



Algebras of tame acyclic cluster type

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ALGEBRAS OF TAME ACYCLIC CLUSTER TYPE

CLAIRE AMIOT AND STEFFEN OPPERMAN

ABSTRACT. In this paper, we study algebras of tame acyclic cluster type. These are algebras of global dimension at most 2 whose generalized cluster category is equivalent to a cluster category of an acyclic quiver of tame representation type. We are particularly interested in their derived equivalence classification. We prove that the algebras which are cluster equivalent to a tree quiver are derived equivalent to the path algebra of this tree. Then we complete the classification by describing explicitly the algebras of cluster type \tilde{A}_n for each possible orientation of \tilde{A}_n . We give an explicit way to read off in which derived equivalence class these algebras are, and describe the Auslander-Reiten quiver of their derived categories.

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1. INTRODUCTION

The classification of finite dimensional algebras up to derived equivalence is a crucial problem in representation theory. It has a complete answer for algebras of global dimension 1 (see [Ha87, Corollary 4.8]): Two finite dimensional k -algebras $\Lambda = kQ$ and $\Lambda' = kQ'$ are derived equivalent if and only if one can pass from the quiver Q to Q' by a sequence of reflections (as introduced in [BGP73]). Therefore it is possible to decide when two hereditary algebras are derived equivalent by simple combinatorial means. The aim of this paper is to apply results of [AO10] in order to generalize this result to certain algebras of global dimension 2.

The notion of reflection of an acyclic quiver has been generalized by Fomin and Zelevinsky [FZ02] to the notion of mutation in their definition of cluster algebras. Since then, categorical interpretations of the mutation have been discovered via 2-Calabi-Yau triangulated categories. These created a link between cluster algebras and representation theory. First, in [BMR⁺06], cluster categories \mathcal{C}_Q associated to acyclic quivers Q were defined as the orbit categories $\mathcal{D}^b(kQ)/\mathbb{S}_2$, where $\mathbb{S}_2 = \mathbb{S}[-2]$ is the second desuspension of the Serre functor \mathbb{S} of the bounded derived category $\mathcal{D}^b(kQ)$. This notion has been generalized in [Ami09] to algebras of global dimension two. In this case the generalized cluster category is defined to be the triangulated hull in the sense of [Kel05] of the orbit category $\mathcal{D}^b\Lambda/\mathbb{S}_2$.

In this paper, we study more explicitly the derived equivalence classes of algebras of global dimension 2 which are of acyclic cluster type, that is algebras whose generalized cluster category is equivalent to some cluster category \mathcal{C}_Q where Q is an acyclic quiver. We strongly use the results and the techniques of [AO10]. In particular we use the notion of graded mutation of graded quiver with potential which is a refinement of the notion of mutation of quiver with potential introduced in [DWZ08]: Associated to an algebra Λ there is a graded Jacobian algebra $\overline{\Lambda}$ [Kel09] whose degree zero subalgebra is Λ . Graded mutation explains how to mutate such graded Jacobian algebras. Then from [AO10] we deduce an analogue of the result for algebras of global dimension 1.

Theorem 1.1 (see Theorem 3.14). *Let Λ_1 and Λ_2 be two finite dimensional algebras of global dimension 2. Assume that Λ_1 is of acyclic cluster type. Then the algebras Λ_1 and Λ_2 are derived equivalent if and only if one can pass from $\overline{\Lambda}_1$ to $\overline{\Lambda}_2$ using a sequence of graded mutations.*

The setup is especially nice when the algebras are of tree cluster type.

Theorem 1.2 (Corollary 3.15). *Let Q be an acyclic quiver whose underlying graph is a tree. If Λ is an algebra of global dimension 2 of cluster type Q , then it is derived equivalent to kQ .*

To get a complete understanding of algebras of tame acyclic cluster type the rest of the paper is focused on the algebras of cluster type $\tilde{A}_{p,q}$. To such an algebra, using graded mutation, we associate an integer that we call weight, which is zero when Λ is hereditary. We prove that two algebras of cluster type $\tilde{A}_{p,q}$ are derived equivalent if and only if they have the same weight (Theorem 4.4). Then a result of [AO10] which shows that two cluster equivalent algebras are graded derived equivalent permits us to compute explicitly the shape of the Auslander-Reiten quiver of the derived category.

Theorem 1.3 (Corollary 5.5). *Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$ and of weight $w \neq 0$. Then the algebra Λ is representation finite and not piecewise hereditary. The Auslander-Reiten quiver of $\mathcal{D}^b(\Lambda)$ has exactly $3|w|$ connected components:*

- $|w|$ components of type $\mathbb{Z}A_\infty^\infty$;
- $2|w|$ components of type $\mathbb{Z}A_\infty$.

Finally we use the explicit description of cluster-tilted algebra of type $\tilde{A}_{p,q}$ of [Bas09] to deduce a description of all algebras of cluster type $\tilde{A}_{p,q}$ in term of quivers with relations.

The paper is organized as follows: Section 2 is devoted to recalling results on generalized cluster categories, on Jacobian algebras, and on 2-Calabi-Yau categories. In Section 3, we apply the results of [AO10] to algebras of acyclic cluster type. The special case of algebras of cluster type $\tilde{A}_{p,q}$ is treated in the last three sections. We introduce the notion of weight and prove that it is an invariant of the derived equivalence class of the algebra in Section 4. We compute the shape of the AR-quiver of the derived category, and we finally describe explicitly these algebras in Section 6. We end the paper by giving (as an example) the complete classification for $\tilde{A}_{2,2}$.

2. BACKGROUND

2.1. Cluster-tilting objects and mutation in 2-Calabi-Yau categories.

Definition 2.1 (Iyama). Let \mathcal{T} be a triangulated category, which is Hom-finite and 2-Calabi-Yau, that is there is a functorial isomorphism $\text{Hom}_{\mathcal{T}}(X, Y[2]) \simeq D\text{Hom}_{\mathcal{T}}(Y, X)$ for all objects X and Y in \mathcal{T} . An object T is called *cluster-tilting* if

$$\text{add}(T) = \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, T[1]) = 0\},$$

where $\text{add}(T) \subset \mathcal{T}$ is the additive closure of T .

The endomorphism algebra of a cluster-tilting object $T \in \mathcal{T}$ is called a *2-Calabi-Yau-tilted algebra*.

Theorem 2.2 ([IY08, Theorem 5.3]). *Let T be a basic cluster-tilting object in a 2-Calabi-Yau triangulated category \mathcal{T} . Let T_i be an indecomposable summand of $T \simeq T_i \oplus T'$. Then there exists a unique object T_i^* non-isomorphic to T_i such that $\mu_i(T) := T' \oplus T_i^*$ is a basic cluster-tilting object in \mathcal{T} . Moreover T_i^* is indecomposable and there exist triangles in \mathcal{T}*

$$T_i \xrightarrow{u} U \longrightarrow T_i^* \longrightarrow T_i[1] \quad \text{and} \quad T_i^* \longrightarrow U' \xrightarrow{u'} T_i \longrightarrow T_i^*[1]$$

where u (resp. u') is a minimal left (resp. right) $\text{add}(T')$ -approximation.

2.2. Generalized cluster categories. Let Λ be a finite dimensional k -algebra of global dimension at most 2. We denote by $\mathcal{D}^b(\Lambda)$ the bounded derived category of finitely generated Λ -modules. It has a Serre functor that we denote \mathbb{S} . We denote by \mathbb{S}_2 the composition $\mathbb{S}[-2]$.

The generalized cluster category \mathcal{C}_Λ of Λ has been defined in [Ami09] as the triangulated hull of the orbit category $\mathcal{D}^b(\Lambda)/\mathbb{S}_2$ (see [Kel05] or [AO10, Section 7] for more details on triangulated hulls). We will denote by π_Λ the triangle functor

$$\pi_\Lambda: \mathcal{D}^b(\Lambda) \longrightarrow \mathcal{D}^b(\Lambda)/\mathbb{S}_2 \hookrightarrow \mathcal{C}_\Lambda$$

We set $\overline{\Lambda} := \text{End}_{\mathcal{C}}(\pi\Lambda) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(\Lambda, \mathbb{S}_2^{-p}\Lambda)$. Therefore this algebra is naturally endowed with a \mathbb{Z} -grading.

Definition 2.3. A finite dimensional k -algebra Λ of global dimension ≤ 2 is said to be τ_2 -finite if the algebra $\text{End}_{\mathcal{C}}(\pi\Lambda) = \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(\Lambda, \mathbb{S}_2^{-p}\Lambda)$ is finite dimensional.

Theorem 2.4 ([Ami09, Theorem 4.10]). *Let Λ be a finite dimensional algebra of global dimension ≤ 2 which is τ_2 -finite. Then \mathcal{C}_{Λ} is a Hom-finite, 2-Calabi-Yau category and the object $\pi(\Lambda) \in \mathcal{C}_{\Lambda}$ is a cluster-tilting object.*

2.3. Jacobian algebras and 2-Calabi-Yau-tilted algebras. Quivers with potentials and the associated Jacobian algebras have been studied in [DWZ08]. Let Q be a finite quiver. For each arrow a in Q , the *cyclic derivative* ∂_a with respect to a is the unique linear map

$$\partial_a: kQ \rightarrow kQ$$

which sends a path p to the sum $\sum_{p=uv} vu$ taken over all decompositions of the path p (where u and v are possibly idempotent elements e_i associated to a vertex i). A *potential* on Q is any linear combination W of cycles in Q . The associated Jacobian algebra is

$$\text{Jac}(Q, W) := k\hat{Q} / \langle \partial_a W \mid a \in Q_1 \rangle,$$

where $k\hat{Q}$ is the completion of the path algebra kQ and $\langle \partial_a W \mid a \in Q_1 \rangle$ is the completion of the ideal generated by $\partial_a W$ for $a \in Q_1$.

Jacobian algebras are closely related to 2-Calabi-Yau-tilted algebras. In particular, we have the following two results:

Proposition 2.5 ([Kel09, Theorem 6.11 a]). *Let $\Lambda = kQ/I$ be an algebra of global dimension ≤ 2 , such that I is generated by a finite minimal set of relations $\{r_i\}$. The relation r_i starts at the vertex $s(r_i)$ and ends at the vertex $t(r_i)$. Then we have an isomorphism of algebras:*

$$\text{End}_{\mathcal{C}}(\pi\Lambda) \simeq \text{Jac}(\overline{Q}, W)$$

where the quiver \overline{Q} is the quiver Q with additional arrows $a_i: t(r_i) \rightarrow s(r_i)$, and the potential W is $\sum_i a_i r_i$.

Definition 2.6. A potential W on a quiver Q is said to be *rigid* if any cycle p of Q is cyclically equivalent to an element in the Jacobian ideal $\langle \partial_a W \mid a \in Q_1 \rangle$.

By [DWZ08] rigidity is stable under mutation, and it implies that the quiver of the Jacobian algebra has no loops nor 2-cycles.

Theorem 2.7 ([BIRS08, Theorem 5.1]). *Let \mathcal{T} be a 2-Calabi-Yau Hom-finite triangulated category with basic cluster-tilting object T . Assume that there exists a quiver with rigid potential (Q, W) such that there is an isomorphism*

$$\text{End}_{\mathcal{T}}(T) \simeq \text{Jac}(Q, W).$$

Let i be a vertex of Q and denote by T_i the indecomposable summand of $T = T_i \oplus T'$ corresponding to i . Then there is an isomorphism

$$\text{End}_{\mathcal{T}}(T' \oplus T_i^*) \simeq \text{Jac}(\mu_i(Q, W)),$$

where T_i^* is defined in Theorem 2.2 and where $\mu_i(Q, W)$ is the mutation at i of (Q, W) as defined in [DWZ08] (see also Subsection 3.2 for definition).

For a sequence $s = (i_1, \dots, i_r)$ of vertices of Q , we denote by μ_s the composition $\mu_{i_r} \circ \mu_{i_{r-1}} \circ \dots \circ \mu_{i_1}$ and by μ_{s^-} the composition $\mu_{i_1} \circ \mu_{i_2} \circ \dots \circ \mu_{i_r}$.

2.4. Classical results on acyclic cluster categories. In this paper we are interested in algebras of global dimension ≤ 2 whose generalized cluster category is equivalent to a cluster category associated to an acyclic quiver. Since we will strongly make use of them, we now recall some results for acyclic cluster categories.

The following theorem follows from a result by Happel and Unger [HU03].

Theorem 2.8 ([BMR⁺06, Prop. 3.5]). *Let Q be an acyclic quiver, and let T be a cluster-tilting object of \mathcal{C}_Q . Then there exists a sequence of mutations linking the cluster-tilting object T to the canonical cluster-tilting object $\pi_Q(kQ)$. In other words, the cluster-tilting graph of the acyclic cluster category is connected*

Another special feature of acyclic cluster categories is that they can be recognized by the quivers of their cluster-tilting objects.

Theorem 2.9 ([KR08]). *Let \mathcal{C} be an algebraic triangulated category, which is Hom-finite and 2-Calabi-Yau. If there exists a cluster-tilting object $T \in \mathcal{C}$ such that $\text{End}_{\mathcal{C}}(T) \simeq kQ$, where Q is an acyclic quiver then there exists a triangle equivalence $\mathcal{C} \simeq \mathcal{C}_Q$ sending T on $\pi_Q(kQ)$.*

Note that an analogue of these results is not known for generalized cluster categories.

If we combine Theorems 2.7, 2.8, and 2.9, we get the following.

Corollary 2.10. *Let T be a cluster-tilting object in a Hom-finite, 2-CY algebraic triangulated category \mathcal{C} . Assume that there exists an acyclic quiver Q and a cluster-tilting object T' in \mathcal{C}_Q such that we have an isomorphism*

$$\text{End}_{\mathcal{C}_Q}(T') \simeq \text{End}_{\mathcal{C}}(T).$$

Then \mathcal{C} is triangle equivalent to \mathcal{C}_Q and this equivalence can be chosen in such a way that it sends T on T' .

Proof. By Theorem 2.8 there exists a sequence of mutation in \mathcal{C}_Q such that $\mu_s(T') = \pi_Q(kQ)$. Let $(Q', W') := \mu_{s^-}(Q, 0)$. The potential W' is rigid and, by Theorem 2.7, the endomorphism algebra $\text{End}_{\mathcal{C}}(T) = \text{End}_{\mathcal{C}_Q}(T')$ is isomorphic to the Jacobian algebra $\text{Jac}(Q', W')$. Now let T'' be the cluster-tilting object $T'' := \mu_s(T)$ in \mathcal{C} . By Theorem 2.7, since W' is rigid, we have $\text{End}_{\mathcal{C}}(T'') \simeq kQ$ and we can apply Theorem 2.9. \square

An algebra of the form $\text{End}_{\mathcal{C}_Q}(T')$, where T' is a cluster-tilting object in \mathcal{C}_Q , is called a *cluster-tilted algebra of type Q* .

3. DERIVED EQUIVALENT ALGEBRAS OF TREE CLUSTER TYPE

In this section we investigate algebras of global dimension at most 2 whose generalized cluster category is a cluster category associated to a tree. We will strongly use results from [AO10].

We start with a definition.

