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RADON TRANSFORM ON SPHERES AND GENERALIZED BESSEL FUNCTION ASSOCIATED WITH DIHEDRAL GROUPS

N. Demni¹

ABSTRACT. Motivated by Dunkl operators theory, we consider a generating series involving a modified Bessel function and a Gegenbauer polynomial, that generalizes a known series already considered by L. Gegenbauer. We actually use inversion formulas for Fourier and Radon transforms to derive a closed formula for this series when the parameter of the Gegenbauer polynomial is a strictly positive integer. As a by-product, we get a relatively simple integral representation for the generalized Bessel function associated with even dihedral groups $D_2(2p)$, $p \geq 1$ when both multiplicities sum to an integer. In particular, we recover a previous result obtained for $D_2(4)$ and we give a special interest to $D_2(6)$. The paper is closed with adapting our method to odd dihedral groups thereby exhausting the list of Weyl dihedral groups.

1. INTRODUCTION

The dihedral group $D_2(n)$ of order $n \geq 2$ is defined as the group of regular n -gone preserving-symmetries ([8]). It figures among reflections groups associated with root systems for which a spherical harmonics theory, generalizing the one of Harish-Chandra on semisimple Lie groups from a discrete to a continuous range of multiplicities, was introduced by C. F. Dunkl in the late eighties (see Ch.I in [3]). Since then, a huge amount of research papers on this new topic and on its stochastic side as well emerged yielding fascinating results (Ch. II, III in [3]). For instance, probabilistic considerations allowed the author to derive the so-called generalized Bessel function associated with dihedral groups ([4]). For even values $n = 2p$, $p \geq 1$, this function depending on two real variables, say $(x, y) \in \mathbb{R}^2$, is expressed in polar coordinates $x = \rho e^{i\phi}$, $y = r e^{i\theta}$, $\rho, r \geq 0$, $\phi, \theta \in [0, \pi/2p]$ as

$$(1) \quad D_k^W(\rho, \phi, r, \theta) = c_{p,k} \left(\frac{2}{r\rho} \right)^\gamma \sum_{j \geq 0} I_{2jp+\gamma}(\rho r) p_j^{l_1, l_0}(\cos(2p\phi)) p_j^{l_1, l_0}(\cos(2p\theta))$$

where

- $k = (k_0, k_1)$ is a positive-valued multiplicity function, $l_i = k_i - 1/2$, $i \in \{1, 2\}$, $\gamma = p(k_0 + k_1)$.
- $I_{2jp+\gamma}$, $p_j^{l_1, l_0}$ are the modified Bessel function of index $2jp + \gamma$ and the j -th orthonormal Jacobi polynomial of parameters l_1, l_0 respectively (the orthogonality (Beta) measure need not to be normalized here. In fact, the normalization only alters the constant $c_{p,k}$ below).

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- The constant $c_{p,k}$ depends on p, k and is such that $D_k^W(0, y) = 1$ for all $y = (r, \theta) \in [0, \infty) \times [0, \pi/2p]$ (see [5])

$$c_{p,k} = 2^{k_0+k_1} \frac{\Gamma(p(k_1+k_0)+1)\Gamma(k_1+1/2)\Gamma(k_0+1/2)}{\Gamma(k_0+k_1+1)}.$$

In a subsequent paper ([5]), the special case $p = 2$ corresponding to the group of square-preserving symmetries was considered. The main ingredient used there was the famous Dijksma-Koornwinder's product formula for Jacobi polynomials ([7]) which may be written in the following way ([5]):

$$c(\alpha, \beta) p_j^{\alpha, \beta}(\cos 2\phi) p_j^{\alpha, \beta}(\cos 2\theta) = (2j + \alpha + \beta + 1) \int \int C_{2j}^{\alpha+\beta+1}(z_{\phi, \theta}(u, v)) \mu^\alpha(du) \mu^\beta(dv)$$

where $\alpha, \beta > -1/2$,

$$c(\alpha, \beta) = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},$$

$$z_{\phi, \theta}(u, v) = u \cos \theta \cos \phi + v \sin \theta \sin \phi,$$

and μ^α is the symmetric Beta probability measure whose density is given by

$$\mu^\alpha(du) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} (1-u^2)^{\alpha-1/2} \mathbf{1}_{[-1,1]}(u) du, \quad \alpha > -1/2.$$

