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Optimal stopping time problem with discontinuous reward

Magdalena Kobylanski* and Marie-Claire Quenez†

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Abstract

We study, for any stopping time S , the optimal stopping time problem $v(S) = \text{ess sup}_{\theta \geq S} E[\phi(\theta) | \mathcal{F}_S]$, where the reward is given by a family $\{\phi(\theta), \theta \in T_0\}$ of positive random variables indexed by stopping times. We solve the problem under the weakest assumptions in terms of the regularity of the reward. More precisely, the reward family $\{\phi(\theta), \theta \in T_0\}$ is supposed to satisfy some compatibility conditions and to be upper-semicontinuous along stopping times in expectation. We give several properties of the value function. We show the existence of an optimal stopping time. Also, we obtain a characterization of the minimal and the maximal optimal stopping times.

Key word: Optimal stopping.

AMS subject classifications: Primary 60G40

Introduction

The optimal stopping time problem has been widely studied in case of *reward* given by a right continuous left limited (RCLL) positive adapted process (ϕ_t) defined on $[0, T]$ (see for example Shiryaev (1978), El Karoui (1981), Karatzas and Shreve (1998) or Peskir and Shiryaev (2006)). If $T > 0$ is the fixed time horizon and if T_0 denotes the set of stopping times θ smaller than T , the problem consists in computing the maximal reward given by

$$v(0) = \sup\{E[\phi_\tau], \tau \in T_0\},$$

in finding conditions for the existence of an optimal stopping time and giving a method to compute these optimal stopping times.

Classically, the *value function* at time $S \in T_0$ is defined by $v(S) = \text{ess sup}\{E[\phi_\tau | \mathcal{F}_S], \tau \in T_0 \text{ and } \tau \geq S \text{ a.s.}\}$. The value function is given by a family of random variable $\{v(S), S \in T_0\}$. By using the right continuity of the reward (ϕ_t) , it can be shown that there exists a RCLL adapted process (v_t) which *aggregates* the family of random variable $\{v(S), S \in T_0\}$ that is such that $v_S = v(S)$ a.s. for each $S \in T_0$. This process is the *Snell envelope* of (ϕ_t) , that is the smallest supermartingale process that dominates ϕ . Moreover, when the reward (ϕ_t) is continuous, the stopping time defined trajectoryally by

$$\bar{\theta}(S) = \inf\{t \geq S, v_t = \phi_t\}$$

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is optimal.

Recall that El Karoui (1981) has introduced the more general notion of a reward given by a family $\{\phi(\theta), \theta \in T_0\}$ of positive random variables which satisfies some compatibility properties. In the recent paper of Kobylanski et al. (2009), this notion appears to be the appropriate one to study the d -multiple optimal stopping time problem. Moreover, in this work, Kobylanski et al. (2009) show that under quite weak assumptions (right and left continuity in expectation along stopping times of the reward), the minimal optimal stopping time for the value function at time S

$$v(S) = \text{ess sup}\{E[\phi(\theta) | \mathcal{F}_S], \theta \in T_0 \text{ and } \theta \geq S \text{ a.s.}\}, \quad (0.1)$$

is given by

$$\theta_*(S) := \text{ess inf}\{\theta \in T_0, \theta \geq S \text{ a.s. and } u(\theta) = \phi(\theta) \text{ a.s.}\}. \quad (0.2)$$

Let us emphasize that the minimal optimal stopping time $\theta_*(S)$ is no longer defined as a hitting time of processes but as an essential infimum of random variables. Also, this result allows to deal with the optimal stopping problem only in terms of admissible families of random variables. It presents the advantage that it does no longer require aggregation results. Indeed, the existence of optimal stopping times as well as the characterization of the minimal one can be done by using only the value function family and the reward family and no longer the aggregated processes. We stress on that in the multiple case, it avoids long and heavy proofs, due to some difficult aggregation problems. It allows to solve the problem under weaker assumptions than before in the unified framework of families of random variables.

In the present work, we consider the case of a one optimal stopping time problem with a discontinuous reward. More precisely, the reward is given by a family of random variables which satisfies some compatibility conditions and which is supposed to be upper-semicontinuous over stopping times in expectation. Note that these assumptions in terms of smoothness of the reward are optimal in order to ensure the existence of an optimal stopping time. Indeed, in the deterministic case, the upper-semicontinuity is the minimal assumption on a function $\phi : [0, T] \rightarrow \mathbb{R}; t \mapsto \phi(t)$ which ensures that the supremum of ϕ is attained on any closed subset of $[0, T]$.

Under these assumptions, we show the existence of an optimal stopping time for the value function (0.1) which is given by the essential infimum $\theta_*(S)$ defined by (0.2). We stress on that the mathematical tools which are used in this proof are not sophisticated tools, as those of the general theory of processes, but just the use of well chosen supermartingale systems and an appropriate construction of penalized stopping times. We also show that $\theta_*(S)$ is the minimal optimal stopping time. Also, the stopping time given by

$$\check{\theta}(S) = \text{ess sup}\{\theta \in T_0, \theta \geq S \text{ a.s. and } E[v(\theta)] = E[v(S)]\},$$

is proven to be the maximal optimal stopping time. Note that an important tool in this work is the use of the family of random variables defined by $v^+(S) = \text{ess sup}\{E[\phi(\theta) | \mathcal{F}_S], \theta \in T_0 \text{ and } \theta > S \text{ a.s.}\}$ for each stopping time S . Some properties and links between v , v^+ and ϕ are studied in this paper.

These new results allow to solve the case of a reward process (ϕ_t) which can be much less regular than in the previous works. For instance, this allows to solve the case of a reward process given by $\phi_t = f(X_t)$, where f is upper-semicontinuous and (X_t) is a RCLL process supposed to be left continuous along stopping times. This opens a way to a large range of applications, for instance in finance.

The paper is organised as follows. In section 1, we give some first properties on v and v^+ . In particular, we have $v(S) = \phi(S) \vee v^+(S)$ a.s. for each $S \in T_0$ and the family $\{v^+(S), S \in T_0\}$ is right continuous along stopping times. In section 2, we show the existence of an optimal

stopping time under some minimal assumptions. We begin by constructing ε -optimal stopping times which are appropriate to this case. Then, these ε -optimal stopping times are shown to tend to $\theta_*(S)$ as ε tends to 0. Moreover, $\theta_*(S)$ is proven to be an optimal stopping time for $v(S)$ and even the minimal one. At last, the stopping time $\check{\theta}(S)$ is proven to be the maximal optimal stopping time. In section 3, we give some strict supermartingale conditions on v which ensure the equality between v and ϕ (locally). Secondly, we give some conditions on v and v^+ which ensure the equality between v and v^+ (for some stopping times which are specified). At last, we give a few applications of the classical Doob-Meyer decomposition in particular when the reward is right continuous in expectation, which allows to use aggregation results. In section 4, we give some examples where the reward is given by an upper semicontinuous function of a RCLL adapted process (X_t) which is left continuous in expectation. We stress on that, except in the last part, all the properties established in this work do not require any result of the general theory of processes.

Let $\mathbb{F} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$ be a probability space equipped with a filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ satisfying the usual conditions of right continuity and augmentation by the null sets $\mathcal{F} = \mathcal{F}_T$. We suppose that \mathcal{F}_0 contains only sets of probability 0 or 1. The time horizon is a fixed constant $T \in]0, \infty[$. We denote by T_0 the collection of stopping times of \mathbb{F} with values in $[0, T]$. More generally, for any stopping times S , we denote by T_S (resp. T_{S+}) the class of stopping times $\theta \in T_0$ with $\theta \geq S$ a.s. (resp. $\theta > S$ a.s. on $\{S < T\}$ and $\theta = T$ a.s. on $\{S = T\}$).

We also define $T_{[S, S']}$ the set of $\theta \in T_0$ with $S \leq \theta \leq S'$ a.s. and $T_{]S, S']}$ the set of $\theta \in T_0$ with $S < \theta \leq S'$ a.s.. Similarly, the set $T_{]S, S']}$ on A will denote the set of $\theta \in T_0$ with $S < \theta \leq S'$ a.s. on A .

