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# CELEBRATING CERCIGNANI'S CONJECTURE FOR THE BOLTZMANN EQUATION

LAURENT DESVILLETES, CLÉMENT MOUHOT & CÉDRIC VILLANI

ABSTRACT. Cercignani's conjecture assumes a linear inequality between the entropy and entropy production functionals for Boltzmann's nonlinear integral operator in rarefied gas dynamics. Related to the field of logarithmic Sobolev inequalities and spectral gap inequalities, this issue has been at the core of the renewal of the mathematical theory of convergence to thermodynamical equilibrium for rarefied gases over the past decade. In this review paper, we survey the various positive and negative results which were obtained since the conjecture was proposed in the 1980s.

*This paper is dedicated to the memory of the late Carlo Cercignani, powerful mind and great scientist, one of the founders of the modern theory of the Boltzmann equation.*

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## 1. THE BOLTZMANN EQUATION AND ITS VARIANTS

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**1.1. The Boltzmann equation.** After Maxwell [57] wrote down the basic equation for the kinetic theory of gases, Boltzmann [14] made such remarkable progress on this equation that its name remained attached to it. The **Boltzmann equation** describes the behavior of a rarefied gas when the only interactions taken into account are binary collisions. This evolution equation reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f), \quad x \in \Omega, \quad v \in \mathbb{R}^d, \quad t \geq 0,$$

where  $\Omega \subset \mathbb{R}^d$  is the spatial domain ( $d \geq 2$ ) and  $f$  is the time-dependent particle distribution function in phase space. In the case when this distribution function is assumed to be independent of the position, equation (1.1) reduces to the **spatially homogeneous Boltzmann equation**

$$(1.1) \quad \frac{\partial f}{\partial t}(t, v) = Q(f, f)(t, v), \quad v \in \mathbb{R}^d, \quad t \geq 0,$$

where  $Q$  is the quadratic **Boltzmann collision operator**, defined by the bilinear form

$$Q(g, f) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} B(|v - v_*|, \cos \theta) (g'_* f' - g_* f) dv_* d\sigma.$$

We have used the shorthands  $f' = f(v')$ ,  $g_* = g(v_*)$  and  $g'_* = g(v'_*)$ , where

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma \quad \text{and} \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma$$

stand for the pre-collisional velocities of particles which after collision have velocities  $v$  and  $v_*$ . Moreover it is usual to introduce the **deviation angle**  $\theta \in [0, \pi]$  between  $v' - v'_*$  and  $v - v_*$ . The function  $B$  is called the **Boltzmann collision kernel** and it is determined by physics (it is related to the physical cross-section  $\Sigma(v - v_*, \sigma)$  by the formula  $B = |v - v_*| \Sigma$ ). On physical grounds (in particular Galilean invariance), it is assumed that  $B \geq 0$  and  $B$  is a function of  $|v - v_*|$  and  $\cos \theta$  only.

**1.2. The collision kernel.** In the theory of Maxwell and Boltzmann, the interaction between particles is reflected in the formula for the collision kernel  $B$ . It may be short-range or long-range. The most important case of short-range interaction is the **hard spheres** model, where particles are spheres interacting by contact. In that case,  $B = |v - v_*|$  in dimension  $d = 3$  (constant cross-section).

Typical models of long-range interactions are given by inverse power-law forces [27]. In dimension  $d = 3$ , if the intermolecular force scales like  $r^{-s}$  with  $s > 2$ , then

$$B(|v - v_*|, \cos \theta) = |v - v_*|^\gamma b(\theta), \quad \theta \in [0, \pi],$$

where  $b$  is smooth except near  $\theta = 0$ ,

$$b(\theta) \sim_{\theta=0} \text{cst } \theta^{-2-\nu},$$

and

$$\gamma = \frac{s-5}{s-1}, \quad \nu = \frac{2}{s-1}.$$

It will be interesting to also consider more general kernels  $B$  which do not necessarily come from microscopic interactions; a remarkable case is  $\gamma = 2$ .

In the important case of Coulomb interaction, the force scales like the inverse of the square of the distance between particles; then the Boltzmann operator does not make sense anymore [72, Annexe 1, Appendice]. This divergence led Landau [54] to introduce in 1936 a “diffusive” version of the Boltzmann collision operator, the **Landau collision operator**, defined by the bilinear form

$$(1.2) \quad Q(g, f) = \nabla_v \cdot \left( \int_{\mathbb{R}^N} \Phi(|v - v_*|) \left\{ |v - v_*|^2 \text{Id} - (v - v_*) \otimes (v - v_*) \right\} \left( (\nabla g) f_* - (\nabla g)_* f \right) dv_* \right)$$

leading to the polar form

$$(1.3) \quad Q(f, f) = \nabla_v \cdot \left( \int_{\mathbb{R}^N} \Phi(|v - v_*|) \left\{ |v - v_*|^2 \text{Id} - (v - v_*) \otimes (v - v_*) \right\} \left( \frac{\nabla f}{f} - \frac{\nabla f_*}{f_*} \right) f f_* dv_* \right).$$

Here  $\Phi(|v - v_*|) = |v - v_*|^{-3}$  and the dimension is  $d = 3$ . As in [36, 37] we can consider any dimension  $d \geq 2$  and more general functions  $\Phi$ , say  $\Phi(|v - v_*|) = |v - v_*|^\gamma$ ,  $\gamma \geq -d$ .

A mathematical way to derive the operator (1.2) is to apply the **grazing collision limit** to the Boltzmann collision operator with collision kernel  $B = \Phi(|v - v_*|) b(\theta)$ , that is, to concentrate on deviation angles  $\theta \simeq 0$  [7, 31, 33, 70, 4]. The case  $\gamma = -3$ ,  $d = 3$  considered by Landau in dimension  $d = 3$  will be called “Landau-Coulomb” for the sake of classification.

