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Causal observability of nonlinear time-delay systems with unknown inputs

G. Zheng , J-P. Barbot , D. Boutat, T. Floquet, J.-P. Richard

***Abstract*— This paper investigates the problem of causal observability of the states and unknown inputs of nonlinear time-delay systems. Using the theory of non-commutative rings, the nonlinear time-delay system is transformed into a suitable canonical form to solve the problem. A necessary and sufficient condition is given to guarantee the existence of a change of coordinate leading to such a form. Then, algorithms are provided to check the possibility of obtaining causal estimations of the states and unknown inputs for the studied systems.**

I. Introduction

Observation and estimation are important issues in control theory. For linear time-delay systems, various aspects of the observability problem have been studied in the literature, using different methods such as the functional analytic approach ([2]) or the algebraic approach ([3], [18], [6]). The theory of non-commutative rings has been applied to analyze nonlinear time-delay systems firstly in [15] for the disturbance decoupling problem of nonlinear time-delay systems, and for observability of nonlinear time-delay systems with known inputs in [19].

In this algebraic framework, the left Ore ring of non-commutative polynomials defined over the field of meromorphic functions is used for the analysis of nonlinear time-delay systems, since the rank of a module over this ring is well defined and can be used to characterize controllability, observability and identifiability of nonlinear time-delay systems.

Based on the algebraic framework proposed in [19], the problem of causal and non-causal observability of the states and unknown inputs for nonlinear time-delay systems has been studied in [20] assuming that the systems can be transformed into a suitable canonical form. This condition is relaxed in this paper by introducing additional information depending on the available measurements. A necessary and sufficient condition is given to guarantee the existence of those new variables.

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II. Algebraic framework

Consider the following nonlinear time-delay system:

$$\begin{cases} \dot{x} = f(x(t-i\tau)) + \sum_{j=0}^{s} g^j(x(t-i\tau))u(t-j\tau) \\ y = h(x(t-i\tau)) = [h_1(x(t-i\tau)), \ldots, h_p(x(t-i\tau))]^T \\ x(t) = \psi(t), u(t) = \varphi(t), t \in [-s\tau, 0] \end{cases} \tag{1}$$

where $x \in W \subset R^n$ denotes the state variables, $u = [u_1, \ldots, u_m]^T \in R^m$ is the unknown admissible input and $y \in R^p$ is the measurable output. Without loss of generality, it can be assumes that $p \geq m$. τ is the basic time delay, and it is assumed that all times delays are multiple times of τ. $i \in S_- = \{0, 1, \ldots, s\}$, a finite set of constant time-delays. f, g^j and h are meromorphic functions[1] where $f(x(t-i\tau)) = f(x, x(t-\tau), \ldots, x(t-s\tau))$. $\psi : [-s\tau, 0] \to R^n$ and $\varphi : [-s\tau, 0] \to R^m$ denote unknown continuous functions of initial conditions. In this work, it is assumed that the system (1) with $u = 0$ is locally observable, and admits a unique solution for initial conditions ψ and φ.

Based on the algebraic framework introduced in [19], let $\mathcal{K}$ be the field of meromorphic functions of a finite number of the variables from $\{x_j(t-i\tau), j \in [1, n], i \in S_-\}$. With the standard differential operator d, define the vector space $\mathcal{E}$ over $\mathcal{K}$:

$$\mathcal{E} = span_{\mathcal{K}}\{d\xi : \xi \in \mathcal{K}\}$$

which is the set of linear combinations of a finite number of one-forms from $dx_j(t-i\tau)$ with row vector coefficients in $\mathcal{K}$. Introduce the backward time-shift operator δ defined by

$$\delta^i \xi(t) = \xi(t-i\tau), \xi(t) \in \mathcal{K}, \text{ for } i \in Z^+ \tag{2}$$

and

$$\begin{aligned} \delta^i\left(a(t)d\xi(t)\right) &= \delta^i a(t)\delta^i d\xi(t) \\ &= a(t-i\tau)d\xi(t-i\tau) \end{aligned} \tag{3}$$

for $a(t)d\xi(t) \in \mathcal{E}$, and $i \in Z^+$.