Definition 3.1. [AO10, Def. 5.1] Two algebras Λ_1 and Λ_2 of global dimension ≤ 2 and τ_2 -finite are said to be *cluster equivalent* if there exists a triangle equivalence $\mathcal{C}_{\Lambda_1} \rightarrow \mathcal{C}_{\Lambda_2}$ between their generalized cluster categories.

If Λ is cluster equivalent to kQ where Q is an acyclic quiver, we will say that Λ is of *cluster type* Q .

Note that since two derived equivalent algebras of global dimension ≤ 2 are cluster equivalent ([AO10, Cor. 7.16]), if the underlying graph of Q is a tree, then the cluster type does not depend on the orientation of Q .

3.1. Graded equivalence and derived equivalence. Cluster equivalence is strongly related with graded equivalence. In this subsection, we will recall some results shown in [AO10].

Let $A = \bigoplus_{p \in \mathbb{Z}} A^p$ be \mathbb{Z} -graded algebra. We denote by d the degree map sending any homogeneous element of A to its degree, and by $\mathbf{gr} A$ the category of finite generated graded modules over A . For a graded module $M = \bigoplus_{p \in \mathbb{Z}} M^p$, we denote by $M\langle q \rangle$ the graded module $\bigoplus_{p \in \mathbb{Z}} M^{p+q}$ (that is, the degree p part of $M\langle q \rangle$ is M^{p+q}). The locally bounded subcategory

$$\mathrm{Cov}(A, d) := \mathrm{add} \{A\langle p \rangle \mid p \in \mathbb{Z}\} \subseteq \mathbf{gr} A$$

is called the \mathbb{Z} -covering of A .

Definition 3.2. Let A_1 and A_2 be two \mathbb{Z} -graded algebras. Assume that A_1 and A_2 are isomorphic as algebras. We will say that A_1 and A_2 are *graded equivalent* (and write $A_1 \sim_{\mathbf{gr}} A_2$) if there exist $r_i \in \mathbb{Z}$ and an isomorphism of \mathbb{Z} -graded algebras

$$A_2 \xrightarrow[\mathbb{Z}]{\sim} \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{Cov}(A_1, \mathbb{Z})}(\bigoplus_{i=1}^n P_i\langle r_i \rangle, \bigoplus_{i=1}^n P_i\langle r_i + p \rangle)$$

where $A_1 = \bigoplus_{i=1}^n P_i$ in $\mathbf{gr} A_1$. This is equivalent to the fact that the coverings $\mathrm{Cov}(A_1, d_1)$ and $\mathrm{Cov}(A_2, d_2)$ are equivalent as categories with \mathbb{Z} -action.

The link between cluster equivalent algebras and graded equivalence algebras is given by the following result.

Theorem 3.3 ([AO10, Thm 5.8]). *Let Λ_1 and Λ_2 be two τ_2 -finite algebras of global dimension ≤ 2 . Assume there is $T \in \mathcal{D}_1$ such that $\pi_1(T)$ is basic cluster tilting in \mathcal{C}_1 , and*

- (1) *there is an isomorphism $\mathrm{End}_{\mathcal{C}_1}(\pi_1 T) \xrightarrow{\sim} \mathrm{End}_{\mathcal{C}_2}(\pi_2 \Lambda_2)$*
- (2) *this isomorphism can be chosen in such a way that the two \mathbb{Z} -gradings defined on $\overline{\Lambda}_2$, given respectively by*

$$\bigoplus_{q \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_2}(\Lambda_2, \mathbb{S}_2^{-q} \Lambda_2) \text{ and } \bigoplus_{p \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_1}(T, \mathbb{S}_2^{-p} T),$$

are equivalent.

Then the algebras Λ_1 and Λ_2 are derived equivalent, and hence cluster equivalent.

3.2. Mutation of a graded quiver with potential. In order to apply Theorem 3.3 we make use of a tool: the left (or right) mutation of a graded quiver with potential which extends the Derksen-Weyman-Zelevinsky mutation of a quiver with potential [DWZ08]. All definitions and results described in this subsection are proved in [AO10, Section 6].

Definition 3.4 ([AO10]). Let (Q, W, d) be a \mathbb{Z} -graded quiver with potential homogeneous of degree 1 (graded QP for short). Let $i \in Q_0$ be a vertex, such that there is no loops nor 2-cycles incident to i . We define $\mu_i^L(Q, W, d) = (Q', W', d')$ the *left graded mutation of (Q, W, d) at vertex i* . The quiver Q' is defined as in [DWZ08] as follows:

- for any subquiver $u \xrightarrow{a} i \xrightarrow{b} v$, with i, u , and v pairwise distinct vertices, we add an arrow $[ba]: u \rightarrow v$;
- we replace all arrows a incident with i by an arrow a^* in the opposite direction.

The potential W is also defined as in [DWZ08] by the sum $[W] + W^*$ where $[W]$ is formed from the potential W replacing all compositions ba through the vertex i by $[ba]$, where W^* is the sum $\sum b^*a^*[ba]$.

The new degree d' is defined as follows:

- $d'(a) = d(a)$ if a is an arrow of Q and Q' ;
- $d'([ba]) = d(b) + d(a)$ if ba is a composition passing through i ;
- $d'(a^*) = -d(a) + 1$ if the target of a is i ;
- $d'(b^*) = -d(b)$ if the source of b is i .

One can check that this operation is compatible with the reduction of a QP (see [AO10]). It is possible to define the right graded mutation μ_i^R by interchanging target and source in the last two items in the definition. We have the following.

Lemma 3.5 ([AO10]). *Let (Q, W, d) be a graded quiver with potential. Then we have an isomorphism of \mathbb{Z} -graded algebras*

$$\text{Jac}(Q, W, d) \simeq_{\mathbb{Z}} \text{Jac}(\mu_i^R \mu_i^L(Q, W, d)).$$

Moreover, this mutation preserves the graded equivalence. More precisely, we have the following.

Proposition 3.6. [AO10] *Let (Q, W, d_1) and (Q, W, d_2) be two \mathbb{Z} -graded QP such that the graded Jacobian algebras $\text{Jac}(Q, W, d_1)$ and $\text{Jac}(Q, W, d_2)$ are graded equivalent. Then the graded Jacobian algebras $\text{Jac}(\mu_i^L(Q, W, d_1))$ and $\text{Jac}(\mu_i^L(Q, W, d_2))$ are graded equivalence.*

Definition 3.7. [AO10] Let Λ be a τ_2 -finite algebra of global dimension ≤ 2 . Let $T = T_1 \oplus \dots \oplus T_n$ be an object in $\mathcal{D}^b(\Lambda)$ such that $\pi(T)$ is a (basic) cluster-tilting object in \mathcal{C}_Λ . Let T_i be an indecomposable summand of $T \simeq T' \oplus T_i$. Define T_i^L as the cone in $\mathcal{D}^b(\Lambda)$ of the minimal left $\text{add}\{\mathbb{S}_2^p T', p \in \mathbb{Z}\}$ -approximation $u: T_i \rightarrow B$ of T_i . We denote by $\mu_i^L(T)$ the object $T_i^L \oplus T'$ and call it the *left mutation of T at T_i* .

It is immediate to check that we have $\pi(\mu_i^L(T)) = \mu_i(\pi(T))$, thus that $\pi(\mu_i^L(T))$ is a cluster-tilting object in \mathcal{C}_Λ .

This (left) mutation in the derived category is reflected by the graded (left) mutation of graded QP in the following sense.

Theorem 3.8 ([AO10, Thm 6.8]). *Let Λ and $T \in \mathcal{D}^b(\Lambda)$ as above. Assume that there exist a \mathbb{Z} -graded QP (Q, W, d) with rigid potential homogeneous of degree 1 such that we have an isomorphism of graded algebras*

$$\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(T, \mathbb{S}_2^p T) \xrightarrow[\mathbb{Z}]{} \text{Jac}(Q, W, d).$$

Let $i \in Q_0$ and T_i be the associated indecomposable summand of $T \simeq T_i \oplus T'$. Then there is an isomorphism of \mathbb{Z} -graded algebras

$$\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}}(T' \oplus T_i^L, \mathbb{S}_2^{-p}(T' \oplus T_i^L)) \xrightarrow[\mathbb{Z}]{\sim} \text{Jac}(\mu_i^L(Q, W, d)).$$

This result is particularly useful when we note that the \mathbb{Z} -grading of $\overline{\Lambda}$ yields a grading on the quiver \overline{Q} .

Definition 3.9. Let (Q, W) be a reduced quiver with potential. A grading d on Q will be said to be a W -grading if

- for all arrows a in Q_1 , $d(a) \in \{0, 1\}$;
- the potential W is homogeneous of degree 1;
- the set of relations $\{\partial_a W \mid d(a) = 1\}$ is linearly independent, in particular for all $a \in Q_1$ such that $d(a) = 1$ then we have $\partial_a W \neq 0$.

Proposition 3.10 ([AO10]). Let $\Lambda = kQ_\Lambda/I$ be an algebra of global dimension ≤ 2 which is τ_2 -finite. Then there exists a quiver with potential (\overline{Q}, W) and a W -grading d such that we have an isomorphism of graded algebras

$$\overline{\Lambda} \simeq_{\mathbb{Z}} \text{Jac}(\overline{Q}, W, d).$$

Moreover, if Λ is of acyclic cluster type, then W is rigid.

The quiver with potential is given by Theorem 2.5 and the arrows of Q_Λ have degree zero and the arrows a_i corresponding to the relations have degree 1. The rigidity comes from Theorem 2.7 and 2.8.

3.3. Application to acyclic cluster type. Let Λ be an algebra of cluster type Q where Q is an acyclic quiver. We have the following diagram

$$\begin{array}{ccc} \mathcal{D}^b(\Lambda) & & \mathcal{D}^b(kQ) \\ \downarrow \pi_\Lambda & & \downarrow \pi_Q \\ \mathcal{C}_\Lambda & \xrightarrow[\sim]{f} & \mathcal{C}_Q \end{array}$$

By Theorem 2.4, the object $f(\pi_\Lambda(\Lambda))$ is a cluster-tilting object in \mathcal{C}_Q . By Theorem 2.8, there exists a sequence of mutations such that the endomorphism algebra $\text{End}_{\mathcal{C}_Q}(\mu_s(f(\pi_\Lambda(\Lambda))))$ is isomorphic to kQ . The endomorphism algebra $\text{End}_{\mathcal{C}_Q}(f(\pi_\Lambda(\Lambda))) \simeq \overline{\Lambda}$ is a graded Jacobian algebra $\text{Jac}(Q_{\overline{\Lambda}}, W, d)$ by Proposition 3.10 with W rigid.

Definition 3.11. The map $d_s: Q \rightarrow \mathbb{Z}$ is called a *grading induced by Λ* if it satisfies

$$\mu_s^L(Q_{\overline{\Lambda}}, W, d) = (Q, 0, d_s),$$

where s is a sequence of mutations as in the situation above.

Proposition 3.12. The grading induced by Λ on Q is unique up to graded equivalence and automorphism of Q .

Proof. Let s and t be two sequences of mutations such that the endomorphism algebras $\text{End}_{\mathcal{C}_Q}(\mu_s(f(\pi_\Lambda(\Lambda))))$ and $\text{End}_{\mathcal{C}_Q}(\mu_t(f(\pi_\Lambda(\Lambda))))$ are isomorphic to kQ . Denote by S (resp. T) the object $\mu_s^L(\Lambda)$ (resp. $\mu_t^L(\Lambda)$) of $\mathcal{D}^b(\Lambda)$. The objects $f(\pi_\Lambda(S))$ and $f(\pi_\Lambda(T))$ are cluster-tilting objects in \mathcal{C}_Q with endomorphism algebra isomorphic to kQ . Then we use the following easy fact.

Lemma 3.13. *Let $R = R_1 \oplus \cdots \oplus R_n$ be a cluster-tilting object in \mathcal{C}_Q such that $\text{End}_{\mathcal{C}_Q}(R) \simeq kQ$, then there exists an automorphism $\varphi: Q \rightarrow Q$ of Q and $p \in \mathbb{Z}$ such that $R_i \simeq \pi_Q(P_{\varphi(i)}[p])$, where P_i is the indecomposable projective $P_i := e_i kQ$.*

We deduce from this lemma that there exists $p \in \mathbb{Z}$ and $\varphi: Q \rightarrow Q$ such that

$$\pi_\Lambda(S_i) \simeq \pi_\Lambda(T_{\varphi(i)}[p]) = \pi_\Lambda(T_{\varphi(i)}[p]).$$

Therefore the automorphism φ induces an equivalence of \mathbb{Z} -graded categories

$$\pi_\Lambda^{-1}(\pi_\Lambda(S)) = \text{add} \{ \mathbb{S}_2^q S \mid q \in \mathbb{Z} \} \simeq \text{add} \{ \mathbb{S}_2^q T[p] \mid q \in \mathbb{Z} \}.$$

The left \mathbb{Z} -category is naturally equivalent (as \mathbb{Z} -category) to the category $\pi_\Lambda^{-1}(\pi_\Lambda(T)) = \text{add} \{ \mathbb{S}_2^q T \mid q \in \mathbb{Z} \}$. Therefore φ gives rise to a graded equivalence

$$\bigoplus_{q \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\Lambda)}(S, \mathbb{S}_2^{-q} S) \sim_{\text{gr}} \bigoplus_{q \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\Lambda)}(T, \mathbb{S}_2^{-q} T).$$

That is, we get a graded equivalence $(kQ, d_s) \sim_{\text{gr}} (kQ, d_t)$ where d_s (resp. d_t) is the grading induced by the sequence s (resp. t). \square

Using the previous results we can prove the following.

Theorem 3.14. *Let Λ_1 and Λ_2 be two algebras of cluster type Q where Q is an acyclic quiver. Let d_1 (resp. d_2) be a grading on kQ induced by Λ_1 (resp. Λ_2). Then the following are equivalent.*

- (1) *The \mathbb{Z} -graded algebras (kQ, d_1) and (kQ, d_2) are graded equivalent.*
- (2) *There exists a derived equivalence $\mathcal{D}^b(\Lambda_1) \rightarrow \mathcal{D}^b(\Lambda_2)$ which makes the following diagram commutative*

$$\begin{array}{ccc} \mathcal{D}^b(\Lambda_1) & \xrightarrow{\sim} & \mathcal{D}^b(\Lambda_2) \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \mathcal{C}_{\Lambda_1} & \xrightarrow[\sim]{f_1} \mathcal{C}_Q \xleftarrow[\sim]{f_2} & \mathcal{C}_{\Lambda_2} \end{array}$$

Proof. For $i = 1, 2$ denote by (Q_i, W_i, ∂_i) the graded QP associated to $\overline{\Lambda}_i$.