**1.3. Conserved quantities and entropy structure.** Boltzmann’s and Landau’s collision operators have the fundamental properties of conserving mass, momentum and energy

$$\int_{\mathbb{R}^d} Q(f, f)(v) \phi(v) dv = 0, \quad \phi(v) = 1, v, |v|^2/2$$

and satisfying (the first part of) Boltzmann's  $H$  theorem, which can be formally written as

$$\mathcal{D}(f) := -\frac{d}{dt} \int_{\mathbb{R}^d} f \log f \, dv = - \int_{\mathbb{R}^d} Q(f, f) \log f \, dv \geq 0.$$

Boltzmann's so-called " $\mathcal{H}$  functional"

$$\mathcal{H}(f) = \int f \log f \, dv$$

is the opposite of the entropy of the gas.

The second part of Boltzmann's  $H$  theorem states that under appropriate boundary conditions, any equilibrium distribution function (that is, such that  $v \cdot \nabla_x f = Q(f, f)$ ) satisfies  $\mathcal{D}(f) = 0$ , or equivalently  $Q(f, f) = 0$ , and takes the form of a Maxwellian distribution

$$M(\rho, u, T)(v) = \frac{\rho}{(2\pi T)^{d/2}} \exp\left(-\frac{|v-u|^2}{2T}\right).$$

The parameters  $\rho = 0$ ,  $u \in \mathbb{R}^d$  and  $T \geq 0$  are interpreted as respectively the density, mean velocity and temperature of the gas:

$$\rho = \int_{\mathbb{R}^d} f(v) \, dv, \quad u = \frac{1}{\rho} \int_{\mathbb{R}^d} v f(v) \, dv, \quad T = \frac{1}{d\rho} \int_{\mathbb{R}^d} |v-u|^2 f(v) \, dv.$$

As a result of the process of entropy production pushing towards local equilibrium combined with the constraints of conservation laws, solutions of the Boltzmann equation are expected to converge to a unique Maxwellian equilibrium (This is the Krasovskii–Lasalle principle in the context of the Boltzmann equation).

Assuming that  $\rho$  and  $T$  are positive, we may rescale  $f := f(v)$  into  $v \mapsto a f(\lambda(v-b))$ , in such a way that the new density, velocity and temperature are  $\rho = 1$ ,  $u = 0$  and  $T = 1$ . So we set  $M(v) = (2\pi)^{-d/2} e^{-|v|^2/2}$  as the Maxwellian equilibrium in the sequel.

**1.4. The linearized collision operators.** Let us first consider the Boltzmann collision operator. We introduce the fluctuation around the Maxwellian equilibrium  $M$  computed above:

$$f = M + Mh, \quad v \mapsto h(v) \in L^2(M)$$

where  $L^2(M)$  denotes the Lebesgue space  $L^2$  on  $\mathbb{R}^d$  with reference measure  $M(v) \, dv$ . Then the linearized collision operator writes

$$Lh = M^{-1} [Q(Mh, M) + Q(M, Mh)],$$

or

$$(1.4) \quad Lh = \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \left[ h' + h'_* - h - h_* \right] B M_* dv_* d\sigma$$

for the same collision kernel  $B$  as in  $Q$ .

It is easy to check that  $L$  is symmetric in the Hilbert space  $L^2(M)$  and that it is non-positive in this space (this is the linearized form of the  $H$  theorem). The dissipation of squared  $L^2$  norm, that is the opposite of the Dirichlet form associated with  $L$ , is

$$D(h) = -\langle h, Lh \rangle_{L^2(M)} = \frac{1}{4} \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S}^{d-1}} \left| h' + h'_* - h - h_* \right|^2 B M M_* dv dv_* d\sigma \geq 0.$$

It is straightforward from this formula that the null space of  $L$  has dimension  $d + 2$ , and is spanned by the so-called **collisional invariants**  $1, v_1, \dots, v_d, |v|^2$ .

In the case of the Landau operator, a similar line of arguments leads to

$$Lh(v) = M^{-1} \nabla_v \cdot \left( \int_{\mathbb{R}^d} \mathbf{a}(v - v_*) \left[ (\nabla h) - (\nabla h)_* \right] M M_* dv_* \right),$$

with

$$\mathbf{a}(z) = |z|^2 \Phi(z) \Pi_{z^\perp}, \quad (\Pi_{z^\perp})_{i,j} = \delta_{i,j} - \frac{z_i z_j}{|z|^2},$$

and the corresponding negative Dirichlet form is

$$(1.5) \quad \begin{aligned} D(h) &= -\langle h, Lh \rangle_{L^2(M)} \\ &= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |v - v_*|^2 \Phi(v - v_*) \left| \Pi_{(v-v_*)^\perp} \left( (\nabla h) - (\nabla h)_* \right) \right|^2 M M_* dv dv_* \geq 0. \end{aligned}$$

**1.5. Comparison with usual differential operators and classification.** The Boltzmann and Landau collision operators are *a priori* extremely intricate, partly due to their integral or integro-differential nature (and partly of course due to their nonlinear nature!). Therefore it is useful, in order to grab an intuition of these operators, to draw a parallel with usual differential operators which are more familiar.

For the Boltzmann collision operators, say with collision kernel of the form  $B = \Phi(|v - v_*|) b(|\theta|)$ , the most important two “parameters” interplaying and determining its behavior are (1) the growth or decay of  $\Phi$ , and (2) the singularity of  $b$  at grazing collisions  $\theta \sim 0$ . To be more precise, let us consider the model case  $\Phi(z) = z^\gamma$ ,  $\gamma \in (-d, +\infty)$  and  $b(|\theta|) \sim \theta^{-(d-1)-\nu}$ ,  $\nu \in (-\infty, 2)$  as  $\theta \sim 0$ , for the Boltzmann collision operator. Then the order of singularity (2) plays the role of the order (highest number of derivatives) in a differential operator. For instance  $\nu < 0$  in the model means a zero order operator, whereas  $\nu \in (0, 2)$  means a fractional differential operator with order  $\nu$ . And the growth or decay of  $\Phi$  (1) plays the role of the

growth or decay of the coefficients in a differential operator. Therefore  $\gamma = 0$  (the so-called Maxwell or pseudo-Maxwell molecules cases) would correspond to a constant coefficients differential operator, and  $\gamma = 1$  (similar to hard spheres) would correspond to unbounded polynomially growing coefficients.