Let $\mathcal{K}(\delta]$ denote the set of polynomials of the form

$$a(\delta] = a_0(t) + a_1(t)\delta + \cdots + a_{r_a}(t)\delta^{r_a} \tag{4}$$

where $a_i(t) \in \mathcal{K}$. The addition in $\mathcal{K}(\delta]$ is defined as usual, but the multiplication is given by

$$a(\delta]b(\delta] = \sum_{k=0}^{r_a+r_b} \sum_{i+j=k}^{i\leq r_a, j\leq r_b} a_i(t)b_j(t-i\tau)\delta^k \tag{5}$$

Note that $\mathcal{K}(\delta]$ satisfies the associative law and is a non-commutative ring (see [19]). However, it is proved that the ring $\mathcal{K}(\delta]$ is a left Ore ring [12], [19], which enables to define

[1] i.e. quotients of convergent power series with real coefficients [4], [19].

the rank of a module over this ring. Let $\mathcal{M}$ denote the left module over $\mathcal{K}(\delta]$: $\mathcal{M} = span_{\mathcal{K}(\delta]}\{d\xi, \xi \in \mathcal{K}\}$, where $\mathcal{K}(\delta]$ acts on $d\xi$ according to (2) and (3).

With the definition of $\mathcal{K}(\delta]$, (1) can be rewritten in a more compact form as follows:

$$\begin{cases} \dot{x} = f(x,\delta) + \sum_{i=1}^{m} G_i u_i(t) \\ y = h(x,\delta) \\ x(t) = \psi(t), u(t) = \varphi(t), t \in [-s\tau, 0] \end{cases} \tag{6}$$

where $f(x,\delta) = f(x(t-i\tau))$ and $h(x,\delta) = h(x(t-i\tau))$ with entries belonging to $\mathcal{K}$, $G_i = \sum_{j=0}^{s} g_i^j \delta^j$ with entries belonging to $\mathcal{K}(\delta]$. It is assumed that $rank_{\mathcal{K}(\delta]} \frac{\partial h}{\partial x} = p$, which implies that $[h_1, \ldots, h_p]^T$ are independent functions of x and its backward time-shifts.

III. Notations and definitions

Observability definitions for nonlinear systems without delays have been given given in [10] and in [5]. A definition of observability for time-delay systems is given in [14]. A more generic definition is stated here as follows:

Definition 1: System (1) is locally observable if the state $x(t) \in W \subset R^n$ can be expressed as a function of the output and its derivatives with their backward and forward shifts as follows:

$$x(t) = \alpha(y(t-j\tau), \ldots, y^{(k)}(t-j\tau)) \tag{7}$$

for $j \in Z$ and $k \in Z^+$. It is locally causally observable if (7) is satisfied for $j \in Z^+$ and $k \in Z^+$, and locally non-causally observer if (7) is satisfied for $j \in Z$ and $k \in Z^+$.

Following the same principle of Definition 1, the following definition for the reconstruction of the unknown inputs is given.

Definition 2: The unknown input $u(t)$ can be estimated if it can be written as a function of the derivatives of the output and its derivatives with backward and forward shifts, i.e.

$$u(t) = \beta(y(t-j\tau), \ldots, y^{(k)}(t-j\tau)) \tag{8}$$

for $j \in Z$ and $k \in Z^+$. It can be causally estimated if (8) is satisfied for $j \in Z^+$ and $k \in Z^+$, and non-causally estimated if (8) is satisfied for $j \in Z$ and $k \in Z^+$.

Definition 3: (Unimodular matrix) [14] A matrix $A \in \mathcal{K}^{n\times n}(\delta]$ is said to be unimodular over $\mathcal{K}(\delta]$ if it has a *left* inverse $A^{-1} \in \mathcal{K}^{n\times n}(\delta]$, such that $A^{-1}A = I_{n\times n}$.

Definition 4: (Change of coordinate) [14] $z = \phi(\delta, x) \in \mathcal{K}^{n\times 1}$ is a causal change of coordinate over $\mathcal{K}$ for system (1) if there exist locally a function $\phi^{-1} \in \mathcal{K}^{n\times 1}$ and some constants $c_1, \cdots, c_n \in N$ such that

$$diag\{\delta^{c_i}\}x = \phi^{-1}(\delta, z).$$

The change of coordinate is bicausal over $\mathcal{K}$ if $max\{c_i\} = 0$, i.e. $x = \phi^{-1}(\delta, z)$.

Note that the relative degree for nonlinear systems without delays is well defined *via* the Lie derivative (see [11]). Then many efforts have been done to extend the classical Lie derivative for nonlinear time-delay systems. Several researchers tried to extend the Lie derivative to nonlinear time-delay systems (see [8], [9], [16], [17]) in the framework of commutative rings. In what follows the relative degree and the observability indices for nonlinear time-delay systems are defined by extending the Lie derivative in the algebraic framework of [19], from the non-commutative rings point of view.

