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To cite this version:
Nizar Demni. Radon transform on spheres and generalized Bessel function associated with dihedral groups. 2010. hal-00519722

HAL Id: hal-00519722
https://hal.science/hal-00519722
Preprint submitted on 21 Sep 2010

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RADON TRANSFORM ON SPHERES AND GENERALIZED
BESSEL FUNCTION ASSOCIATED WITH DIHEDRAL GROUPS

N. Demni

Abstract. Motivated by Dunkl operators theory, we consider a generating series involving a modified Bessel function and a Gegenbauer polynomial, that generalizes a known series already considered by L. Gegenbauer. We actually use inversion formulas for Fourier and Radon transforms to derive a closed formula for this series when the parameter of the Gegenbauer polynomial is a strictly positive integer. As a by-product, we get a relatively simple integral representation for the generalized Bessel function associated with even dihedral groups $D_{2p}$, $p \geq 1$ when both multiplicities sum to an integer. In particular, we recover a previous result obtained for $D_2(4)$ and we give a special interest to $D_2(6)$. The paper is closed with adapting our method to odd dihedral groups thereby exhausting the list of Weyl dihedral groups.

1. Introduction

The dihedral group $D_2(n)$ of order $n \geq 2$ is defined as the group of regular $n$-gon preserving-symmetries ([8]). It figures among reflections groups associated with root systems for which a spherical harmonics theory, generalizing the one of Harish-Chandra on semisimple Lie groups from a discrete to a continuous range of multiplicities, was introduced by C. F. Dunkl in the late eighties (see Ch.I in [3]). Since then, a huge amount of research papers on this new topic and on its stochastic side as well emerged yielding fascinating results (Ch. II, III in [3]). For instance, probabilistic considerations allowed the author to derive the so-called generalized Bessel function associated with dihedral groups ([4]). For even values $n = 2p$, $p \geq 1$, this function depending on two real variables, say $(x, y) \in \mathbb{R}^2$, is expressed in polar coordinates $x = \rho e^{i\phi}, y = r e^{i\theta}, \rho, r \geq 0, \phi, \theta \in [0, \pi/2p]$ as

$$D_k^W(\rho, \phi, r, \theta) = c_{p,k} \left(\frac{2}{\rho r}\right)^{\gamma} \sum_{j \geq 0} I_{2jp+\gamma}(\rho r)p_{j}^{l_1,l_0}(\cos(2p\phi))p_{j}^{l_1,l_0}(\cos(2p\theta))$$

where

- $k = (k_0, k_1)$ is a positive-valued multiplicity function, $l_i = k_i - 1/2, i \in \{1, 2\}, \gamma = p(k_0 + k_1)$.
- $I_{2jp+\gamma}, p_{j}^{l_1,l_0}$ are the modified Bessel function of index $2jp + \gamma$ and the $j$-th orthonormal Jacobi polynomial of parameters $l_1, l_0$ respectively (the orthogonality (Beta) measure need not to be normalized here. In fact, the normalization only alters the constant $c_{p,k}$ below).

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Keywords: Generalized Bessel function, dihedral groups, Jacobi polynomials, Radon Transform.
AMS Classification: 33C52; 33C45; 42C10; 43A85; 43A90.
The constant $c_{p,k}$ depends on $p,k$ and is such that $D^W_k(0,y) = 1$ for all $y = (r,\theta) \in [0,\infty) \times [0,\pi/2p]$ (see [5])

$$
c_{p,k} = 2^{k_0+k_1} \frac{\Gamma(p(k_1+k_0)+1)\Gamma(k_1+1/2)\Gamma(k_0+1/2)}{\Gamma(k_0+k_1+1)}.
$$

In a subsequent paper ([5]), the special case $p = 2$ corresponding to the group of square-preserving symmetries was considered. The main ingredient used there was the famous Dijksma-Koornwinder’s product formula for Jacobi polynomials ([7]) which may be written in the following way ([5]):

$$
c(\alpha,\beta) p_j^\alpha,\beta(\cos 2\phi)p_j^\alpha,\beta(\cos 2\theta) = (2j+\alpha+\beta+1) \int \int C_{2j}^{\alpha+\beta+1}(z_{\phi,\theta}(u,v))\mu^\alpha(du)\mu^\beta(dv)
$$

where $\alpha, \beta > -1/2,$

$$
c(\alpha,\beta) = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},
$$

$z_{\phi,\theta}(u,v) = u \cos \theta \cos \phi + v \sin \theta \sin \phi,$

and $\mu^\alpha$ is the symmetric Beta probability measure whose density is given by

$$
\mu^\alpha(du) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi} \Gamma(\alpha+1/2)} (1-u^2)^{-\alpha-1/2} 1_{[-1,1]}(u)du, \quad \alpha > -1/2.
$$