(1) \Rightarrow (2) For $i = 1, 2$, let s_i be a sequence of mutations such that $\mu_{s_i}^L(Q_i, W_i, \partial_i) = (Q, 0, d_i)$. Thus by assumption, we have a graded equivalence

$$\text{Jac}(\mu_{s_1}^L(Q_1, W_1, \partial_1)) \sim_{\text{gr}} \text{Jac}(\mu_{s_2}^L(Q_2, W_2, \partial_2)).$$

Then by Proposition 3.6 and Lemma 3.5, we immediately get that

$$\text{Jac}(\mu_{s_2}^R \mu_{s_1}^L(Q_1, W_1, \partial_1)) \sim_{\text{gr}} \text{Jac}(Q_2, W_2, \partial_2).$$

Now denote by $T \in \mathcal{D}^b(\Lambda_1)$ the object $\mu_{s_2}^R \mu_{s_1}^L(\Lambda_1)$. By Theorem 3.8 we have an isomorphism of \mathbb{Z} -graded algebra

$$\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\Lambda_1)}(T, \mathbb{S}_2^{-p} T) \simeq_{\mathbb{Z}} \text{Jac}(\mu_{s_2}^R \mu_{s_1}^L(Q_1, W_1, \partial_1))$$

Therefore, we get the result by Theorem 3.3.

(2) \Rightarrow (1) Assume that Λ_1 and Λ_2 are derived equivalent, then there exists a tilting complex T in $\mathcal{D}^b(\Lambda_2)$ such that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{D}^b(\Lambda_1) & \xrightarrow[\sim]{-\overset{L}{\otimes}_{\Lambda_1} T} & \mathcal{D}^b(\Lambda_2) \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ \mathcal{C}_{\Lambda_1} & \xrightarrow[\sim]{f} & \mathcal{C}_{\Lambda_2} \end{array}$$

Since the cluster category \mathcal{C}_{Λ_2} is acyclic, there exists a sequence s such that $\mu_s(\pi_2 \Lambda_2) = \pi_2(T) = f(\pi_1 \Lambda_1)$. Denote by $T' \in \mathcal{D}^b(\Lambda_2)$ the object $T' := \mu_s^L(\Lambda_2)$. Then we have $\pi_2(T) = \pi_2(T')$, thus we have an equivalence of categories with \mathbb{S}_2 -action

$$\text{add}\{\mathbb{S}_2^p T \mid p \in \mathbb{Z}\} \simeq \text{add}\{\mathbb{S}_2^p T' \mid p \in \mathbb{Z}\}$$

This exactly means that there is a graded equivalence

$$\bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\Lambda_2)}(T, \mathbb{S}_2^{-p} T) \underset{\text{gr}}{\sim} \bigoplus_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{D}^b(\Lambda_2)}(T', \mathbb{S}_2^{-p} T').$$

Since $-\overset{L}{\otimes}_{\Lambda_1} T$ is an equivalence, the left term is isomorphic as \mathbb{Z} -graded algebra to $\text{Jac}(Q_1, W_1, \partial_1)$ and by Theorem 3.8 the left term is isomorphic to the \mathbb{Z} -graded algebra $\text{Jac}(\mu_s^L(Q_2, W_2, d_2))$. Therefore we have

$$(*) \quad \text{Jac}(\mu_s^L(Q_2, W_2, d_2)) \underset{\text{gr}}{\sim} \text{Jac}(Q_1, W_1, \partial_1).$$

Now let s' be a sequence such that the cluster-tilting object $T'' := \mu_{s'}(f\pi_1 \Lambda_1)$ has endomorphism algebra isomorphic to kQ . Then we have

$$\mu_{s'}(Q_1, W_1) = (Q, 0) \text{ and } \mu_{s'} \mu_s(Q_2, W_2) = (Q, 0).$$

Now let d_1 and d_2 be the gradings on Q such that we have

$$\mu_{s'}^L(Q_1, W_1, \partial_1) = (Q, 0, d_1) \text{ and } \mu_{s'}^L \mu_s^L(Q_2, W_2, d_2) = (Q, 0, d_2).$$

Now by $(*)$ and Proposition 3.6, we get

$$\text{Jac}(Q, 0, d_1) \underset{\text{gr}}{\sim} \text{Jac}(Q, W, d_2).$$

Now by Proposition 3.12, any such two gradings will be graded equivalent and we get (1). \square

Corollary 3.15. *If Q is a tree, and if Λ is of cluster type Q , then Λ is derived equivalent to kQ .*

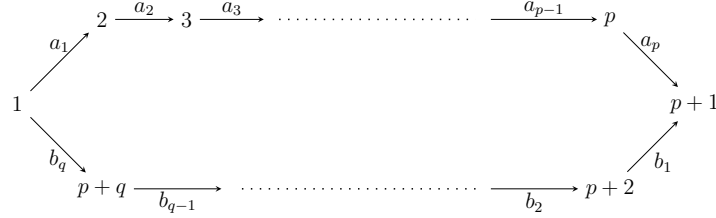
Proof. By Theorem 3.14, it is enough to observe that if Q is a connected tree, all gradings on kQ are equivalent. We consider the map

$$\begin{array}{ccc} \mathbb{Z}^{Q_0} & \rightarrow & \mathbb{Z}^{Q_1} \\ (r_i)_{i \in Q_0} & \mapsto & (r_{t(a)} - r_{s(a)})_{a \in Q_1} \end{array}.$$

Then the grading on Q which are equivalent to the trivial grading are clearly the image of this map. Moreover, the kernel of this map is isomorphic to \mathbb{Z} . If Q is a connected tree, then $\#Q_0 = \#Q_1 + 1$ and so this map is surjective. \square

4. DERIVED EQUIVALENCE CLASSES OF ALGEBRAS OF CLUSTER TYPE $\tilde{A}_{p,q}$

In this section we study the algebras of cluster type $\tilde{A}_{p,q}$, i.e. all algebras which are cluster equivalent to the path algebra H of the acyclic quiver Q_H :



Since Q_H is not a tree, these algebras need not be derived equivalent. In this section we introduce an invariant of an algebra of cluster type Q_H which distinguishes its class of derived equivalence.

4.1. The weight of an algebra of cluster type $\tilde{A}_{p,q}$.

Definition 4.1. Let d be a \mathbb{Z} -grading on Q_H . We define the weight of the grading d by

$$w(H, d) := \sum_{i=1}^p d(a_i) - \sum_{i=1}^q d(b_i)$$

Lemma 4.2. Let d_1 and d_2 be two \mathbb{Z} -gradings on H . Then the following are equivalent:

- (1) (H, d_1) and (H, d_2) are graded equivalent;
- (2) $w(H, d_1) = w(H, d_2)$.

Proof. We consider the map

$$\begin{aligned} \phi: \quad \mathbb{Z}^{(Q_H)_0} &\rightarrow \mathbb{Z}^{(Q_H)_1} \\ (r_i)_{i \in (Q_H)_0} &\mapsto (r_{t(a)} - r_{s(a)})_{a \in (Q_H)_1} \end{aligned}$$

The gradings d_1 and d_2 are equivalent if and only if $d_1 - d_2$ is in the image of ϕ . The kernel of ϕ is isomorphic to \mathbb{Z} . Since Q_H is of type $\tilde{A}_{p,q}$, we have $\sharp(Q_H)_0 = \sharp(Q_H)_1$, therefore the cokernel of this map is also isomorphic to \mathbb{Z} . The weight can be seen as a map $\mathbb{Z}^{(Q_H)_1} \rightarrow \mathbb{Z}$. It is immediate to check that the composition $w\phi$ vanishes. Therefore $w(H, d_1) = w(H, d_2)$ if and only if d_1 and d_2 are equivalent. \square

This result allows us to define the following.

Definition 4.3. Let Λ be an algebra of global dimension at most 2 and of cluster type $\tilde{A}_{p,q}$. Let d be an grading on Q_H induced by Λ . Then define the *weight* of Λ by

$$w(\Lambda) := w(d) = \sum_{i=1}^p d(a_i) - \sum_{i=1}^q d(b_i).$$

If $p > q$, the weight of Λ is well defined. Indeed in this case, there is no automorphism of Q_H . Hence by Proposition 3.12 the grading is unique up to graded equivalence. And by Lemma 4.2 two different induced gradings give the same weight.

Theorem 4.4. Let Λ_1 and Λ_2 be two algebras of cluster type $\tilde{A}_{p,q}$ with $p > q$. Then there is a derived equivalence between Λ_1 and Λ_2 if and only if $w(\Lambda_1) = w(\Lambda_2)$.

Proof. Let Q_H be an orientation of $\tilde{A}_{p,q}$ and let δ_1 (resp. δ_2) be a grading on Q_H induced by Λ_1 (resp. Λ_2). Consider the following statements:

- (a) Λ_1 and Λ_2 are derived equivalent;
- (b) the gradings δ_1 and δ_2 are equivalent;
- (c) $w(\Lambda_1) = w(\Lambda_2)$.

By Theorem 3.14 we have $(a) \Leftrightarrow (b)$. By Lemma 4.2 we have $(b) \Leftrightarrow (c)$. \square

Corollary 4.5. *Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$. Then Λ is piecewise hereditary if and only if $w(\Lambda) = 0$.*

Proof. Let us treat the case where $p > q$. The ‘if’-part of the assertion is a direct consequence of Theorem 4.4 applied for $\Lambda_1 = \Lambda$ and $\Lambda_2 = H$.

The algebra Λ is piecewise hereditary if and only if the derived category $\mathcal{D}^b(\Lambda)$ is equivalent to $\mathcal{D}^b(\mathcal{H})$ for some hereditary category \mathcal{H} . Therefore it implies that the generalized cluster category \mathcal{C}_Λ is equivalent to the cluster category $\mathcal{C}_\mathcal{H}$. Since Λ is of cluster type $\tilde{A}_{p,q}$, we have $\mathcal{C}_\mathcal{H} \simeq \mathcal{C}_{\tilde{A}_{p,q}}$. Hence we get $\mathcal{D}^b(\mathcal{H}) \simeq \mathcal{D}^b(\tilde{A}_{p,q})$. Therefore by Theorem 4.4 we have $w(\Lambda) = 0$.

The case $p = q$ is a direct consequence of the result below. \square

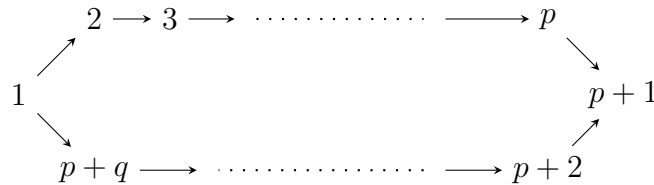
4.2. The case $p = q$. In the case $p = q$ there is a unique non-trivial automorphism of Q_H fixing the vertices 1 and $p + 1$ and interchanging the vertices i and $2p + 2 - i$ for $i = 2, \dots, p$. This automorphism induces a derived equivalence between algebras Λ_1 and Λ_2 of cluster type $\tilde{A}_{p,p}$ such that $w(\Lambda_1) = -w(\Lambda_2)$. Therefore we obtain the following result, whose proof is the same as Theorem 4.4.

Theorem 4.6. *Let Λ_1 and Λ_2 be two algebras of cluster type $\tilde{A}_{p,p}$. Then there is a derived equivalence between Λ_1 and Λ_2 if and only if $|w(\Lambda_1)| = |w(\Lambda_2)|$.*

5. THE AUSLANDER-REITEN QUIVER OF THE DERIVED CATEGORY OF AN ALGEBRA OF CLUSTER TYPE $\tilde{A}_{p,q}$

Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$ and of weight $w \neq 0$. In this section we compute the Auslander-Reiten quiver of the derived category $\mathcal{D}^b(\Lambda)$. In order to do that, we use some results of [AO10, Section 8].

5.1. Graded derived equivalence. Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$. Then, by Proposition 3.10, there exists a graded quiver with reduced potential (\overline{Q}, W, d) such that we have an isomorphism of \mathbb{Z} -graded algebra $\overline{\Lambda} \xrightarrow[\mathbb{Z}]{\simeq} \text{Jac}(\overline{Q}, W, d)$. Let ∂ be a grading induced by Λ on H where Q_H is the following quiver



Then there exists a sequence of mutation s such that $\mu_s(\overline{Q}, W, d) = (Q_H, 0, \partial)$. Now define the \mathbb{Z}^2 -graded quiver with reduced potential $(Q', W', (d', \delta))$ by the following

$$(Q', W', (d', \delta)) := \mu_{s-}^R(Q_H, 0, (\partial, 0)).$$

Then by Lemma 3.5 we have an isomorphism of \mathbb{Z} -graded algebra $\text{Jac}(Q', W', d') \simeq_{\mathbb{Z}} \text{Jac}(\overline{Q}, W, d)$. Since the potentials W and W' are reduced, the quivers \overline{Q} and Q' are isomorphic. By definition of \overline{Q} and d , the quiver Q_Λ of the algebra Λ is the subquiver of \overline{Q} satisfying $(Q_\Lambda)_0 = \overline{Q}_0$ and $(Q_\Lambda)_1 = \{a \in \overline{Q}_1 \mid d(a) = 0\}$. Moreover we have an isomorphism $\Lambda \simeq kQ_\Lambda / \langle \partial_a W, a \in \overline{Q}_1, d(a) = 1 \rangle$. Since we have an isomorphism of \mathbb{Z} -graded algebra $\text{Jac}(Q', W', d') \simeq_{\mathbb{Z}} \text{Jac}(\overline{Q}, W, d)$, we get an isomorphism $\Lambda \simeq kQ_\Lambda / \langle \partial_a W', a \in Q'_1, d'(a) = 1 \rangle$. The grading δ , which is a grading on $Q' \simeq \overline{Q}$, restricts on a grading on Q_Λ . The grading δ makes W' homogeneous of degree 1. Hence the relations $\partial_a W', a \in Q'_1$ with $d'(a) = 1$ are homogeneous with respect to the degree δ . Consequently δ yields a grading on the algebra Λ that we still denote by δ .

Then we have the following direct consequence of [AO10, Theorem 8.3].

Theorem 5.1 ([AO10]). *In the setup above, there exists a triangle equivalence*

$$\mathcal{D}^b(\text{Cov}(\Lambda, \delta)) \xrightarrow{\sim} \mathcal{D}^b(\text{Cov}(H, \partial))$$

Moreover we have an isomorphism of triangle functors $F \circ \langle 1 \rangle_\delta \simeq \mathbb{S}_2^{-1} \circ \langle -1 \rangle_\partial \circ F$.

Remark 5.2. Note that in this situation, the compatibility condition defined in [AO10, Definition 8.5] is automatically satisfied since \overline{Q} is mutation acyclic and since W is rigid. Indeed, two \mathbb{Z} -gradings on a quiver induce a \mathbb{Z}^2 -grading on it. But in general two \mathbb{Z} -gradings on an algebra do not give rise to a \mathbb{Z}^2 -grading on it.

Theorem 5.1 implies the following result which will be very useful to compute explicitly the Auslander-Reiten quiver of $\mathcal{D}^b(\Lambda)$.