From this comparison it becomes natural to consider the Landau collision operator with  $\Phi(z) = z^\gamma$  formally as the limit case  $\nu = 2$  in the above classification. This unified picture of this family of integro-differential operators is summarized in figure 1 below.

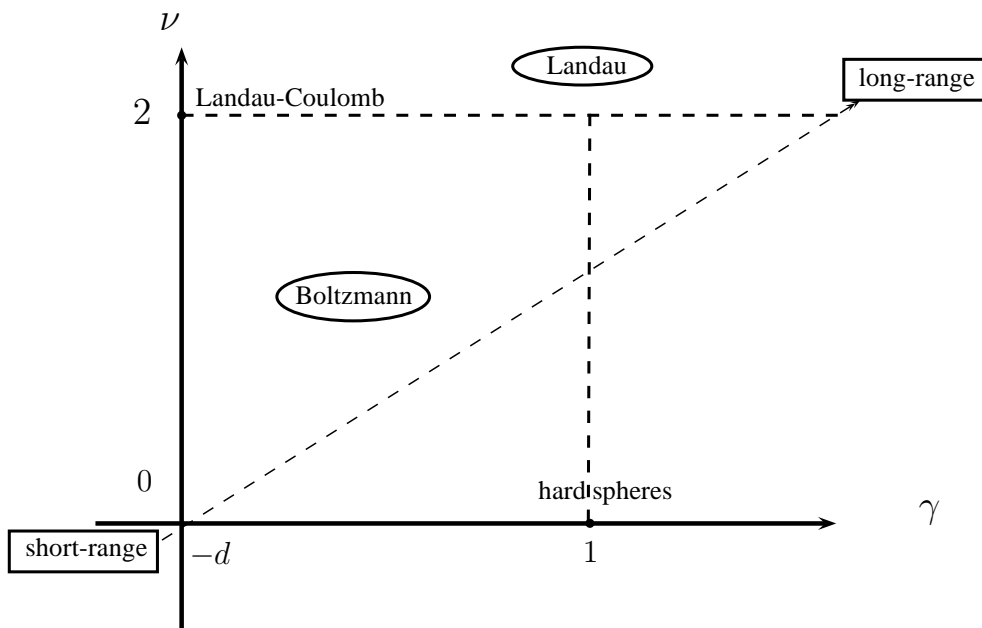


FIGURE 1. Classification of the Boltzmann and Landau operators

## 2. CERCIGNANI'S CONJECTURE

**2.1. Constructive quantitative estimates for the large time behavior.** The relaxation to equilibrium has been studied since the works of Boltzmann and it is at the core of kinetic theory. The motivation is to provide an analytic basis for the second principle of thermodynamics for a statistical physics model of a gas out of equilibrium. Indeed, Boltzmann's famous  $H$  theorem gives an analytic meaning to the entropy production process and identifies possible equilibrium states. In this context, proving convergence towards equilibrium is a fundamental step to justify Boltzmann model, but cannot be fully satisfactory as long as it remains based on

non-constructive arguments. Indeed, as suggested implicitly by Boltzmann when answering critics of his theory based on Poincaré's recurrence Theorem, the validity of the Boltzmann equation breaks for very large time (see [73, Chapter 1, Section 2.5] for a discussion). It is therefore crucial to obtain constructive quantitative informations on the time scale of the convergence, in order to show that this time scale is much smaller than the time scale of validity of the model. Cercignani's conjecture is an attempt to provide such constructive quantitative estimates. In the words of Cercignani: "*The present contribution is intended as a step toward the solution of the first main problem of kinetic theory, as defined by Truesdell and Muncaster, i.e. "to discover and specify the circumstances that give rise to solutions which persist forever".*" It is inspired by the entropy - entropy production method, that we now briefly describe.

**2.2. The entropy - entropy production method.** This method was first used in kinetic theory for the **Fokker-Planck equation** (Cf. [8, 67])

$$\partial_t f = \nabla_v \cdot (\nabla f + v f), \quad v \in \mathbb{R}^d, \quad \int_{\mathbb{R}^d} f(w) dw = 1.$$

In that case, the equilibrium  $M$  is given by the formula

$$M(v) = (2\pi)^{-d/2} e^{-|v|^2/2}$$

and the entropy production is

$$\mathcal{D}_{FP}(f) = \int_{\mathbb{R}^d} f(v) \left| \nabla \log \frac{f}{M} \right|^2 dv.$$

The exponential convergence is then obtained thanks to the **logarithmic Sobolev inequality** (cf. [45]), which exactly means in this setting

$$\mathcal{D}_{FP}(f) \geq 2 \left[ \mathcal{H}(f) - \mathcal{H}(M) \right].$$

Consider the more general case of an equation for which a Lyapunov functional  $\mathcal{H}_*$  exists, that is

$$\mathcal{D}_*(f(t)) := -\frac{d}{dt} \mathcal{H}_*(f(t)) \geq 0$$

and assume that the entropy  $-\mathcal{H}_*$  is maximal for a unique function  $M_*$  (among the functions belonging to a space depending on the conserved quantities in the equation). As seen in the previous section, this structure is provided by the  $H$  theorem for Boltzmann and Landau equations. The **entropy - entropy production method** consists in looking for (functional) estimates like

$$\mathcal{D}_*(f) \geq \Theta(\mathcal{H}_*(f) - \mathcal{H}_*(M_*)),$$



where  $\Theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is a function such that

$$\Theta(x) = 0 \quad \iff \quad x = 0.$$

The stronger  $\Theta$  increases near 0 the better the rate of relaxation to equilibrium, since the differential inequality

$$\frac{d}{dt}(\mathcal{H}_*(f) - \mathcal{H}_*(M_*)) \leq -\Theta(\mathcal{H}_*(f) - \mathcal{H}_*(M_*))$$