Inverting the order of integration, we were in front of the following series

$$(2) \quad \left(\frac{2}{r\rho}\right)^\gamma \sum_{j \geq 0} (2j + k_0 + k_1) I_{2j+\gamma}(\rho r) C_{2j}^{k_0+k_1}(z_{p\phi, p\theta}(u, v))$$

for $(u, v) \in]-1, 1[^2$, which specializes for $p = 2$ to

$$\frac{1}{2} \sum_{j \equiv 0[4]} (j + \gamma) I_{j+\gamma}(\rho r) C_{j/2}^{\gamma/2}(z_{2\phi, 2\theta}(u, v)).$$

Using the identity noticed by Y. Xu ([13]):

$$C_j^\nu(\cos \zeta) = \int C_{2j}^{2\nu} \left(\sqrt{\frac{1 + \cos \zeta}{2}} z \right) \mu^{\nu-1/2}(dz), \quad \nu > -1/2, \zeta \in [0, \pi],$$

we were led to

$$\sum_{j \equiv 0[4]} (j + \gamma) I_{j+\gamma}(\rho r) C_j^\gamma(z_{2\phi, 2\theta}(u, v))$$

which we wrote as

$$\frac{1}{4} \sum_{s=1}^4 \sum_{j \geq 0} (j + \gamma) I_{j+\gamma}(\rho r) C_j^\gamma(z_{2\phi, 2\theta}(u, v)) e^{isj\pi/2}$$

after the use of the elementary identity

$$(3) \quad \frac{1}{n} \sum_{s=1}^m e^{2i\pi s j/m} = \begin{cases} 1 & \text{if } j \equiv 0[m], \\ 0 & \text{otherwise,} \end{cases}$$

valid for any integer $m \geq 1$. Accordingly (Corollary 1.2 in [5])

$$D_k^W(\rho, \phi, r, \theta) = \int \int i_{(\gamma-1)/2} \left(\rho r \sqrt{\frac{1 + z_{2\phi, 2\theta}(u, v)}{2}} \right) \mu^{l_1}(du) \mu^{l_0}(dv)$$

where

$$i_\alpha(x) := \sum_{m=0}^{\infty} \frac{1}{(\alpha+1)_m m!} \left(\frac{x}{2}\right)^{2m}$$

is the normalized modified Bessel function ([8]) and $\gamma = 2(k_0 + k_1) \geq 2$ is even. This is a relatively simple integral representation of D_k^W since the latter function may be expressed as a bivariate hypergeometric function of Bessel-type. Recall also that it follows essentially from closed formulas due to L. Gegenbauer (equations (4), (5), p.369 in [12]):

$$\left(\frac{2}{r\rho}\right)^\gamma \sum_{j \geq 0} (j+\gamma) I_{j+\gamma}(\rho r) C_j^\gamma(\cos \zeta) (\pm 1)^j = \frac{1}{\Gamma(\gamma)} e^{\pm \rho r \cos \zeta}.$$

In this paper, we shall see that a relatively simple integral representation of D_k^W still exists for general integer $p \geq 2$ and integer $\nu := k_0 + k_1 \geq 1^2$. In fact, with regard to (2), one has to derive closed formulas for both series below

$$(4) \quad f_{\nu,p}^\pm(R, \cos \zeta) := \left(\frac{2}{R}\right)^{p\nu} \sum_{j \geq 0} (j+\nu) I_{p(j+\nu)}(R) C_j^\nu(\cos \zeta) (\pm 1)^j$$

with $R = \rho r$ and $\cos \zeta := \cos \zeta(u, v) = z_{p\phi, p\theta}(u, v)$. The obtained formulas reduce to Gegenbauer's results when $p = 1$, $\nu \geq 1$ is an integer, and do not exist up to our knowledge. Moreover, our approach is somewhat geometric since we shall interpret the sequence:

$$(\pm 1)^j I_{p(j+\nu)}(R), \quad j \geq 0$$

for fixed R as the Gegenbauer-Fourier coefficients of $\zeta \mapsto f_{\nu,p}^\pm(R, \cos \zeta)$, and since spherical functions on the sphere viewed as a homogeneous space are expressed by means of Gegenbauer polynomials ([1]). Then, following [1], solving the problem when ν is a strictly positive integer amounts to appropriately use inversion formulas for Fourier and Radon transforms. Our main result is stated as