Corollary 5.3. *There exists a k -linear equivalence*

$$\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\text{Cov}(H, \partial)) / \mathbb{S}_2 \langle 1 \rangle_\partial$$

Proof. By Theorem 5.1 we deduce that there is a k -linear equivalence between the orbit categories

$$\mathcal{D}^b(\text{Cov}(\Lambda, \delta)) / \langle 1 \rangle_\delta \simeq \mathcal{D}^b(\text{Cov}(H, \partial)) / \mathbb{S}_2 \langle 1 \rangle_\partial.$$

Now we can use the following result due to Keller

Theorem 5.4 ([Kel05]). *Let Q_H be an acyclic quiver and ∂ be a \mathbb{Z} -grading on Q_H .*

Let $F := - \overset{L}{\otimes}_{\text{Cov}(H, \partial)} X$ be an auto-equivalence of $\mathcal{D}^b(\text{Cov}(H, \partial))$ for some object $X \in \mathcal{D}(\text{Cov}(H, \partial))$. Assume that

- (1) *For each indecomposable $U \in \text{mod Cov}(H, \partial)$, the set $\{i \in \mathbb{Z} \mid F^i U \in \text{mod Cov}(H, \partial)\}$ is finite.*
- (2) *There exists $N \geq 0$ such that for each indecomposable $X \in \mathcal{D}^b(\text{Cov}(H, \partial))$, there exists $0 \leq n \leq N$ and $i \in \mathbb{Z}$ with $F^i X[-n] \in \text{mod Cov}(H, \partial)$.*

Then the orbit category $\mathcal{D}^b(\text{Cov}(H, \partial)) / F$ admits a natural triangulated structure such that the projection functor $\mathcal{D}^b(\text{Cov}(H, \partial)) \rightarrow \mathcal{D}^b(\text{Cov}(H, \partial)) / F$ is triangulated.

It is already shown in [BMR⁺06] that the functor \mathbb{S}_2 satisfies conditions (1) and (2). Since the functor $\langle 1 \rangle_\partial$ is an auto-equivalence of $\text{mod Cov}(H, \partial)$, then the functor $\mathbb{S}_2 \langle 1 \rangle_\partial$

clearly satisfies again condition (1) and (2). Thus the orbit category $(\mathcal{D}^b(\text{Cov}(\Lambda, \delta)) / \langle 1 \rangle_\delta) \simeq \mathcal{D}^b(\text{Cov}(H, \partial)) / \mathbb{S}_2 \langle 1 \rangle_\partial$ is triangulated and the natural functor

$$\mathcal{D}^b(\text{Cov}(\Lambda, \delta)) \longrightarrow \mathcal{D}^b(\text{Cov}(\Lambda, \delta)) / \langle 1 \rangle_\delta$$

is a triangle functor.

By [AO10, Corollary 7.14], the derived category $\mathcal{D}^b(\Lambda)$ is equivalent to the triangulated hull $(\mathcal{D}^b(\text{Cov}(\Lambda, \delta)) / \langle 1 \rangle_\delta)_\Delta$. Hence we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}^b(\text{Cov}(\Lambda, \delta)) & \xrightarrow{A} & \mathcal{D}^b(\text{Cov}(\Lambda, \delta)) / \langle 1 \rangle_\delta \xrightarrow{B} \mathcal{D}^b(\Lambda) \\ & \searrow C & \nearrow \end{array}$$

where A and C are triangle functors. Therefore B commutes with the shift and sends triangles to triangles. Hence the fully faithful functor B is a k -equivalence

$$\mathcal{D}^b(\text{Cov}(\Lambda, \delta)) / \langle 1 \rangle_\delta \simeq \mathcal{D}^b(\text{Cov}(H, \partial)) / \mathbb{S}_2 \langle 1 \rangle_\partial. \quad \square$$

5.2. The shape of the Auslander-Reiten quiver of $\mathcal{D}^b(\Lambda)$. In this subsection we use Corollary 5.3 to compute explicitly the shape of the Auslander-Reiten quiver of the derived category of an algebra of cluster type $\tilde{A}_{p,q}$.

Throughout this subsection Λ is an algebra of cluster type $\tilde{A}_{p,q}$ and of weight $w \neq 0$. Let ∂ be a grading induced by Λ on Q_H . By Lemma 4.2 we can assume that ∂ is the following grading.



By Corollary 5.3 we have a k -linear equivalence

$$\mathcal{D}^b(\Lambda) \simeq \mathcal{D}^b(\text{Cov}(H, \partial)) / \mathbb{S}_2 \langle 1 \rangle_\partial.$$

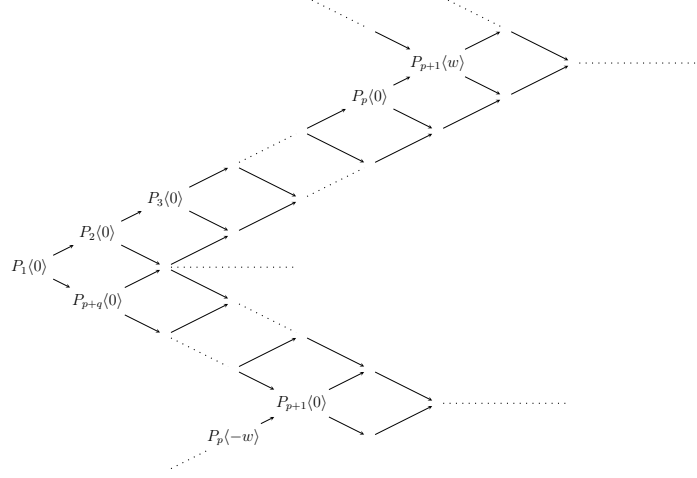
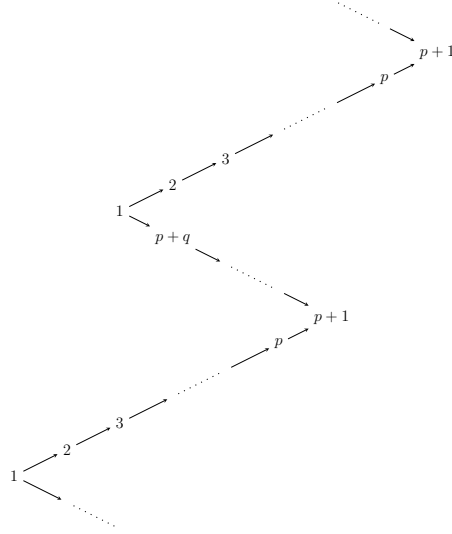


FIGURE 5.1. Shape of the component \mathcal{P}_0 of the Auslander-Reiten quiver of $\text{mod}(\text{Cov}(H, \partial))$.

Therefore the quiver of the subcategory $\text{Cov}(H, \partial) = \text{add}\{H\langle p\rangle_\partial, p \in \mathbb{Z}\} \subset \text{gr } H$ consists of $|w|$ connected components of type A_∞^∞ with the following orientation.



Then it is not hard to compute the Auslander-Reiten quiver of the module category $\text{mod Cov}(H, \partial)$. It has $4|w|$ connected components: \mathcal{P}_i , \mathcal{R}_i^p , \mathcal{R}_i^q , and \mathcal{I}_i with $i \in \mathbb{Z}/w\mathbb{Z}$ satisfying $\mathcal{P}_i = \mathcal{P}_0\langle i\rangle$, $\mathcal{R}_i^p = \mathcal{R}_0^p\langle i\rangle$, $\mathcal{R}_i^q = \mathcal{R}_0^q\langle i\rangle$ and $\mathcal{I}_i = \mathcal{I}_0\langle i\rangle$ for any $i \in \mathbb{Z}/w\mathbb{Z}$.

The component \mathcal{P}_0 is of type NA_∞^∞ and contains the indecomposable projective modules $P_j\langle 0\rangle$ for $j = 1, \dots, p+q$. Its shape is described in Figure 5.1.

The component \mathcal{I}_0 is of type $(-\mathbb{N})A_\infty^\infty$ and contains the indecomposable injective modules $I_j\langle 0\rangle$ for $j = 1, \dots, p+q$. Its shape is described in Figure 5.2.

The components \mathcal{R}_0^p and \mathcal{R}_0^q are of type $\mathbb{Z}A_\infty^\infty$ and contain the regular modules. The shape of the connected components \mathcal{R}_0^p and \mathcal{R}_0^q are described in Figures 5.3 and 5.4. In these figures, $S_i\langle m\rangle$ is the m -degree-shift of the simple module associated with the vertex

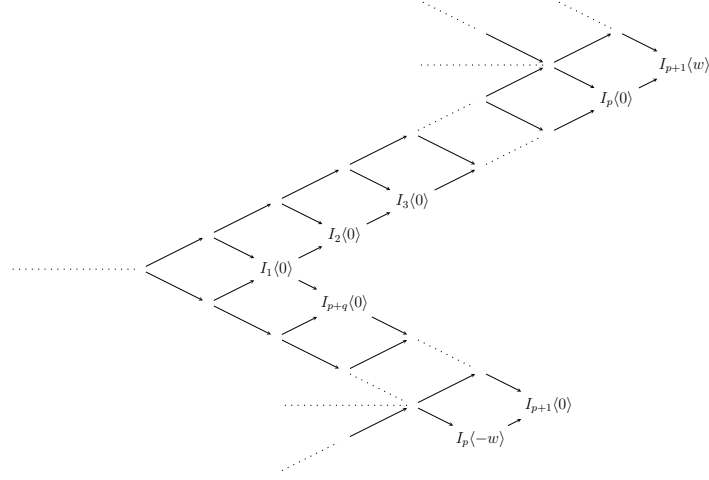


FIGURE 5.2. Shape of the component \mathcal{I}_0 of the Auslander-Reiten quiver of $\text{mod}(\text{Cov}(H, \partial))$.

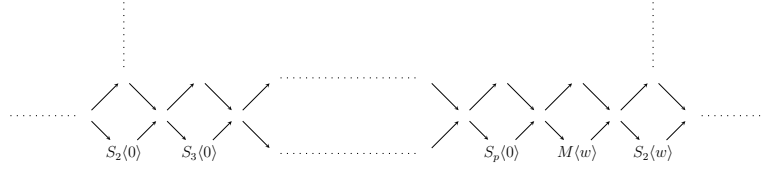


FIGURE 5.3. Shape of the connected component \mathcal{R}_0^p .

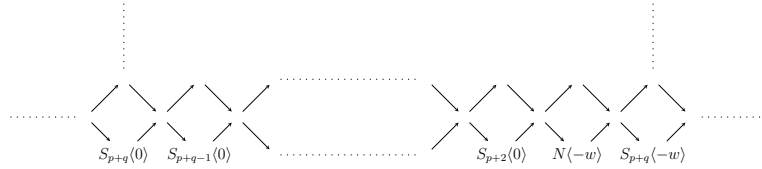


FIGURE 5.4. Shape of the connected component \mathcal{R}_0^q .

i , the object $M\langle 0 \rangle$ is defined by the exact sequence

$$P_p\langle -w \rangle \xrightarrow{a_p} P_{p+1}\langle 0 \rangle \longrightarrow M\langle 0 \rangle \longrightarrow 0 ,$$

and the object $N\langle 0 \rangle$ is defined by the exact sequence

$$P_{p+2}\langle 0 \rangle \xrightarrow{b_q} P_p\langle 0 \rangle \longrightarrow N\langle 0 \rangle \longrightarrow 0 .$$

Under the forgetful functor $\text{mod Cov}(H, \partial) \rightarrow \text{mod } H$, the components \mathcal{P}_i are sent to \mathcal{P} the preprojective component of $\text{mod } H$, the components \mathcal{I}_i are sent to the preinjective component \mathcal{I} , the components \mathcal{R}_j^p are sent to the exceptional tube of rank p and the components \mathcal{R}_j^q are sent to the exceptional tube of rank q . The indecomposable H -modules lying in the homogeneous tubes are not gradable if $w \neq 0$, therefore the description above is complete.

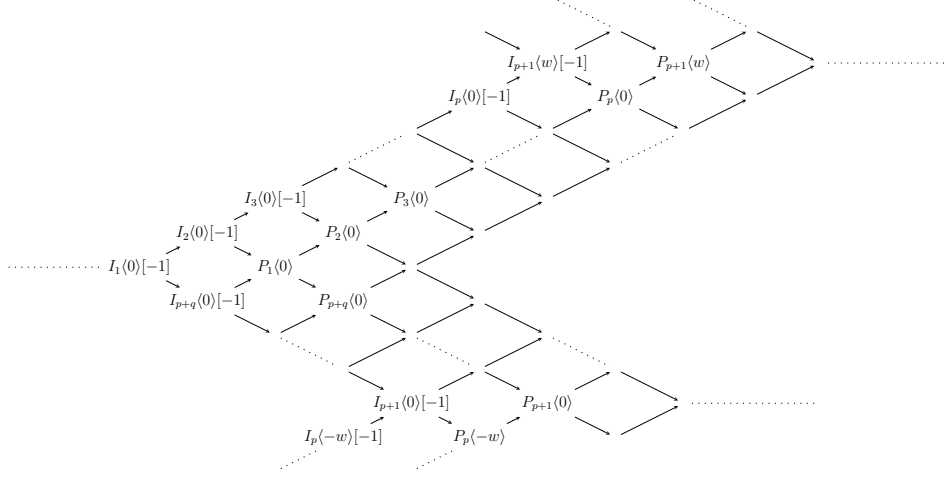


FIGURE 5.5. Shape of the component $\mathcal{P}_{(0,0)}$ of the Auslander-Reiten quiver of $\mathcal{D}^b(\text{Cov}(H, \partial))$.

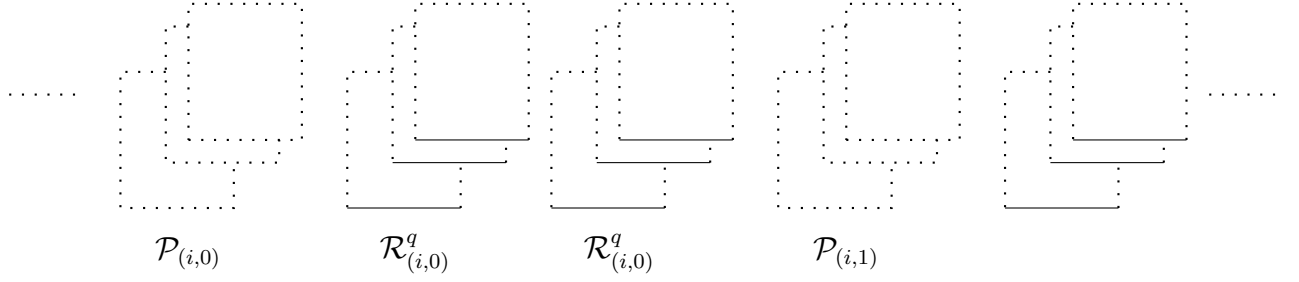


FIGURE 5.6. Auslander-Reiten quiver of $\mathcal{D}^b(\text{Cov}(H, \partial))$

Since $\text{mod Cov}(H, \partial)$ is a hereditary category, one can easily deduce the shape of the Auslander-Reiten quiver of $\mathcal{D}^b(\text{Cov}(H, \partial))$. It has three kinds of components $\mathcal{P}_{(i,n)}$, $\mathcal{R}_{(i,n)}^p$, and $\mathcal{R}_{(i,n)}^q$, with $i \in \mathbb{Z}/w\mathbb{Z}$ and $n \in \mathbb{Z}$. The component $\mathcal{P}_{(0,0)}$ contains \mathcal{P}_0 and $\mathcal{I}_0[-1]$ and is described in Figure 5.5. The component $\mathcal{R}_{(0,0)}^p$ (resp. $\mathcal{R}_{(0,0)}^q$) is \mathcal{R}_0^p (resp. \mathcal{R}_0^q). Moreover we have $\mathcal{P}_{(i,n)} = \mathcal{P}_{(0,0)}\langle i \rangle[n]$, $\mathcal{R}_{(i,n)}^p = \mathcal{R}_{(0,0)}^p\langle i \rangle[n]$, and $\mathcal{R}_{(i,n)}^q = \mathcal{R}_{(0,0)}^q\langle i \rangle[n]$ for $i \in \mathbb{Z}/w\mathbb{Z}$ and $n \in \mathbb{Z}$.