Let $f(x(t-j\tau))$ and $h(x(t-j\tau))$ for $0 \le j \le s$ be an n and p dimensional vectors, respectively, with entries $f_r \in \mathcal{K}$ for $1 \le r \le n$ and $h_i \in \mathcal{K}$ for $1 \le i \le p$. Let

$$\frac{\partial h_i}{\partial x} = \left[\frac{\partial h_i}{\partial x_1}, \cdots, \frac{\partial h_i}{\partial x_n}\right] \in \mathcal{K}^{1\times n}(\delta] \tag{9}$$

where for $1 \le r \le n$:

$$\frac{\partial h_i}{\partial x_r} = \sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)} \delta^j \in \mathcal{K}(\delta]$$

then the Lie derivative for nonlinear systems without delays can be extended to nonlinear time-delay systems in the framework of [19] as follows

$$L_f h_i = \frac{\partial h_i}{\partial x}(f) = \sum_{r=1}^{n}\sum_{j=0}^{s} \frac{\partial h_i}{\partial x_r(t-j\tau)} \delta^j (f_r) \tag{10}$$

For $j = 0$, (10) is the classical definition of the Lie derivative of h along f. For $h_i \in \mathcal{K}$, define

$$L_{G_i} h_i = \frac{\partial h_i}{\partial x}(G_i)$$

Using the above definition of Lie derivative, the relative degree can be defined in the following way:

Definition 5: (Relative degree) System (6) has relative degree $(\nu_1, \cdots, \nu_p)$ in an open set $W \subseteq R^n$ if, for $1 \le i \le p$, the following conditions are satisfied :

1) for all $x \in W$, $L_{G_j} L_f^r h_i(x,\delta) = 0$, for all $1 \le j \le m$ and $0 \le r < \nu_i - 1$;
2) there exists $x \in W$ such that $\exists j \in [1, m]$, $L_{G_j} L_f^{\nu_i - 1} h_i(x,\delta) \neq 0$.

If for $1 \le i \le p$, (1) is satisfied for all $r \ge 0$, then we set $\nu_i = \infty$.

Since (6) is assumed to be locally observable when $u = 0$, then one can define, for (6), the so-called observability indices introduced in [13]. Let

$$\mathcal{F}_k := span_{\mathcal{K}(\delta]}\left\{dh, dL_f h, \cdots, dL_f^{k-1} h\right\}$$

for $1 \le k \le n$. It can be shown that the filtration of $\mathcal{K}(\delta]$-module satisfies

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots \subset \mathcal{F}_n.$$

Then define

$$d_1 = rank_{\mathcal{K}(\delta]} \mathcal{F}_1$$

and

$$d_k = rank_{\mathcal{K}(\delta]} \mathcal{F}_k - rank_{\mathcal{K}(\delta]} \mathcal{F}_{k-1}$$

for $2 \le k \le n$. Let

$$k_i = card\{d_k \ge i, 1 \le k \le n\}.$$

Then $(k_1, \cdots, k_p)$ are the observability indices and

$$\sum_{i=1}^{p} k_i = n$$

since (6) is locally observable with $u = 0$. Reorder, if necessary, the output components of (6), such that

$$rank_{\mathcal{K}(\delta]}\{\frac{\partial h_1}{\partial x}, \cdots, \frac{\partial L_f^{k_1-1}h_1}{\partial x}, \cdots, \frac{\partial h_p}{\partial x}, \cdots, \frac{\partial L_f^{k_p-1}h_p}{\partial x}\} = k_1 + \cdots + k_p = n$$

Since $rank_{\mathcal{K}(\delta]}\frac{\partial h}{\partial x} = p$, the observability indices $(k_1, \cdots, k_p)$ for $(h_1, \cdots, h_p)$ are well defined, but the order may be not unique.

IV. Canonical form and Causal observability

Hereafter are recalled some results given in [20].

Proposition 1: [20] For $1 \leq i \leq p$, denote k_i the observability indices and ν_i the relative degree index for y_i of (6), and note $\rho_i = \min\{\nu_i, k_i\}$. Then there exists a change of coordinate $\phi(x, \delta) \in \mathcal{K}^{n\times 1}$, such that (6) can be transformed into the following form:

$$\dot{z}_{i,j} = z_{i,j+1} \tag{11}$$

$$\dot{z}_{i,\rho_i} = V_i = L_f^{\rho_i} h_i(x, \delta) + \sum_{j=1}^{m} L_{G_j} L_f^{\rho_i - 1} h_i(x, \delta) u_j \tag{12}$$