Proposition 1. *Assume $\nu \geq 1$ is a strictly positive integer, then*

$$\left(\frac{R}{2}\right)^{p\nu} f_{\nu,p}^\pm(R, \cos \zeta) = \frac{1}{2^\nu (\nu-1)!} \left[-\frac{1}{\sin \zeta} \frac{d}{d\zeta} \right]^\nu \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(\zeta+2\pi s)/p]}.$$

A first glance at the main result may be ambiguous for the reader since the LHS depends on $\cos \zeta$ while the RHS depends on $\cos(\zeta/p), p \geq 1$. But $\cos(\zeta/p), p \geq 1$ may be expressed, though in a very complicated way (inverses of linearization formulas), as a function of $\cos \zeta$. For instance, when $p = 2$,

$$\cos(\zeta/2) = \sqrt{\frac{1 + \cos \zeta}{2}}, \quad \zeta \in [0, \pi].$$

One then recovers Corollary 1.2. in [5] after using appropriate formulas for modified Bessel functions. When $p = 3$, one has to solve a special cubic equation. To proceed, we rely on results from analytic function theory and the required solution is expressed by means of Gauss hypergeometric functions ([10]) in contrast to Cardan's solution. Therefore, we get a somewhat explicit formula for the series (2), though much more complicated than the one derived for $p = 2$. The paper is closed with adapting our method to odd dihedral groups, in particular to $D_2(3)$ thereby

²When $p = 2$, this condition is equivalent to γ is even as stated in [5].

exhausting the list of dihedral groups that are Weyl groups ($p = 1$ corresponds to the product group \mathbb{Z}_2^2).

2. PROOF OF THE MAIN RESULT

Recall the orthogonality relation for Gegenbauer polynomials ([8]):

$$\begin{aligned} \int_0^\pi C_j^\nu(\cos \zeta) C_m^\nu(\cos \zeta) (\sin \zeta)^{2\nu} d\zeta &= \delta_{jm} \frac{\pi \Gamma(j+2\nu) 2^{1-2\nu}}{\Gamma^2(\nu) (j+\nu) j!} \\ &= \delta_{jm} \frac{\pi 2^{1-2\nu} \Gamma(2\nu)}{(j+\nu) \Gamma^2(\nu)} C_j^\nu(1) \\ &= \delta_{jm} \nu \frac{\sqrt{\pi} \Gamma(\nu+1/2)}{\Gamma(\nu+1)} \frac{C_j^\nu(1)}{(j+\nu)} \end{aligned}$$

where we used $\Gamma(\nu+1) = \nu \Gamma(\nu)$, the Gauss duplication's formula ([8])

$$\sqrt{\pi} \Gamma(2\nu) = 2^{2\nu-1} \Gamma(\nu) \Gamma(\nu+1/2),$$

and the special value ([8])

$$C_j^\nu(1) = \frac{(2\nu)_j}{j!}.$$

Equivalently, if $\mu^\nu(d \cos \zeta)$ is the image of $\mu^\nu(d\zeta)$ under the map $\zeta \mapsto \cos \zeta$, then

$$(j+\nu) \int C_j^\nu(\cos \zeta) C_m^\nu(\cos \zeta) \mu^\nu(d \cos \zeta) = \nu C_j^\nu(1) \delta_{jm}$$

so that (4) yields

$$(5) \quad \nu(\pm 1)^j \left(\frac{2}{R}\right)^{p\nu} I_{p(j+\nu)}(R) = \int W_j^\nu(\cos \zeta) f_{\nu,p}^\pm(R, \cos \zeta) \mu^\nu(d \cos \zeta)$$

where

$$W_j^\nu(\cos \zeta) := C_j^\nu(\cos \zeta) / C_j^\nu(1)$$

is the j -th normalized Gegenbauer polynomial. Thus, the j -th Gegenbauer-Fourier coefficients of $\zeta \mapsto f_{\nu,p}^\pm(R, \cos \zeta)$ are given by

$$\nu(\pm 1)^j \left(\frac{2}{R}\right)^{p\nu} I_{p(j+\nu)}(R), \quad p \geq 2.$$

