



HAL
open science

Radon transform on spheres and generalized Bessel function associated with dihedral groups

Nizar Demni

► **To cite this version:**

Nizar Demni. Radon transform on spheres and generalized Bessel function associated with dihedral groups. 2010. hal-00519722

HAL Id: hal-00519722

<https://hal.science/hal-00519722>

Preprint submitted on 21 Sep 2010

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

RADON TRANSFORM ON SPHERES AND GENERALIZED BESSEL FUNCTION ASSOCIATED WITH DIHEDRAL GROUPS

N. Demni¹

ABSTRACT. Motivated by Dunkl operators theory, we consider a generating series involving a modified Bessel function and a Gegenbauer polynomial, that generalizes a known series already considered by L. Gegenbauer. We actually use inversion formulas for Fourier and Radon transforms to derive a closed formula for this series when the parameter of the Gegenbauer polynomial is a strictly positive integer. As a by-product, we get a relatively simple integral representation for the generalized Bessel function associated with even dihedral groups $D_2(2p)$, $p \geq 1$ when both multiplicities sum to an integer. In particular, we recover a previous result obtained for $D_2(4)$ and we give a special interest to $D_2(6)$. The paper is closed with adapting our method to odd dihedral groups thereby exhausting the list of Weyl dihedral groups.

1. INTRODUCTION

The dihedral group $D_2(n)$ of order $n \geq 2$ is defined as the group of regular n -gone preserving-symmetries ([8]). It figures among reflections groups associated with root systems for which a spherical harmonics theory, generalizing the one of Harish-Chandra on semisimple Lie groups from a discrete to a continuous range of multiplicities, was introduced by C. F. Dunkl in the late eighties (see Ch.I in [3]). Since then, a huge amount of research papers on this new topic and on its stochastic side as well emerged yielding fascinating results (Ch. II, III in [3]). For instance, probabilistic considerations allowed the author to derive the so-called generalized Bessel function associated with dihedral groups ([4]). For even values $n = 2p$, $p \geq 1$, this function depending on two real variables, say $(x, y) \in \mathbb{R}^2$, is expressed in polar coordinates $x = \rho e^{i\phi}$, $y = r e^{i\theta}$, $\rho, r \geq 0$, $\phi, \theta \in [0, \pi/2p]$ as

$$(1) \quad D_k^W(\rho, \phi, r, \theta) = c_{p,k} \left(\frac{2}{r\rho} \right)^\gamma \sum_{j \geq 0} I_{2jp+\gamma}(\rho r) p_j^{l_1, l_0}(\cos(2p\phi)) p_j^{l_1, l_0}(\cos(2p\theta))$$

where

- $k = (k_0, k_1)$ is a positive-valued multiplicity function, $l_i = k_i - 1/2$, $i \in \{1, 2\}$, $\gamma = p(k_0 + k_1)$.
- $I_{2jp+\gamma}$, $p_j^{l_1, l_0}$ are the modified Bessel function of index $2jp + \gamma$ and the j -th orthonormal Jacobi polynomial of parameters l_1, l_0 respectively (the orthogonality (Beta) measure need not to be normalized here. In fact, the normalization only alters the constant $c_{p,k}$ below).

¹IRMAR, Rennes 1 University, France. E-mail: nizar.demni@univ-rennes1.fr.

Keywords: Generalized Bessel function, dihedral groups, Jacobi polynomials, Radon Transform.
AMS Classification: 33C52; 33C45; 42C10; 43A85; 43A90.

- The constant $c_{p,k}$ depends on p, k and is such that $D_k^W(0, y) = 1$ for all $y = (r, \theta) \in [0, \infty) \times [0, \pi/2p]$ (see [5])

$$c_{p,k} = 2^{k_0+k_1} \frac{\Gamma(p(k_1+k_0)+1)\Gamma(k_1+1/2)\Gamma(k_0+1/2)}{\Gamma(k_0+k_1+1)}.$$

In a subsequent paper ([5]), the special case $p = 2$ corresponding to the group of square-preserving symmetries was considered. The main ingredient used there was the famous Dijksma-Koornwinder's product formula for Jacobi polynomials ([7]) which may be written in the following way ([5]):

$$c(\alpha, \beta) p_j^{\alpha, \beta}(\cos 2\phi) p_j^{\alpha, \beta}(\cos 2\theta) = (2j + \alpha + \beta + 1) \int \int C_{2j}^{\alpha+\beta+1}(z_{\phi, \theta}(u, v)) \mu^\alpha(du) \mu^\beta(dv)$$

where $\alpha, \beta > -1/2$,

$$c(\alpha, \beta) = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)},$$

$$z_{\phi, \theta}(u, v) = u \cos \theta \cos \phi + v \sin \theta \sin \phi,$$

and μ^α is the symmetric Beta probability measure whose density is given by

$$\mu^\alpha(du) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} (1-u^2)^{\alpha-1/2} \mathbf{1}_{[-1,1]}(u) du, \quad \alpha > -1/2.$$