The morphisms satisfy the following:

$$\begin{aligned} \text{Hom}(\mathcal{P}_{(i,n)}, \mathcal{P}_{(j,m)}) &\neq 0 && \text{if and only if } i = j \text{ and } m \in \{n, n+1\} \\ \text{Hom}(\mathcal{P}_{(i,n)}, \mathcal{R}_{(j,m)}^{p,q}) &\neq 0 && \text{if and only if } i = j \text{ and } m = n \\ \text{Hom}(\mathcal{R}_{(i,n)}^{p,q}, \mathcal{P}_{(j,m)}) &\neq 0 && \text{if and only if } i = j \text{ and } m = n+1 \\ \text{Hom}(\mathcal{R}_{(i,n)}^{p,q}, \mathcal{R}_{(j,m)}^{p,q}) &\neq 0 && \text{if and only if } i = j \text{ and } m \in \{n, n+1\} \end{aligned}$$

Now we can compute the Auslander-Reiten quiver of the orbit category $\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle$. We look at the action of the functor $\mathbb{S}_2\langle 1 \rangle$ on the connected components of the AR-quiver of $\mathcal{D}^b(\text{Cov}(H, \partial))$. The functor $\mathbb{S}_2\langle 1 \rangle$ acts on the set $\{\mathcal{P}_{(i,n)}, i \in \mathbb{Z}/w\mathbb{Z}, n \in \mathbb{Z}\}$ by

$$\mathbb{S}_2(\mathcal{P}_{(i,n)})\langle 1 \rangle = \tau(\mathcal{P}_{(i,n)})[-1]\langle 1 \rangle = \mathcal{P}_{(i,n)}[-1]\langle 1 \rangle = \mathcal{P}_{(i+1, n-1)}$$

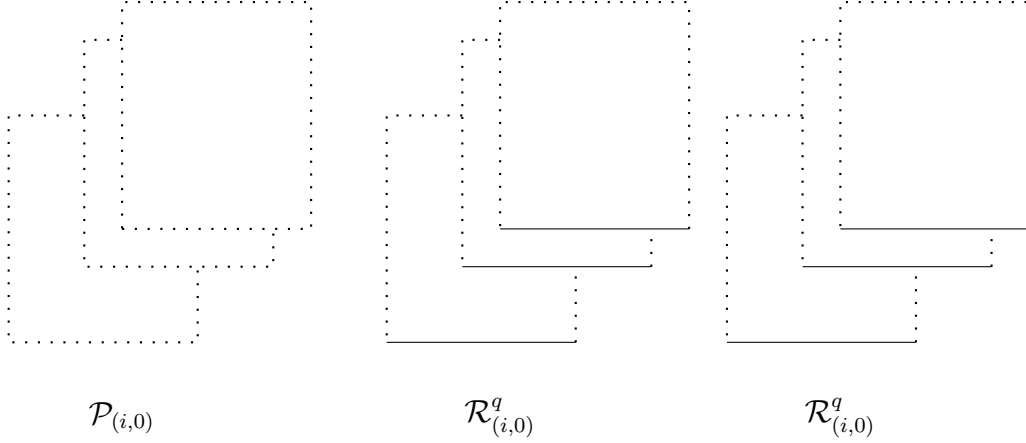


FIGURE 5.7. Auslander-Reiten quiver of the category $\mathcal{D}^b \Lambda$, where Λ is of cluster type $\tilde{A}_{p,q}$ and of weight $w > 0$.

Then there are exactly $|w|$ orbits which are $(\mathbb{S}_2\langle 1 \rangle)^{\mathbb{Z}} \cdot \mathcal{P}_{(i,0)}$ for $i \in \mathbb{Z}/w\mathbb{Z}$. Similarly $\mathbb{S}_2\langle 1 \rangle$ acts on the sets $\{\mathcal{R}^p_{(i,n)}, i \in \mathbb{Z}/w\mathbb{Z}, n \in \mathbb{Z}\}$ and $\{\mathcal{R}^q_{(i,n)}, i \in \mathbb{Z}/w\mathbb{Z}, n \in \mathbb{Z}\}$ by

$$\mathbb{S}_2(\mathcal{R}^p_{(i,n)})\langle 1 \rangle = \mathcal{R}^p_{(i+1,n-1)} \text{ and } \mathbb{S}_2(\mathcal{R}^q_{(i,n)})\langle 1 \rangle = \mathcal{R}^q_{(i+1,n-1)}$$

Therefore we get $2|w|$ orbits which are $(\mathbb{S}_2\langle 1 \rangle)^{\mathbb{Z}} \cdot \mathcal{R}^p_{(i,0)}$ and $(\mathbb{S}_2\langle 1 \rangle)^{\mathbb{Z}} \cdot \mathcal{R}^q_{(i,0)}$ for $i \in \mathbb{Z}/w\mathbb{Z}$.

Finally the shape of the AR-quiver of $\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle$ is described in Figure 5.7. This description can be resumed in the following.

Corollary 5.5. *Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$ and of weight $w \neq 0$. Then the Auslander-Reiten quiver of $\mathcal{D}^b(\Lambda)$ has exactly $3|w|$ connected components:*

- $|w|$ components of type $\mathbb{Z}A_{\infty}^{\infty}$;
- $2|w|$ components of type $\mathbb{Z}A_{\infty}$.

5.3. Consequences of the description of the AR-quiver. From this description we also get the following consequences.

Corollary 5.6. *Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$ and of weight $w \neq 0$. Then Λ is representation-finite.*

Proof. The algebra Λ is derived discrete in the sense of [Vos01]. Hence it is representation-finite. \square

Corollary 5.7. *Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$ and of weight w , then the Coxeter polynomial of Λ is*

$$X^{p+q} - (-1)^w X^{p-w} - (-1)^w X^{q+w} + 1.$$

Proof. The Coxeter matrix is the matrix of the automorphism τ at the level of the Grothendieck group $K_0(\mathcal{D}^b(\Lambda))$ in the basis $\{[S_i] \mid 1 \leq i \leq p+q\}$ consisting of the representatives of the simples. The Coxeter polynomial $C(X)$ is by definition its characteristic polynomial.

In the category $\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle$ we have the isomorphisms

$$\tau^{-p}(S_2\langle 0 \rangle) \simeq S_2\langle w \rangle \simeq \mathbb{S}_2^{-w}(S_2\langle 0 \rangle) \simeq \tau^{-w}S_2\langle 0 \rangle[w].$$

Hence we get the equality $[\tau^{-p+w}S_2] = (-1)^w[S_2]$ in $K_0(\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)$. Moreover from the previous computation (see Figure 5.3), it is immediate to see that the family $([\tau^i S_2], -p+w+1 \leq i \leq 0)$ is linearly independent in $K_0(\mathcal{D}^b(\Lambda))$. Therefore all $(w-p)^{th}$ roots of $(-1)^w$ are eigenvalues of $[\tau]$ and the polynomial $X^{p-w} - (-1)^w$ divides $C(X)$.

Similarly we have $[\tau^{-q-w}S_{p+q}] = (-1)^w[S_{p+q}]$ in $K_0(\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)$. Hence, by the same argument as above, the polynomial $X^{q+w} - (-1)^w$ divides $C(X)$.

Now, since S_2 does not appear in the composition series of $\tau^n S_{p+q}$ for all $n \in \mathbb{Z}$ the product $(X^{q+w} - (-1)^w)(X^{p-w} - (-1)^w)$ divides the Coxeter polynomial $C(X)$. Using the fact that $C(X)$ is of degree $p+q$ and that its constant term is 1, we get the result. \square

Remark 5.8. Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$ and of weight $w \neq 0$. Then the category $\mathcal{D}^b(\Lambda)$ is *locally fractionally Calabi-Yau* of dimensions $\frac{p-2w}{p-w}$ and $\frac{q+2w}{q+w}$ in the sense that there exists objects X and Y such that there are isomorphisms $\mathbb{S}^{p-w}X \simeq X[p-2w]$ and $\mathbb{S}^{q+w}Y \simeq Y[q+2w]$. These kind of algebras are studied in [AO, Section 6].

From the previous section, we also deduce a result for the image of the derived category in the cluster category.

Corollary 5.9. *Let Λ be an algebra of cluster type $\tilde{A}_{(p,q)}$ and of weight $w \neq 0$. Then the quiver of the orbit category $\mathcal{D}^b(\Lambda)/\mathbb{S}_2$ has 3 connected components which are of the forms $\mathbb{Z}\tilde{A}_{p,q}$, $\mathbb{Z}A_\infty/(\tau^p)$ and $\mathbb{Z}A_\infty/(\tau^q)$.*

Proof. From Corollary 5.3 we have the following diagram

$$\begin{array}{ccc} \mathcal{D}^b(\text{Cov}(\Lambda, \delta)) & \xrightarrow{\sim} & \mathcal{D}^b(\text{Cov}(H, \partial)) \\ \downarrow & & \downarrow \\ \mathcal{D}^b(\Lambda) & \xrightarrow{\sim} & \mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle \\ \downarrow & & \downarrow \\ \mathcal{D}^b(\Lambda)/\mathbb{S}_2 & \xrightarrow{\dots\dots\dots} & (\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)/\mathbb{S}_2 \end{array}$$

The upper functor is a triangle functor, hence it commutes with \mathbb{S}_2 . Therefore, the k -equivalence of Corollary 5.3 commutes with \mathbb{S}_2 . Using this we deduce that we have a k -equivalence

$$\mathcal{D}^b(\Lambda)/\mathbb{S}_2 \xrightarrow{\sim} (\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)/\mathbb{S}_2$$

Therefore the AR quiver of $\mathcal{D}^b(\Lambda)/\mathbb{S}_2$ is the same as the AR quiver of the orbit category $(\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)/\mathbb{S}_2$. Note that since $\mathbb{S}_2 \simeq \langle -1 \rangle$ in $(\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)$, then we have just to understand the action of $\langle 1 \rangle$ in the category $(\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)$. We know that the category $(\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)$ has $3|w|$ components which are

$$\mathcal{P}_{(i,0)}, \mathcal{R}_{(i,0)}^p \text{ and } \mathcal{R}_{(i,0)}^q, \text{ for } i \in \mathbb{Z}/w\mathbb{Z}$$

Moreover we have the equality

$$\mathcal{P}_{(i,0)}\langle 1 \rangle = \mathcal{P}_{(i+1,0)}, \mathcal{R}_{(i,0)}^p\langle 1 \rangle = \mathcal{R}_{(i+1,0)}^p, \text{ and } \mathcal{R}_{(i,0)}^q\langle 1 \rangle = \mathcal{R}_{(i+1,0)}^q$$

Thus for each connected component Γ of the AR quiver of $(\mathcal{D}^b(\text{Cov}(H, \partial))/\mathbb{S}_2\langle 1 \rangle)$ we have $\Gamma\langle w \rangle = \Gamma$. Now it is easy to check that for $X \in \mathcal{R}_{(i,0)}$ or we have $X\langle w \rangle \simeq \tau^{-p}X$. The action of $\langle w \rangle$ on the component $\mathcal{P}_{(0,0)}$ makes $\mathcal{P}_{(0,0)}/\langle w \rangle$ isomorphic to $\mathbb{Z}\tilde{A}_{p,q}$. Therefore we get the result. \square

6. EXPLICIT DESCRIPTION OF ALGEBRAS OF CLUSTER TYPE $\tilde{A}_{p,q}$

In this section, we describe explicitly (in term of quiver with relations) the algebras of cluster type $\tilde{A}_{p,q}$, and give a way to compute very easily the weight of such an algebra. The strategy consists of describing first the cluster-tilted algebras of type $\tilde{A}_{p,q}$, following [Bas09], and then showing that all algebras of cluster type $\tilde{A}_{p,q}$ can be seen as the degree zero part of a cluster-tilted algebra of type $\tilde{A}_{p,q}$ with an appropriate grading. We first start by describing the algebras of cluster type A_n using a similar approach.

6.1. Algebras of cluster type A_n . The aim of this section is to describe all algebras of global dimension ≤ 2 which are of cluster type A_n . By Corollary 3.15, we already know that they are the algebras of global dimension ≤ 2 derived equivalent to kQ , where Q is a quiver of type A_n . A description of iterated-tilted algebras of type A_n was done by Assem [Ass82]. Here we use other techniques, based on further developments in cluster-tilting theory.

Definition 6.1. For $n \geq 1$, we define the class \mathcal{M}_n^A of quivers Q that satisfy the following:

- they have n vertices;
- all non-trivial cycles are oriented and of length 3;
- a vertex has valency at most 4;
- if a vertex has valency 4, then two of its adjacent arrows belong to one 3-cycle, and the other two belong to another 3-cycle;
- if a vertex has valency 3, then two of its adjacent arrows belong to a 3-cycle, and the third arrow does not belong to any 3-cycle.

The set \mathcal{M}^A denotes the union of all \mathcal{M}_n^A . For a quiver Q in \mathcal{M}_A we denote by W_Q the sum of all oriented 3-cycles. This is a potential in the sense of [DWZ08].

Theorem 6.2 ([BV08]). *Let Γ be a finite dimensional algebra. Then Γ is a cluster-tilted algebra of type A_n if and only if there exists a quiver Q in \mathcal{M}_n^A such that we have an isomorphism $\Gamma \simeq \text{Jac}(Q, W_Q)$.*

The main result of this subsection is the following.

Theorem 6.3. *Let Q be a quiver in \mathcal{M}_n^A , and let d be W_Q -grading (cf Definition 3.9). It yields a grading on the Jacobian algebra $B := \text{Jac}(Q, W)$. Denote by $\Lambda := B_0$ its degree zero part. Then Λ is an algebra of global dimension ≤ 2 and of cluster type A_n . Moreover each algebra of global dimension ≤ 2 and of cluster type A_n is isomorphic to such a Λ .*

Proof. We divide the proof into several steps for the convenience of the reader.

Step 1: The global dimension of the algebra Λ is at most 2.