Following [1] p.356, the Mehler's integral representation of W_j^ν ([9], p.177)

$$W_j^\nu(\cos \zeta) = 2^\nu \frac{\Gamma(\nu+1/2)}{\Gamma(\nu) \sqrt{\pi}} (\sin \zeta)^{1-2\nu} \int_0^\zeta [\cos(j+\nu)t] (\cos t - \cos \zeta)^{\nu-1} dt$$

valid for real $\nu > 0$, transforms (5) to

$$\begin{aligned} \left(\frac{2}{R}\right)^{p\nu} (\pm 1)^j I_{p(j+\nu)}(R) &= \frac{2^\nu}{\pi} \int_0^\pi f_{\nu,p}^\pm(R, \cos \zeta) \sin \zeta \int_0^\zeta [\cos(j+\nu)t] (\cos t - \cos \zeta)^{\nu-1} dt d\zeta \\ (6) \quad &= \frac{2^\nu}{\pi} \int_0^\pi [\cos(j+\nu)t] \int_t^\pi f_{\nu,p}^\pm(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta dt. \end{aligned}$$

The second integral displayed in the RHS of the second equality is known as the Radon transform of $\zeta \mapsto f_{\nu,p}^\pm(R, \cos \zeta)$ and inversion formulas already exist ([1]). As a matter of fact, we firstly need to express $(\pm 1)^{j+\nu} I_{p(j+\nu)}$, when $\nu \geq 1$ is an integer, as the Fourier-cosine coefficient of order $j+\nu$ of some function. This is a

consequence of the Lemma below. Secondly, we shall use the appropriate inversion formula for the Radon transform.

Lemma. *For any integer $p \geq 1$ and any $t \in [0, \pi]$:*

$$2 \sum_{j \geq 0} (\pm 1)^j I_{pj}(R) \cos(jt) = I_0(R) + \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]}.$$

Proof of the Lemma: we will prove the (+) part, the proof of the (-) part follows the same lines with minor modifications. Write

$$\begin{aligned} 2 \sum_{j \geq 0} I_{pj}(R) \cos(jt) &= \sum_{j \geq 0} I_{pj}(R) [e^{ijt} + e^{-ijt}] \\ &= I_0(R) + \sum_{j \in \mathbb{Z}} I_{pj}(R) e^{ijt} \end{aligned}$$

where used the fact that $I_j(r) = I_{-j}(r), j \geq 0$. Using the identity (3), one obviously gets

$$\sum_{j \in \mathbb{Z}} I_{pj}(R) e^{ijt} = \frac{1}{p} \sum_{s=1}^p \sum_{j \in \mathbb{Z}} I_j(R) e^{ij(t+2\pi s)/p}.$$

The (+) part of the Lemma then follows from the generating series for modified Bessel functions ([12]):

$$e^{(z+1/z)R/2} = \sum_{j \in \mathbb{Z}} I_j(R) z^j, \quad z \in \mathbb{C}.$$

The Lemma yields

$$I_{pj}(R) = I_0(R) \delta_{j0} + \frac{1}{\pi} \int_0^\pi \cos(jt) \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]} dt$$

for any integer $j \geq 0$. Assuming that ν is a strictly positive integer, one has

$$(7) \quad I_{p(j+\nu)}(R) = \frac{1}{\pi} \int_0^\pi \cos((j+\nu)t) \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]} dt.$$

