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# MODULI OF GALOIS $p$-COVERS IN MIXED CHARACTERISTICS

DAN ABRAMOVICH AND MATTHIEU ROMAGNY

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## 1. Introduction

Fix a prime number $p$. The aim of this paper is to define a complete moduli stack of degree-$p$ covers $Y \to \mathcal{X}$, with $Y$ a stable curve which is a $\mathcal{G}$-torsor over $\mathcal{X}$, for a suitable group scheme $\mathcal{G}/\mathcal{X}$. The curve $\mathcal{X}$ is a twisted curve in the sense of [5, 4] but in general not stable. This follows the same general approach as the characteristic-0 paper [1], but diverges from that of [4], where the curve $\mathcal{X}$ is stable, the group scheme $\mathcal{G}$ is assumed linearly reductive, but $Y$ is in general much more singular.

The approach is based on [12, Proposition 1.2.1] of Raynaud, and the more general notion of *effective model* of a group-scheme action due to the second author [13]. The general strategy was outlined in [2] in a somewhat special case.

1.1. **Rigidified group schemes.** The group scheme $\mathcal{G}$ comes with a supplementary structure which we call a *generator*. Before we define this notion, let us briefly recall from Katz-Mazur [10, §1.8] the concept of a *full set of sections*. Let $Z \to S$ be a finite locally free morphism of schemes of degree $N$. Then for all affine $S$-schemes $\mathrm{Spec}(R)$, the $R$-algebra $\Gamma(Z_R, \mathcal{O}_{Z_R})$ is locally free of rank $N$ and has a canonical norm mapping. We say that a set of $N$ sections $x_1, \dots, x_N \in Z(S)$ is a *full set of sections* if and only if for any affine $S$-scheme $\mathrm{Spec}(R)$ and any $f \in \Gamma(Z_R, \mathcal{O}_{Z_R})$, the norm of $f$ is equal to the product $f(x_1) \dots f(x_N)$.

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**Definition 1.2.** Let $G \to S$ be a finite locally free group scheme of order $p$. A *generator* is a morphism of $S$-group schemes $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$ such that the sections $x_i = \gamma(i)$, $0 \le i \le p-1$, are a full set of sections. A *rigidified group scheme* is a group scheme of degree $p$ with a generator.

The notion of generator is easily described in terms of the Tate-Oort classification of group schemes of order $p$. This is explained and complemented in appendix A.

**Remark 1.3.** One can define the stack of rigidified group schemes a bit more directly: consider the Artin stack $\mathfrak{GS}_p$ of group-schemes of degree $p$, and let $\mathcal{G}^u \to \mathfrak{GS}_p$ be the universal group-scheme - an object of $\mathcal{G}^u$ over a scheme $S$ consists of a group-scheme $\mathcal{G} \to S$ with a section $S \to \mathcal{G}$. It has a unique non-zero point over $\mathbb{Q}$ corresponding to $\mathbb{Z}/p\mathbb{Z}$ with the section 1. The stack of rigidified group schemes is canonically isomorphic to the closure of this point.

Of course describing a stack as a closure of a sub-stack is not ideal from the moduli point of view, and we find the definition using Katz–Mazur generators more satisfying.

1.4. **Stable $p$-torsors.** Fix a prime number $p$ and integers $g, h, n \ge 0$ with $2g - 2 + n > 0$.

**Definition 1.5.** A *stable $n$-marked $p$-torsor* of genus $g$ (over some base scheme $S$) is a triple

$$(\mathcal{X}, \mathcal{G}, Y)$$

where

(1) $(\mathcal{X}, \{\Sigma_i\}_{i=1}^n)$ is an $n$-marked twisted curve of genus $h$,
(2) $(Y, \{P_i\}_{i=1}^n)$ is a nodal curve of genus $g$ with étale marking divisors $P_i \to S$, which is stable in the sense of Deligne-Mumford-Knudsen,
(3) $\mathcal{G} \to \mathcal{X}$ is a rigidified group-scheme of degree $p$,
(4) $Y \to \mathcal{X}$ is a $\mathcal{G}$-torsor and $P_i = \Sigma_i \times_{\mathcal{X}} Y$ for all $i$.

Note that as usual the markings $\Sigma_i$ (resp. $P_i$) are required to lie in the smooth locus of $\mathcal{X}$ (resp. $Y$). They split into two groups. In the first group $\Sigma_i$ is twisted and $[P_i : S] = 1$, while in the second group $\Sigma_i$ is a section and $[P_i : S] = p$. The number $m$ of twisted markings is determined by $(2g-2) = p(2h-2) + m(p-1)$ and it is equivalent to fix $h$ or $m$.

The notion of stable marked $p$-torsor makes sense over an arbitrary base scheme $S$. Given stable $n$-marked $p$-torsors $(\mathcal{X}, \mathcal{G}, Y)$ over $S$ and $(\mathcal{X}', \mathcal{G}', Y')$ over $S'$, one defines as usual a morphism $(\mathcal{X}, \mathcal{G}, Y) \to (\mathcal{X}', \mathcal{G}', Y')$ over $S \to S'$ as a fiber diagram. This defines a category fibered over $\operatorname{Spec}\mathbb{Z}$ that we denote $ST_{p,g,h,n}$.

Our main result is:

**Theorem 1.6.** *The category $ST_{p,g,h,n}/\operatorname{Spec}\mathbb{Z}$ is a proper Deligne-Mumford stack with finite diagonal.*

Notice that $ST_{p,g,h,n}$ contains an open substack of étale $\mathbb{Z}/p\mathbb{Z}$-covers. Identifying the closure of this open locus remains an interesting question.

1.7. **Organization.** Section 2 is devoted to Proposition 2.1, in particular showing the algebricity of $ST_{p,g,h,n}$. Section 3 completes the proof of Theorem 1.6 by showing properness. We give simple examples in Section 4. Two appendices are provided - in Appendix A we discuss embeddings of group schemes of order $p$ into smooth group schemes. In Appendix B we recall some facts about the Weil restriction of closed subschemes, and state the representability result in a form useful for us.

1.8. **Acknowledgements.** We thank Sylvain Maugeais for helping us clarify a point in this paper.

## 2. The stack $ST_{p,g,h,n}$

In this section, we review some basic facts on twisted curves and then we show:

**Proposition 2.1.** *The category $ST_{p,g,h,n}/\operatorname{Spec}\mathbb{Z}$ is an algebraic stack of finite type over $\mathbb{Z}$.*

2.2. **Twisted curves and Log twisted curves.** We review some material from Olsson's treatment in [4, Appendix A], with some attention to properness of the procedure of "log twisting".

Recall that a *twisted curve* over a scheme $S$ is a tame Artin stack $\mathcal{C} \to S$ with a collection of gerbes $\Sigma_i \subset \mathcal{C}$ satisfying the following conditions:

(1) The coarse moduli space $C$ of $\mathcal{C}$ is a prestable curve over $S$, and the images $\bar{\Sigma}_i$ of $\Sigma_i$ in $C$ are the images of disjoint sections $\sigma_i : S \to C$ of $C \to S$ landing in the smooth locus.
(2) Étale locally on $S$ there are positive integers $r_i$ such that, on a neighborhood of $\Sigma_i$ we can identify $\mathcal{C}$ with the root stack $C(\sqrt[r_i]{\bar{\Sigma}_i})$.
(3) Near a node $z$ of $C$ write $C^{sh} = \operatorname{Spec}(\mathcal{O}_S^{sh}[x,y]/(xy-t))^{sh}$. Then there exists a positive integer $a_z$ and an element $s \in \mathcal{O}_S^{sh}$ such that $s^{a_z} = t$ and
$$\mathcal{C}^{\mathrm{sh}} = [\operatorname{Spec}\mathcal{O}_S^{sh}[u,v]/(uv-s))^{sh}/\mu_{a_z}],$$
where $\mu_{a_z}$ acts via $(u,v) \mapsto (\zeta u, \zeta^{-1}v)$ and where $x = u^{a_z}$ and $y = v^{a_z}$.

The purpose of [4, Appendix A] was to show that twisted curves form an Artin stack which is locally of finite type over $\mathbb{Z}$. There are two steps involved.

The introduction of the stack structure over the markings is a straightforward step: the stack $\mathfrak{M}_{g,\delta}^{tw}$ of twisted curves with genus $G$ and $n$ markings is the infinite disjoint union $\mathfrak{M}_{g,\delta}^{tw} = \sqcup \mathfrak{M}_{g,\delta}^{r}$, where $r$ runs over the possible *marking indices*, namely vectors of positive integers $r = (r_1, \ldots, r_n)$, and

the stacks $\mathfrak{M}^r_{g,\delta}$ are all isomorphic to each other - the universal family over $\mathfrak{M}^r_{g,\delta}$ is obtained form that over $\mathfrak{M}^{(1,\dots,1)}_{g,\delta}$ by taking the $r_i$-th root of $\bar{\Sigma}_i$.

The more subtle point is the introduction of twisting at nodes. Olsson achieves this using the canonical log structure of prestable curves, and provides an equivalence between twisted curves with $r = (1, \dots, 1)$ and log-twisted curves. A *log twisted curve* over a scheme $S$ is the data of a prestable curve $C/S$ along with a simple extension $\mathcal{M}^S_{C/S} \hookrightarrow \mathcal{N}$. Here $\mathcal{M}^S_{C/S}$ is F. Kato's canonical locally free log structure of the base $S$ of the family of prestable curves $C/S$, and a *simple extension* is an injective morphism $\mathcal{M}^S_{C/S} \hookrightarrow \mathcal{N}$ of locally free log structures of equal rank where an irreducible element is sent to a multiple of an irreducible element up to units.