leads to

$$\mathcal{H}_*(f(t)) - \mathcal{H}_*(M_*) \leq R(t),$$

where  $R$  is the reciprocal of a primitive of  $-1/\Theta$ . Then, if the relative entropy  $\mathcal{H}_*(f) - \mathcal{H}_*(M_*)$  is coercive in the sense that it controls from below some distance or some norm (denoted by  $\|\cdot\|_*$ ) between  $f$  and its associated equilibrium distribution  $M_*$  (for the Boltzmann entropy this is precisely provided by the so-called Czizsar-Kullback-Pinsker inequality, see [29, 53]), we obtain

$$\|f(t) - M_*\|_* \leq S(t),$$

where (generically)

$$S(t) = C R(t)^{1/2}.$$

In the particular case  $\Phi(x) = Cx$  (like in the case of the Fokker-Planck equation), one gets

$$R(t) \leq C e^{-C't},$$

*i.e.*, **exponential convergence towards equilibrium** (with explicit rate). In the slightly worse case  $\Phi(x) = C_\varepsilon x^{1+\varepsilon}$  for some (or all)  $\varepsilon > 0$  we can deduce

$$R(t) \leq C'_\varepsilon t^{-1/\varepsilon},$$

and we thus get **algebraic convergence towards equilibrium** (with explicit rate). When  $\varepsilon > 0$  can be taken as small as one wishes, we speak of **almost exponential convergence**.

**2.3. Cercignani's conjecture.** The original Cercignani's conjecture [28] is written in the following form: for any  $f$  and its associated Maxwellian state  $M$  with same mass, momentum and temperature

$$\mathcal{D}(f) \geq \lambda \rho \left[ \mathcal{H}(f) - \mathcal{H}(M) \right],$$

where  $\rho$  is the density (mass of  $f$ ) and  $\lambda > 0$  is a “*suitable constant*”.

We shall now develop this very general statement into a layer of more specified conjectures. Let us fix  $\rho = 1$  without loss of generality.

In the case when the constant  $\lambda$  only depends on the collision kernel  $B$ , the temperature of  $M$  (or  $f$ ), and some bound on the entropy of  $f$  (*i.e.*, only the

basic physical *a priori* estimates), we shall call this inequality the **strong form of Cercignani's conjecture**.

In the case when the constant  $\lambda$  also depends on some additional bounds on  $f$  (typically of smoothness, moments and lower bounds), we shall call such an inequality the **weak form of Cercignani's conjecture**. Let us point out that it is of crucial importance to know whether the bounds used can be shown to be propagated by the Boltzmann equation, in order to be able to “apply” the weak form of Cercignani's conjecture to the relaxation to equilibrium of its solutions. This of course guides which bounds are natural or not.

In the slightly different case when the following inequality holds

$$\mathcal{D}(f) \geq \lambda_\varepsilon [\mathcal{H}(f) - \mathcal{H}(M)]^{1+\varepsilon}, \quad \varepsilon > 0$$

we shall speak of the  $\varepsilon$ -**polynomial Cercignani's conjecture** and it can be divided again into weak and strong versions according to how much the constant  $\lambda_\varepsilon$  depends on  $f$ .

Finally a strictly similar hierarchy of conjectures can be formulated on the Landau entropy production functional, and we shall call it **Cercignani's conjecture for the Landau equation**.

**2.4. A linearized counterpart to the conjecture.** A natural linearized counterpart of Cercignani's conjecture for the Boltzmann or Landau equation consists in replacing the entropy production functional and the Boltzmann entropy by their linearized approximation, *i.e.*, respectively the Dirichlet form of the collision operators discussed above and the  $L^2(M)$  norm. This spectral gap question was already well-known for a long time and used by Cercignani as an inspiration and supportive argument for his conjecture in [28]. So let us call this the **linearized Cercignani's conjecture**:

$$D(h) \geq \lambda \|h - \Pi(h)\|_{L^2(M)}^2,$$

where  $\Pi$  denotes the orthogonal projector in  $L^2(M)$  onto the null space of the linearized collision operator, and  $\lambda$  only depends on the collision kernel  $B$  and the temperature of  $M$ .

Note that due the linear homogeneity of this relation, no weak version (with constant depending on the function  $h$ ) would make sense.

Again obviously the same question can be asked on the Dirichlet form of the Landau collision operators and leads to the **linearized Cercignani's conjecture for the Landau equation**.

**2.5. Comparison with differential operators.** In the light of the comparison we have made with usual differential operators, a functional inequality interpretation of Cercignani's conjecture is the following. Its nonlinear form is an intricate (because of

strong nonlinearity and average over additional angular variables) amplified version of a logarithmic Sobolev inequality. Its linearized form is an intricate (because again of average over additional angular variables) amplified version of a Poincaré inequality.

**2.6. Use of the collision kernel, weighted forms of the conjecture.** A natural generalization of the conjecture is to know whether an inequality of the form

$$\mathcal{D}(f) \geq \lambda \left[ \mathcal{H}_w(f) - \mathcal{H}_w(M) \right]$$

holds, where

$$\mathcal{H}_w(f) := \int f \log f w(v) dv$$

for some weight function  $w > 0$ . This requires the additional moment normalization

$$\int f w |v|^2 dv = \int M w |v|^2 dv$$

in order for the relative entropy to satisfy

$$\mathcal{H}_w(f) - \mathcal{H}_w(M) \geq 0.$$

We shall call such an inequality a **weighted Cercignani's conjecture**.

In the linearized case, a natural conjecture is similarly

$$D(h) \geq \lambda \|(h - \Pi(h)) w\|_{L^2(M)}^2,$$

for some weight function  $w$ , and we shall call such an inequality a **linearized weighted Cercignani's conjecture**.

**2.7. Semigroup form of the conjecture.** Another natural related question is the following. Cercignani's conjecture at the end of the day is a purely functional inequality, and has nothing to do with the solutions of the Boltzmann equation. However the main application of this conjecture is of course the rate of convergence to equilibrium for the Boltzmann equation. Hence a natural question is whether

$$\forall t \geq 0, \quad \mathcal{D}(f_t) \geq \lambda \left[ \mathcal{H}(f_t) - \mathcal{H}(M) \right]$$

for any solution  $(f_t)_{t \geq 0}$ , or for a subset of solutions to the spatially homogeneous Boltzmann equation. We shall call this the **strong semigroup Cercignani conjecture**.