Note that

$$t \mapsto \int_t^\pi f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta$$

as well as

$$t \mapsto \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]}$$

are even functions. This is true since

$$\zeta \mapsto f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1}$$

is an odd function so that

$$\int_{-t}^t f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta = 0,$$

and since

$$\cos[(-t + 2s\pi)/p] = \cos[(t + 2(p-s)\pi)/p]$$

so that one performs the index change $s \rightarrow p-s$ and notes that the terms corresponding to $s=0$ and $s=p$ are equal. Similar arguments yield the 2π -periodicity

of these functions, therefore, the Fourier-cosine transforms of their restrictions on $(-\pi, \pi)$ coincide with their Fourier transforms on that interval. By injectivity of the Fourier transform and 2π -periodicity,

$$\left(\frac{R}{2}\right)^{p\nu} \int_t^\pi f_{\nu,p}(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta = \frac{1}{2^\nu p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]}$$

for all t since both functions are continuous. Finally, the Proposition follows from Theorem 3.1. p.363 in [1]. \blacksquare

Remark. When $\nu = (d-1)/2$ for some integer $d \geq 1$, the Gegenbauer-Fourier transform is interpreted as the Fourier Transform on the sphere S^{d+1} considered as a homogenous space $SO(d+1)/SO(d)$. More precisely, the spherical functions of this space are given by ([1] p.356):

$$W_j^\nu(\langle z, N \rangle), \quad z \in S^{d+1},$$

where $N = (0, \dots, 0, 1) \in S^{d+1}$ is the north pole and $\langle \cdot, \cdot \rangle$ denotes the Euclidian inner product on \mathbb{R}^{d+1} .

Corollary 1. For any integer $\nu \geq 1$

$$\sum_{j \geq 0} (2j + \nu) I_{p(2j+\nu)}(R) C_{2j}^\nu(\cos \zeta) = \frac{1}{2^\nu \Gamma(\nu)} \left[-\frac{1}{\sin \zeta} \frac{d}{d\zeta} \right]^\nu \frac{1}{p} \sum_{s=1}^p \cosh(R \cos[(\zeta + 2\pi s)/p]).$$

3. WEYL GROUP SETTINGS $p = 2, 3$: EXPLICIT FORMULAS

3.1. **p=2.** Letting $p = 2$ and using the fact that $u \mapsto \cosh u$ is an even function, our main result yields

$$\left(\frac{4}{R^2}\right)^\nu \sum_{j \geq 0} (2j + \nu) I_{2(2j+\nu)}(R) C_{2j}^\nu(\cos \zeta) = \frac{1}{2^\nu \Gamma(\nu)} \left[-\frac{4}{R^2 \sin \zeta} \frac{d}{d\zeta} \right]^\nu \cosh(R \cos(\cdot/2))(\zeta).$$

Noting that

$$-\frac{4}{R^2 \sin \zeta} \frac{d}{d\zeta} \cosh(R \cos(\cdot/2))(\zeta) = \frac{1}{R \cos t/2} \frac{d}{dt} (u \mapsto \cosh u)|_{u=R \cos(\zeta/2)},$$

after the use of the identity $\sin \zeta = 2 \sin \zeta/2 \cos \zeta/2$, it follows that

$$\begin{aligned} \left[-\frac{4}{R^2 \sin \zeta} \frac{d}{d\zeta} \right]^\nu \cosh(R \cos(\cdot/2))(\zeta) &= \left[\frac{1}{u} \frac{d}{du} \right]^\nu (u \mapsto \cosh u)|_{u=R \cos(\zeta/2)} \\ &= \left[\frac{1}{u} \frac{d}{du} \right]^{\nu-1} (u \mapsto \frac{\sinh u}{u})|_{u=R \cos(\zeta/2)} \\ &= \sqrt{\frac{\pi}{2}} \left[\frac{d}{du} \right]^{\nu-1} \left(u \mapsto \frac{I_{1/2}(u)}{\sqrt{u}} \right)|_{u=R \cos(\zeta/2)} \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{u^{\nu-1/2}} I_{\nu-1/2}(u)|_{u=R \cos(\zeta/2)} \\ &= \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu + 1/2)} i_{\nu-1/2}(R \cos(\zeta/2)) \end{aligned}$$

where the fourth equality is a consequence of the differentiation formula (6) p.79 in [12]. With the help of Gauss duplication's formula, one easily gets:

$$\left(\frac{4}{R^2}\right)^\nu \sum_{j \geq 0} (2j + \nu) I_{2(2j+\nu)}(R) C_{2j}^\nu(\cos \zeta) = \frac{1}{2\Gamma(2\nu)} i_{\nu-1/2}(R \cos(\zeta/2))$$

and finally recovers Corollary 1.2 in [5] since $c_{2,k}/c(k_1-1/2, k_0-1/2) = \Gamma(2\nu+1)/\nu$.