We now describe an aspect of this equivalence which is relevant for our main results. Consider a family of prestable curves $C/S$ and denote by $\iota : \operatorname{Sing} C/S \to C$ the embedding of the locus where $\pi : C \to S$ fails to be smooth. A *node function* is a section $a$ of $\pi_*\iota_*\mathbb{N}_{\operatorname{Sing} C/S}$. In other words it gives a positive integer $a_z$ for each singular point $z$ of $C/S$ in a continuous manner. Given a morphism $T \to S$, we say that a twisted curve $\mathcal{C}/T$ with coarse moduli space $C_T$ is *$a$-twisted* over $C/S$ if the index of a node of $\mathcal{C}$ over a node $z$ of $C$ is precisely $a_z$.

**Proposition 2.3.** *Fix a family of prestable curves $C/S$ of genus $g$ with $n$ markings over a noetherian scheme $S$. Further fix marking indices $r = (r_1, \dots, r_n)$ and a node function $a$. Then the category of $a$-twisted curves over $C/S$ with marking indices given by $r$ is a proper and quasi-finite tame stack over $S$.*

*Proof.* The problem is local on $S$, and further it is stable under base change in $S$. So it is enough to prove this when $S$ is a versal deformation space of a prestable curve $C_s$ of genus $g$ with $n$ markings, over a closed geometric point $s \in S$, in such a way that we have a chart $\mathbb{N}^k \to \mathcal{M}^S_{C/S}$ of the log structure, where $k$ is the number of nodes of $C_s$. The image of the $i$-th generator of $\mathbb{N}^k$ in $\mathcal{O}_S$ is the defining equation of the smooth divisor $D_i$ where the $i$-th node persists. Now consider an $a$-twisted curve over $\phi : T \to S$, corresponding to a simple extension $\phi^*\mathcal{M}^S_{C/S} \to \mathcal{N}$ where the image of the $i$-th generator $m_i$ becomes an $a_i$-multiple up to units. This precisely means that $\mathcal{O}^*_{C_T} m_i$, the principal bundle associated to $\mathcal{O}_S(-D_i)$, is an $a_i$-th power. In other words, the stack of $a$-twisted curves over $C/S$ is isomorphic to the stack

$$S(\sqrt[a_1]{D_1} \cdots \sqrt[a_n]{D_n}) \quad = \quad S(\sqrt[a_1]{D_1}) \times_S \cdots \times_S S(\sqrt[a_n]{D_n})$$

encoding $a_i$-th roots of $O_S(D_i)$. This is evidently proper and quasi-finite tame stack over $S$. ♠

We now turn to the index of twisted points in a stable $p$-torsor.

**Lemma 2.4.** *Let $(\mathcal{X}, \mathcal{G}, Y)$ be a stable $p$-torsor. Then the index of a point $x \in \mathcal{X}$ divides $p$.*

*Proof.* Let $r$ be the index of $x$ and $d$ the local degree of $Y \to \mathcal{X}$ at a point $y$ above $x$. Since $Y \to \mathcal{X}$ is finite flat of degree $p$ and $\mathcal{G}$ acts transitively on the fibers, then $d \mid p$. Let $f : \mathcal{X} \to X$ be the coarse moduli space of $\mathcal{X}$. In order to compute $d$, we pass to strict henselizations on $S$, $X$ and $Y$ at the relevant points. Thus $S$ is the spectrum of a strictly henselian local ring $(R, \mathfrak{m})$, and we have two cases to consider.

If $x$ is a smooth point,

- $X \simeq \operatorname{Spec} R[a]^{\mathrm{sh}}$,
- $Y \simeq \operatorname{Spec} R[s]^{\mathrm{sh}}$,
- $\mathcal{X} \simeq [D/\mu_r]$ with $D = \operatorname{Spec} R[u]^{\mathrm{sh}}$ and $\zeta \in \mu_r$ acting by $u \mapsto \zeta u$.

Consider the fibered product $E = Y \times_{\mathcal{X}} D$. The map $E \to Y$ is a $\mu_r$-torsor of the form $E \simeq \operatorname{Spec} \mathcal{O}_Y[w]/(w^r - f)$ for some invertible function $f \in \mathcal{O}_Y^\times$, and $E \to D$ is a $\mu_r$-equivariant map given by $u \mapsto \varphi w$ for some function $\varphi$ on $Y$. Let $\tilde{x} : \operatorname{Spec} k \to D$ be a point mapping to $x$ in $\mathcal{X}$ i.e. corresponding to $u = \mathfrak{m} = 0$, and let $\bar{\varphi}, \bar{f}$ be the restrictions of $\varphi, f$ to $Y_{\tilde{x}}$. The preimage of $\tilde{x}$ under $E \to D$ is a finite $k$-scheme with algebra $k[s][w]/(\bar{\varphi}, w^r - \bar{f})$. We see that $d = r \dim_k k[s]/(\bar{\varphi})$ and hence the index $r$ divides $p$.

If $x$ is a singular point, there exist $\lambda, \mu, \nu$ in $\mathfrak{m}$ such that

- $X \simeq \operatorname{Spec}(R[a,b]/(ab - \lambda))^{\mathrm{sh}}$,
- $Y \simeq \operatorname{Spec}(R[s,t]/(st - \mu))^{\mathrm{sh}}$,
- $\mathcal{X} \simeq [D/\mu_r]$ where $D = \operatorname{Spec}(R[u,v]/(uv - \nu))^{\mathrm{sh}}$,

and $\zeta \in \mu_r$ acts by $u \mapsto \zeta u$ and $v \mapsto \zeta^{-1} v$. The scheme $E = Y \times_{\mathcal{X}} D$ is of the form $E \simeq \operatorname{Spec} \mathcal{O}_Y[w]/(w^r - f)$ for some invertible function $f \in \mathcal{O}_Y^\times$, and the map $E \to D$ is given by $u \mapsto \varphi w$, $v \mapsto \psi w^{-1}$ for some functions $\varphi, \psi$ on $Y$ satisfying $\varphi\psi = \nu$. Let $\tilde{x} : \operatorname{Spec} k \to D$ be a point mapping to $x$ and let $\bar{\varphi}, \bar{\psi}, \bar{f}$ be the restrictions of $\varphi, \psi, f$ to $Y_{\tilde{x}}$. The preimage of $\tilde{x}$ under $E \to D$ is a finite $k$-scheme with algebra $k[s,t][w]/(st, \bar{\varphi}, \bar{\psi}, w^r - \bar{f})$. We see that $d = r \dim_k k[s,t]/(st, \bar{\varphi}, \bar{\psi})$ and hence $r$ divides $p$. ♠

2.5. **Proof of proposition 2.1.** Let $\delta = (\delta_1, \ldots, \delta_n)$ be the sequence of degrees of the markings $P_i$ on the total space of stable $p$-torsors, with each $\delta_i$ equal to 1 or $p$. We build $ST_{p,g,h,n}$ from existing stacks: the stack $\overline{\mathcal{M}}_{g,\delta}$ of Deligne-Mumford-Knudsen stable marked curves (for the family of curves $Y$), the stack $\mathfrak{M}$ of twisted curves (for the family of marked twisted curves $\mathcal{X}$), and Hilbert schemes and $\underline{\mathrm{Hom}}$ stacks for construction of $Y \to \mathcal{X}$ and $\mathcal{G}$.

Bounding the twisted curves. We have an evident forgetful functor $ST_{p,g,h,n} \to \overline{\mathcal{M}}_{g,\delta} \times \mathfrak{M}$. Note that the image $ST_{p,g,h,n} \to \mathfrak{M}$ lies in an open substack $\mathfrak{M}'$ of finite type over $\mathbb{Z}$: the index of the twisted curve $\mathcal{X}$ divides $p$ by Lemma 2.4, and its topological type is bounded by that of $Y$. The stack $\mathfrak{M}'$ parametrizing such twisted curves is of finite type over $\mathbb{Z}$ by [4, Corollary A.8].

Set $M_{Y,\mathcal{X}} = \overline{\mathcal{M}}_{g,\delta} \times \mathfrak{M}'$. This is an algebraic stack of finite type over $\mathbb{Z}$.

The map $Y \to \mathcal{X}$. Consider the universal family $Y \to M_{Y,\mathcal{X}}$ of stable curves of genus $g$ and the universal family $\mathcal{X} \to M_{Y,\mathcal{X}}$ of twisted curves,

with associated family of coarse curves $X \to M_{Y,\mathcal{X}}$. Since Hilbert schemes of fixed Hilbert polynomial are of finite type, there is an algebraic stack $Hom^{\leq p}_{M_{Y,\mathcal{X}}}(Y, X)$, of finite type over $M_{Y,\mathcal{X}}$, parametrizing morphisms $Y_s \to X_s$ of degree $\leq p$ between the respective fibers. By [4, Corollary C.4] the stack $Hom^{\leq p}_{M_{Y,\mathcal{X}}}(Y, \mathcal{X})$ corresponding to maps $Y_s \to \mathcal{X}_s$ with target the twisted curve is of finite type over $Hom^{\leq p}_{M_{Y,\mathcal{X}}}(Y, X)$, hence over $M_{Y,\mathcal{X}}$. There is an open substack $M_{Y\to\mathcal{X}}$ parametrizing flat morphisms of degree precisely $p$. We have an evident forgetful functor $ST_{p,g,h,n} \to M_{Y\to\mathcal{X}}$ lifting the functor $ST_{p,g,h,n} \to \overline{\mathcal{M}}_{g,\delta} \times \mathfrak{M}'$ above.