This conjecture is of course related to the weak functional form of the conjecture in the sense that if appropriate conditions are found on  $f$  for which the latter holds, and these conditions are shown to be propagated by the solutions to the Boltzmann equation, then this strong semigroup form of the conjecture will hold. However it

is possible to imagine weird conditions created by the semigroup that cannot be simply identified at the purely functional level.

Let us first remark that Cercignani's conjecture does not exhaust the issue of the rate of relaxation towards equilibrium, since it is possible that an exponential convergence occurs for the semigroup even without any functional inequality<sup>1</sup>. Hence it is natural to weaken again the semigroup conjecture in the following form. The **weak form of the semigroup Cercignani conjecture** is the following: is it true that

$$\forall t \geq 0, \quad |\mathcal{H}(f_t) - \mathcal{H}(M)| \leq C e^{-C' t}$$

for some positive constants  $C, C' > 0$ , and any solutions  $(f_t)_{t \geq 0}$ , or for a subset of solutions to the spatially homogeneous Boltzmann equation.

Let us add as a final remark that obvious extension to these semigroup forms of the conjecture can be drawn for the spatially homogeneous Landau equation.

**2.8. Relation with mean-field limit.** Let us mention the different but closely related question raised by Kac [51]. In this paper Kac introduced a many-particle jump process whose mean-field limit is the spatially homogeneous Boltzmann equation (he also introduced the mathematical formalization of the by-now well-known notion of propagation of chaos for a many-particles system). One of the goals of such a derivation was the understanding of the asymptotic behavior of the nonlinear Boltzmann equation through the linear many-particles system. Hence it is natural to search for a Cercignani's conjecture **at the level of the many-particles jump process**. Even if this process is linear the conjecture can be searched in nonlinear form or linear form, the difference being the kind of functional which is used for measuring the spectral gap (the Boltzmann entropy or an  $L^2$  norm). The specific new difficulty is to track the dependency of the estimates in terms of the number of particles (since the final goal is originally to pass to the limit). For recent results in the "linear" case (spectral gap in  $L^2$ ) see for instance [40, 50, 22, 56] which solves the problem (note however that since the  $L^2$  norm does not "tensorize" correctly in high dimension, it is not possible to pass to the limit in these estimates), and see [74, 21, 58] for some progress in the "nonlinear" case.

### 3. NEGATIVE RESULTS AT THE FUNCTIONAL LEVEL

**3.1. Counterexample in the Maxwell molecules case with only mass, energy and entropy bounds.** The explicit solutions constructed by Bobylev [10, 11] show in particular that the exponential rate of relaxation can be arbitrarily slow if

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<sup>1</sup>Think for instance of the trivial example of a nonsymmetric matrix  $A$  with only negative eigenvalues: its Dirichlet form will not control anything from below since it has no sign, although the semigroup  $e^{tA}$  will obviously relax exponentially fast towards zero.

one assumes only finite mass, energy and entropy (in the cutoff Maxwell molecules case  $B = 1$ ). The question was then: would the conjecture be true with better growth of the collision kernel than  $B = 1$ , for instance in the important physical case of hard spheres  $B = |v - v_*|$ ?

### 3.2. Counterexample in the hard spheres case with higher moment bounds.

Then Wennberg [78] proved in 1997 that the conjecture was false also in the case of hard spheres. However the counterexample of Wennberg does not have infinitely many moments. And since it was known since [34] that hard potentials with cutoff ( $\gamma > 0$  and no angular singularity) would produce such higher moments at every order for any positive time, the next natural question was whether such counterexamples would hold with infinitely many moments.

**3.3. Counterexample with infinitely many moment bounds.** It was finally shown by Bobylev and Cercignani in [13] that, for hard potentials with angular cutoff and with  $\gamma < 2$ , the estimate

$$\mathcal{D}(f) \geq C(f) \left[ \mathcal{H}(f) - \mathcal{H}(M) \right]$$

does not hold uniformly over any class of distributions  $f$  which have fixed mass, momentum and energy, are bounded below by a uniform Maxwellian distribution, and satisfy

$$\|f\|_{H^k} \leq S_k, \quad \|f\|_{L^1(1+|v|^s)} \leq M_s$$

for some given sequences  $M_s > 0$ ,  $s \in \mathbb{N}$  and  $S_k > 0$ ,  $k \in \mathbb{N}$ . This is still true if an arbitrary number of the constants  $M_s$  and  $S_k$  are chosen arbitrarily close to their equilibrium value (*i.e.*, the value of the corresponding moment or Sobolev norm for the Maxwellian distribution associated with  $f$ ).

## 4. POSITIVE RESULTS AT THE FUNCTIONAL LEVEL

**4.1. Early attempts.** A first result was proved in 1979 by Aizenmann and Bak in [1] on a nonlinear model bearing some reminiscence to the Boltzmann operator, namely a coagulation-fragmentation kernel with constant rate. The proof uses two very cleverly devised convexity inequalities and this is probably the first example of a linear relation between entropy and entropy production for a non-diffusive Boltzmann-like equation.

For the Boltzmann and Landau equations, an inequality linking the entropy dissipation towards an  $L^1$ -type distance to the space of Maxwellian (instead of the entropy!) was established by the first author in [32], for densities bounded below by a Maxwellian.

A decisive improvement was the discovery of a link between the entropy and the entropy dissipation of the Boltzmann equation, due to Carlen and Carvalho in the

beginning of the 90s (Cf. [17, 18]): this link was however still far from linear, the relation was of the form

$$\mathcal{D}(f) \geq \Theta(\mathcal{H}(f) - \mathcal{H}(M)),$$

for some nonlinear function  $\Theta$ , but no information was given on the behavior of the function  $\Theta$  at zero, and it turns out that its behavior was much worse than linear. The inequality was proved in the case of pseudo-Maxwell molecules and its proof made crucial use of the theory of Wild sums. As an elaborated consequence of this breakthrough let us also mention the paper [24] where the authors give (for a constant collision kernel) the first proof of the convergence of the entropy towards its equilibrium value with minimal assumptions on the initial datum (finite mass, energy and entropy).