Inverting the order of integration, we were in front of the following series

$$(2) \quad \left(\frac{2}{r\rho}\right)^\gamma \sum_{j \geq 0} (2j + k_0 + k_1) I_{2j+\gamma}(\rho r) C_{2j}^{k_0+k_1}(z_{p\phi, p\theta}(u, v))$$

for $(u, v) \in]-1, 1[^2$, which specializes for $p = 2$ to

$$\frac{1}{2} \sum_{j \equiv 0[4]} (j + \gamma) I_{j+\gamma}(\rho r) C_{j/2}^{\gamma/2}(z_{2\phi, 2\theta}(u, v)).$$

Using the identity noticed by Y. Xu ([13]):

$$C_j^\nu(\cos \zeta) = \int C_{2j}^{2\nu} \left(\sqrt{\frac{1 + \cos \zeta}{2}} z \right) \mu^{\nu-1/2}(dz), \quad \nu > -1/2, \zeta \in [0, \pi],$$

we were led to

$$\sum_{j \equiv 0[4]} (j + \gamma) I_{j+\gamma}(\rho r) C_j^\gamma(z_{2\phi, 2\theta}(u, v))$$

which we wrote as

$$\frac{1}{4} \sum_{s=1}^4 \sum_{j \geq 0} (j + \gamma) I_{j+\gamma}(\rho r) C_j^\gamma(z_{2\phi, 2\theta}(u, v)) e^{isj\pi/2}$$

after the use of the elementary identity

$$(3) \quad \frac{1}{n} \sum_{s=1}^m e^{2i\pi s j/m} = \begin{cases} 1 & \text{if } j \equiv 0[m], \\ 0 & \text{otherwise,} \end{cases}$$

valid for any integer $m \geq 1$. Accordingly (Corollary 1.2 in [5])

$$D_k^W(\rho, \phi, r, \theta) = \int \int i_{(\gamma-1)/2} \left(\rho r \sqrt{\frac{1 + z_{2\phi, 2\theta}(u, v)}{2}} \right) \mu^{l_1}(du) \mu^{l_0}(dv)$$

where

$$i_\alpha(x) := \sum_{m=0}^{\infty} \frac{1}{(\alpha+1)_m m!} \left(\frac{x}{2}\right)^{2m}$$

is the normalized modified Bessel function ([8]) and $\gamma = 2(k_0 + k_1) \geq 2$ is even. This is a relatively simple integral representation of D_k^W since the latter function may be expressed as a bivariate hypergeometric function of Bessel-type. Recall also that it follows essentially from closed formulas due to L. Gegenbauer (equations (4), (5), p.369 in [12]):

$$\left(\frac{2}{r\rho}\right)^\gamma \sum_{j \geq 0} (j+\gamma) I_{j+\gamma}(\rho r) C_j^\gamma(\cos \zeta) (\pm 1)^j = \frac{1}{\Gamma(\gamma)} e^{\pm \rho r \cos \zeta}.$$

In this paper, we shall see that a relatively simple integral representation of D_k^W still exists for general integer $p \geq 2$ and integer $\nu := k_0 + k_1 \geq 1^2$. In fact, with regard to (2), one has to derive closed formulas for both series below

$$(4) \quad f_{\nu,p}^\pm(R, \cos \zeta) := \left(\frac{2}{R}\right)^{p\nu} \sum_{j \geq 0} (j+\nu) I_{p(j+\nu)}(R) C_j^\nu(\cos \zeta) (\pm 1)^j$$

with $R = \rho r$ and $\cos \zeta := \cos \zeta(u, v) = z_{p\phi, p\theta}(u, v)$. The obtained formulas reduce to Gegenbauer's results when $p = 1$, $\nu \geq 1$ is an integer, and do not exist up to our knowledge. Moreover, our approach is somewhat geometric since we shall interpret the sequence:

$$(\pm 1)^j I_{p(j+\nu)}(R), \quad j \geq 0$$

for fixed R as the Gegenbauer-Fourier coefficients of $\zeta \mapsto f_{\nu,p}^\pm(R, \cos \zeta)$, and since spherical functions on the sphere viewed as a homogeneous space are expressed by means of Gegenbauer polynomials ([1]). Then, following [1], solving the problem when ν is a strictly positive integer amounts to appropriately use inversion formulas for Fourier and Radon transforms. Our main result is stated as