We use the following notation: For $t \in \mathbb{R}$ and for real valued random variables X and X_n , $n \in \mathbb{N}$, “ $X_n \uparrow X$ ” stands for “the sequence (X_n) is nondecreasing and converges to X a.s.”.

1 First properties

Definition 1.1. We say that a family $\{\phi(\theta), \theta \in T_0, \}$ is admissible if it satisfies the following conditions

- 1) for all $\theta \in T_0$ $\phi(\theta)$ is a \mathcal{F}_θ -measurable positive random variable (r.v.),
- 2) for all $\theta, \theta' \in T_0$, $\phi(\theta) = \phi(\theta')$ a.s. on $\{\theta = \theta'\}$.

Remark 1.1. By convention, the positivity property of a random variable means that it takes its values in $\overline{\mathbb{R}}^+$.

Also, note that it is possible to define a family associated with a given process. More precisely, let (ϕ_t) be a progressive process. Then, the family $\{\phi(\theta), \theta \in T_0, \}$ defined by $\phi(\theta) = \phi_\theta$ is admissible.

Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family called *reward*. For $S \in T_0$, the *value function at time S* is given by

$$v(S) = \text{ess sup}_{\theta \in T_S} E[\phi(\theta) | \mathcal{F}_S]. \quad (1.1)$$

Let us introduce the family of random variables $\{v^+(S), S \in T_0\}$ defined for each $S \in T_0$ by

$$v^+(S) = \text{ess sup}_{\theta \in T_{S+}} E[\phi(\theta) | \mathcal{F}_S]. \quad (1.2)$$

where T_{S+} is the class of stopping times $\theta \in T_0$ with $\theta > S$ a.s. on $\{S < T\}$ and $\theta = T$ a.s. on $\{S = T\}$. Note that $v^+(S) = \phi(T)$ a.s. on $\{S = T\}$.

Proposition 1.1. (Admissibility of v and v^+) *The families $\{v(S), S \in T_0\}$ and $\{v^+(S), S \in T_0\}$ defined by (1.1) and (1.2) are admissible families.*

PROOF: The arguments are the same for $\{v(S), S \in T_0\}$ and $\{v^+(S), S \in T_0\}$. We prove the property only for $\{v^+(S), S \in T_0\}$. Property 1 of admissibility for $\{v^+(S), S \in T_0\}$ follows from the existence of the essential supremum (see Appendix Theorem A.7).

Take $S, S' \in T_0$ and let $A = \{S = S'\}$. For each $\theta \in T_{S^+}$ put $\theta_A = \theta \mathbf{1}_A + T \mathbf{1}_{A^c}$. As $A \in \mathcal{F}_S \cap \mathcal{F}_{S'}$, one has a.s. on A , $E[\phi(\theta) | \mathcal{F}_S] = E[\phi(\theta_A) | \mathcal{F}_S] = E[\phi(\theta_A) | \mathcal{F}_{S'}] \leq v^+(S')$, because $\theta_A \in T_{S^+}$. Hence, taking the essential supremum over $\theta \in T_{S^+}$ one has $v^+(S) \leq v^+(S')$ a.s. and by symmetry of S and S' , we have proven property 2 of admissibility. \square

Proposition 1.2. (Optimizing sequences for v and v^+) *There exists a sequence of stopping times $(\theta^n)_{n \in \mathbb{N}}$ with θ^n in T_S (resp. T_{S^+}) such that the sequence $(E[\phi(\theta^n) | \mathcal{F}_S])_{n \in \mathbb{N}}$ is increasing and such that*

$$v(S) \quad (\text{resp. } v^+(S)) \quad = \lim_{n \rightarrow \infty} \uparrow E[\phi(\theta^n) | \mathcal{F}_S] \quad \text{a.s.}$$

PROOF: Again, the arguments are the same for $\{v(S), S \in T_0\}$ and $\{v^+(S), S \in T_0\}$. We prove the property only for $\{v^+(S), S \in T_0\}$. For each $S \in T_0$, one can show that the set $\{E[\phi(\theta) | \mathcal{F}_S], \theta \in T_S\}$ is closed under pairwise maximization. Indeed, let $\theta, \theta' \in T_{S^+}$. Put $A = \{E[\phi(\theta') | \mathcal{F}_S] \leq E[\phi(\theta) | \mathcal{F}_S]\}$. One has $A \in \mathcal{F}_S$. Put $\tau = \theta \mathbf{1}_A + \theta' \mathbf{1}_{A^c}$. $\tau \in T_{S^+}$. It is easy to check that $E[\phi(\tau) | \mathcal{F}_S] = E[\phi(\theta) | \mathcal{F}_S] \vee E[\phi(\theta') | \mathcal{F}_S]$. The result follows by a classical result (see Appendix Theorem A.7). \square

Recall that for each fixed $S \in T_0$, an admissible family $\{h(\theta), \theta \in T_0\}$ is said to be a *supermartingale system* (resp. a *martingale system*) if for any $\theta, \theta' \in T_0$ such that $\theta \geq \theta'$ a.s.,

$$E[h(\theta) | \mathcal{F}_{\theta'}] \leq h(\theta') \quad \text{a.s.}, \quad (\text{resp. } E[h(\theta) | \mathcal{F}_{\theta'}] = h(\theta') \quad \text{a.s.}) \quad (1.3)$$

The families v and v^+ can be shown to be supermartingale systems. Also, the value function is characterized as the Snell envelope system associated with the reward ϕ . More precisely:

Proposition 1.3. *The two following properties hold.*

- *The admissible families $\{v(S), S \in T_0\}$ and $\{v^+(S), S \in T_0\}$ are supermartingale systems.*
- *The value function family $\{v(S), S \in T_0\}$ is characterized as the Snell envelope system associated with $\{\phi(S), S \in T_0\}$, that is the smallest supermartingale system which is greater (a.s.) than $\{\phi(S), S \in T_0\}$.*

PROOF: Let us prove the first part. Let us prove the supermartingale property for v^+ . Fix $S \geq S'$ a.s.. By Proposition 1.2, there exists an optimizing sequence (θ^n) for $v^+(S)$. By the monotone convergence theorem, $E[v^+(S) | \mathcal{F}_{S'}] = \lim_{n \rightarrow \infty} E[\phi(\theta^n) | \mathcal{F}_{S'}]$ a.s.. Now, for each n , since $\theta^n \in T_{(S')^+}$, we have $E[\phi(\theta^n) | \mathcal{F}_{S'}] \leq v^+(S')$ a.s. Hence, $E[v^+(S) | \mathcal{F}_{S'}] \leq v^+(S')$ a.s., which gives the supermartingale property of v^+ . The supermartingale property of v can be proved by using the same arguments.

Let us prove the second point (which is classical). First, we clearly have that $\{v(S), S \in T_0\}$ is a supermartingale system and that for each $S \in T_0$, $v(S) \geq \phi(S)$ a.s. Let us prove that is the smallest. Let $\{v'(S), S \in T_0\}$ be a supermartingale system such that for each $\theta \in T_0$, $v'(\theta) \geq \phi(\theta)$ a.s.. Fix $S \in T_0$. By the properties of v' , for all $\theta \in T_S$, $v'(S) \geq E[v'(\theta) | \mathcal{F}_S] \geq E[\phi(\theta) | \mathcal{F}_S]$ a.s.. Taking the supremum over $\theta \in T_S$, we have $v'(S) \geq v(S)$ a.s.. \square

Recall now the following optimality criterium (see for instance El Karoui (1981)):

Proposition 1.4. (Optimality criterium) *Fix $S \in T_0$ and let $\theta_* \in T_S$ be such that $E[\phi(\theta_*)] < \infty$. The two following assertions are equivalent*

1. θ_* is S -optimal for $v(S)$, that is

$$v(S) = E[\phi(\theta_*) | \mathcal{F}_S] \quad \text{a.s.} \quad (1.4)$$

2. The following equalities hold

$$v(\theta_*) = \phi(\theta_*) \quad \text{a.s.}, \quad \text{and} \quad E[v(S)] = E[v(\theta_*)].$$

3. The following equality holds

$$E[v(S)] = E[\phi(\theta_*)].$$

Remark 1.2. Note that since the value function is a supermartingale system, equality $E[v(S)] = E[v(\theta_*)]$ is equivalent to the fact that the family $\{v(\theta), \theta \in T_{[S, \theta_*]}\}$ is a martingale system, that is for all $\theta, \theta' \in T_0$ such that $S \leq \theta, \theta' \leq \theta_*$ a.s., $v(\theta) = E[v(\theta') | \mathcal{F}_\theta]$ a.s. on $\{\theta \leq \theta'\}$ (which can also be written $\{v((\theta \vee S) \wedge \theta_*), \theta \in T_0\}$ is a martingale system).