$$y_i = C_i z_i = z_{i,1} \tag{13}$$

$$\dot{\xi} = \alpha(z, \xi, \delta) + \beta(z, \xi, \delta) u \tag{14}$$

where

$$\begin{aligned} z_i &= (z_{i,1}, \cdots, z_{i,\rho_i})^T = \left(h_i, \cdots, L_f^{\rho_i-1} h_i\right)^T \in \mathcal{K}^{\rho_i \times 1} \\ \alpha &\in \mathcal{K}^{\mu\times 1}, \beta \in \mathcal{K}^{\mu\times 1}(\delta] \text{ with } \mu = n - \sum_{j=1}^{p} \rho_j \\ C_i &= (1, 0, \cdots, 0) \in R^{1\times\rho_i} \end{aligned}$$

Moreover if $k_i < \nu_i$, one has $V_i = L_f^{\rho_i} h_i = L_f^{k_i} h_i$. ■

For (11), note

$$H(x, \delta) = \Psi(x, \delta) + \Gamma(x, \delta) u \tag{15}$$

with

$$H(x, \delta) = (V_1, \cdots, V_p)^T$$
$$\Psi(x, \delta) = \left(L_f^{\rho_1} h_1, \cdots, L_f^{\rho_p} h_p\right)^T$$

and

$$\Gamma(x, \delta) = \begin{pmatrix} L_{G_1} L_f^{\rho_1-1} h_1 & \cdots & L_{G_m} L_f^{\rho_1-1} h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1} L_f^{\rho_p-1} h_p & \cdots & L_{G_m} L_f^{\rho_p-1} h_p \end{pmatrix} \tag{16}$$

where $H(x, \delta) \in \mathcal{K}^{p\times 1}$, $\Psi(x, \delta) \in \mathcal{K}^{p\times 1}$ and $\Gamma(x, \delta) \in \mathcal{K}^{p\times m}(\delta]$. Since $\Gamma \in \mathcal{K}^{p\times m}(\delta]$ with $m \leq p$, if

$$rank_{\mathcal{K}(\delta]}\Gamma = m,$$

there exists a matrix $\Xi \in \mathcal{K}^{p\times p}(\delta]$ such that

$$\Xi\Gamma = \begin{bmatrix} \bar{\Gamma} \\ \mathbf{0} \end{bmatrix}$$

where $\bar{\Gamma} \in \mathcal{K}^{m\times m}(\delta]$ has full rank m. The set Φ is introduced as follows:

$$\Phi = \{dh_1, \cdots, dL_f^{\rho_1-1}h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p-1}h_p\} \tag{17}$$

Then the following theorem can be stated.

Theorem 1: [20] Consider the system (6) with outputs $(y_1, \ldots, y_p)$ and the corresponding $(\rho_1, \ldots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indexes, respectively. If

$$rank_{\mathcal{K}(\delta]}\Phi = n$$

where Φ is defined in (17), then there exists a change of coordinate $\phi(x, \delta)$ such that (6) can be transformed into (11-14) with dim $\xi = 0$.

Moreover, if the change of coordinate is locally bicausal over $\mathcal{K}$, then the state $x(t)$ of (6) is locally causally observable, and if $\bar{\Gamma} \in \mathcal{K}^{m\times m}(\delta]$ is also unimodular over $\mathcal{K}(\delta]$, then the unknown input $u(t)$ of (6) can be causally estimated as well.■

Due to the particular triangular structure of the system (11)-(12), the whole state of the system, as well as the unknown input, can be estimated in finite time using higher order sliding mode observers [7].

When $rank_{\mathcal{K}(\delta]}\Phi < n$, the problem is to estimate the state ξ. It may be still possible provided that some additional conditions are satisfied. This is detailed in the next Section.

V. Extended case

In [1], a constructive algorithm to solve the above mentioned problem for nonlinear systems without delays has been studied. The result of this section can be seen as an extension of the work [1] to treat the observation problem for time-delay systems with unknown inputs of the form (6). The objective is to generate additional variables from the available measurement and unaffected by the unknown input such that an extended canonical form similar to (11)-(12) can be obtained for the estimation of the remaining state ξ.