Inverting the order of integration, we were in front of the following series

$$
(2) \quad \left( \frac{2}{\gamma p} \right) \sum_{j \geq 0} (2j+k_0+k_1)I_{2jp+\gamma}(pr)C_{2j+k_1}^{k_0+k_1}(z_{p\phi,p\theta}(u,v))
$$

for $(u,v) \in [-1,1]^2$, which specializes for $p = 2$ to

$$
\frac{1}{2} \sum_{j=0}^{[4]} (j+\gamma)I_{j+\gamma}(pr)C_j^{\nu}(z_{2\phi,2\theta}(u,v)).
$$

Using the identity noticed by Y. Xu ([13]):

$$
C_j^{\nu}(\cos \zeta) = \int C_2^{2\nu} \left( \sqrt{\frac{1+\cos \zeta}{2}} z \right) \mu^{\nu-1/2}(dz), \quad \nu > -1/2, \quad \xi \in [0,\pi],
$$

we were led to

$$
\sum_{j=0}^{[4]} (j+\gamma)I_{j+\gamma}(pr)C_j^{\nu}(z_{2\phi,2\theta}(u,v))
$$

which we wrote as

$$
\frac{1}{4} \sum_{s=1}^{4} \sum_{j \geq 0} (j+\gamma)I_{j+\gamma}(pr)C_j^{\nu}(z_{2\phi,2\theta}(u,v))e^{is\pi j/2}
$$

after the use of the elementary identity

$$
(3) \quad \frac{1}{n} \sum_{j=1}^{m} e^{2i\pi x_j/m} = \begin{cases} 1 & \text{if } j \equiv 0[m], \\ 0 & \text{otherwise,} \end{cases}
$$

valid for any integer $m \geq 1$. Accordingly (Corollary 1.2 in [5])

$$
D^W_k(\rho,\phi,r,\theta) = \int \int i^{(\gamma-1)/2} \left( \rho r \sqrt{\frac{1+z_{2\phi,2\theta}(u,v)}{2}} \right) \mu^\rho(du)\mu^\theta(dv).
$$
where

\[ i_\alpha(x) := \sum_{m=0}^{\infty} \frac{1}{(\alpha + 1)m!} \left( \frac{x}{2} \right)^{2m} \]

is the normalized modified Bessel function \((8)\) and \(\gamma = 2(k_0 + k_1) \geq 2\) is even. This is a relatively simple integral representation of \(D_W^k\) since the latter function may be expressed as a bivariate hypergeometric function of Bessel-type. Recall also that it follows essentially from closed formulas due to L. Gegenbauer (equations (4), (5), p.369 in \[12\]):

\[ \left( \frac{2}{rp} \right)^{\gamma} \sum_{j \geq 0} (j + \gamma) I_{j+\gamma}(pr) C_j^\gamma(\cos \zeta)(\pm 1)^j = \frac{1}{\Gamma(\gamma)} e^{\pm pr \cos \zeta}. \]

In this paper, we shall see that a relatively simple integral representation of \(D_W^k\) still exists for general integer \(p \geq 2\) and integer \(\nu := k_0 + k_1 \geq 1\). In fact, with regard to (2), one has to derive closed formulas for both series below

\[ (4) \quad f_{\nu,p}^{\pm}(R, \cos \zeta) := \left( \frac{2}{R} \right)^{\nu} \sum_{j \geq 0} (j + \nu) I_{j+\nu}(R) C_j^{\nu}(\pm 1)^j \]

with \(R = pr\) and \(\cos \zeta := \cos \zeta(u, v) = z_{p\nu, p\phi}(u, v).\) The obtained formulas reduce to Gegenbauer’s results when \(p = 1, \nu \geq 1\) is an integer, and do not exist up to our knowledge. Moreover, our approach is somewhat geometric since we shall interpret the sequence:

\[ (\pm 1)^j I_{j+\nu}(R), j \geq 0 \]

for fixed \(R\) as the Gegenbauer-Fourier coefficients of \(\zeta \mapsto f_{\nu,p}^{\pm}(R, \cos \zeta),\) and since spherical functions on the sphere viewed as a homogeneous space are expressed by means of Gegenbauer polynomials \([1]\). Then, following \[1\], solving the problem when \(\nu\) is a strictly positive integer amounts to appropriately use inversion formulas for Fourier and Radon transforms. Our main result is stated as

**Proposition 1.** Assume \(\nu \geq 1\) is a strictly positive integer, then

\[ \left( \frac{R}{2} \right)^{\nu} f_{\nu,p}^{\pm}(R, \cos \zeta) = \frac{1}{2^{\nu}(\nu - 1)!} \left[ \frac{1}{\sin \zeta} \frac{d}{d\zeta} \right]^{\nu - 1} \sum_{s=1}^{p} \frac{e^{\pm R \cos(\zeta + 2\pi s)/p}}{p}. \]

A first glance at the main result may be ambiguous for the reader since the LHS depends on \(\cos \zeta\) while the RHS depends on \(\cos(\zeta/p), p \geq 1\). But \(\cos(\zeta/p), p \geq 1\) may be expressed, though in a very complicated way (inverses of linearization formulas), as a function of \(\cos \zeta\). For instance, when \(p = 2\),

\[ \cos(\zeta/2) = \sqrt{\frac{1 + \cos \zeta}{2}}, \quad \zeta \in [0, \pi]. \]

One then recovers Corollary 1.2. in \[5\] after using appropriate formulas for modified Bessel functions. When \(p = 3\), one has to solve a special cubic equation. To proceed, we rely on results from analytic function theory and the required solution is expressed by means of Gauss hypergeometric functions \([10]\) in contrast to Cardan’s solution. Therefore, we get a somewhat explicit formula for the series (2), though much more complicated than the one derived for \(p = 2\). The paper is closed with adapting our method to odd dihedral groups, in particular to \(D_2(3)\) thereby

\[ ^2\text{When } p = 2, \text{ this condition is equivalent to } \gamma \text{ is even as stated in } [5]. \]
exhausting the list of dihedral groups that are Weyl groups ($p = 1$ corresponds to the product group $\mathbb{Z}_2^2$).

2. Proof of the main result

Recall the orthogonality relation for Gegenbauer polynomials ([8]):

$$
\int_0^{\pi} C_j^\nu(\cos \zeta) C_m^\nu(\cos \zeta) (\sin \zeta)^{2\nu} d\zeta = \delta_{jm} \frac{\pi \Gamma(\nu)(j+\nu)2^{2-2\nu}}{\Gamma^2(\nu) (j+\nu)!} \frac{\pi^{1-2\nu} \Gamma(2\nu)}{(j+\nu)!} C_j^\nu(1)
$$

$$
= \delta_{jm} \sqrt{\pi} \Gamma(\nu+1/2) \frac{C_j^\nu(1)}{\Gamma(\nu+1)} (j+\nu)
$$


where we used $\Gamma(\nu + 1) = \nu \Gamma(\nu)$, the Gauss duplication’s formula ([8])

$$
\sqrt{\pi} \Gamma(2\nu) = 2^{2\nu-1}\Gamma(\nu)\Gamma(\nu + 1/2),
$$

and the special value ([8])

$$
C_j^\nu(1) = \frac{(2\nu)_j}{j!}.
$$

Equivalently, if $\mu^\nu(d \cos \zeta)$ is the image of $\mu^\nu(d \zeta)$ under the map $\zeta \mapsto \cos \zeta$, then

$$(j + \nu) \int C_j^\nu(\cos \zeta) C_m^\nu(\cos \zeta) \mu^\nu(d \cos \zeta) = \nu C_j^\nu(1) \delta_{jm}
$$

so that (4) yields

$$(5) \quad \nu(\pm 1)^{j} \left( \frac{2}{R} \right)^{\nu} I_{p(j+\nu)}(R) = \int W_j^\nu(\cos \zeta) f_{\nu,p}^{\pm}(R, \cos \zeta) \mu^\nu(d \cos \zeta)
$$