Remark 1.3. Note that by very similar arguments to those used in the previous proof, we can even show that for a fixed $S \in T_0$ and $\theta_* \in T_S$ such that $E[\phi(\theta_*) | \mathcal{F}_S] < \infty$ a.s., then θ_* is S -optimal for $v(S)$ if and only if $v(\theta_*) = \phi(\theta_*)$ a.s. and $v(S) = E[v(\theta_*) | \mathcal{F}_S]$ a.s..

PROOF: Let us show that 1) implies 2). Suppose 1) is satisfied. Since the value function v is a supermartingale system greater than ϕ , we have clearly

$$v(S) \geq E[v(\theta_*) | \mathcal{F}_S] \geq E[\phi(\theta_*) | \mathcal{F}_S] \quad \text{a.s.}$$

Since equality (1.4) holds, this implies that the previous inequalities are actually equalities. In particular, $E[v(\theta_*) | \mathcal{F}_S] = E[\phi(\theta_*) | \mathcal{F}_S]$ a.s. but as inequality $v(\theta_*) \geq \phi(\theta_*)$ holds a.s., and as $E[\phi(\theta_*)] < \infty$, we have $v(\theta_*) = \phi(\theta_*)$ a.s..

Moreover, $v(S) = E[v(\theta_*) | \mathcal{F}_S]$ a.s. which gives $E[v(S)] = E[v(\theta_*)]$. Hence, 2) is satisfied.

Clearly, 2) implies 3). It remains to show that 3) implies 1).

Suppose 3) is satisfied. Since $v(S) \geq E[\phi(\theta_*) | \mathcal{F}_S]$ a.s., this gives $v(S) = E[\phi(\theta_*) | \mathcal{F}_S]$ a.s.. Hence, 1) is satisfied. \square

Remark 1.4. It is clear that by 3) of Proposition 1.4, a stopping time $\theta_* \in T_S$ such that $E[\phi(\theta_*)] < \infty$ is S -optimal for $v(S)$ if and only if it is optimal for the optimal stopping time problem (1.1), that is

$$\sup_{\theta \in T_S} E[\phi(\theta)] = E[\phi(\theta_*)].$$

Note that the following simple property holds.

Lemma 1.1. For each $S, \theta \in T_0$, we have

$$E[\phi(\theta) | \mathcal{F}_S] \mathbf{1}_{\{\theta > S\}} \leq v^+(S) \mathbf{1}_{\{\theta > S\}} \quad \text{a.s.}$$

PROOF: Note first that the r.v. $\bar{\theta}$ defined by $\bar{\theta} := \theta \mathbf{1}_{\{\theta > S\}} + T \mathbf{1}_{\{\theta \leq S\}}$ belongs to T_{S^+} . It follows that $E[\phi(\bar{\theta}) | \mathcal{F}_S] \leq v^+(S)$ a.s. and hence

$$E[\phi(\theta) | \mathcal{F}_S] \mathbf{1}_{\{\theta > S\}} = E[\phi(\bar{\theta}) | \mathcal{F}_S] \mathbf{1}_{\{\theta > S\}} \leq v^+(S) \mathbf{1}_{\{\theta > S\}} \quad \text{a.s.},$$

which ends the proof. \square

Using this lemma, we easily derive the following property (which corresponds to Proposition D.3 in Karatzas and Shreve (1998)):

Proposition 1.5. For all $S \in T_0$, the following equality holds:

$$v(S) = \phi(S) \vee v^+(S) \quad \text{a.s.}$$

PROOF: Note first that $v(S) \geq v^+(S)$ a.s. and that $v(S) \geq \phi(S)$ a.s., which yields the inequality $v(S) \geq \phi(S) \vee v^+(S)$ a.s..

It remains to show the other inequality. Fix $\theta \in T_S$. We have a.s.

$$\begin{aligned} E[\phi(\theta) | \mathcal{F}_S] &= \phi(S) \mathbf{1}_{\{\theta=S\}} + E[\phi(\theta) | \mathcal{F}_S] \mathbf{1}_{\{\theta>S\}} \\ &\leq \phi(S) \mathbf{1}_{\{\theta=S\}} + v^+(S) \mathbf{1}_{\{\theta>S\}}, \end{aligned}$$

where the last inequality follows from Lemma 1.1. Therefore,

$$E[\phi(\theta) | \mathcal{F}_S] \leq \phi(S) \vee v^+(S) \quad \text{a.s.}$$

By taking the essential supremum over $\theta \in T_S$, we derive that $v(S) \leq \phi(S) \vee v^+(S)$ a.s. and the proof is ended. \square

Definition 1.2. An admissible family $\{\phi(\theta), \theta \in T_0\}$ is said to be right continuous along stopping times in expectation (RCE) (resp. right continuous along stopping times (RC)) if for any $\theta \in T_0$ and for any sequence of stopping times $(\theta_n)_{n \in \mathbb{N}}$ such that $\theta_n \downarrow \theta$ a.s. one has $E[\phi(\theta)] = \lim_{n \rightarrow \infty} E[\phi(\theta_n)]$ (resp. $\phi(\theta) = \lim_{n \rightarrow \infty} \phi(\theta_n)$ a.s..)

Remark 1.5. Note that the definition makes sense even if $E[\phi(\theta)] = \infty$.

We now state that $\{v^+(S), S \in T_0\}$ is RCE (without any regularity assumption on the reward ϕ).

Proposition 1.6. (RCE property for v^+) Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family. The following properties hold:

- The associated family $\{v^+(\theta), \theta \in T_0\}$ is RCE.
- Let $S \in T_0$ and $A \in \mathcal{F}_S$. The family $\{v^+(\theta) \mathbf{1}_A, \theta \in T_S\}$ is RCE.

Remark 1.6. Note that the RCE property of $\{v^+(\theta) \mathbf{1}_A, \theta \in T_S\}$ at S gives that for each non increasing sequence of stopping times $(\theta_n)_{n \in \mathbb{N}}$ such that $\theta_n \downarrow S$, we have

$$E[v^+(S) \mathbf{1}_A] = \lim_{n \rightarrow \infty} E[v^+(\theta_n) \mathbf{1}_A].$$

PROOF:

Step 1: Let us prove the first assertion. Since $\{v^+(\theta), \theta \in T_0\}$ is a supermartingale system, the function $\theta \mapsto E[v^+(\theta)]$ is a non increasing functions of stopping times. Suppose it is not RCE at $\theta \in T_0$. Suppose first that $E[v^+(\theta)] < \infty$. Then there exists a constant $\alpha > 0$ and a sequence of stopping times $(\theta_n)_{n \in \mathbb{N}}$ such that $\theta_n \downarrow \theta$ a.s. and

$$\lim_{n \rightarrow \infty} \uparrow E[v^+(\theta_n)] + \alpha \leq E[v^+(\theta)]. \quad (1.5)$$

Now one can easily show, by using an optimizing sequence of stopping time for $v^+(\theta)$ (Proposition 1.2) that $E[v^+(\theta)] = \sup_{\tau \in T_{S^+}} E[\phi(\tau)]$. Therefore there exists $\theta' \in T_{\theta^+}$ such that

$$\lim_{n \rightarrow \infty} \uparrow E[v^+(\theta_n)] + \frac{\alpha}{2} \leq E[\phi(\theta')]. \quad (1.6)$$