For this, consider Φ defined in (17). If $rank_{\mathcal{K}(\delta]}\Phi = j$, one can select j linearly independent vectors over $\mathcal{K}(\delta]$ from Φ, denoted as

$$\Phi = \{dz_1, \cdots, dz_j\}.$$

Note

$$\pounds = span_{R[\delta]}\{z_1, \cdots, z_j\}$$

where $R[\delta]$ is the commutative ring of polynomials of δ with coefficients belonging to the field R and let $\pounds(\delta]$ be the set of polynomials of δ with coefficients over $\pounds$. Define the module spanned by the elements of Φ over $\pounds(\delta]$ as follows

$$\Omega = span_{\pounds(\delta]}\{\xi, \xi \in \Phi\}. \tag{18}$$

Define also

$$\mathcal{G} = span_{R[\delta]}\{G_1, \ldots, G_m\}$$

and its *left* annihilator

$$\mathcal{G}^{\perp} = span_{R[\delta]}\{\omega \in \Omega \mid \omega g = 0, \forall g \in \mathcal{G}\}.$$

Based on the above definitions, let us state the main result of the paper.

Theorem 2: Consider the system (6) with outputs $y = (y_1, \cdots, y_p)^T$ and the corresponding $(\rho_1, \ldots, \rho_p)$ with $\rho_i = \min\{k_i, \nu_i\}$ where k_i and ν_i are the observability indices and the relative degree indexes, respectively. Suppose $rank_{\mathcal{K}(\delta]}\Phi < n$ where Φ is defined in (17). There exist l new independent outputs over $\mathcal{K}$ suitable to the causal estimation problem if and only if

$$rank_{\mathcal{K}}\mathcal{H} = l$$

where

$$\mathcal{H} = span_{R[\delta]}\{\omega \in \mathcal{G}^{\perp} \cap \Omega \mid \omega f \notin \pounds\}. \tag{19}$$

Moreover, the l additional outputs, denoted $\bar{y}_i$, $1 \leq i \leq l$, are given by:

$$\bar{y}_i = \omega_i f \mod \pounds$$

where $\omega_i \in \mathcal{H}$.

Proof:

Denote $Q_i = \left[q_1^i, \cdots, q_p^i\right]$ as a $1 \times p$ vector with $q_j^i \in \mathcal{K}(\delta]$ for $1 \leq j \leq p$. One has

$$\begin{aligned} Q_i\Gamma &= Q_i \begin{pmatrix} L_{G_1}L_f^{\rho_1-1}h_1 & \cdots & L_{G_m}L_f^{\rho_1-1}h_1 \\ \vdots & \ddots & \vdots \\ L_{G_1}L_f^{\rho_p-1}h_p & \cdots & L_{G_m}L_f^{\rho_p-1}h_p \end{pmatrix} \\ &= \left(Q_i \begin{bmatrix} \frac{\partial L_f^{\rho_1-1}h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p-1}h_p}{\partial x} \end{bmatrix} \right) [G_1, \cdots, G_m] \end{aligned}$$

because of the associativity law over $\mathcal{K}(\delta]$. Then according to the definition (9), one gets

$$Q_i\Gamma = \omega_i \left[G_1, \cdots, G_m\right] = \omega_i G$$

where $\omega_i = \sum_{c=1}^{n}\sum_{j=1}^{p} q_j^i \frac{\partial L_f^{\rho_j-1}h_j}{\partial x_c} dx_c$.

Moreover, one can check that

$$\begin{aligned} \omega_i f &= \left(Q_i \begin{bmatrix} \frac{\partial L_f^{\rho_1-1}h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p-1}h_p}{\partial x} \end{bmatrix} \right) f \\ &= Q_i \left(\begin{bmatrix} \frac{\partial L_f^{\rho_1-1}h_1}{\partial x} \\ \vdots \\ \frac{\partial L_f^{\rho_p-1}h_p}{\partial x} \end{bmatrix} f \right) \\ &= Q_i \begin{bmatrix} L_f^{\rho_1}h_1 \\ \vdots \\ L_f^{\rho_p}h_p \end{bmatrix} = Q_i\Psi. \end{aligned}$$

According to (15), one has

$$Q_iH = Q_i(\Psi + \Gamma u) = \omega_i f + \omega_i G u \tag{20}$$

where $H = (V_1, \cdots, V_p)^T$ is a vector which can be estimated in finite time but which is affected by the unknown input. Thus, one can generate l additional information suitable to solve the estimation problem if and only if one can find l independent $\{Q_1, \cdots, Q_l\}$ vectors over $\mathcal{K}(\delta]$ such that for each $Q_i = \{q_1^i, \cdots, q_p^i\}$ satisfying $q_j^i \in \pounds(\delta]$, one has $Q_i\Gamma = 0$ and $Q_iH \notin \pounds$,

Thus, one has to prove that the following conditions are equivalent:

1) there exist l row vectors $Q_i = \left[q_1^i, \cdots, q_p^i\right]$, with $q_j^i \in \pounds(\delta]$, such that $rank_{\mathcal{K}(\delta]}\{Q_1, \cdots, Q_l\} = l$, $Q_i\Gamma = 0$ and $Q_iH \notin \pounds$.
2) $rank_{\mathcal{K}}\mathcal{H} = l$ with $\mathcal{H}$ defined in (19).