where

$$
W_j^\nu(\cos \zeta) := C_j^\nu(\cos \zeta)/C_j^\nu(1)
$$

is the $j$-th normalized Gegenbauer polynomial. Thus, the $j$-th Gegenbauer-Fourier coefficients of $\zeta \mapsto f_{\nu,p}^{\pm}(R, \cos \zeta)$ are given by

$$
\nu(\pm 1)^{j} \left( \frac{2}{R} \right)^{\nu} I_{p(j+\nu)}(R), \quad p \geq 2.
$$

Following [1] p.356, the Mehler’s integral representation of $W_j^\nu$ ([9], p.177)

$$
W_j^\nu(\cos \zeta) = 2^{\nu} \Gamma(\nu + 1/2) \left( \sin \zeta \right)^{1-2\nu} \int_0^{\zeta} (\cos(j + \nu)t)(\cos t - \cos \zeta)^{\nu-1} dt
$$

valid for real $\nu > 0$, transforms (5) to

$$
\left( \frac{2}{R} \right)^{\nu} (\pm 1)^{j} I_{p(j+\nu)}(R) = \frac{2^{\nu}}{\pi} \int_0^{\pi} f_{\nu,p}^{\pm}(R, \cos \zeta) \sin \zeta \int_0^{\zeta} (\cos(j + \nu)t)(\cos t - \cos \zeta)^{\nu-1} dtd\zeta
$$

$$
= \frac{2^{\nu}}{\pi} \int_0^{\pi} [\cos(j + \nu)t] \int_0^{\zeta} f_{\nu,p}^{\pm}(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} dtd\zeta.
$$

The second integral displayed in the RHS of the second equality is known as the Radon transform of $\zeta \mapsto f_{\nu,p}^{\pm}(R, \cos \zeta)$ and inversion formulas already exist ([1]). As a matter of fact, we firstly need to express $(\pm 1)^{j+\nu} I_{p(j+\nu)}$, when $\nu \geq 1$ is an integer, as the Fourier-cosine coefficient of order $j + \nu$ of some function. This is a
consequence of the Lemma below. Secondly, we shall use the appropriate inversion formula for the Radon transform.

**Lemma.** For any integer \( p \geq 1 \) and any \( t \in [0, \pi] \):

\[
2 \sum_{j \geq 0} (\pm 1)^j I_{pj}(R) \cos(jt) = I_0(R) + \frac{1}{p} \sum_{s=1}^{p} e^{\pm R \cos([t+2\pi s]/p)}.
\]

**Proof of the Lemma:** We will prove the \((+)\) part, the proof of the \((-)\) part follows the same lines with minor modifications. Write

\[
2 \sum_{j \geq 0} I_{pj}(R) \cos(jt) = \sum_{j \geq 0} I_{pj}(R)[e^{jt} + e^{-jt}]
\]

\[
= I_0(R) + \sum_{j \in \mathbb{Z}} I_{pj}(R)e^{jt}
\]

where used the fact that \( I_j(r) = I_{-j}(r), j \geq 0 \). Using the identity (3), one obviously gets

\[
\sum_{j \in \mathbb{Z}} I_{pj}(R)e^{jt} = \frac{1}{p} \sum_{s=1}^{p} \sum_{j \in \mathbb{Z}} I_{j}(R)e^{ij(t+2\pi s)/p}.
\]

The \((+)\) part of the Lemma then follows from the generating series for modified Bessel functions ([12]):

\[
e^{(z+1/z)R/2} = \sum_{j \in \mathbb{Z}} I_{j}(R)z^j, \ z \in \mathbb{C}.
\]

The Lemma yields

\[
I_{pj}(R) = I_0(R)\delta_{j,0} + \frac{1}{\pi} \int_{0}^{\pi} \cos(jt)\frac{1}{p} \sum_{s=1}^{p} e^{\pm R \cos([t+2\pi s]/p)} dt
\]

for any integer \( j \geq 0 \). Assuming that \( \nu \) is a strictly positive integer, one has

\[
I_{p(j+\nu)}(R) = \frac{1}{\pi} \int_{0}^{\pi} \cos((j + \nu)t)\frac{1}{p} \sum_{s=1}^{p} e^{\pm R \cos([t+2\pi s]/p)} dt.
\]