Let us first consider the simpler case where $\theta < T$ a.s.. Since $\theta' \in T_{\theta^+}$, $\theta' > \theta$ a.s.; one has $\{\theta' > \theta\} = \bigcup_{n \in \mathbb{N}} \uparrow \{\theta' > \theta_n\}$. Therefore we have $E[\phi(\theta')] = \lim_{n \rightarrow \infty} \uparrow E[\mathbf{1}_{\{\theta' > \theta_n\}} \phi(\theta')]$. Hence, there exists n_0 such that

$$\lim_{n \rightarrow \infty} \uparrow E[v^+(\theta_n)] + \frac{\alpha}{4} \leq E[\mathbf{1}_{\{\theta' > \theta_{n_0}\}} \phi(\theta')].$$

Let $\bar{\theta}$ be the following stopping time:

$$\bar{\theta} := \theta' \mathbf{1}_{\{\theta' > \theta_{n_0}\}} + T \mathbf{1}_{\{\theta' \leq \theta_{n_0}\}}. \quad (1.7)$$

One has $\bar{\theta} > \theta_{n_0}$ a.s.. Hence, $E[\mathbf{1}_{\{\theta' > \theta_{n_0}\}} \phi(\theta')] \leq E[\phi(\bar{\theta})] \leq E[v^+(\theta_{n_0})]$. Finally,

$$E[v^+(\theta_{n_0})] + \frac{\alpha}{4} \leq \lim_{n \rightarrow \infty} \uparrow E[v^+(\theta_n)] + \frac{\alpha}{4} \leq E[v^+(\theta_{n_0})]. \quad (1.8)$$

which gives the expected contradiction.

Let us now consider the general case where θ is not supposed to satisfy necessarily $S < T$ a.s.. Since $\theta' \in T_{\theta^+}$, we have

$$E[\phi(\theta')] = E[\phi(\theta') \mathbf{1}_{\{T > \theta\}}] + E[\phi(T) \mathbf{1}_{\{\theta = T\}}].$$

Since $\theta' > \theta$ a.s. on $\{T > \theta\}$, it follows that $E[\phi(\theta') \mathbf{1}_{\{T > \theta\}}] = \lim_{n \rightarrow \infty} \uparrow E[\mathbf{1}_{\{\theta' > \theta_n\} \cap \{T > \theta\}} \phi(\theta')]$. This with (2.12) imply that there exists n_0 such that

$$\lim_{n \rightarrow \infty} \uparrow E[v^+(\theta_n)] + \frac{\alpha}{4} \leq E[\mathbf{1}_{\{\theta' > \theta_{n_0}\} \cap \{T > \theta\}} \phi(\theta')] + E[\phi(T) \mathbf{1}_{\{\theta = T\}}].$$

Put $\bar{\theta} = \theta' \mathbf{1}_{\{\theta' > \theta_{n_0}\} \cap \{T > \theta\}} + T \mathbf{1}_{\{\theta' \leq \theta_{n_0}\} \cap \{T > \theta\}} + T \mathbf{1}_{\{\theta = T\}}$. One has $\bar{\theta} \in T_{\theta^+}$. Hence, $E[\mathbf{1}_{\{\theta' > \theta_{n_0}\} \cap \{T > \theta\}} \phi(\theta')] + E[\phi(T) \mathbf{1}_{\{\theta = T\}}] \leq E[\phi(\bar{\theta})] \leq E[v^+(\theta_{n_0})]$. Finally, we derive again (1.8) which gives the expected contradiction.

In the case where $E[v^+(\theta)] = \infty$, by similar arguments, one can show that $\lim_{n \rightarrow \infty} E[v^+(\theta_n)]$ cannot be finite for $\theta_n \downarrow \theta$ a.s..

Step 2: Let us show the second assertion.

The proof is based on the same arguments as those used in step 1 with $\{\phi(\tau), \tau \in T_S\}$ replaced by $\{\phi(\tau) \mathbf{1}_A, \tau \in T_S\}$ and $\{v^+(\tau), \tau \in T_S\}$ replaced by $\{v^+(\tau) \mathbf{1}_A, \tau \in T_S\}$ and by noticing that for each $\theta \in T_S$, $E[v^+(\theta) \mathbf{1}_A] = \sup_{\tau \in T_{\theta^+}} E[\phi(\tau) \mathbf{1}_A]$.

The second assertion can also be derived by applying the RCE continuity property to V^+ at $\theta \in T_S$, where V^+ is the ‘‘strict value function’’ associated with the reward $\{\phi(\tau) \mathbf{1}_A, \tau \in T_S\}$, that is

$$V^+(\theta) = \text{ess sup}_{\tau \in T_{\theta^+}} E[\phi(\tau) \mathbf{1}_A \mid \mathcal{F}_{\theta}],$$

for each $\theta \in T_S$. Indeed, one can show that $v^+(\theta) \mathbf{1}_A = V^+(\theta)$ a.s. for each $\theta \in T_S$. \square

Remark 1.7. Note that this proof is based on similar arguments to those used in the proof of Proposition D.3 in Karatzas and Shreve (1998).

We now state a useful lemma.

Lemma 1.2. Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family.

For each $\theta, \tau \in T_0$, we have:

$$E[v(\theta) \mid \mathcal{F}_{\tau}] \leq v^+(\tau) \text{ a.s. on } \{\theta > \tau\}.$$

PROOF OF LEMMA 1.2 : Recall that there exists an optimizing sequence of stopping times (θ^n) with θ^n in T_θ such that

$$v(\theta) = \lim_{n \rightarrow \infty} \uparrow E[\phi(\theta^n) | \mathcal{F}_\theta] \quad \text{a.s.}$$

By taking the conditional expectation, we derive that a.s. on $\{\theta > \tau\}$,

$$E[v(\theta) | \mathcal{F}_\tau] = E[\lim_{n \rightarrow \infty} \uparrow E[\phi(\theta^n) | \mathcal{F}_\theta] | \mathcal{F}_\tau] = \lim_{n \rightarrow \infty} \uparrow E[\phi(\theta^n) | \mathcal{F}_\tau],$$

where the second equality follows from the monotone convergence theorem for conditional expectation.

Now, on $\{\theta > \tau\}$, $\theta^n \geq \theta > \tau$ a.s. hence, by Lemma 1.1, we have

$$E[\phi(\theta^n) | \mathcal{F}_\tau] \leq v^+(\tau), \quad \text{a.s. on } \{\theta > \tau\}.$$

This with the previous equality gives that $E[v(\theta) | \mathcal{F}_\tau] \leq v^+(\tau)$ a.s. on $\{\theta > \tau\}$. \square

In the particular case where $\{\phi(\theta), \theta \in T_0\}$ is supposed to be RCE, we have:

Proposition 1.7. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family which is RCE. Then, we have:*

- *The value function $\{v(S), S \in T_0\}$ is RCE.*
- *If $v(0) = \sup_{\theta \in T_0} E[\phi(\theta)] < \infty$, then for each $S \in T_0$, $v(S) = v^+(S)$ a.s..*

Remark 1.8. *Note that the first property is classical (see Lemma 2.13 in El Karoui (1981)). Moreover, in the particular case of a reward given by a RCLL adapted process, the second property corresponds to Corollary D.4 in Karatzas and Shreve (1998).*

PROOF: For the sake of completeness, we give the proof of the RCE property of the value function (which can be found in El Karoui (1981)).

Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family which is RCE. Note first that since the associated value function $\{v(S), S \in T_0\}$ is a supermartingale system, the function $S \mapsto E[v(S)]$ is a nonincreasing function of stopping times. Suppose it is not RCE at $S \in T_0$. If $E[v(S)] < \infty$, there exists a constant $\alpha > 0$ and a sequence of stopping times $(S_n)_{n \in \mathbb{N}}$ such that $S_n \downarrow S$ a.s. and such that

$$\lim_{n \rightarrow \infty} \uparrow E[v(S_n)] + \alpha \leq E[v(S)]. \quad (1.9)$$

Now, recall that $E[v(S)] = \sup_{\theta \in T_S} E[\phi(\theta)]$. Hence, there exists $\theta' \in T_\theta$ such that

$$\sup_{n \in \mathbb{N}} \sup_{\theta \in T_{S_n}} E[\phi(\theta)] + \frac{\alpha}{2} \leq E[\phi(\theta')].$$

It follows that for all $n \in \mathbb{N}$, $E[\phi(\theta' \vee S_n)] + \frac{\alpha}{2} \leq E[\phi(\theta')]$. Since $\theta' \vee S_n \downarrow \theta'$ a.s., by taking the limit as $n \rightarrow \infty$ and by using the RCE property of ϕ , we derive that

$$E[\phi(\theta')] + \frac{\alpha}{2} \leq E[\phi(\theta')],$$

which gives the expected contradiction in the case $E[v(S)] < \infty$.