Denote by $Q^{(0)}$ the subquiver of Q which is defined by $Q_0^{(0)} := Q_0$ and $Q_1^{(0)} := \{a \in Q_1, d(a) = 0\}$. Then we have

$$\Lambda \simeq kQ^{(0)} / \langle \partial_a W_Q, d(a) = 1 \rangle.$$

Note that since W is a sum of 3-cycles, the ideal of relations $\langle \partial_a W_Q, d(a) = 1 \rangle$ is included in $kQ_2^{(0)}$, that is every relation is of length 2. Let $i \in Q_0$ be a vertex. Then the projective resolution of the simple Λ -module associated to i is given by

$$\bigoplus_{b, s(b)=i, d(b)=1} e_{t(b)} \Lambda \xrightarrow{(f_{a,b})} \bigoplus_{a, t(a)=i, d(a)=0} e_{s(a)} \Lambda \longrightarrow e_i \Lambda \longrightarrow S_i \longrightarrow 0$$

Let $b: i \rightarrow l$ be an arrow of degree 1, and $a: j \rightarrow i$ such that $f_{a,b} \neq 0$. Then there exists a 3-cycle:

$$\begin{array}{ccc} & j & \\ c \nearrow & & \searrow a \\ l & \xleftarrow{b} & i \end{array}$$

and the map $f_{a,b}: e_l \Lambda \rightarrow e_j \Lambda$ is induced by c . The arrow c does not belong to another 3-cycle, therefore there is no relation between some predecessor of l and j . Hence the map $f_{a,b}: e_l \Lambda \rightarrow e_j \Lambda$ is a monomorphism.

Let a be an arrow with $t(a) = i$ and of degree 0, then a belongs to at most one cycle. Therefore there exists at most one arrow b of degree 1 with $s(b) = i$ such that $f_{a,b}$ does not vanish. Therefore the map

$$\bigoplus_{b, s(b)=i, d(b)=1} e_{t(b)} \Lambda \xrightarrow{(f_{a,b})} \bigoplus_{a, t(a)=i, d(a)=0} e_{s(a)} \Lambda$$

is injective, and the projective dimension of S_i is at most 2.

Step 2: We have an isomorphism $\text{Jac}(Q, W_Q) \simeq \text{End}_{\mathcal{C}_\Lambda}(\pi \Lambda)$.

Since d is a W_Q -grading the set $\{\partial_a W_Q, d(a) = 1\}$ is a set of minimal relations for Λ . Therefore by Proposition 2.5, we have $\overline{Q}^{(0)} = Q$ and $W_\Lambda = W_Q$ and hence Step 2.

Step 3: There is a triangle equivalence $\mathcal{C}_\Lambda \simeq \mathcal{C}_{A_n}$

By Theorem 6.2 there exists a cluster-tilting object in \mathcal{C}_{A_n} whose endomorphism algebra is $\text{Jac}(Q, W_Q)$. Therefore we are done by Corollary 2.10.

Step 4: Each algebra of cluster type A_n and of global dimension ≤ 2 is isomorphic to the degree zero part of a graded Jacobian algebra $\text{Jac}(Q, W_Q, d)$ where $Q \in \mathcal{M}_n^A$ and d is a W_Q -grading.

Let $\Lambda \simeq kQ_\Lambda/I$ be an algebra of global dimension ≤ 2 and of cluster type A_n . Denote by $f: \mathcal{C}_\Lambda \rightarrow \mathcal{C}_Q$ the triangle equivalence. By Proposition 3.10 there exists a graded QP (Q_Λ, W, d) , where W is rigid and d is a W -grading, such that we have

$$\overline{\Lambda} \simeq_{\mathbb{Z}} \text{Jac}(Q_\Lambda, W, d).$$

Now the object $f(\pi_\Lambda(\Lambda))$ is a cluster-tilting object in \mathcal{C}_{A_n} . Thus by Theorem 6.2, there exists $Q \in \mathcal{M}_n^A$ such that we have an isomorphism

$$\overline{\Lambda} \simeq \text{Jac}(Q, W_Q).$$

It is clear that we have $Q = Q_\Lambda$. We then conclude using the following lemma. □

Lemma 6.4. *Let (Q, W, d) be a graded QP where $Q \in \mathcal{M}_n^A$ and W is rigid. Then d makes W_Q homogeneous of degree 1 and we have an isomorphism of \mathbb{Z} -graded algebras*

$$\text{Jac}(Q, W_Q, d) \simeq_{\mathbb{Z}} \text{Jac}(Q, W, d).$$

Before proving this lemma, we give a definition.

Definition 6.5. Let Q be a quiver. A cycle $a_1 \dots a_r$ in Q is called *irreducible* if for all $1 \leq i \neq j \leq r$ we have $s(a_i) \neq s(a_j)$. All cycles which are not irreducible are called *reducible*.

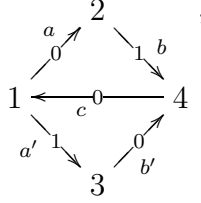
Note that if d is a grading on Q , such that with respect to this grading all irreducible cycles are of degree 1, then all reducible cycles are of degree at least 2.

Proof of Lemma 6.4. Denote by C_1, \dots, C_l the oriented 3-cycles of Q , so that we have $W_Q = \sum_{i=1}^r C_i$. One easily checks that the irreducible oriented cycles of the quiver Q are exactly the C_i 's (up to cyclic equivalence). Therefore we can assume that

$$W = \sum_{i=1}^r \lambda_i C_i + \text{extra terms}$$

where the extra terms are reducible. If there exists i with $\lambda_i = 0$ then the cycle C_i and all the cycles cyclically equivalent to it are not in the Jacobian ideal since there are no cycle of length ≤ 2 in the quiver Q . Therefore since W is rigid, we have $\lambda_i \neq 0$ for all $i = 1, \dots, r$. Now the existence of the degree d implies that $d(C_i) = 1$ for all $i = 1, \dots, r$. Hence the extra terms in the potential have to be zero and we have $W = \sum_{i=1}^r \lambda_i C_i$ with $\lambda_i \neq 0$. \square

Remark 6.6. Note that Theorem 6.3 is not true for the other Dynkin type. Let (Q, d) be the following graded quiver

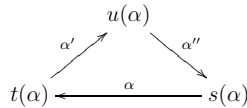


and $W := cba + cb'a'$. The Jacobian algebra $\text{Jac}(Q, W)$ is a cluster-tilted algebra of type D_4 . The grading d is a W -grading but the degree zero part of the graded algebra $\text{Jac}(Q, W, d)$ is an iterated-tilted algebra of type A_4 and of global dimension 3, so it cannot be of cluster type D_4 .

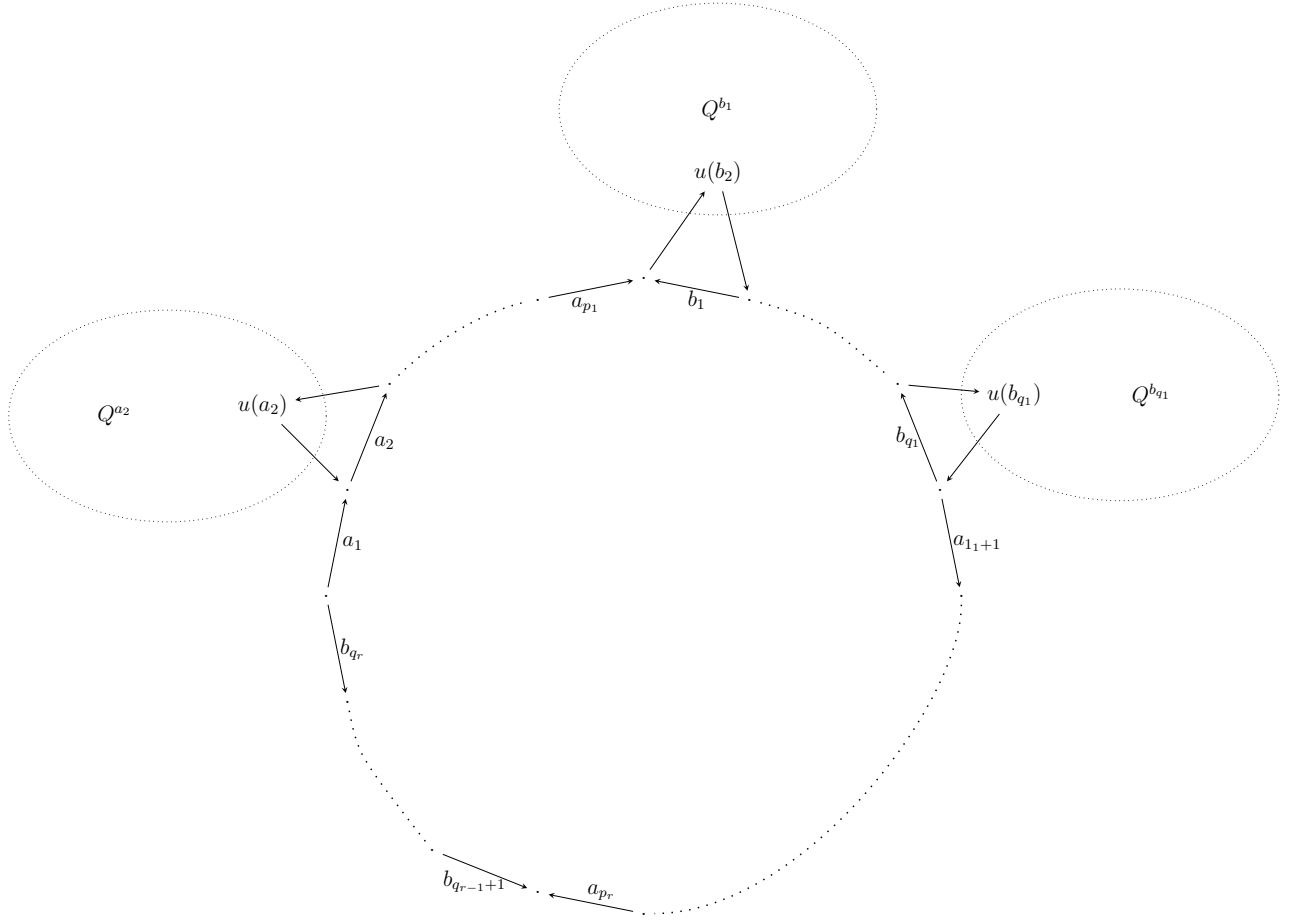
6.2. Cluster-tilted algebras of type $\tilde{A}_{p,q}$.

Definition 6.7 ([Bas09]). For $p \geq q \geq 1$, we define the class $\mathcal{M}_{p,q}^{\tilde{A}}$ of quivers Q that satisfy the following conditions:

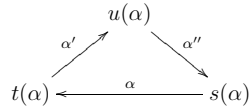
- Q has $p + q$ vertices;
- there exist integers $1 \leq p_1 < p_2 < \dots < p_r \leq p$ and $1 \leq q_1 < q_2 < \dots < q_r \leq q$ such that Q contains precisely one full subquiver C which is a non-oriented cycle of type $(p_1, q_1, p_2 - p_1, q_2 - q_1, \dots, p_r - p_{r-1}, q_r - q_{r-1})$ (that is C is the composition of p_1 arrows going in one direction with q_1 arrows going in the other direction, with $p_2 - p_1$ arrows going the first direction, etc...). We denote by a_1, \dots, a_{p_r} the arrows of C going in one direction and we call them the p -arrows. We denote by b_1, \dots, b_{q_r} the arrows going in the opposite direction and we call them the q -arrows.
- each arrow connecting C to a vertex not in C is in exactly one 3-cycle of Q of the form



where α is in C . We denote by $u(\alpha)$ the connecting vertex. It has valency at most 4. When its valency is 4, the adjacent arrows which are not α' and α'' belong to exactly one 3-cycle, and when it has valency 3, the third arrow does not belong to any 3-cycle. Moreover, the subquiver containing C , α' and α'' is a full subquiver of Q . Hence we cannot have $u(\alpha) = u(\beta)$ for $\alpha \neq \beta$.

FIGURE 6.1. Shape of a quiver in $\mathcal{M}_{p,q}^{\tilde{A}}$.

- the full subquiver of Q whose vertices are not in C is a disjoint union of quivers $Q^\alpha \in \mathcal{M}^A$ where α is an arrow of the non-oriented cycle. The quiver Q^α is empty if there is no 3-cycle containing α , and the quiver Q^α contains the vertex $u(\alpha)$ if there is a 3-cycle



- We have the equalities

$$p = \sum_{l=1}^{p_r} \#Q_0^{a_l} + p_r \quad \text{and} \quad q = \sum_{l=1}^{q_r} \#Q_0^{b_l} + q_r.$$

For a quiver Q in $\mathcal{M}_{p,q}^{\tilde{A}}$ we denote by W_Q the sum of all oriented 3-cycles. More precisely we define

$$W_Q := \sum_{\alpha \in C} (\alpha'' \alpha' \alpha + W_{Q^\alpha})$$

where C is the non-oriented cycle of Q . This is a rigid potential in the sense of [DWZ08].

Theorem 6.8 ([Bas09]). *Let Γ be a finite dimensional algebra. Then Γ is a cluster-tilted algebra of type $\tilde{A}_{p,q}$ if and only if there exists a quiver Q in $\mathcal{M}_{p,q}^{\tilde{A}}$ such that we have an isomorphism*

$$\Gamma \simeq \text{Jac}(Q, W_Q).$$

6.3. Algebras of cluster type $\tilde{A}_{p,q}$. We have then the same kind of result as for the A_n case.

Theorem 6.9. *Let Q be a quiver in $\mathcal{M}_{p,q}^{\tilde{A}}$, and let d be W_Q -grading. It yields a grading on the Jacobian algebra $B := \text{Jac}(Q, W)$. Denote by $\Lambda := B_0$ its degree zero part. Then Λ is an algebra of global dimension ≤ 2 and of cluster type $\tilde{A}_{p,q}$. Moreover each algebra of global dimension ≤ 2 and of cluster type $\tilde{A}_{p,q}$ is isomorphic to such a Λ .*

The proof of the first part is exactly the same as in the proof of Theorem 6.3. For the proof of the second part, we will need the following.

Lemma 6.10. *Let (Q, W, d) be a graded quiver with reduced potential such that:*

- *the quiver Q is in $\mathcal{M}_{p,q}^{\tilde{A}}$,*
- *the potential W is rigid,*
- *the grading d is W -grading.*

Then there exists an algebra isomorphism $\varphi: k\hat{Q} \rightarrow k\hat{Q}$ (where $k\hat{Q}$ is the completion of the path algebra kQ) such that φ is the identity on the vertices, and such that $\varphi(W_Q)$ is cyclically equivalent to W . Moreover there exists a W_Q -grading d' on Q such that $\varphi: (k\hat{Q}, d') \rightarrow (k\hat{Q}, d)$ is an isomorphism of graded algebras.

Remark 6.11. This means that the graded QP (Q, W, d) and (Q, W_Q, d') are graded right equivalent in the sense of [AO10, Def. 6.3].

Proof. We denote by C_1, \dots, C_l the oriented cycles such that $W_Q = \sum_{i=1}^l C_i$, by a_1, \dots, a_{p_r} the p -arrows, and by b_1, \dots, b_{q_r} the q -arrows of Q . These are irreducible cycles of Q . But contrary to the A_n -case, there might be other irreducible cycles in the quiver Q . We treat here the most complicated case.

Assume that for all $i = 1, \dots, p_r$, and all $j = 1, \dots, q_r$ we have $Q^{a_i} \neq \emptyset$ and $Q^{b_j} \neq \emptyset$.