3.2. $\mathbf{p=3}$. The corresponding dihedral group $D_2(6)$ is isomorphic to the Weyl group of type G_2 ([2]). Let $\zeta \in]0, \pi[$ and start with the linearization formula:

$$4 \cos^3(\zeta/3) = \cos \zeta + 3 \cos(\zeta/3).$$

Thus, we are led to find a root lying in $[-1, 1]$ of the cubic equation

$$Z^3 - (3/4)Z - (\cos \zeta)/4 = 0$$

for $|Z| < 1$. Set $Z = (\sqrt{-1}/2)T$, $|T| < 2$, the above cubic equation transforms to

$$T^3 + 3T - 2\sqrt{-1} \cos \zeta = 0.$$

The obtained cubic equation already showed up in analytic function theory in relation to the local inversion Theorem ([10] p.265-266). Amazingly (compared to Cardan's formulas), its real and both complex roots are expressed through the Gauss Hypergeometric function ${}_2F_1$. Since we are looking for real $Z = (\sqrt{-1}/2)T$, we shall only consider the complex roots (see the bottom of p. 266 in [10]):

$$T^\pm = \pm \sqrt{-1} \left[\sqrt{3} {}_2F_1 \left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; \cos^2 \zeta \right) - \frac{1}{3} \cos \zeta {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; \cos^2 \zeta \right) \right]$$

so that

$$Z^\pm = \pm \left[\frac{\sqrt{3}}{2} {}_2F_1 \left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; \cos^2 \zeta \right) - \frac{1}{6} \cos \zeta {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; \cos^2 \zeta \right) \right].$$

Since for $\zeta = \pi/2$, $\cos \zeta/3 = \cos \pi/6 = \sqrt{3}/2$, it follows that

$$\cos(\zeta/3) = \left[\frac{\sqrt{3}}{2} {}_2F_1 \left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; \cos^2 \zeta \right) - \frac{1}{6} \cos \zeta {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; \cos^2 \zeta \right) \right]$$

for all $\zeta \in (0, \pi)$. Now, write $Z = Z(\cos \zeta)$ so that

$$\begin{aligned} \cos[(\zeta + 2s\pi)/3] &= \cos(2s\pi/3) \cos(\zeta/3) - \sin(2s\pi/3) \sqrt{1 - \cos^2(\zeta/3)} \\ &= \cos(2s\pi/3) Z(\cos \zeta) - \sin(2s\pi/3) \sqrt{1 - Z^2(\cos \zeta)} \end{aligned}$$

for any $1 \leq s \leq 3$. It follows that

$$f_{\nu,3}(R, \cos \zeta) = \frac{1}{3\Gamma(\nu)} \left[-\frac{4}{R^3 \sin \zeta} \frac{d}{d\zeta} \right]^\nu \sum_{s=1}^3 g_s(RZ(\cos \zeta))$$

where

$$g_s(u) = \cosh \left[\left(\cos(2s\pi/3)u - \sin(2s\pi/3) \sqrt{R^2 - u^2} \right) \right], u \in (-1, 1).$$

Finally,

$$f_{\nu,3}(R, \cos \zeta) = \frac{1}{3\Gamma(\nu)} \left[\frac{4}{R^3} \frac{d}{du} \right]^\nu \sum_{s=1}^3 h_s(u)|_{u=\cos \zeta}$$

where $h_s(u) := g_s(RZ(u))$, $1 \leq s \leq 3$. For instance, let $\nu = 1$, then it is not difficult to see that

$$\frac{d}{du} h_s(u)|_{u=\cos \zeta} = \frac{R}{\sin \zeta/3} \frac{dZ}{du}|_{u=\cos \zeta} \sin\left(\frac{\xi + 2\pi s}{3}\right) \sinh\left[\sin\left(\frac{\xi + 2\pi s}{3}\right)\right]$$

for any $s \in \{1, 2, 3\}$ and the derivative of $u \mapsto Z(u)$ is computed using the differentiation formula for ${}_2F_1$:

$$\frac{d}{du} {}_2F_1(a, b, c; u) = \frac{ab}{c} {}_2F_1(a+1, b+1, c+1; u), \quad |u| < 1, c \neq 0.$$

As the reader may conclude, formulas are cumbersome compared to the ones derived for $p = 2$.