Necessity: Suppose item 1) is satisfied, then according to (20), one has

$$Q_i\Gamma = \omega_i G = 0 \Rightarrow \omega_i \in \mathcal{G}^{\perp}$$

and

$$Q_iH = \omega_i f \notin \pounds.$$

Thus $rank_{\mathcal{K}(\delta]}\{Q_1, \cdots, Q_l\} = l$ implies $rank_{\mathcal{K}}\{\omega_1, \cdots, \omega_l\} = l$.

Since $L_f^{\rho_i-1}h_i \in \pounds$ and $q_i \in \pounds(\delta]$, then $\omega_i = \sum_{c=1}^{n}\sum_{j=1}^{p} q_j^i \frac{\partial L_f^{\rho_j-1}h_j}{\partial x_c} dx_c \in \Omega$. Thus, $\omega_i \in \mathcal{H}$ defined in (19).

Sufficiency: Suppose $rank_{\mathcal{K}}\mathcal{H} = l$. Then one can find l independent 1-forms over $\mathcal{K}$: $\{\omega_1, \cdots, \omega_l\}$ with $\omega_i \in \mathcal{G}^{\perp} \cap \Omega$ which implies there exist l independent vectors over $\mathcal{K}(\delta]$: $\{Q_1, \cdots, Q_l\}$ with entries belonging to $\pounds(\delta]$ such that $rank_{\mathcal{K}(\delta]}\{Q_1, \cdots, Q_l\} = l$, since for each Q_i one has

$$Q_i\Gamma = \omega_i G = 0$$

and

$$Q_iH = \omega_i f \notin \pounds.$$

The variable

$$\bar{y}_i = Q_iH = \omega_i f \mod \pounds$$

can be used as an additional output since

1) H can be estimated in finite time;
2) Q_i has entries in $\pounds(\delta]$;
3) $\bar{y}_i$ do not belong to the current set $\pounds$ of measured variables.

■

Theorem 2 means that if $rank_{\mathcal{K}}\mathcal{H} = l$, one can define an extended output $\left[y^T, \bar{y}\right]^T \in \mathbb{R}^{p+l}$ with observability indices $\bar{k}_i$ and relative degree indexes $\bar{\nu}_i$. Then, similarly to (17), a new $\bar{\Phi}$ can be defined. If $rank_{\mathcal{K}(\delta]}\bar{\Phi} = n$, the state and the unknown inputs can be estimated in finite time. Otherwise, if $rank_{\mathcal{K}(\delta]}\bar{\Phi} < n$ and if Theorem 2 is still valid, the procedure is iterated until Theorem 1 is true, resulting in the causal observation of the states and unknown inputs for nonlinear time-delay systems of the form (6).

In the tables Algorithm 1 and 2, two algorithms are proposed in order to check the possible causal estimation of the states and unknown inputs. Algorithm 2 is used to determine whether the studied system can be transformed into the proposed canonical form, by using Theorem 2. If it

is, Algorithm 1 then checks whether the states and unknown inputs of the system are causally observable.

Let us note that it is the locally bicausal change of coordinate which makes the state of system locally causally observable. And it is the unimodular characteristic of Γ over $\mathcal{K}(\delta]$ which guarantees the causal reconstruction of unknown inputs. The non-causal case was studied in [20].