Proposition 1. *Assume $\nu \geq 1$ is a strictly positive integer, then*

$$\left(\frac{R}{2}\right)^{p\nu} f_{\nu,p}^\pm(R, \cos \zeta) = \frac{1}{2^\nu (\nu-1)!} \left[-\frac{1}{\sin \zeta} \frac{d}{d\zeta} \right]^\nu \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(\zeta+2\pi s)/p]}.$$

A first glance at the main result may be ambiguous for the reader since the LHS depends on $\cos \zeta$ while the RHS depends on $\cos(\zeta/p), p \geq 1$. But $\cos(\zeta/p), p \geq 1$ may be expressed, though in a very complicated way (inverses of linearization formulas), as a function of $\cos \zeta$. For instance, when $p = 2$,

$$\cos(\zeta/2) = \sqrt{\frac{1 + \cos \zeta}{2}}, \quad \zeta \in [0, \pi].$$

One then recovers Corollary 1.2. in [5] after using appropriate formulas for modified Bessel functions. When $p = 3$, one has to solve a special cubic equation. To proceed, we rely on results from analytic function theory and the required solution is expressed by means of Gauss hypergeometric functions ([10]) in contrast to Cardan's solution. Therefore, we get a somewhat explicit formula for the series (2), though much more complicated than the one derived for $p = 2$. The paper is closed with adapting our method to odd dihedral groups, in particular to $D_2(3)$ thereby

²When $p = 2$, this condition is equivalent to γ is even as stated in [5].

exhausting the list of dihedral groups that are Weyl groups ($p = 1$ corresponds to the product group \mathbb{Z}_2^2).

2. PROOF OF THE MAIN RESULT

Recall the orthogonality relation for Gegenbauer polynomials ([8]):

$$\begin{aligned} \int_0^\pi C_j^\nu(\cos \zeta) C_m^\nu(\cos \zeta) (\sin \zeta)^{2\nu} d\zeta &= \delta_{jm} \frac{\pi \Gamma(j+2\nu) 2^{1-2\nu}}{\Gamma^2(\nu) (j+\nu) j!} \\ &= \delta_{jm} \frac{\pi 2^{1-2\nu} \Gamma(2\nu)}{(j+\nu) \Gamma^2(\nu)} C_j^\nu(1) \\ &= \delta_{jm} \nu \frac{\sqrt{\pi} \Gamma(\nu+1/2)}{\Gamma(\nu+1)} \frac{C_j^\nu(1)}{(j+\nu)} \end{aligned}$$

where we used $\Gamma(\nu+1) = \nu \Gamma(\nu)$, the Gauss duplication's formula ([8])

$$\sqrt{\pi} \Gamma(2\nu) = 2^{2\nu-1} \Gamma(\nu) \Gamma(\nu+1/2),$$

and the special value ([8])

$$C_j^\nu(1) = \frac{(2\nu)_j}{j!}.$$

Equivalently, if $\mu^\nu(d \cos \zeta)$ is the image of $\mu^\nu(d\zeta)$ under the map $\zeta \mapsto \cos \zeta$, then

$$(j+\nu) \int C_j^\nu(\cos \zeta) C_m^\nu(\cos \zeta) \mu^\nu(d \cos \zeta) = \nu C_j^\nu(1) \delta_{jm}$$

so that (4) yields

$$(5) \quad \nu(\pm 1)^j \left(\frac{2}{R}\right)^{p\nu} I_{p(j+\nu)}(R) = \int W_j^\nu(\cos \zeta) f_{\nu,p}^\pm(R, \cos \zeta) \mu^\nu(d \cos \zeta)$$

where

$$W_j^\nu(\cos \zeta) := C_j^\nu(\cos \zeta) / C_j^\nu(1)$$

is the j -th normalized Gegenbauer polynomial. Thus, the j -th Gegenbauer-Fourier coefficients of $\zeta \mapsto f_{\nu,p}^\pm(R, \cos \zeta)$ are given by

$$\nu(\pm 1)^j \left(\frac{2}{R}\right)^{p\nu} I_{p(j+\nu)}(R), \quad p \geq 2.$$