The rigidified group scheme $\mathcal{G}$. The scheme $Y_2 = Y \times_{\mathcal{X}} Y$ is flat of degree $p$ over $Y$. Giving it the structure of a group scheme over $Y$ with unit section equal to the diagonal $Y \to Y_2$ is tantamount to choosing structure $Y$-arrows $m : Y_2 \times_Y Y_2 \to Y_2$ and $i' : Y_2 \to Y_2$, which are parametrized by a $Hom$-scheme, and passing to the closed subscheme where these give a group-scheme structure (that this condition is closed follows from representability of the Weil restriction, see the discussion in the appendix and in particular Corollary B.4). Giving a group scheme $\mathcal{G}$ over $\mathcal{X}$ with isomorphism $\mathcal{G}\times_{\mathcal{X}} Y \simeq Y_2$ is tantamount to giving descent data for $Y_2$ with its chosen group-scheme structure. This is again parametrized by a suitable $Hom$-scheme. Finally requiring that the projection $Y_2 \to Y$ correspond to an action of $\mathcal{G}$ on $Y$ is a closed condition (again by Weil restriction, see Corollary B.4).

Passing to a suitable $Hom$-stack we can add a homomorphism $\mathbb{Z}/p\mathbb{Z} \to \mathcal{G}$, giving a section $\mathcal{X} \to \mathcal{G}$ (equivalently a morphism $\mathcal{X} \to \mathcal{G}^u$, see remark 1.3). By [10, corollary 1.3.5], the locus of the base where this section is a generator is closed. Since $Y_2 \to Y$ and $Y \to \mathcal{X}$ are finite, all the necessary $Hom$ stacks are in fact of finite type.

The resulting stack is clearly isomorphic to $ST_{p,g,h,n}$.

## 3. Properness

Since $ST_{p,g,h,n} \to \operatorname{Spec}\mathbb{Z}$ is of finite type, we need to prove the valuative criterion for properness.

We have the following situation:

(1) $R$ is a discrete valuation ring with spectum $S = \operatorname{Spec} R$, fraction field $K$ with corresponding generic point $\eta = \operatorname{Spec} K$, and residue field $\kappa$ with corresponding special point $s = \operatorname{Spec}\kappa$.
(2) $(\mathcal{X}_\eta, \mathcal{G}_\eta, Y_\eta)$ a stable marked $p$-torsor of genus $g$ over $\eta$.

By an *extension* of $(\mathcal{X}_\eta, \mathcal{G}_\eta, Y_\eta)$ across $s$ we mean

(1) a local extension $R \to R'$ with $K'/K$ finite,
(2) a stable marked $p$-torsor $(\mathcal{X}', \mathcal{G}', Y')$ of genus $g$ over $S' = \operatorname{Spec} R'$, and
(3) an isomorphism $(\mathcal{X}', \mathcal{G}', Y')'_\eta \simeq (\mathcal{X}_\eta, \mathcal{G}_\eta, Y_\eta) \times_\eta \eta'$.

We have

**Proposition 3.1.** *An extension exists. When extension over $S'$ exists, it is unique up to a unique isomorphism.*

*Proof.* Extension of $Y_\eta$. Since $\overline{\mathcal{M}}_{g,\delta}$ is proper, there is a stable marked curve $Y'$ extending $Y_\eta$ over some $S'$, and this extension is unique up to a unique isomorphism. We replace $S$ by $S'$, and assume that there is $Y$ over $S$ with generic fiber $Y_\eta$.

Coarse extension of $\mathcal{X}_\eta$. By unicity, the action of $G$ on $Y_\eta$ induced by the map $G_{\mathcal{X}_\eta} \to \mathcal{G}_\eta$ extends to $Y$. There is a finite extension $K'/K$ such that the intersection points of the orbits of geometric irreducible components of $Y_\eta$ under the action of $G$ are all $K'$-rational. We may and do replace $S$ by the spectrum of the integral closure of $R$ in $K'$. Let us call $Y_1, \dots, Y_m$ the orbits of irreducible components of $Y$ and $\{y_{i,j}\}_{1\le i,j\le m}$ their intersections, which is a set of disjoint sections of $Y$. For each $i = 1, \dots, m$ we define a morphism $\pi_i : Y_i \to X_i$ as follows. If the action of $G$ on $Y_i$ is nontrivial we put $X_i := Y_i/G$ and $\pi_i$ equal to the quotient morphism. If the action of $G$ on $Y_i$ is trivial, note that we must have $\mathrm{char}(K) = p$, since the map from $Y_i$ to its image in $\mathcal{X}$ is a $\mathcal{G}$-torsor while $G_{\mathcal{X}} \to \mathcal{G}$ is an isomorphism in characteristic 0. Then we consider the Frobenius twist $X_i := Y_i^{(p)}$ and we define $\pi_i : Y_i \to X_i$ to be the relative Frobenius. Finally we let $X$ be the scheme obtained by glueing the $X_i$ along the sections $x_{i,j} = \pi_i(y_{i,j}) \in X_i$ and $x_{j,i} = \pi_j(y_{i,j}) \in X_j$. There are markings $\Sigma_i^X \subset X$ given by the closures in $X$ of the generic markings $\Sigma_i^{X_\eta}$. It is clear that the morphisms $\pi_i$ glue to a morphism $\pi : Y \to X$.

Extension of $\mathcal{X}_\eta$ and $Y_\eta \to \mathcal{X}_\eta$ along generic nodes and markings. In the following two lemmas we extend the stack structure of $\mathcal{X}_\eta$, and then the map $Y_\eta \to \mathcal{X}_\eta$, along the generic nodes and the markings:

**Lemma 3.2.** *There is a unique extension $\overline{\mathcal{X}}$ of the twisted curve $\mathcal{X}_\eta$ over $X$, such that $\overline{\mathcal{X}} \to X$ is an isomorphism away from the generic nodes and the markings.*

*Proof.* We follow [4, proof of proposition 4.3]. First, let $\Sigma_{i,\eta}^{\mathcal{X}_\eta}$ be a marking on $\mathcal{X}_\eta$ and let $P_{i,\eta} \subset Y_\eta$ be its preimage. There are extensions $P_i \subset Y$ and $\Sigma_i^X \subset X$. Let $r$ be the index of $\mathcal{X}_\eta$ at $\Sigma_{i,\eta}^{\mathcal{X}_\eta}$. Then we define $\mathcal{X}$ to be the stack of $r$-th roots of $\Sigma_i^X$ on $X$. This extension is unique by the separatedness of stacks of $r$-th roots.

Now let $x_\eta \in X_\eta$ be a node with index $r$ and let $x \in X_s$ be its reduction. Locally in the étale topology, around $x$ the curve $X$ looks like the spectrum of $R[u,v]/(uv)$. Let $B_u$ resp. $B_v$ be the branches at $x$ in $X$. The stacks of $r$-th roots of the divisor $u = 0$ in $B_u$ an of the divisor $v = 0$ in $B_v$ are isomorphic and glue to give a stack $\overline{\mathcal{X}}$. By definition of $r$ we have $\overline{\mathcal{X}}_\eta \simeq \mathcal{X}_\eta$. This extension is unique by the separatedness of stacks of $r$-th roots, so the construction of $\overline{\mathcal{X}}$ descends to $X$. ♠

**Lemma 3.3.** *There is a unique lifting $Y \to \overline{\mathcal{X}}$.*

*Proof.* We need to check that there is a lifting at any point $y \in Y_s$ which either lies on a marking or is the reduction of a generic node. We can apply the purity lemma [4, Lemma 4.4] provided that the local fundamental group of $Y$ at $y$ is trivial and the local Picard group of $Y$ at $y$ is torsion-free. In order to see this, we replace $R$ by its strict henselization and $Y$ by the spectrum of the strict henselization of the local ring at $y$. We let $U = Y \smallsetminus \{y\}$.

If $y$ lies on a marking then $Y$ is isomorphic to the spectrum of $R[a]^{\mathrm{sh}}$. Since this ring is local regular of dimension 2, the scheme $U$ has trivial fundamental group by the Zariski-Nagata purity theorem, and trivial Picard group by Auslander-Buchsbaum. Hence the purity lemma applies.

If $y$ is the reduction of a generic node, then $Y$ is isomorphic to the strict henselization of $R[a,b]/(ab)$. Let $B_a = \mathrm{Spec}(R[a]^{\mathrm{sh}})$ resp. $B_b = \mathrm{Spec}(R[b]^{\mathrm{sh}})$ be the branches at $y$ and $U_a = U \cap B_a$, $U_b = U \cap B_b$.

The schemes $U_a$ and $U_b$ have trivial fundamental group by Zariski-Nagata, and they intersect in $Y$ in a single point of the generic fibre. Moreover the map $U_a \sqcup U_b \to U$, being finite surjective and finitely presented, is of effective descent for finite étale coverings [9, Exp. IX, cor. 4.12]. It then follows from the Van Kampen theorem [9, Exp. IX, th. 5.1] that $\pi_1(U) = 1$.

For the computation of the local Picard group, first notice that since $B_a, B_b$ are local regular of dimension 2 we have $\mathrm{Pic}(U_a) = \mathrm{Pic}(U_b) = 0$, and moreover it is easy to see that $H^0(U_a, \mathcal{O}_{U_a}^\times) = R^\times$ and similarly for $U_b$. Now we consider the long exact sequence in cohomology associated to the short exact sequence

$$0 \to \mathcal{O}_U^\times \to i_{a,*}\mathcal{O}_{U_a}^\times \oplus i_{a,*}\mathcal{O}_{U_a}^\times \to i_{ab,*}\mathcal{O}_{U_{ab}}^\times \to 0$$

where the symbols $i_?$ stand for the obvious closed immersions. We obtain

$$\begin{aligned}\mathrm{Pic}(U) &= \mathrm{coker}\big(H^0(U_a, \mathcal{O}_{U_a}^\times) \oplus H^0(U_b, \mathcal{O}_{U_a}^\times) \to H^0(U_{ab}, \mathcal{O}_{U_{ab}}^\times)\big)\\ &= K^\times/R^\times = \mathbb{Z},\end{aligned}$$

which is torsion-free as desired. ♠

Note that we still need to introduce stack structure over special nodes of $\overline{\mathcal{X}}$.