Otherwise, we have instead of (1.9), $\lim_{n \rightarrow \infty} \uparrow E[v(S_n)] \leq C$ for some constant $C > 0$ and similar arguments to the finite case lead to a contradiction as well.

Let now show the second point of Proposition 1.7. Suppose $\{\phi(\theta), \theta \in T_0\}$ is RCE with $v(0) = \sup_{\theta \in T_0} E[\phi(\theta)] < \infty$. Fix $S \in T_0$.

For each $n \in \mathbb{N}^*$, put $S_n := (S + \frac{1}{n}) \wedge T$. The sequence $(S_n)_{n \in \mathbb{N}}$ is clearly a non increasing

sequence of stopping times of T_0 such that, for each n , $S_n > S$ a.s. on $\{S < T\}$ and which satisfies $S_n \downarrow S$.

Let us first consider the simpler case where $S < T$ a.s.. By Lemma 1.2, for each $n \in \mathbb{N}$,

$$E[v^+(S_n)|\mathcal{F}_S] \leq E[v(S_n)|\mathcal{F}_S] \leq v^+(S) \text{ a.s.} \quad (1.10)$$

By taking the expectation, we have

$$E[v^+(S_n)] \leq E[v(S_n)] \leq E[v^+(S)]. \quad (1.11)$$

Now, by the first point, $\{v(\theta), \theta \in T_0\}$ is RCE and by Proposition 1.6, $\{v^+(\theta), \theta \in T_0\}$ is also RCE. Hence, by letting n tend to ∞ , it follows that $E[v(S)] = E[v^+(S)]$ which implies that $v^+(S) = v(S)$ a.s. because $v(0) < \infty$.

Let us now consider the general case. In this case inequality 1.10 holds a.s. on $\{S < T\}$. By taking the expectation, we have

$$E[v^+(S_n)\mathbf{1}_{\{S < T\}}] \leq E[v(S_n)\mathbf{1}_{\{S < T\}}] \leq E[v^+(S)\mathbf{1}_{\{S < T\}}]. \quad (1.12)$$

Now, note that on $\{S = T\}$, we have that for each n , $S_n = T$ a.s. as well as $v^+(S_n) = v(S_n) = v^+(S) = \phi(T)$ a.s..

Therefore, by adding the term $E[\phi(T)\mathbf{1}_{\{S=T\}}]$ in (1.12), we derive inequalities (1.11) which leads to the desired result (as in the first case). \square

2 Optimal stopping times

The main aim of this section is to prove the existence of an optimal stopping time under some minimal assumptions. We begin by constructing ε -optimal stopping times or rather $(1 - \lambda)$ -optimal stopping times with $\lambda \in]0, 1[$. They are appropriate to the case of a reward family which is not necessarily supposed to be right-continuous in expectation. Then, the existence of an optimal stopping time will be derived by letting ε tend to 0 or equivalently λ tend to 1.

We stress on that no aggregation result is used to prove these properties.

2.1 Existence of $(1 - \lambda)$ -optimal stopping times

In the sequel, we will show that if the reward is right upper-semicontinuous over stopping times (resp. in expectation), then there exists a $(1 - \lambda)$ -optimal stopping time for $v(S)$ (resp. a “ $(1 - \lambda)$ -optimal” stopping time for $E[v(S)]$).

Let us now precise the construction of these stopping times. Fix $S \in T_0$. For $\lambda \in]0, 1[$, let us introduce the following \mathcal{F}_S -measurable random variable

$$\theta^\lambda(S) := \text{ess inf}\{\theta \in T_S, \lambda v^+(\theta) \leq \phi(\theta) \text{ a.s.}\} \quad (2.1)$$

First, let us precise some properties of these random variables.

Proposition 2.8. *For each $S \in T_0$, $\theta^\lambda(S)$ is a stopping time and satisfies inequality $\theta^\lambda(S) \geq S$ a.s., that is $\theta^\lambda(S) \in T_S$.*

Moreover for each $S, S' \in T_0$, if $S \leq S'$ a.s., then $\theta^\lambda(S) \leq \theta^\lambda(S')$ a.s..

PROOF: The set $\mathbb{T}_S = \{\theta \in T_S, \lambda v^+(\theta) \leq \phi(\theta) \text{ a.s.}\}$ is clearly stable by pairwise minimization. Therefore there exists a minimizing sequence (θ^n) in \mathbb{T}_S such that $\lambda v^+(\theta^n) \leq \phi(\theta^n)$ a.s. for each n and such that $\theta^\lambda(S) = \lim_{n \rightarrow \infty} \downarrow \theta^n$ a.s.. In particular $\theta^\lambda(S)$ is a stopping time and $\theta^\lambda(S) \geq S$ a.s.. Note that $\theta^\lambda(S) = S$ a.s. on $\{v(S) = \phi(S)\}$.

Also, the map $S \mapsto \theta^\lambda(S)$ is non decreasing. Indeed, if $S' \in T_0$ with $S' \geq S$ a.s., then $\mathbb{T}_{S'} \subset \mathbb{T}_S$. Hence, $\text{ess inf } \mathbb{T}_{S'} \geq \text{ess inf } \mathbb{T}_S$ a.s. which gives the inequality $\theta^\lambda(S') \geq \theta^\lambda(S)$ a.s.. \square

Let us introduce the following definition

Definition 2.3. *An admissible family $\{\phi(\theta), \theta \in T_0\}$ is said to be right (resp. left) upper-semicontinuous in expectation along stopping times (right (resp. left) USCE) if for all $\theta \in T_0$ and for all sequences of stopping times (θ_n) such that $\theta^n \downarrow \theta$ a.s., (resp. $\theta^n \uparrow \theta$ a.s.)*

$$E[\phi(\theta)] \geq \limsup_{n \rightarrow \infty} E[\phi(\theta_n)]. \quad (2.2)$$

An admissible family $\{\phi(\theta), \theta \in T_0\}$ is said to be upper-semicontinuous in expectation along stopping times (USCE) if it is right and left-USCE.

Also, an admissible family $\{\phi(\theta), \theta \in T_0\}$ is said to be right upper-semicontinuous along stopping times (right-USC) if (2.2) is satisfied without expectation.

The following Theorem holds:

Theorem 2.1.

- Suppose the reward $\{\phi(\theta), \theta \in T_0\}$ is right-USC with $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$. Fix $S \in T_0$.

For each $\lambda \in]0, 1[$, $\theta^\lambda(S)$ is a $(1 - \lambda)$ -optimal stopping time for $v(S)$ that is

$$\lambda v(S) \leq E[\phi(\theta^\lambda(S)) | \mathcal{F}_S] \text{ a.s..}$$

- Suppose the reward $\{\phi(\theta), \theta \in T_0\}$ is right-USCE with $v(0) = \sup_{\theta \in T_0} E[\phi(\theta)] < \infty$. Fix $S \in T_0$.

For each $\lambda \in]0, 1[$, the stopping time $\theta^\lambda(S)$ is a “ $(1 - \lambda)$ -optimal stopping time” for $E[v(S)] = \sup_{\theta \in T_S} E[\phi(\theta)]$, that is

$$\lambda E[v(S)] \leq E[\phi(\theta^\lambda(S))]. \quad (2.3)$$

Remark 2.9. *Note that by using Fatou’s lemma, one can easily show that if $\{\phi(\theta), \theta \in T_0\}$ is a right-USC admissible family with $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$, then it is right-USCE. Of course, the converse does not hold.*

Remark 2.10. *Actually, the second assertion of Theorem 2.1 will be sufficient to obtain the existence of an optimal stopping time. Also, we stress on that the proof of this second assertion does not require any result of the general theory of processes.*

We will see later that the existence of an optimal stopping time for $v(S)$ will be obtained by letting λ tend to 1 in (2.3).