Note that

\[
t \mapsto \int_{t}^{\pi} f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta
\]

as well as

\[
t \mapsto \frac{1}{p} \sum_{s=1}^{p} e^{\pm R \cos([t+2\pi s]/p)}
\]

are even functions. This is true since

\[
\zeta \mapsto f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1}
\]

is an odd function so that

\[
\int_{-t}^{t} f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta = 0,
\]

and since

\[
\cos([-t + 2s\pi]/p) = \cos([t + 2(p - s)\pi]/p)
\]

so that one performs the index change \( s \to p - s \) and notes that the terms corresponding to \( s = 0 \) and \( s = p \) are equal. Similar arguments yield the \( 2\pi \)-periodicity
of these functions, therefore, the Fourier-cosine transforms of their restrictions on \((-\pi, \pi)\) coincide with their Fourier transforms on that interval. By injectivity of the Fourier transform and \(2\pi\)-periodicity,

\[
\left(\frac{R}{2}\right)^{\nu} \int_{-\pi}^{\pi} f_{\nu,p}(R,\cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta = \frac{1}{2^{\nu+1}} \sum_{s=1}^{p} e^{\pm R\cos((t+2\pi s)/p)}
\]

for all \(t\) since both functions are continuous. Finally, the Proposition follows from Theorem 3.1. p.363 in [1]. ■

Remark. When \(\nu = (d - 1)/2\) for some integer \(d \geq 1\), the Gegenbauer-Fourier transform is interpreted as the Fourier Transform on the sphere \(S^{d+1}\) considered as a homogenous space \(SO(d+1)/SO(d)\). More precisely, the spherical functions of this space are given by ([1] p.356):

\[
W_j^\nu((z,N)), z \in S^{d+1},
\]

where \(N = (0, \ldots, 0, 1) \in S^{d+1}\) is the north pole and \(\langle \cdot, \cdot \rangle\) denotes the Euclidian inner product on \(\mathbb{R}^{d+1}\).

Corollary 1. For any integer \(\nu \geq 1\)

\[
\sum_{j \geq 0} (2j + \nu) I_{p(2j+\nu)}(R) C^\nu_{2j}(\cos \zeta) = \frac{1}{2^{\nu+1}} \left[ -\frac{d}{\sin \zeta} \right]^{\nu} \frac{1}{p} \sum_{s=1}^{p} \cosh (R \cos((\zeta + 2\pi s)/p)) \nu.
\]

3. Weyl group settings \(p = 2, 3\): explicit formulas

3.1. \(p=2\). Letting \(p = 2\) and using the fact that \(u \mapsto \cosh u\) is an even function, our main result yields

\[
\left(\frac{4}{R^2}\right)^{\nu} \sum_{j \geq 0} (2j + \nu) I_{2(2j+\nu)}(R) C^\nu_{2j}(\cos \zeta) = \frac{1}{2^{\nu+1}} \left[ -\frac{d}{\sin \zeta} \right]^{\nu} \cosh (R \cos(\zeta/2)) \nu.
\]

Noting that

\[
-\frac{4}{R^2 \sin \zeta} \frac{d}{d\zeta} \cosh (R \cos(\zeta/2)) \nu = \frac{1}{R \cos t/2} \frac{d}{dt} (u \mapsto \cosh u) |_{u = R \cos(\zeta/2)},
\]

after the use of the identity \(\sin \zeta = 2 \sin \zeta/2 \cos \zeta/2\), it follows that

\[
\left[ -\frac{4}{R^2 \sin \zeta} \frac{d}{d\zeta} \right]^{\nu} \cosh (R \cos(\zeta/2)) \nu = \left[ \frac{1}{u \mapsto \cosh u} \right]^{\nu-1} \left( u \mapsto I_{1/2}(u) / \sqrt{u} \right) |_{u = R \cos(\zeta/2)}
\]

and

\[
= \sqrt{\pi} \left[ \frac{1}{u^{\nu-1/2}} I_{-1/2}(u) u |_{u = R \cos(\zeta/2)}
\]

\[
= \sqrt{\pi} \frac{1}{2^{\nu+1/2}} I_{\nu-1/2}(R \cos(\zeta/2))
\]