Extension of $\mathcal{G}_\eta$ over generic points of $\overline{\mathcal{X}}_s$. Let $\xi$ be the generic point of a component of $\overline{\mathcal{X}}_s$. Let $U$ be the localization of $\overline{\mathcal{X}}$ at $\xi$ and $V$ be its inverse image in $Y$. Consider the closure $\mathcal{G}_\xi$ of $\mathcal{G}_\eta$ in $\mathrm{Aut}_U V$.

**Proposition 3.4.** *The scheme $\mathcal{G}_\xi \to U$ is a finite flat group scheme of degree $p$, and $V \to U$ is a $\mathcal{G}_\xi$-torsor.*

*Proof.* This is a generalization of [12, Proposition 1.2.1], see [13, Theorem 4.3.5]. ♠

Extension of $\mathcal{G}_\eta$ over the smooth locus of $\overline{\mathcal{X}}/S$. Quite generally, for a stable $p$-torsor $(\mathcal{X}, \mathcal{G}, Y)$ over a scheme $T$, by $\mathrm{Aut}_{\mathcal{X}} Y$ we denote the algebraic stack whose objects over an $T$-scheme $U$ are pairs $(u, f)$ with

$u \in \mathcal{X}(U)$ and $f$ a $U$-automorphism of $Y \times_{\mathcal{X}} U$. Now consider $\overline{\mathcal{X}}^{\mathrm{sm}}$, the smooth locus of $\overline{\mathcal{X}}/S$, and its inverse image $Y^{\mathrm{sm}}$ in $Y$. Then $Y^{\mathrm{sm}} \to \overline{\mathcal{X}}^{\mathrm{sm}}$ is flat. Let $\mathcal{G}^{\mathrm{sm}}$ be the closure of $\mathcal{G}_\eta$ in $\mathrm{Aut}_{\overline{\mathcal{X}}^{\mathrm{sm}}} Y^{\mathrm{sm}}$.

**Proposition 3.5.** *The scheme $\mathcal{G}^{\mathrm{sm}} \to \overline{\mathcal{X}}^{\mathrm{sm}}$ is a finite flat group scheme of degree $p$, and $Y^{\mathrm{sm}} \to \overline{\mathcal{X}}^{\mathrm{sm}}$ is a $\mathcal{G}^{\mathrm{sm}}$-torsor.*

*Proof.* Given Proposition 3.4, and since $\overline{\mathcal{X}}^{\mathrm{sm}}$ has local charts $U \to \overline{\mathcal{X}}^{\mathrm{sm}}$ with $U$ regular 2-dimensional, this follows from [2, Propositions 2.2.2 and 2.2.3]. ♠

Extension of $\mathcal{G}^{\mathrm{sm}}$ over generic nodes of $\mathcal{X}/S$. Consider the complement $\overline{\mathcal{X}}^0$ of the isolated nodes of $\overline{\mathcal{X}}_s$, and its inverse image $Y^0$ in $Y$.

**Lemma 3.6.** *The morphism $Y^0 \to \overline{\mathcal{X}}^0$ is flat.*

*Proof.* It is enough to verify the claim at the reduction $x_s$ of an arbitrary generic node $x_\eta \in X_\eta$. Since generic nodes remain distinct in reduction, it is enough to prove that $Y \to \mathcal{X}$ is flat at a chosen point $y_s \in Y$ above $x_s$. Since the branches at $y_s$ are not exchanged by $\mathcal{G}$, étale locally $Y$ and $\mathcal{X}$ are the union of two branches which are flat over $S$ and the restriction of $Y \to \mathcal{X}$ to each of the branches at $x_s$ is flat. Since proper morphisms descend flatness ([8], IV.11.5.3) it follows that $Y \to \mathcal{X}$ is flat at $y_s$. ♠

Let $\mathcal{G}^0$ be the closure of $\mathcal{G}^{\mathrm{sm}}$ in $\mathrm{Aut}_{\overline{\mathcal{X}}^0} Y^0$.

**Proposition 3.7.** *The stack $\mathcal{G}^0 \to \overline{\mathcal{X}}^0$ is a finite flat group scheme of degree $p$, and $Y^0 \to \overline{\mathcal{X}}^0$ is a $\mathcal{G}^0$ torsor.*

*Proof.* We only have to look around the closure of a generic node. Again since proper morphisms descend flatness, it is enough to prove the claim separately on the two branches. Then the result follows again from [2, Propositions 2.2.2 and 2.2.3] by the same reason as in the proof of 3.5. ♠

Twisted structure at special nodes. Let $P$ be a special node of $X$. By [2, Section 3.2] there is a canonical twisted structure $\mathcal{X}$ at $P$ determined by the local degree of $Y/X$. If near a given node $Y_\eta/X_\eta$ is inseparable, then this degree is $p$. Otherwise $Y/X$ has an action of $\mathbb{Z}/p\mathbb{Z}$ which is nontrivial near $P$, and therefore the local degree is either 1 or $p$. Then $\mathcal{X}$ is twisted with index $p$ at $P$ whenever this local degree is $p$. These twisted structures at the various nodes $P$ glue to give a twisted curve $\mathcal{X}$.

We claim that this $\mathcal{X}$ is unique up to a unique isomorphism. This follows from Proposition 2.3 below. Indeed, let $a$ be the node function which to a node $P$ of $X$ gives the local degree of $Y/X$ at $Y$, and let $r_i$ be the fixed indices at the sections. Then the stack of $a$-twisted curves over $X/S$ with markings of indices $r_i$ is proper over $S$, hence $\mathcal{X}$ is uniquely determined by $\mathcal{X}_\eta$ up to unique isomorphism.

By [2, Lemma 3.2.1], there is a unique lifting $Y \to \mathcal{X}$, and by [2, Theorem 3.2.2] the group scheme $\mathcal{G}^0$ extends uniquely to $\mathcal{G} \to \mathcal{X}$ such that $Y$ is a $\mathcal{G}$-torsor. The rigidification extends immediately by taking the closure, since $\mathcal{G} \to \mathcal{X}$ is finite. ♠

## 4. Examples

4.1. **First, some non-examples.** Consider a smooth projective curve $X$ of genus $h > 1$ in characteristic $p$ and and a $p$-torsion point in its Jacobian, corresponding to a $\mu_p$-torsor $Y' \to X$. This is *not* a stable $p$-torsor in the sense of Definition 1.5: the curve $Y'$ is necessarily singular. In fact, $Y' \to X$ may be described by a locally logarithmic differential form $\omega$ on $X$, such that if locally $\omega = df/f$ for some $f \in \mathcal{O}_X^\times$ then $Y'$ is given by an equation $z^p = f$. Since the genus $h > 1$, all differentials on $X$ have zeroes, and each zero of $\omega$ (i.e. a zero of the derivative of $f$ with respect to a coordinate) contributes to a unibranch singularity on $Y'$.

Now consider a ramified $\mathbb{Z}/p\mathbb{Z}$-cover $Y \to X$ of smooth projective curves over a field. Let $y \in Y$ be a fixed point for the action of $\mathbb{Z}/p\mathbb{Z}$ and let $x \in X$ be its image. In characteristic 0, since the stabilizer of $y$ is a multiplicative group, the curve $X$ may be twisted at $x$ to yield a stable $\mathbb{Z}/p\mathbb{Z}$-torsor $Y \to \mathcal{X}$. However in characteristic $p$ the stabilizer is additive and the result is not a $\mathbb{Z}/p\mathbb{Z}$-torsor. Hence ramified covers of smooth curves in characteristic $p$ do not provide stable $\mathbb{Z}/p\mathbb{Z}$-torsors.

However something else does occur in both examples: the torsor $Y' \to X$ of the first example, and the branched cover $Y \to X$ in the second, lift to characteristic 0. The reduction back to characteristic $p$ of the corresponding stable torsor "contains the original cover" in the following sense: there is a unique component $\mathcal{X}$ whose coarse moduli space is isomorphic to $X$. In particular that component $\mathcal{X}$ is necessarily a twisted curve, and the group scheme over it has to degenerate to $\alpha_p$ over the twisted points. We see a manifestation of this in the next example.

4.2. **Limit of a $p$-isogeny of elliptic curves.** Now consider the case where $X$ is an elliptic curve, with a marked point $x$, over a discrete valuation ring $R$ of characteristic 0 and residue characteristic $p$. For simplicity assume that $R$ contains $\mu_p$; let $\eta$ be the generic point of $\operatorname{Spec} R$ and $s$ the closed point of $\operatorname{Spec} R$. Given a $p$-torsion point on $X$ with non-trivial reduction, we obtain a corresponding nontrivial $\mu_p$-isogeny $Y' \to X$. Over the generic point $\eta$ we can make $Y'_\eta$ stable by marking the fiber $P_\eta$ over $x_\eta$. But note that the reduction of $P_\eta$ in $Y$ is not étale, hence something must modified. Since our stack is proper, a stable $p$-torsor $Y \to \mathcal{X}$ limiting $Y'_\eta \to X_\eta$ exists, at least over a base change of $R$. Here is how to describe it.