4. ODD DIHEDRAL GROUPS

Let $n \geq 3$ be an odd integer. For odd dihedral groups $D_2(n)$, the generalized Bessel function reads ([4] p.157):

$$D_k^W(\rho, \phi, r, \theta) = c_{n,k} \left(\frac{2}{r\rho}\right)^{nk} \sum_{j \geq 0} I_{n(2j+k)}(\rho r) p_j^{-1/2, l_0}(\cos(2n\phi)) p_j^{-1/2, l_0}(\cos(2n\theta))$$

where $k \geq 0, \rho, r \geq 0, \theta, \phi \in [0, \pi/n]$, and

$$c_{n,k} = 2^k \Gamma(nk + 1) \frac{\sqrt{\pi} \Gamma(k + 1/2)}{\Gamma(k + 1)}.$$

In order to adapt our method to those groups, we need to write down the product formula for orthonormal Jacobi polynomials in the limiting case $\alpha = -1/2$ or equivalently $k_1 = 0$. But note that, from an analytic point of view, this generalized Bessel function is obtained from the one associated with even dihedral groups via the substitutions $k_1 = 0, p = n$. Hence one expects the product formula for orthonormal Jacobi polynomials still holds in the limiting case. Indeed, the required limiting formula was derived in [7] p.194 using implicitly the fact that the Beta distribution μ^α converges weakly to the dirac mass δ_1 . In order to fit it into our normalizations, we proceed as follows: use the well-known quadratic transformation ([8]):

$$P_j^{-1/2, k-1/2}(1-2\sin^2(n\theta)) = (-1)^j P_j^{k-1/2, -1/2}(2\sin^2(n\theta)-1) = (-1)^j \frac{(1/2)_j}{(k)_j} C_{2j}^k(\sin(n\theta))$$

where $P_j^{\alpha, \beta}$ is the (non orthonormal) j -th Jacobi polynomial, together with $\cos(2n\theta) = 1 - 2\sin^2(n\theta)$ to obtain

$$P_j^{-1/2, k-1/2}(\cos(2n\theta)) P_j^{-1/2, k-1/2}(\cos(2n\phi)) = \left[\frac{(1/2)_j}{(k)_j}\right]^2 C_{2j}^k(\sin(n\theta)) C_{2j}^k(\sin(n\phi)).$$

Now, let $k > 0$ and recall that the squared L^2 -norm of $P_j^{-1/2, k-1/2}$ is given by ([8])

$$\frac{2^k}{2j+k} \frac{\Gamma(j+1/2)\Gamma(j+k+1/2)}{j!\Gamma(j+k)} = \frac{2^k \sqrt{\pi} \Gamma(k+1/2)}{\Gamma(k)} \frac{(1/2)_j}{(k)_j} \frac{(k+1/2)_j}{(2j+k)j!}.$$

Recall also the special value

$$C_{2j}^k(1) = \frac{(2k)_{2j}}{(2j)!} = \frac{2^{2k}}{\Gamma(2k)} \frac{\Gamma(k+j)\Gamma(k+j+1/2)}{\Gamma(j+1/2)j!} = 2 \frac{(k)_j (k+1/2)_j}{(1/2)_j j!}$$

where we use Gauss duplication formula twice to derive both the second and the third equalities. It follows that

$$\begin{aligned} c(k)p_j^{-1/2,k-1/2}(\cos(2n\theta))p_j^{-1/2,k-1/2}(\cos(2n\phi)) &= \frac{(1/2)_j}{(k)_j} \frac{(2j+k)j!}{(k+1/2)_j} C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)) \\ &= \frac{(2j+k)}{C_{2j}^k(1)} C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)) \\ &= (2j+k) \int C_{2j}^k(z_{n\phi,n\theta}(u,1)) \mu^k(du), \end{aligned}$$

according to [7] p.194, where

$$c(k) := \frac{2^{k+1}\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k)}.$$