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where the fourth equality is a consequence of the differentiation formula (6) p.79 in [12]. With the help of Gauss duplication’s formula, one easily gets:

\[
\left(\frac{4}{R^2}\right)^\nu \sum_{j=0}^\nu (2j+\nu)I_{2(2j+\nu)}(R)C_{\nu,j}^2(\cos \zeta) = \frac{1}{2\Gamma(2\nu)}i_{\nu-1/2}(R\cos(\zeta/2))
\]

and finally recovers Corollary 1.2 in [5] since \(c_{2,k}/c(k_1-1/2, k_0-1/2) = \Gamma(2\nu+1)/\nu\).

3.2. \(p=3\). The corresponding dihedral group \(D_2(6)\) is isomorphic to the Weyl group of type \(G_2\) ([2]). Let \(\zeta \in [0, \pi]\) and start with the linearization formula:

\[
4\cos^3(\zeta/3) = \cos \zeta + 3\cos(\zeta/3).
\]

Thus, we are led to find a root lying in \([-1, 1]\) of the cubic equation

\[
Z^3 - (3/4)Z - (\cos \zeta)/4 = 0
\]

for \(|Z| < 1\). Set \(Z = (\sqrt{-1}T)/|T| < 2\), the above cubic equation transforms to

\[
T^3 + 3T - 2\sqrt{-1}\cos \zeta = 0.
\]

The obtained cubic equation already showed up in analytic function theory in relation to the local inversion Theorem ([10] p.265-266). Amazingly (compared to Cardan’s formulas), its real and both complex roots are expressed through the Gauss Hypergeometric function \(\text{$_2F_1$}\). Since we are looking for real \(Z = (\sqrt{-1}/2)T\), we shall only consider the complex roots (see the bottom of p. 266 in [10]):

\[
T^\pm = \pm \sqrt{-1}\left[\sqrt{3}\text{$_2F_1$}\left(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; \cos^2 \zeta\right) - \frac{1}{3}\cos \zeta \text{$_2F_1$}\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; \cos^2 \zeta\right)\right]
\]

so that

\[
Z^\pm = \pm \left[\sqrt{\frac{3}{2}}\text{$_2F_1$}\left(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; \cos^2 \zeta\right) - \frac{1}{6}\cos \zeta \text{$_2F_1$}\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; \cos^2 \zeta\right)\right].
\]

Since for \(\zeta = \pi/2\), \(\cos \zeta/3 = \cos \pi/6 = \sqrt{3}/2\), it follows that

\[
\cos(\zeta/3) = \left[\sqrt{\frac{3}{2}}\text{$_2F_1$}\left(-\frac{1}{6}, \frac{1}{6}; \frac{1}{2}; \cos^2 \zeta\right) - \frac{1}{6}\cos \zeta \text{$_2F_1$}\left(\frac{1}{3}, \frac{2}{3}; \frac{3}{2}; \cos^2 \zeta\right)\right]
\]

for all \(\zeta \in (0, \pi)\). Now, write \(Z = Z(\cos \zeta)\) so that

\[
\cos[(\zeta + 2s\pi)/3] = \cos(2s\pi/3)\cos(\zeta/3) - \sin(2s\pi/3)\sqrt{1 - \cos^2(\zeta/3)}
\]

\[
= \cos(2s\pi/3)Z(\cos \zeta) - \sin(2s\pi/3)\sqrt{1 - Z^2(\cos \zeta)}
\]

for any \(1 \leq s \leq 3\). It follows that

\[
f_{\nu,3}(R, \cos \zeta) = \frac{1}{3\Gamma(\nu)} \left[ -\frac{4}{R^3 \sin \zeta} \frac{d}{d\zeta} \sum_{s=1}^{\nu} g_s(RZ(\cos \zeta)) \right]
\]

where

\[
g_s(u) = \cosh \left[ (\cos(2s\pi/3)u - \sin(2s\pi/3)\sqrt{R^2 - u^2}) \right], u \in (-1, 1).
\]