Consider the completed local ring $\mathcal{O}_{Y',O} \simeq R[[Z]]$ at the origin $O \in Y'_s$ and its spectrum $\mathbb{D}$. Then $\mathbb{D}_\eta$ is identified with an open $p$-adic disk modulo Galois action. Write $P_\eta = \{P_{\eta,1}, \ldots, P_{\eta,p}\}$ as a sum of points permuted by the $\mu_p$-action. Then the $P_{\eta,i}$ induce $K$-rational points of $\mathbb{D}_\eta$ which moreover are

$\pi$-adically equidistant, i.e. the valuation $v = v_\pi(P_{\eta,i} - P_{\eta,j})$ is independent of $i, j$. It follows that after blowing-up the closed subscheme with ideal $(\pi^v, Z)$ these points reduce to $p$ distinct points in the exceptional divisor. Thus after twisting at the node, the fiber $Y_s \to \mathcal{X}_s$ over the special point $s$ of $R$ is described as follows:

$$\begin{tikzcd} Y_s \arrow[r, equal] \arrow[d] & Y'_s \cup \mathbb{P}^1 \arrow[d] & P \arrow[l, hook'] \arrow[d] \\ X'_s \arrow[r, equal] & E \cup Q & \{0\} \arrow[l, hook'] \end{tikzcd}$$

Here

- $Y_s$ is a union of two components $Y'_s \cup \mathbb{P}^1$, attached at the origin of $Y'_s$.
- $\mathcal{X}_s$ is a twisted curve with two components $E \cup Q$
- Here $E = X_s(\sqrt[p]{x})$ and $Q = \mathbb{P}^1(\sqrt[p]{\infty})$, with the twisted points attached.
- The map $Y_s \to \mathcal{X}_s$ decomposes into $Y'_s \to E$ and $\mathbb{P}^1 \to Q$.
- $\mathbb{P}^1 \to Q$ is an Artin–Schreier cover ramified at $\infty$.
- The curve is marked by the inverse image of $0 \in Q$ in $\mathbb{P}^1$, which is a $\mathbb{Z}/p\mathbb{Z}$-torsor $P \subset \mathbb{P}^1$.
- The map $Y'_s \to E$ is a lift of $Y'_s \to X_s$.
- The group scheme $\mathcal{G} \to \mathcal{X}$ is generically étale on $Q$ and generically $\mu_p$ on $E$, but the fiber over the node is $\alpha_p$.

Notice that we can view $Y'_s \to E$, marked by the origin on $Y'_s$, as a twisted torsor as well, but this twisted torsor does not lift to characteristic 0 simply because the marked point on $Y'_s$ can not be lifted to an invariant divisor. This is an example of the phenomenon described at the end of Section 4.1 above.

A very similar picture occurs when the cover $Y'_\eta \to X_\eta$ degenerates to an $\alpha_p$-torsor. If, however, the reduction of the cover is a $\mathbb{Z}/p\mathbb{Z}$-torsor, then $Y' \to X$, marked by the fiber over the origin, is already stable and new components do not appear.

4.3. **The double cover of $\mathbb{P}^1$ branched over 4 points.** Consider an elliptic double cover $Y$ over $\mathbb{P}^1$ in characteristic 0 given by the equation $y^2 = x(x-1)(x-\lambda)$. Marked by the four branched points, it becomes a stable $\mu_2$-torsor over the twisted curve $Q = \mathbb{P}^1(\sqrt{0, 1, \infty, \lambda})$. What is its reduction in characteristic 2? We describe here one case, the others can be described in a similar way.

If the elliptic curve $Y$ has good ordinary reduction $E_s$, the picture is as follows: $Y_s$ has three components $\mathbb{P}^1 \cup E_s \cup \mathbb{P}^1$. The twisted curve $\mathcal{X}_s$ also has three rational components $Q_1 \cup Q_2 \cup Q_3$. The map splits as $\mathbb{P}^1 \to Q_1$, $E_s \to Q_2$ and $\mathbb{P}^1 \to Q_3$, where the first and last are generically $\mu_2$-covers, and $E_s \to Q_2$ is a lift of the hyperelliptic cover $E_s \to \mathbb{P}^1$. The fibers of $\mathcal{G}$ at

the nodes of $X_s$ are both $\alpha_2$. The points $0, 1, \infty, \lambda$ reduce to two pairs, one pair on each of the two $\mathbb{P}^1$ components, for instance:

$$\begin{tikzcd} & \mathbb{P}^1 \cup E_s \cup \mathbb{P}^1 \arrow[d] & \\ \{0,1\} \arrow[r, hook] & Q_1 \cup Q_2 \cup Q_3 & \{\lambda, \infty\}. \arrow[l, hook'] \end{tikzcd}$$

## Appendix A. Group schemes of order $p$

In this appendix, we give some complements on group schemes of order $p$. The main topic is the construction of an embedding of a given group scheme of order $p$ into an affine smooth one-dimensional group scheme (an analogue of Kummer or Artin-Schreier theory). Although not strictly necessary in the paper, this result highlights the nature of our stable torsors in two respects : firstly because the original definition of generators in [10, § 1.4] involves a smooth ambient group scheme, and secondly because the short exact sequence given by this embedding induces a long exact sequence in cohomology that may be useful for computations of torsors.

Anyway, let us now state the result.

**Definition A.1.** Let $G \to S$ be a finite locally free group scheme of order $p$.

(1) A *generator* is a morphism of $S$-group schemes $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$ such that the sections $x_i = \gamma(i)$, $0 \leq i \leq p-1$, are a full set of sections.
(2) A *cogenerator* is a morphism of $S$-group schemes $\kappa : G \to \mu_{p,S}$ such that the Cartier dual $(\mathbb{Z}/p\mathbb{Z})_S \to G^\vee$ is a generator.

We will prove the following.

**Theorem A.2.** *Let $S$ be a scheme and let $G \to S$ be a finite locally free group scheme of order $p$. Let $\kappa : G \to \mu_{p,S}$ be a cogenerator. Then $\kappa$ can be canonically inserted into a commutative diagram with exact rows*

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d, "\kappa"] & \mathscr{G} \arrow[r, "\varphi_\kappa"] \arrow[d] & \mathscr{G}' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mu_{p,S} \arrow[r] & \mathbb{G}_{m,S} \arrow[r, "p"] & \mathbb{G}_{m,S} \arrow[r] & 0 \end{tikzcd}$$

*where $\varphi_\kappa : \mathscr{G} \to \mathscr{G}'$ is an isogeny between affine smooth one-dimensional $S$-group schemes with geometrically connected fibres.*

In order to obtain this, we introduce two categories of invertible sheaves with sections: one related to groups with a cogenerator and one related to groups defined as kernels of isogenies, and we compare these categories.

**Remark A.3.** Not all group schemes of order $p$ can be embedded into an affine smooth group scheme as in the theorem. For example, assume that there exists a closed immersion from $G = (\mathbb{Z}/p\mathbb{Z})_{\mathbb{Q}}$ to some affine smooth

one-dimensional geometrically connected $\mathbb{Q}$-group scheme $\mathscr{G}$. Then $\mathscr{G}$ is a form of $\mathbb{G}_{m,\mathbb{Q}}$ and $G$ is its $p$-torsion subgroup. Since $\mathscr{G}$ is trivialized by a quadratic field extension $K/\mathbb{Q}$, we obtain $G_K \simeq \mu_{p,K}$. This implies that $K$ contains the $p$-th roots of unity, which is impossible for $p > 3$. Similar examples can be given for $\mathbb{Z}/p\mathbb{Z}$ over the Tate-Oort ring $\Lambda \otimes \mathbb{Q}$.

A.4. **Tate-Oort group schemes.** We recall the notations and results of the Tate-Oort classification of group schemes of degree $p$ over the ring $\Lambda$ (section 2 of [15]). We introduce two fibered categories:

- a $\Lambda$-category $TG$ of *triples* encoding *groups*,
- a $\Lambda$-category $TGC$ of *triples* encoding *groups with a cogenerator*.

Let $\chi : \mathbb{F}_p \to \mathbb{Z}_p$ be the unique multiplicative section of the reduction map, that is $\chi(0) = 0$ and if $m \in \mathbb{F}_p^\times$ then $\chi(m)$ is the $(p-1)$-st root of unity with residue equal to $m$. Set

$$\Lambda = \mathbb{Z}[\chi(\mathbb{F}_p), \frac{1}{p(p-1)}] \cap \mathbb{Z}_p.$$

There is in $\Lambda$ a particular element $w_p$ equal to $p$ times a unit.

**Definition A.5.** The category $TG$ is the category fibered over $\operatorname{Spec}\Lambda$ whose fiber categories over a $\Lambda$-scheme $S$ are as follows.

- Objects are the triples $(L, a, b)$ where $L$ is an invertible sheaf and $a \in \Gamma(S, L^{\otimes(p-1)})$, $b \in \Gamma(S, L^{\otimes(1-p)})$ satisfy $a \otimes b = w_p 1_{\mathcal{O}_S}$.
- Morphisms between $(L, a, b)$ and $(L', a', b')$ are the morphisms of invertible sheaves $f : L \to L'$, viewed as global sections of $L^{\otimes -1} \otimes L'$, such that $a \otimes f^{\otimes p} = f \otimes a'$ and $b' \otimes f^{\otimes p} = f \otimes b$.

The main result of [15] is an explicit description of a covariant equivalence of fibered categories between $TG$ and the category of finite locally free group schemes of order $p$. The group scheme associated to a triple $(L, a, b)$ is denoted $G^L_{a,b}$. Its Cartier dual is isomorphic to $G^{L^{-1}}_{b,a}$.