Let us now prove Theorem 2.1. The proof relies on two lemmas.

Lemma 2.3.

- Suppose the reward $\{\phi(\theta), \theta \in T_0\}$ is right-USC. Then, for each $\lambda \in]0, 1[$, the stopping time $\theta^\lambda(S)$ satisfies

$$\lambda v(\theta^\lambda(S)) \leq \phi(\theta^\lambda(S)) \text{ a.s.};$$

- Suppose the reward $\{\phi(\theta), \theta \in T_0\}$ is right-USCE with $v(0) = \sup_{\theta \in T_0} E[\phi(\theta)] < \infty$. Then, for each $\lambda \in]0, 1[$, the stopping time $\theta^\lambda(S)$ satisfies

$$\lambda E[v(\theta^\lambda(S))] \leq E[\phi(\theta^\lambda(S))]. \quad (2.4)$$

PROOF: Fix $S \in T_0$. In order to simplify notation, let us denote $\theta^\lambda(S)$ by θ^λ .

Recall that there exists a minimizing sequence (θ^n) in \mathbb{T}_S such that $\theta^\lambda = \lim_{n \rightarrow \infty} \downarrow \theta^n$ a.s. and such that for each n ,

$$\lambda v^+(\theta^n) \leq \phi(\theta^n) \text{ a.s.} \quad (2.5)$$

Step a: Suppose first that ϕ is right-USC.

Note first that on $\{v(\theta^\lambda) = \phi(\theta^\lambda)\}$, we have obviously that $\lambda v(\theta^\lambda) \leq v(\theta^\lambda) \leq \phi(\theta^\lambda)$ a.s..

Second, on $\{v(\theta^\lambda) > \phi(\theta^\lambda)\}$, we have $v(\theta^\lambda) = v^+(\theta^\lambda)$ a.s..

By aggregation results (see Appendix A.4), v^+ is right continuous (RC) along stopping times and since $\theta^n \downarrow \theta^\lambda$ a.s. one has $v^+(\theta^\lambda) = \lim_{n \rightarrow \infty} v^+(\theta^n)$ a.s.. Hence,

$$\lambda v^+(\theta^\lambda) = \lim_{n \rightarrow \infty} v^+(\theta^n) \leq \liminf_{n \rightarrow \infty} \phi(\theta^n) \text{ a.s.} \quad (2.6)$$

where the last inequality follows from the fact, for each n , $\lambda v^+(\theta^n) \leq \phi(\theta^n)$ a.s. (see (2.5)). Now, since ϕ is USC, $\limsup_{n \rightarrow \infty} \phi(\theta^n) \leq \phi(\theta^\lambda)$ a.s..

It follows that on $\{v(\theta^\lambda) > \phi(\theta^\lambda)\}$, we have a.s.

$$\lambda v(\theta^\lambda) = \lambda v^+(\theta^\lambda) \leq \limsup_{n \rightarrow \infty} \phi(\theta^n) \leq \phi(\theta^\lambda).$$

Thus, the proof of the first assertion of the theorem is ended.

Step b: Let us now deal with the case where ϕ is supposed to be right-USCE. By step a (see 2.6), we clearly have

$$\begin{aligned} \lambda v(\theta^\lambda) &= \lambda v^+(\theta^\lambda) \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}} + \lambda \phi(\theta^\lambda) \mathbf{1}_{\{v(\theta^\lambda) = \phi(\theta^\lambda)\}} \\ &\leq \liminf_{n \rightarrow \infty} \phi(\theta^n) \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}} + \phi(\theta^\lambda) \mathbf{1}_{\{v(\theta^\lambda) = \phi(\theta^\lambda)\}} \text{ a.s.} \end{aligned} \quad (2.7)$$

We derive that

$$\lambda v(\theta^\lambda) \leq \liminf_{n \rightarrow \infty} \phi(\bar{\theta}^n) \text{ a.s.}, \quad (2.8)$$

where for each $n \in \mathbb{N}$,

$$\bar{\theta}^n := \theta^n \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}} + \theta^\lambda \mathbf{1}_{\{v(\theta^\lambda) = \phi(\theta^\lambda)\}}. \quad (2.9)$$

Note that $(\bar{\theta}^n)$ is a non increasing sequence of stopping times such that $\bar{\theta}^n \downarrow \theta^\lambda$. By taking the expectation in (2.8) and by applying Fatou's lemma, we obtain that

$$\lambda E[v(\theta^\lambda)] \leq \liminf_{n \rightarrow \infty} E[\phi(\bar{\theta}^n)] \leq \limsup_{n \rightarrow \infty} E[\phi(\bar{\theta}^n)] \leq E[\phi(\theta^\lambda)],$$

where the last inequality follows from the right-USCE property of ϕ . Hence, the desired inequality $\lambda E[v(\theta^\lambda)] \leq E[\phi(\theta^\lambda)]$ holds. This ends the proof of the second assertion.

Let us now give another proof of the second assertion which does not use any aggregation result. Note first that

$$\lambda E[v(\theta^\lambda)] = \lambda E[v^+(\theta^\lambda) \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}}] + \lambda E[\phi(\theta^\lambda) \mathbf{1}_{\{v(\theta^\lambda) = \phi(\theta^\lambda)\}}] \quad (2.10)$$

Let us consider the first term of the right member of this inequality. The Remark 1.6 applied to $S = \theta^\lambda$ and $A = \{v(\theta^\lambda) > \phi(\theta^\lambda)\}$ yields the following equality

$$\lambda E[v^+(\theta^\lambda) \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}}] = \lambda \lim_{n \rightarrow \infty} E[v^+(\theta^n) \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}}].$$

Now, for each n , $\lambda v^+(\theta^n) \leq \phi(\theta^n)$ a.s. (see (2.5)). It follows that

$$\lambda E[v^+(\theta^\lambda) \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}}] \leq \limsup_{n \rightarrow \infty} E[\phi(\theta^n) \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}}].$$

Consequently, by equality (2.10), we have

$$\begin{aligned}\lambda E[v(\theta^\lambda)] &\leq \limsup_{n \rightarrow \infty} E[\phi(\theta^n) \mathbf{1}_{\{v(\theta^\lambda) > \phi(\theta^\lambda)\}}] + E[\phi(\theta^\lambda) \mathbf{1}_{\{v(\theta^\lambda) = \phi(\theta^\lambda)\}}] \\ &\leq \limsup_{n \rightarrow \infty} E[\phi(\bar{\theta}^n)],\end{aligned}$$

where for each n , $\bar{\theta}^n$ is defined by (2.9). At last, the right-USCE property of ϕ leads to the desired inequality $\lambda E[v(\theta^\lambda)] \leq E[\phi(\theta^\lambda)]$. \square

We now state the second lemma:

Lemma 2.4. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family with $v(0) = \sup_{\theta \in T_0} E[\phi(\theta)] < \infty$. For each $\lambda \in]0, 1[$ and for each $S \in T_0$,*

$$v(S) = E[v(\theta^\lambda(S)) | \mathcal{F}_S] \quad \text{a.s.} \quad (2.11)$$

Remark 2.11. *Note first that the strict inequality $\lambda < 1$ is necessary to ensure equality (2.11). Note also that equality (2.11) is equivalent to the martingale property of the family $\{v(\theta), \theta \in T_{[S, \theta^\lambda(S)]}\}$. In other words, $\{v((\theta \vee S) \wedge \theta^\lambda(S)), \theta \in T_0\}$ is a martingale system.*

PROOF: Let us define for each $S \in T_0$, the random variable $J_\lambda(S) = E[v(\theta^\lambda(S)) | \mathcal{F}_S]$. It is sufficient to show that $J_\lambda(S) = v(S)$ a.s..