Finally,

\[
f_{\nu,3}(R, \cos \zeta) = \frac{1}{3\Gamma(\nu)} \left[ 4 \frac{d}{du} \sum_{s=1}^{\nu} h_s(u)|_{u=\cos \zeta} \right]
\]
where \( h_s(u) := g_s(RZ(u)), 1 \leq s \leq 3. \) For instance, let \( \nu = 1, \) then it is not difficult to see that
\[
\frac{d}{du} h_s(u)_{u = \cos \zeta} = \frac{R}{\sin \zeta/3} \frac{dZ}{du} \sin \left( \frac{\xi + 2\pi s}{3} \right) \sinh \left[ \sin \left( \frac{\xi + 2\pi s}{3} \right) \right]
\]
for any \( s \in \{1, 2, 3\} \) and the derivative of \( u \mapsto Z(u) \) is computed using the differentiation formula for \( _2F_1: \)
\[
\frac{d}{du} _2F_1(a, b, c; u) = \frac{ab}{c} _2F_1(a + 1, b + 1, c + 1; u), \quad |u| < 1, c \neq 0.
\]
As the reader may conclude, formulas are cumbersome compared to the ones derived for \( p = 2. \)

4. Odd Dihedral Groups

Let \( n \geq 3 \) be an odd integer. For odd dihedral groups \( D_2(n), \) the generalized Bessel function reads ([4] p.157):
\[
D^W_{nk}(\rho, \phi, r, \theta) = c_{n,k} \left( \frac{2}{\rho p} \right)^{nk} \sum_{j \geq 0} I_{n(2j+k)}(\rho r)p^{-1/2,l}j(\cos(2n\phi))p_j^{-1/2,l}j(\cos(2n\theta))
\]
where \( k \geq 0, \rho, r \geq 0, \theta, \phi \in [0, \pi/n], \) and
\[
c_{n,k} = 2^k \Gamma(nk + 1) \frac{\sqrt{\pi} \Gamma(k + 1/2)}{\Gamma(k + 1)}.
\]
In order to adapt our method to those groups, we need to write down the product formula for orthonormal Jacobi polynomials in the limiting case \( \alpha = -1/2 \) or equivalently \( k_1 = 0. \) But note that, from an analytic point of view, this generalized Bessel function is obtained from the one associated with even dihedral groups via the substitutions \( k_1 = 0, p = n. \) Hence one expects the product formula for orthonormal Jacobi polynomials still holds in the limiting case. Indeed, the required limiting formula was derived in [7] p.194 using implicitly the fact that the Beta distribution \( \mu^\alpha \) converges weakly to the dirac mass \( \delta_1. \) In order to fit it into our normalizations, we proceed as follows: use the well-known quadratic transformation ([8]):
\[
P_j^{-1/2,k-1/2}(1 - 2\sin^2(n\theta)) = (-1)^j P_j^{k-1/2,-1/2}(2\sin^2(n\theta) - 1) = (-1)^j \frac{(1/2)_j}{(k)_j} C_{2j}^k(\sin(n\theta))
\]
where \( P_j^{\alpha,\beta} \) is the (non orthonormal) \( j \)-th Jacobi polynomial, together with \( \cos(2n\theta) = 1 - 2\sin^2(n\theta) \) to obtain
\[
P_j^{-1/2,k-1/2}(\cos(2n\theta))P_j^{1/2,k-1/2}(\cos(2n\phi)) = \left[ \frac{(1/2)_j}{(k)_j} \right]^2 C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)).
\]
Now, let \( k > 0 \) and recall that the squared \( L^2 \)-norm of \( P_j^{-1/2,k-1/2} \) is given by ([8])
\[
\frac{2^k}{2j + k} \frac{\Gamma(j + 1/2)\Gamma(j + k + 1/2)}{\Gamma(j)\Gamma(k)} = \frac{2^k \sqrt{\pi} \Gamma(k + 1/2)}{\Gamma(k)} \frac{(1/2)_j}{(k)_j} (k + 1/2) \frac{(1/2)_j}{(k)_j} (2j + k) j!.
\]
Recall also the special value
\[
C_{2j}^k(1) = \frac{(2k)_{2j}}{(2j)!} = \frac{2^k}{\Gamma(2k)} \frac{\Gamma(k + j)\Gamma(k + j + 1/2)}{\Gamma(j + 1/2)j!} = 2 \frac{(k)_{j+1/2}}{(1/2)_j j!}.
\]
where we use Gauss duplication formula twice to derive both the second and the third equalities. It follows that
\[ c(k)p_j^{-1/2}k^{-1/2}(\cos(2n\theta))p_j^{-1/2}k^{-1/2}(\cos(2n\phi)) = \frac{(1/2)_j}{(k)} \frac{(2j+k)}{(k+1/2)_j} C^k_{2j}(\sin(n\theta))C^k_{2j}(\sin(n\phi)) \]
\[ = \frac{(2j+k)}{C^k_{2j}(1)} C^k_{2j}(\sin(n\theta))C^k_{2j}(\sin(n\phi)) \]
\[ = (2j+k) \int C^k_{2j}(z_{n\phi,n\theta}(u,1)) \mu^k(du), \]
according to [7] p.194, where
\[ c(k) := 2^{k+1} \sqrt{\pi} \Gamma(k+1/2) \Gamma(k). \]
As a matter of fact, we are led again to series of the form
\[ \left( \frac{2}{R} \right)^n \sum_{j \geq 0} (2j+k) I_{n(2j+k)}(R) C^k_{2j}(\cos \zeta) = \frac{1}{2} [f^+_k + f^-_k](R, \cos \zeta). \]