**Examples A.6.** We have $(\mathbb{Z}/p\mathbb{Z})_S = G^{\mathcal{O}_S}_{1,w_p}$ and $\mu_{p,S} = G^{\mathcal{O}_S}_{w_p,1}$. Moreover if $G = G^L_{a,b}$ then a morphism $(\mathbb{Z}/p\mathbb{Z})_S \to G$ is given by a global section $u \in \Gamma(S, L)$ such that $u^{\otimes p} = u \otimes a$ and a morphism $G \to \mu_{p,S}$ is given by a global section $v \in \Gamma(S, L^{-1})$ such that $v^{\otimes p} = v \otimes b$.

**Lemma A.7.** *Let $S$ be a $\Lambda$-scheme and let $G = G^L_{a,b}$ be a finite locally free group scheme of rank $p$ over $S$. Then:*

*(1) Let $\gamma : (\mathbb{Z}/p\mathbb{Z})_S \to G$ be a morphism of $S$-group schemes given by a section $u \in \Gamma(S, L)$ such that $u^{\otimes p} = u \otimes a$. Then $\gamma$ is a generator if and only if $u^{\otimes(p-1)} = a$.*

*(2) Let $\kappa : G \to \mu_{p,S}$ be a morphism of $S$-group schemes given by a section $v \in \Gamma(S, L^{-1})$ such that $v^{\otimes p} = v \otimes b$. Then $\kappa$ is a cogenerator if and only if $v^{\otimes(p-1)} = b$.*

*Proof.* The proof of (2) follows from (1) by Cartier duality so we only deal with (1). The claim is local on $S$ so we may assume that $S$ is affine equal to $\operatorname{Spec}(R)$ and $L$ is trivial. It follows from [15] that $G = \operatorname{Spec} R[x]/(x^p - ax)$ and the section $\gamma(i) : \operatorname{Spec}(R) \xrightarrow{i} (\mathbb{Z}/p\mathbb{Z})_R \to G$ is given by the morphism of algebras $R[x]/(x^p - ax) \to R$, $x \mapsto \chi(i)u$. Thus $\gamma$ is a generator if and only if $\operatorname{Norm}(f) = \prod f(\chi(i)u)$ for all functions $f = f(x)$. In particular for $f = 1+x$ one finds $\operatorname{Norm}(f) = (-1)^p a+1$ and $\prod(1+\chi(i)u) = (-1)^p u^{p-1}+1$. Therefore if $\gamma$ is a generator then $u^{p-1} = a$. Conversely, assuming that $u^{p-1} = a$ we want to prove that $\operatorname{Norm}(f) = \prod f(\chi(i)u)$ for all $f$. It is enough to prove this in the universal case where $R = \Lambda[a, b, u]/(ab - w_p, u^p - u)$. Since $a$ is not a zerodivisor in $R$, it is in turn enough to prove the equality after base change to $K = R[1/a]$. Then $G_K$ is étale and the morphism

$$K[x]/(x^p - ax) = K[x]/\prod(x - \chi(i)u) \to K^p$$

taking $f$ to the tuple $(f(\chi(i)u))_{0\le i\le p-1}$ is an isomorphism of algebras. Since the norm in $K^p$ is the product of the coordinates, the result follows. ♠

**Definition A.8.** The category $TGC$ is the category fibered over $\operatorname{Spec}\Lambda$ whose fibers over a $\Lambda$-scheme $S$ are as follows.

- Objects are the triples $(L, a, v)$ where $L$ is an invertible sheaf and $a \in \Gamma(S, L^{\otimes(p-1)})$, $v \in \Gamma(S, L^{\otimes -1})$ satisfy $a \otimes v^{\otimes(p-1)} = w_p 1_{\mathcal{O}_S}$.
- Morphisms between $(L, a, v)$ and $(L', a', v')$ are the morphisms of invertible sheaves $f : L \to L'$, viewed as global sections of $L^{\otimes -1} \otimes L'$, such that $a \otimes f^{\otimes p} = f \otimes a'$ and $v' \otimes f = v$.

By lemma A.7, the category $TGC$ is equivalent to the category of group schemes with a cogenerator. The functor from group schemes with a cogenerator to group schemes that forgets the cogenerator is described in terms of categories of invertible sheaves by the functor $\omega : TGC \to TG$ given by $\omega(L, a, v) = (L, a, v^{\otimes(p-1)})$.

Note also that lemma A.7 tells us that for any locally free group scheme $G$ over a $\Lambda$-scheme $S$, there exists a finite locally free morphism $S' \to S$ of degree $p - 1$ such that $G \times_S S'$ admits a generator or a cogenerator.

A.9. **Congruence group schemes.** Here, we introduce and describe a $\mathbb{Z}$-category $TCG$ of *triples* encoding *congruence groups*.

Let $R$ be ring with a discrete valuation $v$ and let $\lambda \in R$ be such that $(p - 1)v(\lambda) \le v(p)$. In [14] are introduced some group schemes $H_\lambda = \operatorname{Spec} R[x]/(((1+\lambda x)^p - 1)/\lambda^p)$ with multiplication $x_1 \star x_2 = x_1 + x_2 + \lambda x_1 x_2$. (The notation in *loc. cit.* is $\mathscr{N}$.) Later Raynaud called them *congruence groups of level* $\lambda$ and we will follow his terminology. We now define the analogues of these group schemes over a general base. The objects that are the input of the construction constitute the following category.

**Definition A.10.** The category $TCG$ is the category fibered over $\operatorname{Spec}\mathbb{Z}$ whose fibers over a scheme $S$ are as follows.

- Objects are the triples $(M, \lambda, \mu)$ where $M$ is an invertible sheaf over $S$ and the global sections $\lambda \in \Gamma(S, M^{-1})$ and $\mu \in \Gamma(S, M^{p-1})$ are subject to the condition $\lambda^{\otimes(p-1)} \otimes \mu = p1_{\mathcal{O}_S}$.
- Morphisms between $(M, \lambda, \mu)$ and $(M', \lambda', \mu')$ are morphisms of invertible sheaves $f : M \to M'$ viewed as sections of $M^{-1} \otimes M'$ such that $f \otimes \lambda' = \lambda$ and $f^{\otimes(p-1)} \otimes \mu = \mu'$.

We will exhibit a functor $(M, \lambda, \mu) \rightsquigarrow H^M_{\lambda,\mu}$ from $TCG$ to the category of group schemes, with $H^M_{\lambda,\mu}$ defined as the kernel of a suitable isogeny.

First, starting from $(M, \lambda)$ we construct a smooth affine one-dimensional group scheme denoted $\mathscr{G}^{(M,\lambda)}$, or simply $\mathscr{G}^{(\lambda)}$. We see $\lambda$ as a morphism $\lambda : \mathbb{V}(M) \to \mathbb{G}_{a,S}$ of (geometric) line bundles over $S$, where $\mathbb{V}(M) = \operatorname{Spec}\operatorname{Sym}(M^{-1})$ is the (geometric) line bundle associated to $M$. We define $\mathscr{G}^{(\lambda)}$ as a scheme by the fibered product

$$\begin{tikzcd} \mathscr{G}^{(\lambda)} \arrow[r, "1+\lambda"] \arrow[d] & \mathbb{G}_{m,S} \arrow[d] \\ \mathbb{V}(M) \arrow[r, "1+\lambda"] & \mathbb{G}_{a,S} \end{tikzcd} .$$

The points of $\mathscr{G}^{(\lambda)}$ with values in an $S$-scheme $T$ are the global sections $u \in \Gamma(T, M \otimes \mathcal{O}_T)$ such that $1 + \lambda \otimes u$ is invertible. We endow $\mathscr{G}^{(\lambda)}$ with a multiplication given on the $T$-points by

$$u_1 \star u_2 = u_1 + u_2 + \lambda \otimes u_1 \otimes u_2 \ .$$

The zero section of $\mathbb{V}(M)$ sits in $\mathscr{G}^{(\lambda)}$ and is the unit section for the law just defined. The formula

$$(1 + \lambda \otimes u_1)(1 + \lambda \otimes u_2) = 1 + \lambda \otimes (u_1 \star u_2)$$

shows that $1+\lambda : \mathscr{G}^{(\lambda)} \to \mathbb{G}_{m,S}$ is a morphism of group schemes. Moreover, if the locus where $\lambda : \mathbb{V}(M) \to \mathbb{G}_{a,S}$ is an isomorphism is scheme-theoretically dense, then $\star$ is the unique group law on $\mathscr{G}^{(\lambda)}$ for which this holds. This construction is functorial in $(M, \lambda)$: given a morphism of invertible sheaves $f : M \to M'$, in other words a global section of $M^{-1} \otimes M'$, such that $f \otimes \lambda' = \lambda$, there is a morphism $f : \mathscr{G}^{(\lambda)} \to \mathscr{G}^{(\lambda')}$ making the diagram

$$\begin{tikzcd} \mathscr{G}^{(\lambda)} \arrow[r, "1+\lambda"] \arrow[d, "f"'] & \mathbb{G}_{m,S} \\ \mathscr{G}^{(\lambda')} \arrow[ur, "1+\lambda'"'] & \end{tikzcd}$$

commutative. The notation is coherent since that morphism is indeed induced by the extension of $f$ to the sheaves of symmetric algebras.