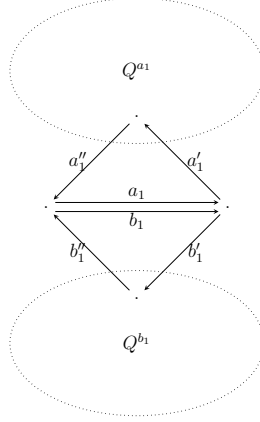
Denote by C_a a cycle containing exactly once the arrows $a_i, i = 1, \dots, p_r$ and $b'_j, b''_j, j = 1, \dots, q_r$ and by C_b a cycle containing exactly once the arrows $b_j, j = 1, \dots, q_r$ and $a'_i, a''_i, i = 1, \dots, p_r$. Then one can check that the irreducible cycles of Q are C_a, C_b and the C_i 's up to cyclic equivalence. Therefore we can write

$$W = \sum_{i=1}^l \lambda_i C_i + \alpha C_a + \beta C_b + \text{extra terms},$$

where $\lambda_i, \alpha, \beta \in k$ and the extra terms are linear combination of reducible cycles.

First we show that we can assume $\lambda_i \neq 0$ for all $i = 1, \dots, l$.

If $p_r + q_r \geq 3$, then we have immediately $\lambda_i \neq 0$ for $i = 1, \dots, l$ by the rigidity of W . Assume $p_r = q_r = 1$. Then Q is of this form:



We can assume (up to renumbering) that C_1 is cyclically equivalent to $a_1 a'_1 a''_1$ and C_2 is cyclically equivalent to $b_1 b'_1 b''_1$, and we have $C_a = a_1 b'_1 b''_1$ and $C_b = b_1 a'_1 a''_1$. Then it is easy to see that the rigidity of W implies that $\lambda_i \neq 0$ for $i = 3, \dots, l$ and that $\lambda_1 \lambda_2 - \alpha \beta \neq 0$. Therefore, up to the automorphism of Q exchanging a_1 and b_1 we can assume that $\lambda_1 \lambda_2 \neq 0$.

Since the potential W is homogeneous of degree 1 we have

$$(*) \quad d(a_i) + d(a'_i) + d(a''_i) = d(b_j) + d(b'_j) + d(b''_j) = 1 \quad \text{for all } i = 1, \dots, p_r, \quad j = 1, \dots, q_r$$

By definition we have

$$(\dagger) \quad d(C_a) = \sum_{i=1}^{p_r} d(a_i) + \sum_{j=1}^{q_r} (d(b'_j) + d(b''_j)) \quad \text{and} \quad d(C_b) = \sum_{j=1}^{q_r} d(b_j) + \sum_{i=1}^{p_r} (d(a'_i) + d(a''_i))$$

Hence combining $(*)$ and (\dagger) we get

$$(\ddagger) \quad d(C_a) + d(C_b) = p_r + q_r.$$

Using (\ddagger) and the fact that $d(C_a)$ and $d(C_b)$ are non-negative (since d is a map $Q_1 \rightarrow \{0, 1\}$), we divide the proof into 4 subcases.

Case 1: $d(C_a) \geq 2$ and $d(C_b) \geq 2$.

In this case, since W is homogeneous of degree 1, we have $\alpha = \beta = 0$, and there is no extra term in the potential W . For $i = 1, \dots, l$ we denote by c_i the arrow such that $C_i = c''_i c'_i c_i$. Then we define φ on Q_1 by

$$\varphi(x) = \begin{cases} \lambda_i c_i & \text{if } x = c_i \\ x & \text{otherwise} \end{cases}$$

It is then clear than φ is an isomorphism of the graded algebra (kQ, d) sending W_Q on W .

Case 2: $d(C_a) = 1$ and $d(C_b) \geq 2$.

In this case, since W is homogeneous of degree 1, we have $\beta = 0$ and there is no extra term in the potential W . Up to cyclic equivalence we can assume that $C_1 = b''_1 b'_1 b_1$ and

$C_a = b_1'' b_1' C_a'$. For $i = 2, \dots, l$ we denote by c_i the arrow such that $C_i = c_i'' c_i' c_i$. Now we define φ on Q_1 by

$$\varphi(x) = \begin{cases} \lambda_1 b_1 + \alpha C_a' & \text{if } x = b_1 \\ \lambda_i c_i & \text{if } x = c_i, \text{ and } i \geq 2 \\ x & \text{otherwise} \end{cases}$$

Since $C_a = b_1'' b_1' C_a'$ and $C_1 = b_1'' b_1' b_1$ are oriented cycles, the path C_a' has the same source and the same target as b_1 , thus φ is an algebra morphism. Moreover, since $\lambda_i \neq 0$, φ is an isomorphism of the completion $k\hat{Q}$. Now, we have

$$d(C_a') = d(C_a) - d(b_1') - d(b_1'') = 1 - d(b_1') - d(b_1'') = d(b_1).$$

Therefore φ is an isomorphism of the graded algebra $(k\hat{Q}, d)$ sending W_Q on W .

Case 3: $d(C_a) = 1$ and $d(C_b) = 1$.

From (\ddagger) we have automatically $p_r = q_r = 1$. All the cycles $C_1 = a_1'' a_1' a_1$, $C_2 = b_1'' b_1' b_1$, $C_b = a_1'' a_1' b_1$, $C_a = b_1'' b_1' a_1$, C_3, \dots, C_l are homogeneous of degree 1. Since W is rigid we have $\lambda_1 \lambda_2 - \alpha \beta \neq 0$. And since the grading d makes W homogeneous of degree 1, there is no extra term in the potential W and we have $d(a_1) = d(b_1)$. Then we can define:

$$\varphi(x) = \begin{cases} \lambda_1 a_1 + \beta b_1 & \text{if } x = a_1 \\ \lambda_2 b_1 + \alpha a_1 & \text{if } x = b_1 \\ x & \text{otherwise} \end{cases}$$

Since $\lambda_1 \lambda_2 - \alpha \beta = 0$, this algebra morphism is an isomorphism. Moreover, since $d(a_1) = d(b_1)$, $\varphi: (kQ, d) \rightarrow (kQ, d)$ is a isomorphism of graded algebras. By construction it sends W_Q on W .

Case 4: $d(C_a) = 0$.

From (\ddagger) , we have $d(C_b) \geq 2$. Hence, since W is homogeneous of degree 1, we have $\alpha = \beta = 0$. In this case, since the degree of C_a is 0, the oriented cycles of the quiver Q which are homogeneous of degree 1 are cyclically equivalent to something of the form $C_i C_a^n$ where $n \in \mathbb{N}$. Thus we can write up to cyclic equivalence:

$$W = \sum_{i=1}^l C_i P_i(C_a),$$

where $P_i \in k[X]$ is a polynomial with constant term $\lambda_i \neq 0$. For each $i = 1, \dots, l$ we write $C_i = c_i'' c_i' c_i$ and we define

$$\varphi(x) = \begin{cases} c_i P_i(C_a) & \text{if } x = c_i \\ x & \text{otherwise} \end{cases}$$

Then φ is an automorphism of the completion $k\hat{Q}$ since $\lambda_i \neq 0$ for all $i = 1, \dots, l$. Since $d(C_a) = 0$, this automorphism is an automorphism of the graded algebra $(k\hat{Q}, d)$.

The other cases, namely *there exists* $1 \leq i \leq p_r$ and $1 \leq j \leq q_r$ *such that* $Q^{a_i} = \emptyset$ *and* $Q^{b_j} = \emptyset$, *and for all* $i = 1, \dots, p_r$, *we have* $Q^{a_i} \neq \emptyset$ *and there exists* j *such that* $Q^{b_j} = \emptyset$ are simpler since in these cases there are less irreducible cycles. The proof is let to the reader. \square

- Remark 6.12.* (1) Note that the first part of this lemma can be deduced directly from [DWZ08]. Indeed since W is rigid the quiver with potential (Q, W) is right equivalent in the sense of [DWZ08] to (Q, W_Q) (there exists a sequence s such that $\mu_s(Q)$ is acyclic, therefore $\mu_s(Q, W)$ is right equivalent to $\mu_s(Q, W_Q)$). By definition, this implies that there exists an automorphism $\varphi: kQ \rightarrow kQ$ which is the identity on the vertices, and such that $\varphi(W_Q)$ is cyclically equivalent to W . However, we have proved this lemma constructing explicitly the isomorphism φ .
- (2) In the case $p_r = q_r = 1$, it might happen that $\lambda_1 \lambda_2 = 0$, then the automorphism φ constructed above will exchange the arrows a_1 and b_1 . The degree map d' will satisfy $d'(a_1) = d(b_1)$ and $d'(b_1) = d(a_1)$. This is the only case where d and d' are not the same.

From Lemma 3.10 we deduce the following result, which is a restatement of the second part of Theorem 6.9 and finish the proof of Theorem 6.9.

Corollary 6.13. *Let Λ be an algebra of global dimension at most 2 and of cluster type $\tilde{A}_{p,q}$. Then there exists a quiver $Q \in \mathcal{M}_{p,q}^{\tilde{A}}$ and a W_Q -grading d' such that Λ is isomorphic to the degree zero part of $\text{Jac}(Q, W_Q, d)$.*

Proof. By Proposition 3.10, there exists a reduced graded quiver with potential (Q, W, d) , such that $\bar{\Lambda} \sim_{\mathbb{Z}} \text{Jac}(Q, W, d)$. Moreover, W is rigid and d is a W -grading. By Theorem 6.8, the quiver Q is in $\mathcal{M}_{p,q}^{\tilde{A}}$. By Lemma 6.10, there exists a W_Q -grading d' such that we have $\text{Jac}(Q, W, d) \sim_{\mathbb{Z}} \text{Jac}(Q, W_Q, d')$. Therefore Λ is isomorphic to the degree zero part of $\text{Jac}(Q, W_Q, d')$. \square

- Remark 6.14.* (1) This corollary implies that an algebra Λ of cluster type $\tilde{A}_{p,q}$ is always isomorphic to an algebra of the form kQ_{Λ}/I , where the relations are paths and difference of two paths. This fact is a consequence of [BGRS85] when Λ is representation-finite. However, since we dispose of a direct and explicit proof for this fact, we decided not to use [BGRS85].
- (2) This corollary gives a description of the iterated tilted algebras of global dimension ≤ 2 of type $\tilde{A}_{p,q}$. A description of all iterated tilted algebras (not distinguishing with respect to their global dimension) of type $\tilde{A}_{p,q}$ has been given in [AS87].

Corollary 6.15. *There are only finitely many algebras (up to isomorphism) of global dimension ≤ 2 and of cluster type $\tilde{A}_{p,q}$.*

Proof. There are only finitely many quivers in the set $\mathcal{M}_{p,q}^{\tilde{A}}$. And given $Q \in \mathcal{M}_{p,q}^{\tilde{A}}$ there are finitely many W_Q -gradings. \square

6.4. An alternative description of the weight. In this subsection, we give an explicit way to compute the weight of an algebra of cluster type $\tilde{A}_{p,q}$.

Definition 6.16. Let Q be a quiver in $\mathcal{M}_{p,q}^{\tilde{A}}$ and d be a \mathbb{Z} -grading on Q such that the potential W_Q is homogeneous of degree 1. Then let a_1, \dots, a_{p_r} be the p -arrows and b_1, \dots, b_{q_r} be the q -arrows of Q . The *weight* of the graded quiver (Q, d) is defined to be

$$w(Q, W_Q, d) := \sum_{l=1}^{p_r} d(a_l) - \sum_{l=1}^{q_r} d(b_l).$$

The aim of the subsection is to show the following.

Proposition 6.17. *Let $Q \in \mathcal{M}_{p,q}^{\tilde{A}}$ and d be a W_Q grading. Let Λ be the degree zero part of the graded algebra $\text{Jac}(Q, W_Q, d)$. Then Λ is an algebra of global dimension at most 2 and of cluster type $\tilde{A}_{p,q}$ and we have*

$$w(\Lambda) = w(Q, W_Q, d).$$

Proof. The first part of the statement follows from Theorem 6.9.

Let s be a sequence of mutation such that $\mu_s(Q, W_Q) = (H, 0)$ where Q_H is an acyclic quiver of type $\tilde{A}_{p,q}$. Define a grading ∂ on Q_H by $\mu_s^L(Q, W_Q, d) = (H, 0, \partial)$. By definition we have $w(\Lambda) = w(H, 0, \partial)$. Hence the proof of the proposition comes directly from the following technical lemma. \square

Lemma 6.18. *Let Q be a quiver in $\mathcal{M}_{p,q}^{\tilde{A}}$ and d be a \mathbb{Z} -grading on Q such that W_Q is homogeneous of degree 1. Let i be a vertex of Q . Then we have*

$$w(\mu_i^L(Q, W_Q, d)) = w(Q, W_Q, d).$$

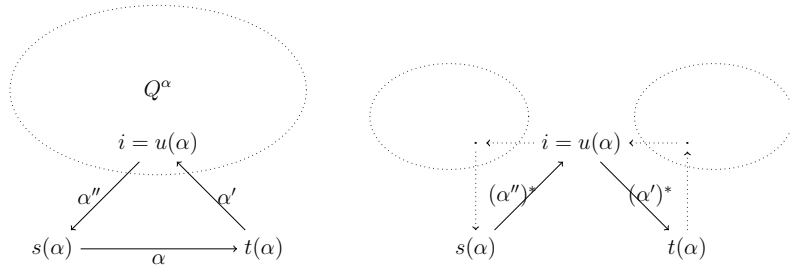
Proof. We define the grading d' on the quiver $\mu_i(Q)$ by $\mu_i^L(Q, W, d) = (\mu_i(Q, W), d')$. We distinguish the following different cases.

Case 1: there exists α in the non-oriented cycle of Q such that $i \in Q^\alpha$ and $i \neq u(\alpha)$

Then the vertices adjacent to i are not in the non-oriented cycle. Therefore the mutation of Q at i does not affect the non-oriented cycle, and so the weight clearly remains the same.

Case 2: $i = u(\alpha)$ for some arrow α which is on the non-oriented cycle

Assume that α is a p -arrow. There exists a 3-cycle $\alpha''\alpha'\alpha$ which is summand of W .



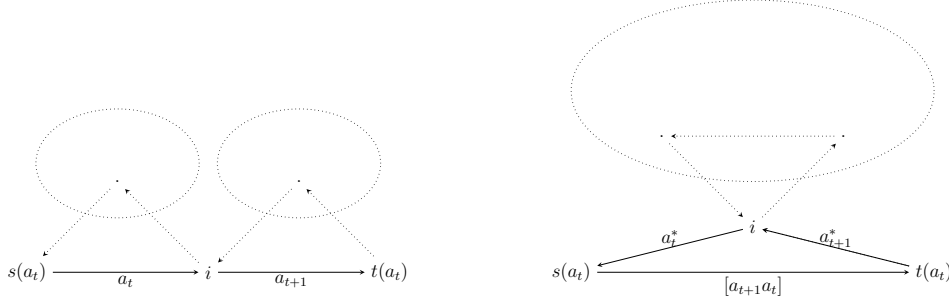
The new arrows $(\alpha')^*$ and $(\alpha'')^*$ become p -arrows in the non-oriented cycle. Therefore we have

$$\begin{aligned} w(\mu_i^L(Q, W, d)) &= w(Q, W, d) - d(\alpha) + d'((\alpha')^*) + d'((\alpha'')^*) \\ &= w(Q, W, d) - d(\alpha) - d(\alpha') - d(\alpha'') + 1 \\ &= w(Q, W, d). \end{aligned}$$

The last equality holds since $\alpha''\alpha'\alpha$ is a summand in the potential W , and hence we have $d(\alpha) + d(\alpha') + d(\alpha'') = 1$.