Algorithm 1 Check the causal estimation of the states and unknown inputs

Input: TDS with $x \in R^n$, $y \in R^p$ and $u \in R^m$.
Output: "Failed" or ("OK", $\hat{x}, \hat{u}$).
Main: ▷ Check observability of states and inputs
$\mathcal{Y} \leftarrow \{y_1, \cdots, y_p\}$
for each $y_i \in \mathcal{Y}$ **do**
  $\nu_i \leftarrow \mathsf{RelaDeg}(y_i)$; ▷ Get relative degree for y_i
  $k_i \leftarrow \mathsf{ObsInd}(y_i)$; ▷ Get observability indices for y_i
  $\rho_i \leftarrow \max\{\nu_i, k_i\}$;
end for
$\Phi \leftarrow \{dh_1, \cdots, dL_f^{\rho_1} h_1, \cdots, dh_p, \cdots, dL_f^{\rho_p} h_p\}$;
$\rho \leftarrow \{\rho_i, \cdots, \rho_p\}$;
$j \leftarrow rank_{\mathcal{K}(\delta]}\Phi$;
result $\leftarrow$ $\mathsf{Search_Diffeo}(\Phi, \mathcal{Y}, \rho, j)$;
if result=("Failed") **then**
  return ("Failed");
end if
$(\Phi, \mathcal{Y}, \rho) \leftarrow$ result;
$z = \phi(x, \delta) \leftarrow (\Phi, \mathcal{Y}, \rho)$; ▷ Get the change of coordinate
if ϕ is bicausal over $\mathcal{K}$ **then**
  $\hat{x} = \phi^{-1}(\delta, z) \in \mathcal{K}^{n\times 1}$; ▷ Causally observable
else
  $\hat{x} \leftarrow \mathsf{Null}$; ▷ Non-causal observation
end if
Calculate $\Gamma(x, \delta)$ ▷ According to Eq. (16)
if $rank_{\mathcal{K}(\delta]}\Gamma = m$ **then**
  $\exists \Xi$ s.t. $\Xi\Gamma = \begin{bmatrix} \bar{\Gamma} \\ \mathbf{0} \end{bmatrix}$ with $rank_{\mathcal{K}(\delta]}\bar{\Gamma} = m$;
  if $\bar{\Gamma}$ is unimodular over $\mathcal{K}(\delta]$ **then**
    $\hat{u} = \beta(\delta, y, \cdots, y^{(n)})$ ▷ Causal estimation
  else
    $\hat{u} \leftarrow \mathsf{Null}$; ▷ Non-causal estimation
  end if
else
  $\hat{u} \leftarrow \mathsf{Null}$; ▷ Might not be estimated
end if
return ("OK", $\hat{x}$, $\hat{u}$);

Algorithm 2 Search the change of coordinate to transform TDS into the canonical form

Function $\mathsf{Search_Diffeo}(\Phi, \mathcal{Y}, \rho, j)$
if $j = n$ **then** ▷ The rank of Φ over $\mathcal{K}(\delta]$ is equal to the dimension of the system
  return (Φ, $\mathcal{Y}$, ρ);
end if
while $j < n$ **do**
  $\Phi \leftarrow \{dz_1, \cdots, dz_j\}$; ▷ Select j linearly independent vectors over $\mathcal{K}$
  $\pounds \leftarrow span_{R[\delta]}\{z_1, \cdots, z_j\}$;
  $\Omega \leftarrow span_{\pounds(\delta]}\{\xi, \xi \in \Phi\}$;
  $\mathcal{H} \leftarrow span_{R[\delta]}\{\omega \in \mathcal{G}^{\perp} \cap \Omega \mid \omega f \notin \pounds\}$;
  $l \leftarrow rank_{\mathcal{K}}\mathcal{H}$;
  if $l > 0$ **then** ▷ Theorem 2 is satisfied
    $\exists l$ 1-forms, s.t. $\mathcal{H} = span_{R[\delta]}\{\omega_1, \cdots \omega_l\}$;
    $\bar{\mathcal{Y}} \leftarrow \{\omega_i f \mod \pounds, 1 \le i \le l\}$; ▷ New outputs
    for each $\bar{y}_i \in \bar{\mathcal{Y}}$ **do**
      $\nu_i \leftarrow \mathsf{RelaDeg}(y_i)$;
      $k_i \leftarrow \mathsf{ObsInd}(y_i)$;
      $\rho_i \leftarrow \max\{\nu_i, k_i\}$;
      $\rho \leftarrow \rho \cup \{\rho_i\}$;
      $\mathcal{Y} \leftarrow \mathcal{Y} \cup \{\bar{y}_i\}$;
      $\Phi \leftarrow \Phi \cup \{d\bar{y}_i, \cdots, dL_f^{\rho_i}\bar{y}_i\}$;
    end for
    $j \leftarrow rank_{\mathcal{K}(\delta]}\Phi$;
  else
    return ("Failed");
  end if
end while
return($\Phi, \mathcal{Y}, \rho$);

VI. Illustrative example

Consider the following dynamical system:

$$\begin{cases} \dot{x}_1 = -\delta x_1 + \delta x_4 u_1 \\ \dot{x}_2 = -\delta x_3 + x_4 \\ \dot{x}_3 = x_2 - \delta x_4 u_1 \\ \dot{x}_4 = u_2 \\ y_1 = x_1 \\ y_2 = \delta x_1 + x_3 \end{cases} \tag{21}$$

One can check that

$$\rho_1 = \rho_2 = 1$$

and $\Phi = \{dx_1, \delta dx_1 + dx_3\} which gives rank_{\mathcal{K}(\delta]}\Phi = 2 < n$.