**4.2. Landau equation with (over)-Maxwellian molecules.** The first positive result obtained with a linear dependence between entropy and entropy dissipation (for Landau or Boltzmann operators) is due to the first and third author of this paper, and concerns Landau's operator in the case when  $\Phi(|v - v_*|) \geq C_\Phi > 0$  (sometimes called "over-Maxwellian"):

$$\mathcal{D}(f) \geq \lambda \left[ \mathcal{H}(f) - \mathcal{H}(M) \right],$$

with  $\lambda > 0$  depending on  $C_\Phi$  and the dimension  $d$  only (for all  $f$  with fixed mass, momentum, energy and upper bound on the entropy). In other words, the strong conjecture holds for Landau equation with over-Maxwellian molecules.

This result is obtained by observing that (for some  $\lambda' > 0$ )

$$\mathcal{D}(f) \geq \lambda' \mathcal{D}_{FP}(f),$$

and then by using the logarithmic Sobolev inequality.

The relationship between  $\mathcal{D}(f)$  and  $\mathcal{D}_{FP}(f)$  can be obtained (Cf. [37]) either thanks to an almost explicit computation in which spherical coordinates are used, or thanks to a variant of the estimates of [32] (in which differential operators have been replaced by integral operators).

**4.3. Boltzmann equation: "almost linear" relation.** Successive results by Toscani and the third author of this survey (Cf. [68, 69, 74]) then led to a result valid for the Boltzmann equation with any reasonable collision kernel. That is, when  $B \geq C \min\{|v - v_*|^\gamma, |v - v_*|^{-\beta}\} > 0$ , with  $\beta, \gamma \geq 0$  (with or without cutoff, that is for any  $\nu \in (-\infty, 2)$ ) the following "almost linear link" exists between entropy and entropy production:

$$\mathcal{D}(f) \geq \lambda_\varepsilon(f) \left[ \mathcal{H}(f) - \mathcal{H}(M) \right]^{1+\varepsilon},$$

where  $\lambda_\varepsilon(f)$  (possibly going to 0 as  $\varepsilon \rightarrow 0$ ) depends on  $\beta, \gamma, C, \|f\|_{H^{s(\varepsilon)}}, \|f(1 + |v|)^{k(\varepsilon)}\|_{L^1}$  for some  $s(\varepsilon)$  and  $k(\varepsilon)$  depending on  $\varepsilon$  (and blowing up as  $\varepsilon \rightarrow 0$ ), and some Gaussian (or better) lower bounds. In other terms, the (weak)  $\varepsilon$ -polynomial Cercignani's conjecture holds for the Boltzmann operator with all reasonable collision kernels.

**4.4. Boltzmann equation: linear relation.** Finally, in the special (and unfortunately non physical) case when  $\Phi(|v - v_*|) \geq C_\Phi (1 + |v - v_*|^2) > 0$ , the original (strong) Cercignani's conjecture was proved to hold by the third author of this paper (Cf. [74]), that is:

$$\mathcal{D}(f) \geq \lambda \left[ \mathcal{H}(f) - \mathcal{H}(M) \right],$$

with  $\lambda > 0$  only depending on  $C_\Phi$  and the energy and entropy of  $f$ .

**4.5. Spectral gaps of linearized operators.** The origin of the study of the spectral properties of the linearized Boltzmann collision operator can be traced back all the way to Hilbert [49] in his work on the integral operators. The integral collision operator of Boltzmann was a key example and motivation from physics there, and Hilbert introduced (in the case of hard spheres, *i.e.*,  $\gamma = 1$  and  $\nu < 0$  in dimension  $d = 3$ ) the splitting between the local and non-local parts of the linearized operator, and proved the “complete continuity” (compactness in today's words) of its non-local part. This result was important in the construction, together with the Fredholm theory, of what is now called the “Hilbert expansions”.

The second key step is due to Carleman [16]. In this book he introduced (still in the hard spheres case) the use of Weyl's theorem (stating that under certain assumptions, the essential spectrum is unchanged by a compact perturbation of the operator), in order to prove the existence of a spectral gap of the linearized collision operator. Grad [43] then generalized this result to a broader class of collision kernels (the so-called hard potentials with cutoff,  $\gamma \in (0, 1], \nu < 0$ ). Further generalizations of this non constructive approach were provided by Caffisch [15] and Golse and Poupaud [42] who proved that in the case of soft potentials with angular cutoff ( $\gamma \in (-d, 0)$  and  $\nu < 0$ ) there is no spectral gap but still a weighted form of the conjecture holds (in weaker norms than the ambient norm).

Another direction of research was investigated by Wang-Chang and Uhlenbeck [76] and Bobylev [11], who obtained a complete diagonalization of the linearized collision operator in the Maxwell molecules case ( $\gamma = 0$ ).

In the paper [9], by Baranger and the second author of this survey, the first constructive estimates were given for the spectral gap of the hard spheres model, by introducing a geometric method and the idea of using “intermediate” collisions in the regions where the collision kernel vanishes, and then rely on the explicit diagonalization of the Maxwell case above (for a constant collision kernel). The

paper [60] then developed a whole series of constructive coercivity estimates, most of them sharp, and gives a full answer to the linearized Cercignani's conjecture in the cutoff case (and some ellipticity estimate in the non-cutoff case, see below).

The interested reader can find more details in the review paper [62].