Case 3: i is on the non-oriented cycle between two p -arrows

Then the quiver Q looks locally like



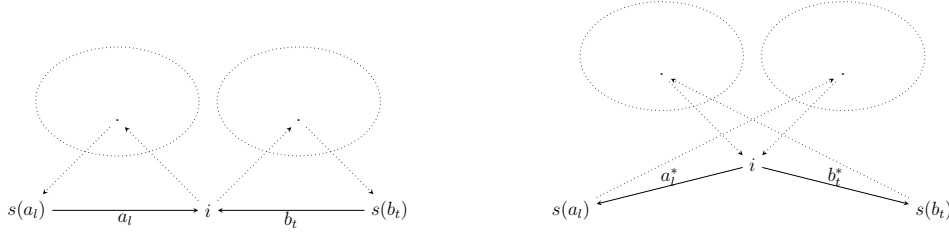
The arrows a_t and a_{t+1} are replaced by the new p -arrow $[a_{t+1}a_t]$ in the non-oriented cycle. Hence we have

$$\begin{aligned} w(\mu_i^L(Q, W, d)) &= w(Q, W, d) - d(a_t) - d(a_{t+1}) + d'([a_{t+1}a_t]) \\ &= w(Q, W, d) \end{aligned}$$

The case where i is between two q arrows is similar.

Case 4: i is a sink on the non-oriented cycle

Assume that i is the target of the p -arrow a_l and the target of the q -arrow b_t .



Then the arrows a_l and b_t are replaced by the arrows a_l^* and b_t^* and we have

$$\begin{aligned} w(\mu_i^L(Q, W, d)) &= w(Q, W, d) - d(a_l) + d(b_t) + d'(b_t^*) - d'(a_l^*) \\ &= w(Q, W, d) - d(a_l) + d(b_t) + (-d(b_t) + 1) - (-d(a_l) + 1) \\ &= w(Q, W, d). \end{aligned}$$

The case where i is the source of one p -arrow and of one q -arrow is similar. □

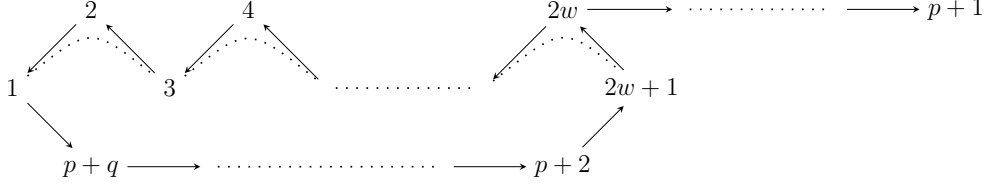
This alternative description of the weight gives us the following consequences.

Corollary 6.19. *The number of derived equivalence classes of algebras of cluster type $\tilde{A}_{p,q}$ is $\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$ if $p \neq q$ and $\lfloor \frac{p}{2} \rfloor + 1$ if $p = q$.*

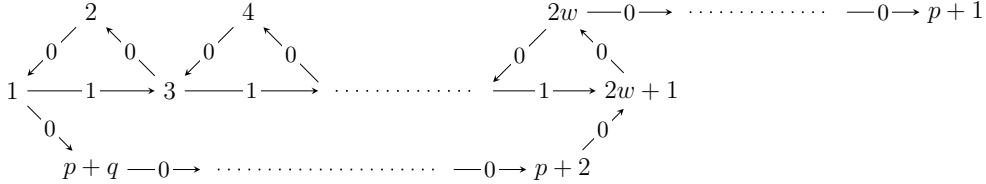
Proof. Let Q be a quiver in $\mathcal{M}_{p,q}^{\tilde{A}}$. It is clear from the definition that w is maximal when the W_Q -grading satisfies $d(a_l) = 1$ for $l = 1, \dots, p_r$ and $d(b_j) = 0$ for $j = 1, \dots, q_r$. In this case the weight is equal to $p_r = p - \sum_{l=1}^{p_r} \#Q_0^{a_l}$. But since d is a W_Q -grading, $d(a_l) = 1$ implies that the quiver Q^{a_l} is non-empty. Then we have $w = p_r = p - \sum_{l=1}^{p_r} \#Q_0^{a_l} \leq p - p_r$. Thus we have $w \leq \lfloor \frac{p}{2} \rfloor$. For the same reason, we have $w \geq -\lfloor \frac{q}{2} \rfloor$. Now it is easy to see that any values $-\lfloor \frac{q}{2} \rfloor, -\lfloor \frac{q}{2} \rfloor + 1, \dots, \lfloor \frac{p}{2} \rfloor$ can occur. □

Corollary 6.20. *Let Λ be an algebra of cluster type $\tilde{A}_{p,q}$, then there exists a tilting object T in $\mathcal{D}^b(\Lambda)$ such that the quiver of $\text{End}_{\mathcal{D}^b(\Lambda)}(T)$ has an oriented cycle.*

Proof. Let $-\frac{q}{2} \leq w \leq \frac{p}{2}$ be a non-zero integer. We construct an algebra B cluster type $\tilde{A}_{p,q}$ of weight w such that Q_B has an oriented cycle. Without loss of generality we can assume $w > 0$. We define B as follows.



It is clear that B is the degree 0 part of the Jacobian algebra $\text{Jac}(Q, W_Q, d)$ with the graded quiver



Then by Theorem 4.4 this algebra B is the endomorphism algebra of some tilting complex $T \in \mathcal{D}^b(\Lambda)$. \square

We end this subsection by giving a result linking the weight of an algebra of cluster type $\tilde{A}_{p,q}$ with the set of cluster-tilting objects of $\mathcal{C}_{\tilde{A}_{p,q}}$ coming from tilting complexes in $\mathcal{D}^b(\Lambda)$. We start with some notation.

Let Λ be an algebra of global dimension at most 2 which is τ_2 -finite. We define a subset \mathcal{T}_Λ of the set of tilting complexes of $\mathcal{D}^b(\Lambda)$ by

$$\mathcal{T}_\Lambda := \{T \in \mathcal{D}^b(\Lambda) \text{ tilting} \mid \text{gldim}(\text{End}_{\mathcal{D}^b(\Lambda)}(T)) \leq 2\}$$

By Theorem 2.4, the set $\pi_\Lambda(\mathcal{T}_\Lambda) \subset \mathcal{C}_\Lambda$ is a subset of the cluster-tilting objects of \mathcal{C}_Λ . Moreover, if Λ_1 and Λ_2 are derived equivalent, they are cluster equivalent and we clearly have $\pi_{\Lambda_1}(\mathcal{T}_{\Lambda_1}) = \pi_{\Lambda_2}(\mathcal{T}_{\Lambda_2})$. We prove here the converse in the case of algebras of cluster type $\tilde{A}_{p,q}$, and explain what inclusions still hold in the other case.

Proposition 6.21. *Let Λ_1 and Λ_2 be two algebras of global dimension 2 and of cluster type $\tilde{A}_{p,q}$. Then we have*

- $0 \leq w(\Lambda_1) < w(\Lambda_2) \Rightarrow \pi_2(\mathcal{T}_{\Lambda_2}) \subsetneq \pi_1(\mathcal{T}_{\Lambda_1})$
- $w(\Lambda_2) < w(\Lambda_1) \leq 0 \Rightarrow \pi_2(\mathcal{T}_{\Lambda_2}) \subsetneq \pi_1(\mathcal{T}_{\Lambda_1})$
- $w(\Lambda_1)w(\Lambda_2) < 0 \Rightarrow \pi_1(\mathcal{T}_{\Lambda_1}) \setminus \pi_2(\mathcal{T}_{\Lambda_2}) \neq \emptyset \neq \pi_2(\mathcal{T}_{\Lambda_2}) \setminus \pi_1(\mathcal{T}_{\Lambda_1})$

Proof. Without loss of generality, we can assume that $w(\Lambda_2) = w(\Lambda_1) + 1 > 0$. It is enough to show that $\pi_2(\Lambda_2) \in \pi_1(\mathcal{T}_{\Lambda_1})$. We denote by H some hereditary algebra of type $\tilde{A}_{p,q}$ and by π_2 a triangle functor $\pi_2: \mathcal{D}^b(\Lambda_2) \rightarrow \mathcal{C}_H$.

Let (Q, W_Q, d) be a graded quiver with potential such that we have isomorphisms of \mathbb{Z} -graded algebras $\text{End}_{\mathcal{C}_H}(\pi_2 \Lambda_2) \simeq \text{Jac}(Q, W_Q, d)$. Since $w(\Lambda_2) \geq 1$, there exists a p -arrow $a_i \in Q$ such that $d(a_i) = 1$. Since d is a W_Q -grading, the subquiver Q^{a_i} is not empty. More precisely, there exists arrows a'_i and a''_i such that $a_i a'_i a''_i$ is an oriented triangle in Q .

Define a new degree d' on Q by:

$$d'(x) = \begin{cases} 0 & \text{if } x = a_i \\ 1 & \text{if } x = a'_i \\ d(x) & \text{otherwise} \end{cases}$$

It is immediate to see that d' is a W_Q -grading. Define the algebra Λ_3 as the degree 0 part of the graded Jacobian algebra $\text{Jac}(\tilde{Q}, W_Q, d')$. By Theorem 6.9, it is an algebra of global dimension 2 which is of cluster type $\tilde{A}_{p,q}$, and by Corollary 2.10 we can assume $\pi_2(\Lambda_2) = \pi_3(\Lambda_3)$ where π_3 is a triangle functor $\pi_3: \mathcal{D}^b(\Lambda_3) \rightarrow \mathcal{C}_{\Lambda_3}$. Moreover by Proposition 6.17, we have $w(\Lambda_3) = w(\Lambda_2) - 1 = w(\Lambda_1)$. Therefore by Theorem 4.4 the algebra Λ_3 is derived equivalent to Λ_1 . The image of Λ_3 through this equivalence is clearly an object X in \mathcal{T}_{Λ_1} which satisfies $\pi_1(X) \simeq \pi_2(\Lambda_2)$.

For any $0 \leq w \leq [\frac{p}{2}]$, one can easily construct a quiver Q such that Q admits a W_Q -grading of weight w but no W_Q -grading of weight $w + 1$. Therefore the inclusion is strict.

The second point holds by symmetry.

For the third point, it is enough to see that for any $w > 0$ the quiver Q constructed in the proof of Corollary 6.20 satisfies the following: there exists a W_Q -grading of weight w and for any $w' < 0$ there is no W_Q -grading of weight w' . The argument for $w < 0$ holds by symmetry. \square

We end this subsection by asking the following intriguing questions:

Question 6.22. • Let Λ_1 and Λ_2 be τ_2 -finite algebras of global dimension ≤ 2 which are cluster equivalent. Do we have the implication

$$\pi_1(\mathcal{T}_{\Lambda_1}) = \pi_2(\mathcal{T}_{\Lambda_2}) \Rightarrow \mathcal{D}^b(\Lambda_1) \simeq \mathcal{D}^b(\Lambda_2) ?$$

- Let Λ be a τ_2 -finite algebra of global dimension ≤ 2 . Do the following implication hold?

$$\pi_\Lambda(\mathcal{T}_\Lambda) = \{X \in \mathcal{C}_\Lambda \mid X \text{ cluster-tilting}\} \Rightarrow \Lambda \text{ piecewise hereditary}$$

6.5. Example. In this subsection we compute explicitly all algebras (up to isomorphisms) of global dimension ≤ 2 and of cluster type $\tilde{A}_{2,2}$ and organize them according to their derived equivalence classes.

The strategy consists of first describing all quivers (up to isomorphism of quivers) which are in the set $\mathcal{M}_{2,2}^{\tilde{A}}$. In our case an easy computation or [Kel] shows that there are only 4 different quivers in $\mathcal{M}_{2,2}^{\tilde{A}}$ which are:



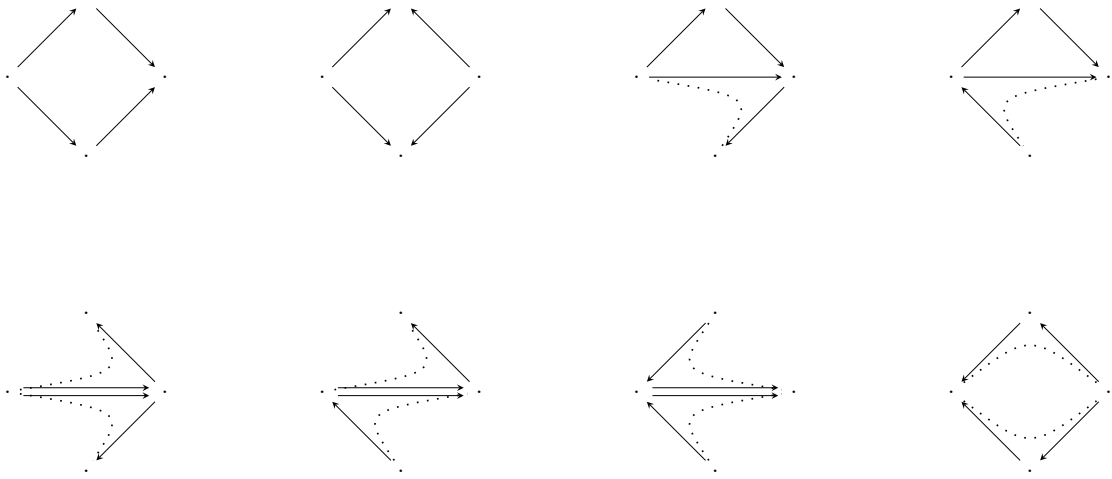
Note that since $p = q$ there is an isomorphism between the two quivers corresponding to $p_r = 2, q_r = 1$ and $p_r = 1, q_r = 2$



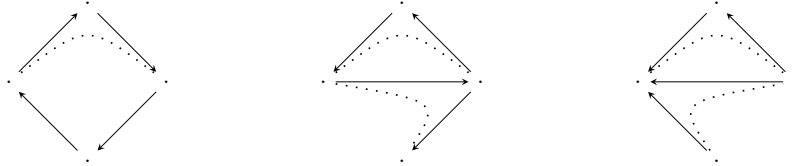
Then one can easily check that there are 11 graded quivers (Q, d) with $Q \in \mathcal{M}_{p,q}^{\tilde{A}}$ and d a W_Q -grading up to isomorphism of graded quiver, Therefore there are exactly 11 non-isomorphic algebras of global dimension 2 and of cluster type $\tilde{A}_{2,2}$.

The only possible weights are $|w| = 0$ or $|w| = 1$. Therefore these eleven algebras are divided into two derived equivalence classes.

There are 8 algebras of global dimension ≤ 2 which are derived equivalent to $\tilde{A}_{2,2}$:



There are 3 algebras of global dimension ≤ 2 which are cluster type $\tilde{A}_{2,2}$ and not derived equivalent to $\tilde{A}_{2,2}$. There are all derived equivalent to each other and not piecewise hereditary:



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