{U}^+(\bar{t},\bar{\nu}) - \mathbf{U}_n(s,\mu) - \frac{1}{\epsilon}\mathbf{d}^2(\mu,\bar{\nu}) - \frac{1}{\epsilon}(\bar{t}-s)^2 - \delta\mathbf{d}(\mu,\bar{\mu}) \ ,\end{aligned}$$

to get

$$\mathbf{U}_n(s,\mu) + \frac{1}{\epsilon}\mathbf{d}^2(\mu,\bar{\nu}) + \delta\mathbf{d}(\mu,\bar{\mu}) + \frac{1}{\epsilon}(\bar{t}-s)^2 \geq \mathbf{U}_n(\bar{s},\bar{\mu}) + \frac{1}{\epsilon}\mathbf{d}^2(\bar{\mu},\bar{\nu}) + \frac{1}{\epsilon}(\bar{t}-\bar{s})^2 \ .$$

If we set $\varphi(s,\mu) = -\frac{1}{\epsilon}\mathbf{d}^2(\mu,\bar{\nu}) - \delta\mathbf{d}(\mu,\bar{\mu}) - \frac{1}{\epsilon}(\bar{t}-s)^2$, then the function $\mathbf{U}_n - \varphi$ has a minimum at $(\bar{s},\bar{\mu})$. From Lemma (5.3), this implies that:

$$-\mathbf{U}_n(\bar{s},\bar{\mu}) + \frac{2}{\epsilon}(\bar{t}-\bar{s}) + H_n(\bar{\mu},\frac{2}{\epsilon}\mathbf{p}) \leq \delta\|f\|_\infty + O\left(\frac{1}{n}\right)$$

where $\mathbf{p}$ is defined by:

$$\int_{\mathbb{R}^{2N}} \langle \xi(y), x-y \rangle d\pi(x,y) = \int_{\mathbb{R}^N} \langle \xi(y), \mathbf{p}(y) \rangle d\bar{\mu}(y) \qquad \forall \xi \in L^2_{\bar{\mu}}(\mathbb{R}^N, \mathbb{R}^N) \ . \tag{27}$$

The same argument applied to

$$\varphi^n_\epsilon(\bar{s}, \bar{\mu}, \bar{t}, \bar{\nu}) \geq \varphi^n_\epsilon(\bar{s}, \bar{\mu}, t, \nu) - \delta \mathbf{d}(\nu, \bar{\nu})$$

leads to

$$\mathbf{U}^+(\bar{t}, \bar{\nu}) - \frac{2}{\epsilon}(\bar{t} - \bar{s}) - H(\bar{\nu}, -\frac{2}{\epsilon}\mathbf{q}) \leq \delta \|f\|_\infty$$

where $\mathbf{q}$ satisfies for $\bar{\pi} \in \Pi_{opt}(\bar{\mu}, \bar{\nu})$:

$$\int_{\mathbb{R}^{2N}} \langle \xi(y), x-y \rangle d\bar{\pi}(x,y) = \int_{\mathbb{R}^N} \langle \xi(x), \mathbf{q}(x) \rangle d\bar{\nu}(x) \qquad \forall \xi \in L^2_{\bar{\nu}}(\mathbb{R}^N, \mathbb{R}^N) \ .$$

If we take for $\bar{\pi}$ the optimal transport plan defined through:

$$\int_{\mathbb{R}^{2N}} \varphi(x,y) d\bar{\pi}(x,y) = \int_{\mathbb{R}^N} \varphi(y,x) d\pi(x,y) \qquad \forall \varphi \in L^2_\pi(\mathbb{R}^{2N}, \mathbb{R}^{2N})$$

we have:

$$\int_{\mathbb{R}^{2N}} \langle \xi(y), x-y \rangle d\pi(x,y) = \int_{\mathbb{R}^N} \langle \xi(x), -\mathbf{q}(x) \rangle d\bar{\nu}(x) \qquad \forall \xi \in L^2_{\bar{\nu}}(\mathbb{R}^N, \mathbb{R}^N) \ .$$