As a matter of fact, we are led again to series of the form

$$\left(\frac{2}{R}\right)^{nk} \sum_{j \geq 0} (2j+k) I_{n(2j+k)}(R) C_{2j}^k(\cos \zeta) = \frac{1}{2} [f_{k,n}^+ + f_{k,n}^-](R, \cos \zeta).$$

5. TWO REMARKS

The first remark is concerned with $D_2(4)$ which coincides with the B_2 -type Weyl group ([8]). Recall from ([6]) that D_k^W may be expressed through a bivariate hypergeometric function as

$$D_k^W(x, y) = {}_1F_0^{(1/k_1)} \left(\frac{\gamma+1}{2}, \frac{x^2}{2}, \frac{y^2}{2} \right),$$

where we set $x^2 := (x_1^2, x_2^2) = (\rho^2 \cos^2 \phi, \rho^2 \sin^2 \phi)$ and similarly for y^2 . This series is defined via Jack polynomials:

$${}_1F_0^{(1/r)}(a, x, y) = \sum_{\tau} (a)_{\tau} \frac{J_{\tau}^{1/r}(x) J_{\tau}^{1/r}(y)}{J_{\tau}^{1/r}(\mathbf{1}) |\tau|!}$$

where $\mathbf{1} = (1, 1)$, $\tau = (\tau_1, \tau_2)$ is a partition of length 2, $|\tau| = \tau_1 + \tau_2$ is its weight and $(a)_{\tau}$ is the generalized Pochhammer symbol (see [6] for definitions). But those polynomials, known also as Jack polynomials of type A_1 , may be expressed through Gegenbauer polynomials, a result due to M. Lassalle (see for instance formula 4.10 in [11]):

$$J_{\tau}^{1/r}(x^2) = \frac{(\tau_1 - \tau_2)!}{2^{|\tau|(r)} (r)_{\tau_1 - \tau_2}} \sin^{|\tau|}(2\phi) C_{\tau_1 - \tau_2}^r \left(\frac{1}{\sin(2\phi)} \right)$$

where $(r)_{\tau_1 - \tau_2}$ is the (usual) Pochhammer symbol. As a matter fact, one wonders if it is possible to come from the hypergeometric series to Corollary 1.2 in [5] and vice-versa.

The second remark comes in the same spirit of the first one. Consider the odd dihedral system $I_2(3) = \{\pm e^{-i\pi/2} e^{i\pi l/3}, 1 \leq l \leq 3\}$ ([8]). It is isomorphic to the A_2 -type root system defined by

$$\{\pm(1, -1, 0), \pm(1, 0, -1), \pm(0, 1, -1)\} \subset \mathbb{R}^3$$

which spans the hyperplane $(1, 1, 1)^\perp$. The isomorphism is given by

$$(z_1, z_2, z_3) \mapsto \frac{1}{\sqrt{2}} \left(\sqrt{\frac{3}{2}} z_2, \frac{z_3 - z_1}{\sqrt{2}} \right)$$

subject to $z_1 + z_2 + z_3 = 0$ and for the A_2 -type root system, the generalized Bessel function is given by the trivariate hypergeometric series ${}_0F_0^{(1/k)}$ (see [6] for the definition). Is it possible to relate this function to

$$\frac{c_{3,k}}{c(k)} \int [f_{k,3}^+ + f_{k,3}^-](\rho r, z_{3\phi, 3\theta}(u, 1)) \mu^k(du) = \frac{3\Gamma(3k)}{4} \int [f_{k,3}^+ + f_{k,3}^-](\rho r, z_{3\phi, 3\theta}(u, 1)) \mu^k(du)$$

in the same way the ${}_0F_1^{1/k_1}$ is related to the integral representation derived for $p = 2$?

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