Following [1] p.356, the Mehler's integral representation of W_j^ν ([9], p.177)

$$W_j^\nu(\cos \zeta) = 2^\nu \frac{\Gamma(\nu+1/2)}{\Gamma(\nu) \sqrt{\pi}} (\sin \zeta)^{1-2\nu} \int_0^\zeta [\cos(j+\nu)t] (\cos t - \cos \zeta)^{\nu-1} dt$$

valid for real $\nu > 0$, transforms (5) to

$$\begin{aligned} \left(\frac{2}{R}\right)^{p\nu} (\pm 1)^j I_{p(j+\nu)}(R) &= \frac{2^\nu}{\pi} \int_0^\pi f_{\nu,p}^\pm(R, \cos \zeta) \sin \zeta \int_0^\zeta [\cos(j+\nu)t] (\cos t - \cos \zeta)^{\nu-1} dt d\zeta \\ (6) \quad &= \frac{2^\nu}{\pi} \int_0^\pi [\cos(j+\nu)t] \int_t^\pi f_{\nu,p}^\pm(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta dt. \end{aligned}$$

The second integral displayed in the RHS of the second equality is known as the Radon transform of $\zeta \mapsto f_{\nu,p}^\pm(R, \cos \zeta)$ and inversion formulas already exist ([1]). As a matter of fact, we firstly need to express $(\pm 1)^{j+\nu} I_{p(j+\nu)}$, when $\nu \geq 1$ is an integer, as the Fourier-cosine coefficient of order $j+\nu$ of some function. This is a

consequence of the Lemma below. Secondly, we shall use the appropriate inversion formula for the Radon transform.

Lemma. *For any integer $p \geq 1$ and any $t \in [0, \pi]$:*

$$2 \sum_{j \geq 0} (\pm 1)^j I_{pj}(R) \cos(jt) = I_0(R) + \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]}.$$

Proof of the Lemma: we will prove the (+) part, the proof of the (-) part follows the same lines with minor modifications. Write

$$\begin{aligned} 2 \sum_{j \geq 0} I_{pj}(R) \cos(jt) &= \sum_{j \geq 0} I_{pj}(R) [e^{ijt} + e^{-ijt}] \\ &= I_0(R) + \sum_{j \in \mathbb{Z}} I_{pj}(R) e^{ijt} \end{aligned}$$

where used the fact that $I_j(r) = I_{-j}(r), j \geq 0$. Using the identity (3), one obviously gets

$$\sum_{j \in \mathbb{Z}} I_{pj}(R) e^{ijt} = \frac{1}{p} \sum_{s=1}^p \sum_{j \in \mathbb{Z}} I_j(R) e^{ij(t+2\pi s)/p}.$$

The (+) part of the Lemma then follows from the generating series for modified Bessel functions ([12]):

$$e^{(z+1/z)R/2} = \sum_{j \in \mathbb{Z}} I_j(R) z^j, \quad z \in \mathbb{C}.$$

The Lemma yields

$$I_{pj}(R) = I_0(R) \delta_{j0} + \frac{1}{\pi} \int_0^\pi \cos(jt) \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]} dt$$

for any integer $j \geq 0$. Assuming that ν is a strictly positive integer, one has

$$(7) \quad I_{p(j+\nu)}(R) = \frac{1}{\pi} \int_0^\pi \cos((j+\nu)t) \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]} dt.$$

Note that

$$t \mapsto \int_t^\pi f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta$$

as well as

$$t \mapsto \frac{1}{p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]}$$

are even functions. This is true since

$$\zeta \mapsto f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1}$$

is an odd function so that

$$\int_{-t}^t f(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta = 0,$$

and since

$$\cos[(-t + 2s\pi)/p] = \cos[(t + 2(p-s)\pi)/p]$$

so that one performs the index change $s \rightarrow p-s$ and notes that the terms corresponding to $s=0$ and $s=p$ are equal. Similar arguments yield the 2π -periodicity

of these functions, therefore, the Fourier-cosine transforms of their restrictions on $(-\pi, \pi)$ coincide with their Fourier transforms on that interval. By injectivity of the Fourier transform and 2π -periodicity,

$$\left(\frac{R}{2}\right)^{p\nu} \int_t^\pi f_{\nu,p}(R, \cos \zeta) \sin \zeta (\cos t - \cos \zeta)^{\nu-1} d\zeta = \frac{1}{2^\nu p} \sum_{s=1}^p e^{\pm R \cos[(t+2\pi s)/p]}$$

for all t since both functions are continuous. Finally, the Proposition follows from Theorem 3.1. p.363 in [1]. \blacksquare

Remark. When $\nu = (d-1)/2$ for some integer $d \geq 1$, the Gegenbauer-Fourier transform is interpreted as the Fourier Transform on the sphere S^{d+1} considered as a homogenous space $SO(d+1)/SO(d)$. More precisely, the spherical functions of this space are given by ([1] p.356):

$$W_j^\nu(\langle z, N \rangle), \quad z \in S^{d+1},$$

where $N = (0, \dots, 0, 1) \in S^{d+1}$ is the north pole and $\langle \cdot, \cdot \rangle$ denotes the Euclidian inner product on \mathbb{R}^{d+1} .