Finally combining the last two inequalities we obtain:

$$\mathbf{U}^+(\bar{t}, \bar{\nu}) - \mathbf{U}_n(\bar{s}, \bar{\mu}) \leq H(\bar{\nu}, -\frac{2}{\epsilon}\mathbf{q}) - H_n(\bar{\mu}, \frac{2}{\epsilon}\mathbf{p}) + 2\delta \|f\|_\infty + O\left(\frac{1}{n}\right) \ . \tag{28}$$

Our next step consists in comparing H and H_n:

Lemma 6.2. *We have, for any $\mu \in \mathcal{W}$ and any $\mathbf{p} \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N)$:*

$$0 \leq H(\mu, \mathbf{p}) - H_n(\mu, \mathbf{p}) \leq \gamma_n \|\mathbf{p}\|_{L^2_\mu} \ ,$$

where

$$\gamma_n = \sup_{x,u,|v_1 - v_2| \leq 1/n} |f(x,u,v_1) - f(x,u,v_2)| \ .$$

Remark 6.3. Note that $\gamma_n \to 0$ as $n \to +\infty$ because f is uniformly continuous on $\mathbb{R}^N \times U \times V$.

Proof. By definition we have $H(\mu, \mathbf{p}) \geq H_n(\mu, \mathbf{p})$. Let $\bar{\mathbf{v}}$ be $\epsilon-$optimal for $H(\mu, \mathbf{p})$ and $\Pi : V \to V_n$ be a Borel measurable selection of the projection map from V onto V_n. Then, by construction of V_n, we have $|v - \Pi(v)| \leq 1/n$ and

$$\begin{aligned} H_n(\mu, \mathbf{p}) &\leq \int_{\mathbb{R}^N} \inf_{\mathbf{u} \in \Delta(U)} \int_{U \times V} \langle f(x, u, v), \mathbf{p}(x) \rangle d\mathbf{u}(u) d\Pi\sharp\bar{\mathbf{v}}(v) \; d\mu(x) \\ &\leq \int_{\mathbb{R}^N} \inf_{\mathbf{u} \in \Delta(U)} \int_{U \times V} \langle f(x, u, v), \mathbf{p}(x) \rangle d\mathbf{u}(u) d\bar{\mathbf{v}}(v) \; d\mu(x) + \gamma_n \int_{\mathbb{R}^N} |\mathbf{p}(x)| d\mu(x) \\ &\leq H(\mu, \mathbf{p}) + \epsilon + \gamma_n \|\mathbf{p}\|_{L^2_\mu} \end{aligned}$$

□

From (27), we have

$$\|\mathbf{p}\|_{L^2_\mu} \leq \mathbf{d}(\bar{\mu}, \bar{\nu}) \; .$$

Combining the continuity of the Hamiltonian stated in Lemma 4.2, the bound on the distance between H and H_n given in Lemma 6.2 and inequality (28) we get:

$$\mathbf{U}^+(\bar{t}, \bar{\nu}) - \mathbf{U}_n(\bar{s}, \bar{\mu}) \leq \frac{2K}{\epsilon} \mathbf{d}^2(\bar{\nu}, \bar{\mu}) + \frac{2\gamma_n}{\epsilon} \mathbf{d}(\bar{\nu}, \bar{\mu}) + 2\delta \|f\|_\infty + O\left(\frac{1}{n}\right) \; .$$

Putting together the estimates (25) and (26) and letting $\epsilon, \delta \to 0$ finally gives

$$\mathbf{V}^+ \leq \mathbf{V}_n + C\gamma_n + O\left(\frac{1}{n}\right) \; .$$

This implies that $\liminf_{n \to \infty} \mathbf{V}_n(t_0, \mu_0) \geq \mathbf{V}^+(t_0, \mu_0)$ and completes the proof. □

Appendix

The following statement is a slight modification of Ekeland's variational Lemma.