**4.6. Elliptic estimates in the long-range case.** A different but related subject is the issue of proving ellipticity estimate on the entropy production functional in the long-range case  $\nu \in (0, 2)$ . The first results are due to Lions [55] and then to the first and third authors of this paper, together with Alexandre and Wennberg ([71] and [2]) and can be roughly stated as the following inequality

$$\mathcal{D}(f) \geq C \left( \|f\|_{H_{loc}^{\nu/2}} - \|f(1 + |v|^s)\|_{L^1} \right)$$

for some  $s \in \mathbb{N}$  and some constant  $C > 0$ , and where the exponent  $\nu/2$  of regularization is related to the singularity of the collision kernel. These regularity estimates were used in a crucial way in the papers [3, 4] by Alexandre and the third author of this survey in the Cauchy theory of the long-ranged Boltzmann equation and its grazing collision limit towards the Landau equation. Earlier an estimate based on the entropy production had already been used in a crucial way in the paper [70] in the Cauchy theory of the spatially homogeneous long-ranged Boltzmann with soft potentials.

The corresponding (and sharper) regularity estimates for the Landau equation were derived in the papers [36, 37] by the first and third authors of this survey.

Now if we consider the linearized case, the first related result for the Boltzmann collision operator is due to Pao [66] who proved the compactness of the resolvent in the case of inverse power-law interaction force  $F(z) = z^{-s}$  with  $s > 3$  in dimension  $d = 3$  (this paper was hard to read because of the use of pseudo-differential calculus, and somewhat controversial, see [52]). However, some more general and constructive versions of these results were recovered by the new estimates proven by Strain and the second author of this paper in [65], confirming that the latter paper was fully correct. In short the paper [65] proves that a “gain” of  $\nu$  polynomial weight occurs in the long-ranged case due to the ellipticity of the operator: roughly speaking the local regularization improves the behavior for large velocities. The recent work by Gressman and Strain [44] finally gives a sharp (although intricate) characterization of the elliptic norm associated with the Dirichlet form, and answers a conjecture raised in [65] about the spectral gap in the non-cutoff case (see below).

For the linearized Landau operator, the non-constructive approach based on Weyl's theorem was used successfully by Degond and Lemou [30] to prove an elliptic coercivity estimate. The sharp (non-isotropic) norm was then devised by Guo in [47] and some constructive versions of this coercivity result were given in [65] (see also [48] for related sharp hypoellipticity results).



## 5. RESULTS ON THE SEMIGROUP CERCIGNANI CONJECTURE

Since up to this date no results are known on the strong semigroup Cercignani's conjecture, we shall review the results on the weak semigroup conjecture (about the rate of decay of the semigroup).

The first paper which tackled the issue of exponential relaxation to equilibrium for data not necessarily close to equilibrium is [6] where a non constructive result is given by Arkeryd, Esposito and Pulvirenti in the hard spheres case for general solutions in weighted  $L^\infty$  spaces in the spatially homogeneous case, or solutions weakly inhomogeneous (close to the latter setting). This approach was generalized to an  $L^p$  setting in [77] by Wennberg.

In the papers [26, 23], Carlen, Carvalho and Lu give sharp lower (and upper) bounds on the rate of relaxation to equilibrium for the semigroup of the spatially homogeneous Boltzmann equation in the case of Maxwell molecules and soft potentials. Let us also mention the related important works by Carlen, Carvalho and Gabetta [19, 20], more focused on the specific issue of the Wild sums.

The paper [25] by Carlen, Gabetta and Toscani then gave a proof of exponential convergence towards equilibrium in the particular case of Maxwell molecules with cutoff, assuming moments and regularity bounds on the initial datum.

In the Maxwell molecules case with angular cutoff ( $\gamma = 0$ ,  $\nu < 0$  in the classification above), Carlen and Lu proved in [26] that when only 2 moments are assumed on the initial datum, then a rate of convergence algebraic and as slow as wanted can be explicitly obtained, and it depends very precisely on the integrability of the initial datum, in the sense that it depends on the function  $\varphi = \varphi(v) > 0$  (going to infinity as  $v$  goes to infinity) such that

$$\int f_{in} (1 + |v|^2) \varphi dv < +\infty.$$

This was the first example of a solution to the Boltzmann equation that *actually relaxes slower than exponentially towards equilibrium*.

In the soft potential case with or without angular cutoff ( $\gamma \in (-d, 0)$  and  $\nu < 2$ ), Carlen, Carvalho and Lu [23] proved (among other things) that for solutions with only  $2 + |\gamma|$  moments then again the convergence (in  $L^1$ ) can be algebraic and arbitrarily slow.

More recently a whole program of research has been carried out by the first and third author in order to develop a theory of relaxation to equilibrium in the large for inhomogeneous kinetic equations, and in particular for the full Boltzmann equation in confined domain under *a priori* smoothness and moments assumptions (see in particular the papers [38, 39]). The rate is not exponential, but still almost exponential (in the sense of a polynomial convergence for any polynomial). These works have

provided new insights on how transport effects combined with the thermalization process in velocity can yield convergence towards the global equilibrium and they have given birth to the new *hypocoercivity* theory.

Finally the last episode in this long research line on the rate of decay in the Boltzmann  $H$  theorem is the development of new tools in spectral theory for non-symmetric operators in order to systematically enlarge the functional space of the decay on a linearized semigroup (this is used for connecting the linearized theory [9, 63] in small  $L^2(M^{-1})$ -type spaces with the nonlinear theory [74, 39] in  $L^1$  type spaces). A “final answer” (constructive exponential relaxation of the relative entropy under reasonable *a priori* estimates) was given in [61] for hard spheres in the spatially homogeneous case, and then in the preprint by Gualdani, Mischler and the second author of this survey [46] for the inhomogeneous case (confined in the torus).

## 6. NEW CONJECTURES ABOUT THE CONJECTURE

We shall now present a few conjectures related to the results presented in this paper, some of them already formulated, some of them new. The work in progress [35] aims at giving partial answers to some of these conjectures.

**6.1. The use of grazing collisions.** In [74] the following conjecture is formulated:

**Conjecture 1 (Villani).** *For a collision kernel with growth of order  $\gamma \in (-d, +\infty)$  and singularity of order  $\nu \in [0, 2]$  (where  $\nu = 2$  formally plays the role of the Landau collision operator), the strong form of Cercignani's conjecture is true if and only if  $\gamma + \nu \geq 2$ .*

This conjecture formally extends the result proved in [74] for “superquadratic” collision kernels (formally  $\nu = 0$  and  $\gamma \geq 2$ ), as well as the result obtained in [37] for the Landau collision operator with hard potential (formally  $\nu = 2$  and  $\gamma \geq 0$ ).