Corollary 1. For any integer $\nu \geq 1$

$$\sum_{j \geq 0} (2j + \nu) I_{p(2j+\nu)}(R) C_{2j}^\nu(\cos \zeta) = \frac{1}{2^\nu \Gamma(\nu)} \left[-\frac{1}{\sin \zeta} \frac{d}{d\zeta} \right]^\nu \frac{1}{p} \sum_{s=1}^p \cosh(R \cos[(\zeta + 2\pi s)/p]).$$

3. WEYL GROUP SETTINGS $p = 2, 3$: EXPLICIT FORMULAS

3.1. **p=2.** Letting $p = 2$ and using the fact that $u \mapsto \cosh u$ is an even function, our main result yields

$$\left(\frac{4}{R^2}\right)^\nu \sum_{j \geq 0} (2j + \nu) I_{2(2j+\nu)}(R) C_{2j}^\nu(\cos \zeta) = \frac{1}{2^\nu \Gamma(\nu)} \left[-\frac{4}{R^2 \sin \zeta} \frac{d}{d\zeta} \right]^\nu \cosh(R \cos(\cdot/2))(\zeta).$$

Noting that

$$-\frac{4}{R^2 \sin \zeta} \frac{d}{d\zeta} \cosh(R \cos(\cdot/2))(\zeta) = \frac{1}{R \cos t/2} \frac{d}{dt} (u \mapsto \cosh u)|_{u=R \cos(\zeta/2)},$$

after the use of the identity $\sin \zeta = 2 \sin \zeta/2 \cos \zeta/2$, it follows that

$$\begin{aligned} \left[-\frac{4}{R^2 \sin \zeta} \frac{d}{d\zeta} \right]^\nu \cosh(R \cos(\cdot/2))(\zeta) &= \left[\frac{1}{u} \frac{d}{du} \right]^\nu (u \mapsto \cosh u)|_{u=R \cos(\zeta/2)} \\ &= \left[\frac{1}{u} \frac{d}{du} \right]^{\nu-1} (u \mapsto \frac{\sinh u}{u})|_{u=R \cos(\zeta/2)} \\ &= \sqrt{\frac{\pi}{2}} \left[\frac{d}{du} \right]^{\nu-1} \left(u \mapsto \frac{I_{1/2}(u)}{\sqrt{u}} \right)|_{u=R \cos(\zeta/2)} \\ &= \sqrt{\frac{\pi}{2}} \frac{1}{u^{\nu-1/2}} I_{\nu-1/2}(u)|_{u=R \cos(\zeta/2)} \\ &= \frac{\sqrt{\pi}}{2^\nu \Gamma(\nu + 1/2)} i_{\nu-1/2}(R \cos(\zeta/2)) \end{aligned}$$

where the fourth equality is a consequence of the differentiation formula (6) p.79 in [12]. With the help of Gauss duplication's formula, one easily gets:

$$\left(\frac{4}{R^2}\right)^\nu \sum_{j \geq 0} (2j + \nu) I_{2(2j+\nu)}(R) C_{2j}^\nu(\cos \zeta) = \frac{1}{2\Gamma(2\nu)} i_{\nu-1/2}(R \cos(\zeta/2))$$

and finally recovers Corollary 1.2 in [5] since $c_{2,k}/c(k_1-1/2, k_0-1/2) = \Gamma(2\nu+1)/\nu$.

3.2. $\mathbf{p=3}$. The corresponding dihedral group $D_2(6)$ is isomorphic to the Weyl group of type G_2 ([2]). Let $\zeta \in]0, \pi[$ and start with the linearization formula:

$$4 \cos^3(\zeta/3) = \cos \zeta + 3 \cos(\zeta/3).$$

Thus, we are led to find a root lying in $[-1, 1]$ of the cubic equation

$$Z^3 - (3/4)Z - (\cos \zeta)/4 = 0$$

for $|Z| < 1$. Set $Z = (\sqrt{-1}/2)T$, $|T| < 2$, the above cubic equation transforms to