Since $\{v(S), S \in T_0\}$ is a supermartingale system and since $\theta^\lambda(S) \geq S$ a.s., we have that

$$J_\lambda(S) = E[v(\theta^\lambda(S)) | \mathcal{F}_S] \leq v(S) \quad \text{a.s.}$$

It remains to show the reverse inequality.

Step 1: Let us show that the family

$$\{J_\lambda(S), S \in T_0\}$$

is a supermartingale system.

Fix $S, S' \in T_0$ such that $S' \geq S$ a.s.. We have $\theta^\lambda(S') \geq \theta^\lambda(S)$ a.s.

Hence, $E[J_\lambda(S') | \mathcal{F}_S] = E[v(\theta^\lambda(S')) | \mathcal{F}_S] = E[E[v(\theta^\lambda(S')) | \mathcal{F}_{\theta^\lambda(S)}] | \mathcal{F}_S]$ a.s.. Now, since $\{v(S), S \in T_0\}$ is a supermartingale system, $E[v(\theta^\lambda(S')) | \mathcal{F}_{\theta^\lambda(S)}] \leq v(\theta^\lambda(S))$ a.s.. Consequently,

$$E[J_\lambda(S') | \mathcal{F}_S] \leq E[v(\theta^\lambda(S)) | \mathcal{F}_S] = J_\lambda(S) \quad \text{a.s.},$$

which ends the proof of step 1.

Step 2: Let us show that for each $S \in T_0$, and each $\lambda \in]0, 1[$,

$$\lambda v(S) + (1 - \lambda)J_\lambda(S) \geq \phi(S) \quad \text{a.s.}$$

Fix $S \in T_0$ and $\lambda \in]0, 1[$. On $A := \{\lambda v(S) \leq \phi(S)\}$, we have $\lambda v^+(S) \leq \phi(S)$ a.s.

Let us show that this implies that $\theta^\lambda(S) = S$ a.s. on A . For this, put $\bar{S} = S \mathbf{1}_A + T \mathbf{1}_{A^c}$. Note that $\bar{S} \in \mathbb{T}_S$. Therefore, $\theta^\lambda(S) = \text{ess inf } \mathbb{T}_S \leq \bar{S}$ a.s. which clearly gives $\theta^\lambda(S) \mathbf{1}_A \leq \bar{S} \mathbf{1}_A = S \mathbf{1}_A$ a.s.. Therefore, $\theta^\lambda(S) = S$ a.s. on A . Hence, $J_\lambda(S) = E[v(\theta^\lambda(S)) | \mathcal{F}_S] = E[v(S) | \mathcal{F}_S] = v(S)$ a.s. on A , which yields the inequality

$$\lambda v(S) + (1 - \lambda)J_\lambda(S) = v(S) \geq \phi(S) \quad \text{a.s. on } A.$$

Furthermore, since $A^c = \{\lambda v(S) > \phi(S)\}$ and since $J_\lambda(S)$ is nonnegative,

$$\lambda v(S) + (1 - \lambda)J_\lambda(S) \geq \lambda v(S) \geq \phi(S) \quad \text{a.s. on } A^c.$$

The proof of step 2 is complete.

Note now that, by convex combination, the family $\{\lambda v(S) + (1 - \lambda)J_\lambda(S), S \in T_0\}$ is a supermartingale system. By step 2, it dominates $\{\phi(S), S \in T_0\}$. Consequently, by the characterization of $\{v(S), S \in T_0\}$ as the smallest supermartingale system which dominates $\{\phi(S), S \in T_0\}$, we have

$$\lambda v(S) + (1 - \lambda)J_\lambda(S) \geq v(S) \quad \text{a.s.}$$

Hence, $J_\lambda(S) \geq v(S)$ a.s. because $v(S) < \infty$ a.s. and because $\lambda < 1$ (note that the strict inequality is necessary here). Consequently, for each $S \in T_0$, $J_\lambda(S) = v(S)$ a.s. The proof of Lemma 2.4 is ended. \square

PROOF OF THEOREM 2.1: By Lemma 2.4,

$$v(S) = E[v(\theta^\lambda(S)) | \mathcal{F}_S] \quad \text{a.s.} \quad (2.12)$$

Step a: Suppose first that the reward $\{\phi(\theta), \theta \in T_0\}$ is right-USC with $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$.

Then, by (2.12) and by taking the conditional expectation in inequality $\lambda v(\theta^\lambda(S)) \leq \phi(\theta^\lambda(S))$ a.s. (see the first point of Lemma 2.3), it follows that

$$\lambda v(S) = \lambda E[v(\theta^\lambda(S)) | \mathcal{F}_S] \leq E[\phi(\theta^\lambda(S)) | \mathcal{F}_S] \quad \text{a.s.}$$

In other words, $\theta^\lambda(S)$ is a $(1 - \lambda)$ -optimal stopping time for $v(S)$.

Step b: Let us now consider the case where ϕ satisfies the weaker assumption: it is right-USCE with $v(0) = \sup_{\theta \in T_0} E[\phi(\theta)] < \infty$. Then, taking the expectation in (2.12) and using inequality $\lambda E[v(\theta^\lambda(S))] \leq E[\phi(\theta^\lambda(S))]$ (see the second point of Lemma 2.3), it follows that

$$\lambda E[v(S)] = \lambda E[v(\theta^\lambda(S))] \leq E[\phi(\theta^\lambda(S))]. \quad (2.13)$$

In other words, $\theta^\lambda(S)$ is “ $(1 - \lambda)$ -optimal” for $E[v(S)]$. \square

Remark 2.12. *Note that by using similar arguments, one can show that the first assertion of Theorem 2.1 still holds in the case where ϕ is supposed to be right-USC in conditional expectation given \mathcal{F}_S , that is satisfies (2.2) where the expectation is replaced by the conditional expectation.*

In the next section, under the additional assumption of left-USCE property of the reward, we will show that the “ $(1 - \lambda)$ -optimal” stopping times $\theta^\lambda(S)$ tend to an optimal stopping time for $v(S)$ as $\lambda \uparrow 1$.

2.2 Existence Theorem and minimal optimal stopping times

In this section, we will show that if the reward is both right-USCE and left-USCE, then there exists an optimal stopping time for $v(S)$.

2.2.1 Existence Theorem

Theorem 2.2. (Existence of optimal stopping times)

Suppose the reward $\{\phi(\theta), \theta \in T_0\}$ is such that $v(0) < \infty$ and is USCE. Fix $S \in T_0$.

Then, the limit of the “ $(1 - \lambda)$ -optimal” stopping times given by

$$\hat{\theta}(S) := \lim_{\lambda \uparrow 1} \uparrow \theta^\lambda(S) \quad (2.14)$$

is an optimal stopping time for $v(S)$.

Remark 2.13. *The above theorem generalizes the classical existence result of optimal stopping times to the case of a reward given by an admissible family of r.v. which is upper-semicontinuous in expectation (instead of a RCLL adapted process which is left-continuous in expectation).*

PROOF: Fix $S \in T_0$.

Clearly for all $S \in T_0$, the function $\lambda \mapsto \theta^\lambda(S)$ is non decreasing on $]0, 1[$. Hence, the r.v. $\hat{\theta}(S)$ is well defined by (2.14) and is a stopping time. Let us now prove that it is optimal for $v(S)$.

By Theorem 2.1, $\lambda E[v(S)] \leq E[\phi(\theta^\lambda(S))]$ for each $\lambda \in]0, 1[$. Letting $\lambda \uparrow 1$ in this last inequality, and using the left upper-semicontinuous property in expectation of the reward ϕ , we derive that $E[v(S)] \leq E[\phi(\hat{\theta}(S))]$. Hence,

$$E[v(S)] = \sup_{\theta \in T_S} E[\phi(\theta)] = E[\phi(\hat{\theta}(S))]$$

By Remark 1.4, this implies that $\hat{\theta}(S)$ is optimal for $v(S)$. \square

In the next subsection, we will show that $\hat{\theta}(S)$ is the minimal optimal stopping time for $v(S)$ and is also characterized as the essential infimum of the stopping times greater than S such that the value function attains the reward.