In [65] it is conjectured the following

**Conjecture 2 (Mouhot-Strain).** *For a collision kernel with growth of order  $\gamma \in (-d, +\infty)$  and singularity of order  $\nu \in [0, 2]$  (where  $\nu = 2$  formally plays the role of the Landau collision operator), the strong form of linearized Cercignani's conjecture (existence of a spectral gap for the linearized operator) is true if and only if  $\gamma + \nu \geq 0$ .*

The direct implication in this conjecture was proved in [65], and recently the work [44] (and the preprints related to this announcement note) has solved this conjecture by providing a sharp characterization of the norm associated with the Dirichlet form of the operator. Let us also mention as an example of the fertility of Cercignani's conjecture that this conjecture also has inspired the work [64] about fractional Poincaré inequalities.

An interesting open question which now calls for further understanding is why there is such a “gap” between the condition  $\gamma + \nu \geq 2$  for a “nonlinear” spectral gap, and the condition  $\gamma + \nu \geq 0$  for a linearized spectral gap (this is somehow reminiscent of the gap between the conditions for a logarithmic Sobolev inequality to hold or a Poincaré inequality to hold).

**6.2. Use of (general) collision kernels.** In view of the preceding conjectures about the influence of the parameters  $\gamma$  and  $\nu$  (both at the nonlinear and linearized levels), it now seems natural to conjecture the following generalized weighted form of Cercignani’s conjecture:

**Conjecture 3.** *For a collision kernel with growth of order  $\gamma \in (-d, +\infty)$  and singularity of order  $\nu \in [0, 2]$  (where  $\nu = 2$  formally plays the role of the Landau collision operator), one has*

$$\mathcal{D}(f) \geq \lambda \int f \log \frac{f}{M} (1 + |v|)^{\gamma + \nu - 2} dv$$

at the nonlinear level and

$$D(h) \geq \lambda' \|(h - \Pi h)(1 + |v|)^{\gamma + \nu}\|_{L^2(M)}$$

at the linearized level (where  $h = (f - M)/M$  denotes the rescaled fluctuation around the equilibrium  $M$ , and  $\Pi$  the  $L^2(M)$ -orthogonal projection onto the collision invariants  $1, v_1, \dots, v_d, |v|^2$ ).

**6.3. Use of the tail of the distribution.** In view of the condition for a measure  $\mu = e^{-V}$  to satisfy a Poincaré inequality (essentially that  $V$  grows faster than linearly at infinity) or a logarithmic Sobolev inequality (essentially that  $V$  grows faster than quadratically at infinity), see for instance [75], it is natural to ask whether some bounds on the tail of the distribution with faster decay than polynomial could help in order to obtain Cercignani’s conjecture (this would be compatible with the counterexamples of [13]):

**Conjecture 4.** *For a collision kernel with growth of order  $\gamma \in (-d, +\infty)$  and singularity of order  $\nu \in [0, 2]$  (where  $\nu = 2$  formally plays the role of the Landau collision operator), and for a distribution  $f$  with exponential tail*

$$f(v) \sim C e^{-a|v|^\eta}, \quad v \sim \infty$$

for some constants  $a, C > 0$  and  $\eta \in (0, 2]$ , one has

$$\mathcal{D}(f) \geq \lambda \int f \log \frac{f}{M} (1 + |v|)^{\gamma + \nu + \eta - 2} dv$$

at the nonlinear level.

Hence if this conjecture were true (and taking for granted Villani's conjecture), it would mean that Cercignani's conjecture would hold for hard spheres and inverse power-laws interactions in dimension  $d = 3$  for initial data with strong enough exponential tail (a Gaussian tail would always work). These exponential tails are known to be propagated by the spatially homogeneous nonlinear Boltzmann equation for hard spheres and hard potentials with cutoff [12, 61] (see also [41] in the case of Gaussian tails).

Concerning general initial data, one has to turn to the theory of appearance of exponential moments, as developed in [61, 59]. However it only provides at the moment an appearance of an  $L^1$  decay of the form  $e^{-a|v|^\eta}$  with  $\eta = \gamma/2$ . Even with the improvement to  $\eta = \gamma$  in the recent work in progress [5], note that the hard spheres case (say  $\gamma = 0$  and  $\nu = 0$ ) is exactly borderline for Cercignani's conjecture, and for inverse power-laws in dimension  $d = 3$  where  $\gamma = (s - 5)/(s - 1)$  and  $\nu = 2/(s - 1)$ , then  $2\gamma + \nu = 2 - 6/(s - 1)$  is always less than 2, and hence this would not be sufficient for Cercignani's conjecture.

Of course a natural further question would be to know whether the grazing collisions  $\nu > 0$  can help for the appearance of exponential moments (for instance with  $\eta = \gamma + \nu$  instead of  $\eta = \gamma$ )... As the reader sees, the story opened by Cercignani's conjecture is far from finished, and it is likely that many more nice surprises are to come.

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L. DESVILLETES

CMLA, ENS CACHAN, CNRS, PRES UNIVERSUD & IUF  
61, AVENUE DU PRÉSIDENT WILSON  
94235 CACHAN CEDEX, FRANCE

E-MAIL: desville@cmla.ens-cachan.fr

C. MOUHOT

UNIVERSITY OF CAMBRIDGE, DPMMS  
WILBERFORCE ROAD, CB3 0WA, UK

E-MAIL: C.Mouhot@dpmms.cam.ac.uk



C. VILLANI

INSTITUT HENRI POINCARÉ & UNIVERSITÉ CLAUDE BERNARD LYON 1  
11 RUE PIERRE ET MARIE CURIE 75230 PARIS CEDEX 05, FRANCE

E-MAIL: [villani@ihp.jussieu.fr](mailto:villani@ihp.jussieu.fr)