$$T^3 + 3T - 2\sqrt{-1} \cos \zeta = 0.$$

The obtained cubic equation already showed up in analytic function theory in relation to the local inversion Theorem ([10] p.265-266). Amazingly (compared to Cardan's formulas), its real and both complex roots are expressed through the Gauss Hypergeometric function ${}_2F_1$. Since we are looking for real $Z = (\sqrt{-1}/2)T$, we shall only consider the complex roots (see the bottom of p. 266 in [10]):

$$T^\pm = \pm \sqrt{-1} \left[\sqrt{3} {}_2F_1 \left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; \cos^2 \zeta \right) - \frac{1}{3} \cos \zeta {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; \cos^2 \zeta \right) \right]$$

so that

$$Z^\pm = \pm \left[\frac{\sqrt{3}}{2} {}_2F_1 \left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; \cos^2 \zeta \right) - \frac{1}{6} \cos \zeta {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; \cos^2 \zeta \right) \right].$$

Since for $\zeta = \pi/2$, $\cos \zeta/3 = \cos \pi/6 = \sqrt{3}/2$, it follows that

$$\cos(\zeta/3) = \left[\frac{\sqrt{3}}{2} {}_2F_1 \left(-\frac{1}{6}, \frac{1}{6}, \frac{1}{2}; \cos^2 \zeta \right) - \frac{1}{6} \cos \zeta {}_2F_1 \left(\frac{1}{3}, \frac{2}{3}, \frac{3}{2}; \cos^2 \zeta \right) \right]$$

for all $\zeta \in (0, \pi)$. Now, write $Z = Z(\cos \zeta)$ so that

$$\begin{aligned} \cos[(\zeta + 2s\pi)/3] &= \cos(2s\pi/3) \cos(\zeta/3) - \sin(2s\pi/3) \sqrt{1 - \cos^2(\zeta/3)} \\ &= \cos(2s\pi/3) Z(\cos \zeta) - \sin(2s\pi/3) \sqrt{1 - Z^2(\cos \zeta)} \end{aligned}$$

for any $1 \leq s \leq 3$. It follows that

$$f_{\nu,3}(R, \cos \zeta) = \frac{1}{3\Gamma(\nu)} \left[-\frac{4}{R^3 \sin \zeta} \frac{d}{d\zeta} \right]^\nu \sum_{s=1}^3 g_s(RZ(\cos \zeta))$$

where

$$g_s(u) = \cosh \left[\left(\cos(2s\pi/3)u - \sin(2s\pi/3) \sqrt{R^2 - u^2} \right) \right], u \in (-1, 1).$$

Finally,

$$f_{\nu,3}(R, \cos \zeta) = \frac{1}{3\Gamma(\nu)} \left[\frac{4}{R^3} \frac{d}{du} \right]^\nu \sum_{s=1}^3 h_s(u)|_{u=\cos \zeta}$$

where $h_s(u) := g_s(RZ(u))$, $1 \leq s \leq 3$. For instance, let $\nu = 1$, then it is not difficult to see that

$$\frac{d}{du} h_s(u)|_{u=\cos \zeta} = \frac{R}{\sin \zeta/3} \frac{dZ}{du}|_{u=\cos \zeta} \sin\left(\frac{\xi + 2\pi s}{3}\right) \sinh\left[\sin\left(\frac{\xi + 2\pi s}{3}\right)\right]$$

for any $s \in \{1, 2, 3\}$ and the derivative of $u \mapsto Z(u)$ is computed using the differentiation formula for ${}_2F_1$:

$$\frac{d}{du} {}_2F_1(a, b, c; u) = \frac{ab}{c} {}_2F_1(a+1, b+1, c+1; u), \quad |u| < 1, c \neq 0.$$

As the reader may conclude, formulas are cumbersome compared to the ones derived for $p = 2$.

4. ODD DIHEDRAL GROUPS

Let $n \geq 3$ be an odd integer. For odd dihedral groups $D_2(n)$, the generalized Bessel function reads ([4] p.157):

$$D_k^W(\rho, \phi, r, \theta) = c_{n,k} \left(\frac{2}{r\rho}\right)^{nk} \sum_{j \geq 0} I_{n(2j+k)}(\rho r) p_j^{-1/2, l_0}(\cos(2n\phi)) p_j^{-1/2, l_0}(\cos(2n\theta))$$

where $k \geq 0, \rho, r \geq 0, \theta, \phi \in [0, \pi/n]$, and

$$c_{n,k} = 2^k \Gamma(nk + 1) \frac{\sqrt{\pi} \Gamma(k + 1/2)}{\Gamma(k + 1)}.$$