Lemma 6.4. *Let $F : [0, T] \times \mathcal{W} \to \mathbb{R}$ be a continuous function which is bounded from below. Then for any $\epsilon > 0$, there exists $(\bar{t}, \bar{\mu}) \in X$ such that for all $(t, \mu) \in X$:*

$$F(t, \mu) \geq F(\bar{t}, \bar{\mu}) - \epsilon \mathbf{d}(\mu, \bar{\mu}) \qquad \text{and} \qquad F(\bar{t}, \bar{\mu}) \leq \inf_X F + \epsilon \; .$$

Proof. Let (t_0, μ_0) be such that

$$F(t_0, \mu_0) \leq \inf_X F + \epsilon \ .$$

Then we build the sequence (t_n, μ_n) by induction, such that, if (t_n, μ_n) is known, then

- if for all $(t, \mu) \in X$, $F(t, \mu) \geq F(t_n, \mu_n) - \epsilon \mathbf{d}(\mu, \mu_n)$, then we set $(t_{n+1}, \mu_{n+1}) = (t_n, \mu_n)$,
- if, on the contrary, there is $(t, \mu) \in X$ such that $F(t, \mu) < F(t_n, \mu_n) - \epsilon \mathbf{d}(\mu, \mu_n)$, then we set

  $$S_n = \{(t, \mu) \in X \text{ such that } F(t, \mu) < F(t_n, \mu_n) - \epsilon \mathbf{d}(\mu, \mu_n) \ \}$$

  and choose $(t_{n+1}, \mu_{n+1}) \in S_n$ such that $F(t_{n+1}, \mu_{n+1}) \leq (F(t_n, \mu_n) + \inf_{S_n} F)/2$.

Note that by construction $F(t_n, \mu_n) \leq \inf_X F + \epsilon$ for any n. We prove that the sequence (μ_n) is a Cauchy sequence. Indeed, it is either stationary, or we have $\epsilon \mathbf{d}(\mu_n, \mu_{n+1}) < F(t_n, \mu_n) - F(t_{n+1}, \mu_{n+1})$. Therefore for all (n, p), $n \geq p$,

$$\epsilon \mathbf{d}(\mu_n, \mu_p) < F(t_p, \mu_p) - F(t_n, \mu_n) \ . \tag{29}$$

The sequence $F(t_n, \mu_n)$ being decreasing and bounded from below, it has a limit and inequality (29) shows that (μ_n) is a Cauchy sequence. Let $\bar{\mu}$ be the limit of the (μ_n) and let us consider any cluster point $\bar{t}$ of the (t_n). We now assume for a while that the is some $(\bar{s}, \bar{\nu}) \in X$ with

$$F(\bar{s}, \bar{\nu}) < F(\bar{t}, \bar{\mu}) - \epsilon \mathbf{d}(\bar{\nu}, \bar{\mu}) \ . \tag{30}$$

Consider some subsequence (t_{n_i}, μ_{n_i}) converging to $(\bar{t}, \bar{\mu})$. Letting $n \to +\infty$ in (29) gives

$$F(\bar{t}, \bar{\mu}) \leq F(t_{n_i}, \mu_{n_i}) - \epsilon \mathbf{d}(\mu_{n_i}, \bar{\mu}) \ .$$

Therefore, we have

$$F(\bar{s}, \bar{\nu}) < F(t_{n_i}, \mu_{n_i}) - \epsilon \mathbf{d}(\bar{\nu}, \mu_{n_i}) \ ,$$

which means that $(\bar{s}, \bar{\nu}) \in S_{n_i}$ for all i. This implies that

$$2F(t_{n_i+1}, \mu_{n_i+1}) - F(t_{n_i}, \mu_{n_i}) \leq \inf_{S_{n_i}} F \leq F(\bar{s}, \bar{\nu}) \ .$$

The sequence $F(t_n, \mu_n)$ being decreasing, we get: $2F(t_{n_{i+1}}, \mu_{n_{i+1}}) - F(t_{n_i}, \mu_{n_i}) \leq F(\bar{s}, \bar{\nu})$. Passing to the limit as $i \to \infty$ gives $F(\bar{t}, \bar{\mu}) \leq F(\bar{s}, \bar{\nu})$, which is in contradiction with (30). Therefore we have

$$F(t, \mu) \geq F(\bar{t}, \bar{\mu}) - \epsilon \mathbf{d}(\mu, \bar{\mu}) \qquad \forall (t, \mu) \in X \ .$$

□

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