Then, we use the section $\mu \in \Gamma(S, M^{p-1})$ and the relation $\lambda^{\otimes(p-1)} \otimes \mu = p1_{\mathcal{O}_S}$ to define an isogeny $\varphi$ fitting into a commutative diagram

$$\begin{tikzcd} \mathscr{G}^{(\lambda)} \arrow[r, "\varphi"] \arrow[d, "1+\lambda"'] & \mathscr{G}^{(\lambda^{\otimes p})} \arrow[d, "1+\lambda^{\otimes p}"] \\ \mathbb{G}_{m,S} \arrow[r, "\wedge p"] & \mathbb{G}_{m,S} \end{tikzcd} .$$

The formula for $\varphi$ is given on the $T$-points $u \in \Gamma(T, M \otimes \mathcal{O}_T)$ by

$$\varphi(u) = u^{\otimes p} + \sum_{i=1}^{p-1} \{{p \atop i}\}\, \lambda^{\otimes(i-1)} \otimes \mu \otimes u^{\otimes i}$$

where $\{{p \atop i}\} = \frac{1}{p}\binom{p}{i}$ is the binomial coefficient divided by $p$. In order to check that the diagram is commutative and that $\varphi$ is an isogeny, we may work locally on $S$ hence we may assume that $S$ is affine and that $M = \mathcal{O}_S$. In this case, the two claims follow from the universal case i.e. from points (1) and (2) in the following lemma.

**Lemma A.11.** *Let $\mathcal{O} = \mathbb{Z}[E, F]/(E^{p-1}F - p)$ and let $\lambda, \mu \in \mathcal{O}$ be the images of the indeterminates $E, F$. Then, the polynomial*

$$P(X) = X^p + \sum_{i=1}^{p-1} \{{p \atop i}\}\, \lambda^{i-1}\mu X^i \ \in \mathcal{O}[X]$$

*satisfies:*

*(1) $1 + \lambda^p P(X) = (1 + \lambda X)^p$, and*
*(2) $P(X + Y + \lambda XY) = P(X) + P(Y) + \lambda^p P(X)P(Y)$.*

*Proof.* Point (1) follows by expanding $(1 + \lambda X)^p$ and using the fact that $p = \lambda^{p-1}\mu$ in $\mathcal{O}$. Then we compute:

$$\begin{aligned} 1 + \lambda^p P(X + Y + \lambda XY) &= (1 + \lambda(X + Y + \lambda XY))^p \\ &= (1 + \lambda X)^p (1 + \lambda Y)^p \\ &= (1 + \lambda^p P(X))(1 + \lambda^p P(Y)) \\ &= 1 + \lambda^p (P(X) + P(Y) + \lambda^p P(X)P(Y)). \end{aligned}$$

Since $\lambda$ is a nonzerodivisor in $\mathcal{O}$, point (2) follows. ♠

**Definition A.12.** We denote by $H^M_{\lambda,\mu}$ the kernel of $\varphi$, and call it the *congruence group scheme* associated to $(M, \lambda, \mu)$.

This construction is functorial in $(M, \lambda, \mu)$. Precisely, consider two triples $(M, \lambda, \mu)$ and $(M', \lambda', \mu')$ and a morphism of invertible sheaves $f : M \to M'$ viewed as a section of $M^{-1} \otimes M'$ such that $f \otimes \lambda' = \lambda$ and $f^{\otimes(p-1)} \otimes \mu = \mu'$. Then we have morphisms $f : \mathscr{G}^{(\lambda)} \to \mathscr{G}^{(\lambda')}$ and $f^{\otimes p} : \mathscr{G}^{(\lambda^{\otimes p})} \to \mathscr{G}^{(\lambda'^{\otimes p})}$ compatible with the isogenies $\varphi$ and $\varphi'$, and $f$ induces a morphism $H^M_{\lambda,\mu} \to$

$H^{M'}_{\lambda',\mu'}$. Note also that the image of $H^M_{\lambda,\mu}$ under $1+\lambda : \mathscr{G}^{(\lambda)} \to \mathbb{G}_{m,S}$ factors through $\mu_{p,S}$, so that by construction $H^M_{\lambda,\mu}$ comes embedded into a diagram

$$\begin{tikzcd} 0 \arrow[r] & H^M_{\lambda,\mu} \arrow[r] \arrow[d, "\kappa"] & \mathscr{G}^{(\lambda)} \arrow[r] \arrow[d, "1+\lambda"] & \mathscr{G}^{(\lambda^{\otimes p})} \arrow[r] \arrow[d, "1+\lambda^{\otimes p}"] & 0 \\ 0 \arrow[r] & \mu_{p,S} \arrow[r] & \mathbb{G}_{m,S} \arrow[r] & \mathbb{G}_{m,S} \arrow[r] & 0. \end{tikzcd}$$

The formation of this diagram is also functorial.

**Lemma A.13.** *The morphism $\kappa : H^M_{\lambda,\mu} \to \mu_{p,S}$ is a cogenerator.*

*Proof.* We have to show that the dual map $(\mathbb{Z}/p\mathbb{Z})_S \to (H^M_{\lambda,\mu})^\vee$ is a generator. This means verifying locally on $S$ certain equalities of norms. Hence we may assume that $S$ is affine and that $M$ is trivial, then reduce to the universal case where $S$ is the spectrum of the ring $\mathcal{O}$ with elements $\lambda, \mu$ satisfying $\lambda^{p-1}\mu = p$ as in lemma A.11, and finally restrict to the schematically dense open subscheme $S' = D(\lambda) \subset S$. Since $\mathscr{G}^{(\lambda)} \times_S S' \to \mathbb{G}_{m,S'}$ is an isomorphism, then $H^M_{\lambda,\mu} \times_S S' \to \mu_{p,S'}$ and the dual morphism also are. The claim follows immediately. ♠

A.14. **Equivalence between $TGC$ and $TCG \otimes_{\mathbb{Z}} \Lambda$.** The results of the previous subsection imply that for a $\Lambda$-scheme $S$, a triple $(M, \lambda, \mu) \in TCG(S)$ gives rise in a functorial way to a finite locally free group scheme with cogenerator $\kappa : H^M_{\lambda,\mu} \to \mu_{p,S}$, that is, an object of $TGC(S)$.

**Theorem A.15.** *The functor*

$$F : TCG \otimes_{\mathbb{Z}} \Lambda \to TGC$$

*defined above is an equivalence of fibered categories over $\Lambda$. If $(M, \lambda, \mu)$ has image $(L, a, v)$ then $H^M_{\lambda,\mu} \simeq G^L_{a,v^{\otimes(p-1)}}$.*

*Proof.* The main point is to describe $F$ in detail using the Tate-Oort classification, and to see that it is essentially surjective. The description of the action of $F$ on morphisms and the verification that it is fully faithful offers no difficulty and will be omitted.

Let $(M, \lambda, \mu)$ be a triple in $TCG(S)$ and let $G = H^M_{\lambda,\mu}$. We use the notations of section 2 of [15], in particular the structure of the group $\mu_p$ is described by a function $z$, the sheaf of $\chi$-eigensections $J = y\mathcal{O}_S \subset \mathcal{O}_{\mu_p}$ with distinguished generator $y = (p-1)e_1(1-z)$, and constants

$$w_1 = 1, w_2, \ldots, w_{p-1}, w_p = pw_{p-1} \in \Lambda.$$

The augmentation ideal of the algebra $\mathcal{O}_G$ is the sheaf $I$ generated by $M^{-1}$, and by [15] the subsheaf of $\chi$-eigensections is the sheaf $I_1 = e_1(I)$ where $e_1$ is the $\mathcal{O}_S$-linear map defined in [15]. It is an invertible sheaf and $L$ is (by definition) its inverse.

We claim that in fact $I_1 = e_1(M^{-1})$. In order to see this, we may work locally. Let $x$ be a local generator for $M^{-1}$ and let

$$t := (p-1)e_1(-x) \in I_1.$$

Let us write $\lambda = \lambda_0 x$ for some local function $\lambda_0$. We first prove that

$$(\star) \qquad x = \frac{1}{1-p}\big(t + \frac{\lambda_0 t^2}{w_2} + \cdots + \frac{\lambda_0^{p-2} t^{p-1}}{w_{p-1}}\big).$$

In fact, by construction the map $\mathcal{O}_{\mu_p} \to \mathcal{O}_G$ is given by $z = 1 + \lambda_0 x$, so we get $y = (p-1)e_1(1-z) = \lambda_0 t$. In order to check the expression for $x$ in terms of $t$, we can reduce to the universal case (lemma A.11). Then $\lambda_0$ is not a zerodivisor and we can harmlessly multiply both sides by $\lambda_0$. In this form, the equality to be proven is nothing else than the identity (16) in [15]. Now write $t = \alpha t^*$ with $t^*$ a local generator for $I_1$ and $\alpha$ a local function. Using $(\star)$ we find that $x = \alpha x^*$ for some $x^* \in \mathcal{O}_G$. Since $x$ generates $M^{-1}$ in the fibres over $S$, this proves that $\alpha$ is invertible. Finally $t$ is a local generator for $I_1$ and this finishes the proof that $I_1 = e_1(M^{-1})$.