In order to adapt our method to those groups, we need to write down the product formula for orthonormal Jacobi polynomials in the limiting case $\alpha = -1/2$ or equivalently $k_1 = 0$. But note that, from an analytic point of view, this generalized Bessel function is obtained from the one associated with even dihedral groups via the substitutions $k_1 = 0, p = n$. Hence one expects the product formula for orthonormal Jacobi polynomials still holds in the limiting case. Indeed, the required limiting formula was derived in [7] p.194 using implicitly the fact that the Beta distribution μ^α converges weakly to the dirac mass δ_1 . In order to fit it into our normalizations, we proceed as follows: use the well-known quadratic transformation ([8]):

$$P_j^{-1/2, k-1/2}(1-2\sin^2(n\theta)) = (-1)^j P_j^{k-1/2, -1/2}(2\sin^2(n\theta)-1) = (-1)^j \frac{(1/2)_j}{(k)_j} C_{2j}^k(\sin(n\theta))$$

where $P_j^{\alpha, \beta}$ is the (non orthonormal) j -th Jacobi polynomial, together with $\cos(2n\theta) = 1 - 2\sin^2(n\theta)$ to obtain

$$P_j^{-1/2, k-1/2}(\cos(2n\theta)) P_j^{-1/2, k-1/2}(\cos(2n\phi)) = \left[\frac{(1/2)_j}{(k)_j}\right]^2 C_{2j}^k(\sin(n\theta)) C_{2j}^k(\sin(n\phi)).$$

Now, let $k > 0$ and recall that the squared L^2 -norm of $P_j^{-1/2, k-1/2}$ is given by ([8])

$$\frac{2^k}{2j+k} \frac{\Gamma(j+1/2)\Gamma(j+k+1/2)}{j!\Gamma(j+k)} = \frac{2^k \sqrt{\pi} \Gamma(k+1/2)}{\Gamma(k)} \frac{(1/2)_j}{(k)_j} \frac{(k+1/2)_j}{(2j+k)j!}.$$

Recall also the special value

$$C_{2j}^k(1) = \frac{(2k)_{2j}}{(2j)!} = \frac{2^{2k}}{\Gamma(2k)} \frac{\Gamma(k+j)\Gamma(k+j+1/2)}{\Gamma(j+1/2)j!} = 2 \frac{(k)_j (k+1/2)_j}{(1/2)_j j!}$$

where we use Gauss duplication formula twice to derive both the second and the third equalities. It follows that

$$\begin{aligned} c(k)p_j^{-1/2,k-1/2}(\cos(2n\theta))p_j^{-1/2,k-1/2}(\cos(2n\phi)) &= \frac{(1/2)_j}{(k)_j} \frac{(2j+k)j!}{(k+1/2)_j} C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)) \\ &= \frac{(2j+k)}{C_{2j}^k(1)} C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)) \\ &= (2j+k) \int C_{2j}^k(z_{n\phi,n\theta}(u,1)) \mu^k(du), \end{aligned}$$

according to [7] p.194, where

$$c(k) := \frac{2^{k+1}\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k)}.$$

As a matter of fact, we are led again to series of the form

$$\left(\frac{2}{R}\right)^{nk} \sum_{j \geq 0} (2j+k) I_{n(2j+k)}(R) C_{2j}^k(\cos \zeta) = \frac{1}{2} [f_{k,n}^+ + f_{k,n}^-](R, \cos \zeta).$$

5. TWO REMARKS

The first remark is concerned with $D_2(4)$ which coincides with the B_2 -type Weyl group ([8]). Recall from ([6]) that D_k^W may be expressed through a bivariate hypergeometric function as

$$D_k^W(x, y) = {}_1F_0^{(1/k_1)} \left(\frac{\gamma+1}{2}, \frac{x^2}{2}, \frac{y^2}{2} \right),$$

where we set $x^2 := (x_1^2, x_2^2) = (\rho^2 \cos^2 \phi, \rho^2 \sin^2 \phi)$ and similarly for y^2 . This series is defined via Jack polynomials:

$${}_1F_0^{(1/r)}(a, x, y) = \sum_{\tau} (a)_{\tau} \frac{J_{\tau}^{1/r}(x) J_{\tau}^{1/r}(y)}{J_{\tau}^{1/r}(\mathbf{1}) |\tau|!}$$

where $\mathbf{1} = (1, 1)$, $\tau = (\tau_1, \tau_2)$ is a partition of length 2, $|\tau| = \tau_1 + \tau_2$ is its weight and $(a)_{\tau}$ is the generalized Pochhammer symbol (see [6] for definitions). But those polynomials, known also as Jack polynomials of type A_1 , may be expressed through Gegenbauer polynomials, a result due to M. Lassalle (see for instance formula 4.10 in [11]):

$$J_{\tau}^{1/r}(x^2) = \frac{(\tau_1 - \tau_2)!}{2^{|\tau|(r)} (r)_{\tau_1 - \tau_2}} \sin^{|\tau|}(2\phi) C_{\tau_1 - \tau_2}^r \left(\frac{1}{\sin(2\phi)} \right)$$

where $(r)_{\tau_1 - \tau_2}$ is the (usual) Pochhammer symbol. As a matter fact, one wonders if it is possible to come from the hypergeometric series to Corollary 1.2 in [5] and vice-versa.