Set $\mathcal{G} = span_{R[\delta]}\{G_1, \ldots, G_m\}$, one has

$$\mathcal{G}^{\perp} = span_{R[\delta]}\{dx_1 + dx_3, dx_2\}$$

Since $rank_{\mathcal{K}(\delta]}\Phi = 2$, $\pounds = span_{R[\delta]}\{x_1, \delta x_1 + x_3\}$ and

$$\Omega = span_{\pounds(\delta]}\{\xi, \xi \in \Phi\} = span_{\pounds(\delta]}\{dx_1, dx_3\}.$$

One obtains

$$\begin{aligned}\Omega \cap \mathcal{G}^{\perp} &= span_{\pounds(\delta]}\{dx_1, dx_3\} \\ &\quad \cap span_{R[\delta]}\{dx_1 + dx_3, dx_2\} \\ &= span_{\pounds(\delta]}\{dx_1 + dx_3\}\end{aligned}$$

From the definition of $\mathcal{H}$ in (19), one can check that $rank_{\mathcal{K}}\mathcal{H} = 1$, which gives the one-form $\omega = dx_1 + dx_3$, satisfying $\omega \in \Omega \cap \mathcal{G}^{\perp}$ and $\omega f = -\delta x_1 + x_2 \notin \pounds$. Thus, a new output $\bar{y}_1 = h_3$ is given by

$$\begin{aligned}\bar{y}_1 &= h_3 = \omega f \mod \pounds \\ &= x_2 = \delta y_1 + (1-\delta)\dot{y}_1 + \dot{y}_2\end{aligned} \tag{22}$$

which yields $\nu_3 = 2$ and $k_3 = 2$, thus $\rho_3 = 2$. Finally one has the new Φ as follows:

$$\Phi = \{dx_1, \delta dx_1 + dx_3, dx_2, -\delta dx_3 + dx_4\}$$

It can be checked that $rank_{\mathcal{K}(\delta]}\Phi = 4$, and the new $\pounds$ is

$$\pounds \quad = span_{R[\delta]}\{x_1, \delta x_1 + x_3, x_2, -\delta x_3 + x_4\}$$

This gives the following change of coordinate

$$z = \phi(x, \delta) = (x_1, \delta x_1 + x_3, x_2, -\delta x_3 + x_4)^T$$

it is easy to check that it is bicausal over $\mathcal{K}(\delta]$, since

$$x = \phi^{-1} = \left(z_1, z_3, z_2 - \delta z_1, z_4 + \delta z_2 - \delta^2 z_1\right)^T$$

and one gets

$$\begin{cases} x_1 = y_1 \\ x_2 = y_3 \\ x_3 = y_2 - \delta y_1 \\ x_4 = -\delta^2 y_1 + \delta y_2 + \dot{y}_3 \end{cases}$$

where the new output y_3 is defined in (22).

Moreover, the matrix Γ with the new output y_3 is

$$\Gamma = \begin{pmatrix} \delta x_4 & 0 \\ \delta^2 x_4 \delta - \delta x_4 & 0 \\ \delta^2 x_4 \delta & 1 \end{pmatrix}$$

and

$$rank_{\mathcal{K}(\delta]}\Gamma = 2.$$

One can find matrices

$$\Xi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\bar{\Gamma}^{-1} = \begin{pmatrix} \frac{1}{\delta x_4} & 0 \\ -\delta & 1 \end{pmatrix}$$

such that

$$\begin{bmatrix} \bar{\Gamma}^{-1} & \mathbf{0} \end{bmatrix} \Xi \Gamma = I_{2\times 2}$$

Consequently, according to Theorem 1, u_1 and u_2 are causally observable. A straightforward computation yields

$$\begin{cases} u_1 = \frac{\dot{y}_1}{-\delta^3 y_1 + \delta^2 y_2 + \delta \dot{y}_3} \\ u_2 = -\delta \dot{y}_1 + \ddot{y}_3 \end{cases}$$

VII. Conclusion

In this paper, causal observation of nonlinear time-delay systems with unknown inputs has been studied. The system has to be transformed in a canonical form suitable for the design of a finite time observer. A necessary and sufficient condition for the existence of a change of coordinate has been given. A constructive algorithm has been proposed to check the possibility to obtain causal estimations of the states and unknown inputs for the studied systems.

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