Let $x^\vee$ be the local generator for $M$ dual to $x$ and write $\mu = \mu_0 (x^\vee)^{\otimes(p-1)}$ for some local function $\mu_0$ such that $(\lambda_0)^{p-1}\mu_0 = p$. Let $t^\vee$ be the local generator for $L$ dual to $t$. We define a local section $a$ of $L^{\otimes(p-1)}$ by

$$a = w_{p-1}\mu_0 (t^\vee)^{\otimes(p-1)}$$

and a local section $v$ of $L^{-1}$ by

$$v = \lambda_0 t.$$

These sections are independent of the choice of the local generator $x$, because if $x' = \alpha x$ then

$$(x')^\vee = \alpha^{-1}x^\vee \ ; \ t' = \alpha t \ ; \ (t')^\vee = \alpha^{-1}t^\vee \ ; \ \lambda_0' = \alpha^{-1}\lambda_0 \ ; \ \mu_0' = \alpha^{p-1}\mu_0$$

so that

$$a' = w_{p-1}\mu_0'(t'^\vee)^{\otimes(p-1)} = w_{p-1}\alpha^{p-1}\mu_0\alpha^{1-p}(t^\vee)^{\otimes(p-1)} = a$$

and

$$v' = \lambda_0' t' = \alpha^{-1}\lambda_0\alpha t = v.$$

They glue to global sections $a$ and $v$ satisfying

$$a \otimes v^{\otimes(p-1)} = w_p 1_{\mathcal{O}_S}.$$

Let us prove that $a$ and $v$ are indeed the sections defining $G$ and the cogenerator in the Tate-Oort classification. The verification for $a$ amounts to checking that the relation

$$t^p = w_{p-1}\mu_0 t$$

holds in the algebra $\mathcal{O}_G$. This may be seen in the universal case where $\lambda_0$ is not a zerodivisor, hence after multiplying by $(\lambda_0)^p$ this follows from the equality $y^p = w_p y$ from [15]. The verification for $v$ amounts to noting that the cogenerator $G \to \mu_{p,S}$ is indeed given by $y \mapsto v$.

This completes the description of $F$ on objects. Finally we prove that $F$ is essentially surjective. Assume given $(L, a, v)$ and let $t$ be a local generator for $I_1 = L^{-1}$. Write $a = w_{p-1}\mu_0(t^\vee)^{\otimes(p-1)}$, $v = \lambda_0 t$ and define an element $x \in \mathcal{O}_G$ by the expresion $(\star)$ above. If we change the generator $t$ to another $t' = \alpha t$, then $\lambda_0' = \alpha^{-1}\lambda_0$ and $x' = \alpha x$. It follows that the subsheaf of $\mathcal{O}_G$ generated by $x$ does not depend on the choice of the generator for $I_1$, call it $N$. Reducing to the universal case as before, we prove that $t = (p-1)e_1(-x)$. This shows that in fact $N$ is an invertible sheaf and we take $M$ to be its inverse. Finally we define sections $\lambda \in \Gamma(S, M^{-1})$ and $\mu \in \Gamma(S, M^{\otimes(p-1)})$ by the local expressions $\lambda = \lambda_0 x$ and $\mu = \mu_0(x^\vee)^{\otimes(p-1)}$. It is verified like in the case of $a, v$ before that they do not depend on the choice of $t$ and hence are well-defined global sections. The equality $\lambda^{\otimes(p-1)} \otimes \mu = p1_{\mathcal{O}_S}$ holds true and the proof is now complete. ♠

A.16. **Proof of theorem A.2.** We are now in a position to prove theorem A.2. We keep its notations. Since the construction of the isogeny $\varphi_\kappa$ and the whole commutative diagram is canonical, if we perform it after fppf base change $S' \to S$ then it will descend to $S$. We choose $S' = S_1 \amalg S_2$ where $S_1 = S \otimes_{\mathbb{Z}} \mathbb{Z}[1/p]$ and $S_2 = S \otimes_{\mathbb{Z}} \Lambda$. Over $S_1$ the group scheme $G$ is étale and the cogenerator is an isomorphism by [10, lemma 1.8.3]. We take $\mathscr{G} = \mathscr{G}' = \mathbb{G}_{m,S}$ and $\varphi_\kappa$ is the $p$-th power map. Over $S_2$ we use theorem A.15 which provides a canonical isomorphism between $\kappa$ and $H^M_{\lambda,\mu}$ with its canonical cogenerator, embedded into a diagram of the desired form. This completes the proof.

## Appendix B. Weil restriction of closed subschemes

Let $Z \to X$ be a morphism of $S$-schemes (or algebraic spaces) and denote by $h : X \to S$ the structure map. The Weil restriction $h_*Z$ of $Z$ along $h$ is the functor on $S$-schemes defined by $(h_*Z)(T) = Hom_X(X \times_S T, Z)$. It may be seen as a left adjoint to the pullback along $h$, or as the functor of sections of $Z \to X$.

If $Z \to X$ is a closed immersion of schemes (or algebraic spaces) of finite presentation over $S$, there are two main cases where $h_*Z$ is known to be representable by a closed subscheme of $S$. As is well-known, this has applications to representability of various equalizers, kernels, centralizers, normalizers, etc. These two cases are :

(i) if $X \to S$ is proper flat and $Z \to S$ is separated, by the Grothendieck-Artin theory of the Hilbert scheme,

(ii) if $X \to S$ is essentially free, by [7, Exp. VIII, th. 6.4].

In this appendix, we want to prove that $h_*Z$ is representable by a closed subscheme of $S$ in a case that includes both situations and is often easier to check in practice, namely the case where $X \to S$ is flat and pure.

B.1. **Essentially free and pure morphisms.** We recall the notions of essentially free and pure morphisms and check that essentially free morphisms and proper morphisms are pure.

In [7, Exp. VIII, section 6], a morphism $X \to S$ is called *essentially free* if and only if there exists a covering of $S$ by open affine subschemes $S_i$, and for each $i$ an affine faithfullay flat morphism $S'_i \to S_i$ and a covering of $X'_i = X \times_S S'_i$ by open affine subschemes $X'_{i,j}$ such that the function ring of $X'_{i,j}$ is free as a module over the function ring of $S'_i$.

In fact, the proof of theorem 6.4 in [7, Exp. VIII] works just as well with a slightly weaker notion than freeness of modules. Namely, for a module $M$ over a ring $A$, let us say that $M$ is *good* if the canonical map $M \to M^{\vee\vee}$ from $M$ to its linear bidual is injective after any change of base ring $A \to A'$. It is a simple exercise to see that this is equivalent to $M$ being a submodule of a product module $A^I$ for some set $I$, over $A$ and after any base change $A \to A'$. For instance, free modules, projective modules, product modules are good. This gives rise to a notion of *essentially good* morphism, and in particular *essentially projective* morphism. Then inspection of the proof of theorem 6.4 of [7, Exp. VIII] shows that it remains valid for these morphisms.

In [11, 3.3.3], a morphism locally of finite type $X \to S$ is called *pure* if and only if for all points $s \in S$, with henselization $(\tilde{S}, \tilde{s})$, and all points $\tilde{x} \in \tilde{X}$ where $\tilde{X} = X \times_S \tilde{S}$, if $\tilde{x}$ is an associated point in its fibre then its closure in $\tilde{X}$ meets the special fibre. Examples of pure morphisms include proper morphisms (by the valuative criterion for properness) and morphisms locally of finite type and flat, with geometrically irreducible fibres without embedded components ([11, 3.3.4]).

Finally if $X \to S$ is locally of finite presentation and essentially free, then it is pure. Indeed, with the notations above for an essentially free morphism, one sees using [11, 3.3.7] that it is enough to see that for each $i, j$ the scheme $X'_{i,j}$ is pure over $S'_i$. But since the function ring of $X'_{i,j}$ is free over the function ring of $S'_i$, this follows from [11, 3.3.5].

B.2. **Representability of $h_*Z$.**

**Proposition B.3.** *Let $h : X \to S$ be a morphism of finite presentation, flat and pure, and let $Z \to X$ be a closed immersion. Then the Weil restriction $h_*Z$ is representable by a closed subscheme of $S$.*

*Proof.* The question is local for the étale topology on $S$. Let $s \in S$ be a point and let $\mathcal{O}^h$ be the henselization of the local ring at $s$. By [11, 3.3.13], for each $x \in X$ lying over $s$, there exists an open affine subscheme $U^h_x$ of $X \times_S \operatorname{Spec}(\mathcal{O}^h)$ containing $x$ and whose function ring is free as an $\mathcal{O}^h$-module. Since $X_s$ is quasi-compact, there is a finite number of points $x_1, \ldots, x_n$ such that the open affines $U^h_i = U^h_{x_i}$ cover it. Since $X$ is locally of finite presentation, after restricting to an étale neighbourhood $S' \to S$ of $s$, there exist affine open subschemes $U_i$ of $X$ inducing the $U^h_i$. According to [11, 3.3.8], the locus of the base scheme $S$ where $U_i \to S$ is pure is open, so

after shrinking $S$ we may assume that for each $i$ the affine $U_i$ is flat and pure. This means that its function ring is projective by [11, 3.3.5]. In other words, the union $U = U_1 \cup \cdots \cup U_n$ is essentially projective over $S$ in the terms of the comments in B.1. If $k : U \to X$ denotes the structure map, it follows from theorem 6.4 of [7, Exp. VIII] that $k_*(Z \cap U)$ is representable by a closed subscheme of $S$. On the other hand, according to [13, 3.1.7], replacing $S$ again by a smaller neighbourhood of $s$, the open immersion $U \to X$ is $S$-universally schematically dense. One deduces immediately that the natural morphism $h_*Z \to k_*(Z \cap U)$ is an isomorphism. This finishes the proof. ♠

This proposition has a long list of corollaries and applications listed in [7, Exp. VIII, section 6]. In particular let us mention the following :

**Corollary B.4.** *Let $X \to S$ be a morphism of finite presentation, flat and pure and $Y \to S$ a separated morphism. Consider two morphisms $f, g : X \to Y$. Then the condition $f = g$ is represented by a closed subscheme of $S$.*

*Proof.* Apply the previous proposition to the pullback of the diagonal of $Y$ along $(f, g) : X \to Y \times_S Y$. ♠

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