The second remark comes in the same spirit of the first one. Consider the odd dihedral system $I_2(3) = \{\pm e^{-i\pi/2} e^{i\pi l/3}, 1 \leq l \leq 3\}$ ([8]). It is isomorphic to the A_2 -type root system defined by

$$\{\pm(1, -1, 0), \pm(1, 0, -1), \pm(0, 1, -1)\} \subset \mathbb{R}^3$$

which spans the hyperplane $(1, 1, 1)^\perp$. The isomorphism is given by

$$(z_1, z_2, z_3) \mapsto \frac{1}{\sqrt{2}} \left(\sqrt{\frac{3}{2}} z_2, \frac{z_3 - z_1}{\sqrt{2}} \right)$$

subject to $z_1 + z_2 + z_3 = 0$ and for the A_2 -type root system, the generalized Bessel function is given by the trivariate hypergeometric series ${}_0F_0^{(1/k)}$ (see [6] for the definition). Is it possible to relate this function to

$$\frac{c_{3,k}}{c(k)} \int [f_{k,3}^+ + f_{k,3}^-](\rho r, z_{3\phi, 3\theta}(u, 1)) \mu^k(du) = \frac{3\Gamma(3k)}{4} \int [f_{k,3}^+ + f_{k,3}^-](\rho r, z_{3\phi, 3\theta}(u, 1)) \mu^k(du)$$

in the same way the ${}_0F_1^{1/k_1}$ is related to the integral representation derived for $p = 2$?

Acknowledgment: the author is grateful to Professor C.F. Dunkl who made him aware of the hypergeometric formulas for the roots of the cubic equation.

REFERENCES

- [1] A. Abouelaz, R. Dhafer. Sur la transformation de Radon de la sphère S^d . *Bull. Soc. Math. France*. **121**, 1993. 353-382.
- [2] J. C. Baez. The octonions. *Bull. Amer. Math. Soc. (N. S.)* **39**, no. 2. 2002, 145-205.
- [3] O. Chybiryakov, N. Demni, L. Gallardo, M. Rösler, M. Voit, M. Yor. Harmonic and Stochastic Analysis of Dunkl Processes. Ed. P. Graczyk, M. Rösler, M. Yor, Collection Travaux en Cours, Hermann.
- [4] N. Demni. Radial Dunkl processes associated with Dihedral systems. *Séminaire de Probabilités*, **XLII**. 153-169.
- [5] N. Demni. Product formula for Jacobi polynomials, spherical harmonics and generalized Bessel function of dihedral type. *Integ. Trans. Special Funct.* **21**, No. 2. 2010, 105-123.
- [6] N. Demni. Generalized Bessel function of type D . *SIGMA, Symmetry Integrability Geom. Methods. Appl.* **4**, 2008, paper 075, 7pp.
- [7] A. Dijksma, T. H. Koornwinder. Spherical Harmonics and the product of two Jacobi polynomials. *Indag. Math.* **33**, 1971, 191-196.
- [8] C. F. Dunkl, Y. Xu. Orthogonal Polynomials of Several Variables. *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press. 2001.
- [9] A. Erdelyi, W. Magnus, F. Oberhettinger, F. G. Tricomi. Tables of Integral Transforms. **Vol. 3**. McGraw-Hill, New-York, 1954.
- [10] E. Hille. Analytic Function Theory. **Vol. 1**. *Introduction to Higher Mathematics*, Ginn and Company. 1959.
- [11] V. V. Mangazeev. An analytic formula for the A_2 -Jack polynomials. *SIGMA, Symmetry Integrability Geom. Methods. Appl.* **3**, 2007, paper 014, 11pp.
- [12] G. N. Watson. A treatise on the theory of Bessel functions. *Cambridge Mathematical Library edition*. 1995.
- [13] Y. Xu. A product formula for Jacobi polynomials. *Proceedings of the International Workshop Special Functions. Hong-Kong, June 21-25, 1999*.