5. Two Remarks

The first remark is concerned with \( D_2(4) \) which coincides with the \( B_2 \)-type Weyl group ([8]). Recall from ([6]) that \( D^W_k(x, y) = 1F_0(1/k) \left( \gamma + 1, x^2, y^2 \right) \), where we set \( x^2 := (x^1, x^2) = (\rho^2 \cos^2 \phi, \rho^2 \sin^2 \phi) \) and similarly for \( y^2 \). This series is defined via Jack polynomials:

\[ 1F_0(1/\tau) (a, x, y) = \sum_{\tau}(a) \frac{J^{1/\tau}_x(x)J^{1/\tau}_y(y)}{J^{1/\tau}_x(1)!}, \]

where \( 1 = (1, 1), \tau = (\tau_1, \tau_2) \) is a partition of length 2, \( |\tau| = \tau_1 + \tau_2 \) is its weight and \( (a) \) is the generalized Pochhammer symbol (see [6] for definitions). But those polynomials, known also as Jack polynomials of type \( A_1 \), may be expressed through Gegenbauer polynomials, a result due to M. Lassalle (see for instance formula 4.10 in [11]):

\[ J^{1/\tau}_x(x^2) = \frac{(\tau_1 - \tau_2)!!}{2^{(|\tau|)}(\tau_1 - \tau_2)^{(|\tau|)}} \sin(|\tau|(2\phi)) C^r_{\tau_1 - \tau_2} \left( \frac{1}{\sin(2\phi)} \right) \]

where \( (r)_{\tau_1 - \tau_2} \) is the (usual) Pochhammer symbol. As a matter fact, one wonders if it is possible to come from the hypergeometric series to Corollary 1.2 in [5] and vice-versa.

The second remark comes in the same spirit of the first one. Consider the odd dihedral system \( I_2(3) = \{ \pm e^{-in/2}e^{im/3}, 1 \leq l \leq 3 \} \) ([8]). It is isomorphic to the \( A_2 \)-type root system defined by

\[ \{ \pm(1, -1, 0), \pm(1, 0, -1), \pm(0, 1, -1) \} \subset \mathbb{R}^3 \]
which spans the hyperplane \((1, 1, 1)\). The isomorphism is given by

\[
(z_1, z_2, z_3) \mapsto \frac{1}{\sqrt{2}} \left( \sqrt{\frac{3}{2}} z_2, \frac{z_3 - z_1}{\sqrt{2}} \right)
\]

subject to \(z_1 + z_2 + z_3 = 0\) and for the \(A_2\)-type root system, the generalized Bessel function is given by the trivariate hypergeometric series \(\tilde{\mathcal{H}}^{1/k}(\cdot)\) (see [6] for the definition). Is it possible to relate this function to \(c_{3,k}\)?

Acknowledgment: the author is grateful to Professor C.F. Dunkl who made him aware of the hypergeometric formulas for the roots of the cubic equation.

References