2.2.2 Minimal optimal stopping times

For each $S \in T_0$, let us define the following random variable $\theta_*(S) := \text{ess inf}\{\theta \in T_S, v(\theta) = \phi(\theta) \text{ a.s.}\}$. In the following, we will show that $\theta_*(S) = \hat{\theta}(S)$ a.s. and that it is the minimal optimal stopping time for $v(S)$. More precisely, we state the following Theorem:

Theorem 2.3. (Characterization of the minimal optimal stopping time)

Suppose $\{\phi(\theta), \theta \in T_0\}$ is USCE with $v(0) < \infty$. Fix $S \in T_0$.

Then, the stopping time $\theta_(S)$ given by*

$$\theta_*(S) = \text{ess inf}\{\theta \in T_S, v(\theta) = \phi(\theta) \text{ a.s.}\}. \quad (2.15)$$

is the minimal optimal stopping time for $v(S)$. Also, it is the limit of the “ $(1 - \lambda)$ -optimal” stopping times, that is $\theta_(S) = \lim_{\lambda \uparrow 1} \theta^\lambda(S)$ a.s..*

PROOF: First it is just routine to check that the set $\mathbb{T}_S = \{\theta \in T_S, v(\theta) = \phi(\theta) \text{ a.s.}\}$ is closed by pairwise minimization. Hence, there exists a nonincreasing sequence of stopping times (θ^n) such that $\theta^n \in \mathbb{T}_S$ for all $n \in \mathbb{N}$ and $\theta^n \downarrow \theta_*(S)$ a.s.. In particular $\theta_*(S)$ is a stopping time.

Step 1: Let us show that $\hat{\theta}(S) \leq \theta_*(S)$ a.s..

It is clear that $\theta_*(S) = \text{ess inf}\{\theta \in T_S, v^+(\theta) \leq \phi(\theta) \text{ a.s.}\}$ because $v(\theta) = v^+(\theta) \vee \phi(\theta)$ a.s..

Hence, since $\theta^\lambda(S) = \text{ess inf}\{\theta \in T_S, \lambda v^+(\theta) \leq \phi(\theta) \text{ a.s.}\}$, it is clear that $\theta^\lambda(S) \leq \theta_*(S)$ a.s..

By letting λ tend to 1, it follows that

$$\hat{\theta}(S) = \lim_{\lambda \uparrow 1} \theta^\lambda(S) \leq \theta_*(S) \text{ a.s.,}$$

which ends step 1.

Step 2: Let us show that $\hat{\theta}(S) = \theta_*(S)$ a.s. and that it is the minimal optimal stopping time.

By the optimality criterium (Proposition 1.4), if $\theta \in T_0$ is optimal for $v(S)$, then $v(\theta) = \phi(\theta)$ a.s.. Therefore we have

$$\theta_*(S) \leq \text{ess inf}\{\theta \in T_S, \theta \text{ optimal for } v(S)\} \text{ a.s.} \quad (2.16)$$

Now, we have already proven that $\hat{\theta}(S)$ is optimal for $v(S)$ (see Theorem 2.2). Therefore, by step 1 and (2.16), we have that

$$\hat{\theta}(S) = \theta_*(S) = \text{ess inf}\{\theta \in T_S, \theta \text{ optimal for } v(S)\} \text{ a.s.,}$$

which ends the proof. \square

Remark 2.14. Note that by definition (2.2.3), the map $S \mapsto \theta_*(S)$ is non decreasing that is for each $S, S' \in T_0$ s.t. $S \leq S'$ a.s., we have $\theta_*(S) \leq \theta_*(S')$ a.s..

2.2.3 Regularity of the value function

Note first that, without any assumption on the reward family, the value function is right-USCE. Indeed, from the supermartingale property of $\{v(\theta), \theta \in T_0\}$, we clearly have the following property: for each $S \in T_0$ and each sequence of stopping times (S_n) such that $S_n \downarrow S$ a.s., $\lim_{n \rightarrow \infty} \uparrow E[v(S_n)] \leq E[v(S)]$.

Define now the property of *left continuity along stopping times* and the *left continuity in expectation along stopping times* (LCE property) similarly to the RCE property (see Definition 1.2) with $\theta_n \uparrow \theta$ instead of $\theta_n \downarrow \theta$.

Using Remark 2.14, we derive the following regularity property of the value function:

Proposition 2.9. *If $\{\phi(\theta), \theta \in T_0\}$ is USCE and $v(0) < \infty$, then $\{v(\theta), \theta \in T_0\}$ is left-continuous in expectation (LCE).*

PROOF: Fix $S \in T_0$. Let (S_n) be a sequence of stopping times such that $S_n \uparrow S$ a.s. Let us show that $\lim_{n \rightarrow \infty} E[v(S_n)] = E[v(S)]$.

First, we have that for each n , $E[v(S_n)] \geq E[v(S)]$. Hence, $\lim_{n \rightarrow \infty} \downarrow E[v(S_n)] \geq E[v(S)]$.

Suppose by contradiction that $\lim_{n \rightarrow \infty} \downarrow E[v(S_n)] \neq E[v(S)]$. Then, there exists $\alpha > 0$ such that for all n , one has $E[v(S_n)] \geq E[v(S)] + \alpha$. By Theorem 2.3, for each n , the stopping time $\theta^*(S_n) \in T_{S_n}$ (defined by (2.2.3)) is optimal for $v(S_n)$. It follows that for each n , $E[\phi(\theta^*(S_n))] \geq E[v(S)] + \alpha$. Now, the sequence of stopping times $(\theta^*(S_n))$ is clearly non decreasing. Put $\bar{\theta} := \lim_{n \rightarrow \infty} \uparrow \theta^*(S_n)$. $\bar{\theta}$ is clearly a stopping time. Using the USCE property of ϕ , we obtain

$$E[\phi(\bar{\theta})] \geq E[v(S)] + \alpha.$$

Now, for each n , $\theta_*(S_n) \geq S_n$ a.s.. Hence, by letting n tend to ∞ , it clearly follows that $\bar{\theta} \geq S$ a.s., which provides the expected contradiction. \square

Consequently, the following corollary holds.

Corollary 2.1. *If $\{\phi(\theta), \theta \in T_0\}$ is USCE and $v(0) < \infty$, then $\{v(\theta), \theta \in T_0\}$ is USCE.*

Application to an optimal double stopping problem

Let us consider the following simple optimal double stopping problem. We suppose that $T = \infty$ and let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family. The stopping times τ_1, τ_2 are separated by a fixed amount of time $\delta > 0$ (sometimes called “refracting time”). The value function is given by

$$v(S) = \text{ess sup}_{(\tau_1, \tau_2) \in \mathcal{S}_S} E[\phi(\tau_1) + \phi(\tau_2) | \mathcal{F}_S],$$

where $\mathcal{S}_S = \{\tau_1, \tau_2 \in T_S \text{ s.t. } \tau_2 \in T_{\tau_1 + \delta}\}$.

Note that it corresponds to a swing option in the bidimensional case (for the d -dimensional case, see Carmona and Dayanik (2008) and also section 3.6 in Kobylanski et al. (2010)). We clearly have that

$$v(S) = \text{ess sup}_{\tau_1 \in T_S} E[\phi(\tau_1) + \text{ess sup}_{\tau_2 \in T_{\tau_1 + \delta}} E[\phi(\tau_2) | \mathcal{F}_{\tau_1}] | \mathcal{F}_S]. \quad (2.17)$$

Let $\{u(\theta), \theta \in T_0\}$ be the value function associated with the reward ϕ defined for each stopping time θ by

$$u(\theta) = \text{ess sup}_{\tau \in T_{\theta + \delta}} E[\phi(\tau) | \mathcal{F}_\theta].$$

By (2.17), we have

$$v(S) = \text{ess sup}_{\theta \in T_S} E[\phi(\theta) + u(\theta) | \mathcal{F}_S]. \quad (2.18)$$

Suppose now that $\{\phi(\theta), \theta \in T_0\}$ is USCE and satisfies $u(0) < \infty$. By Corollary 2.1, the associated value function $\{u(\theta), \theta \in T_0\}$ is USCE. Also, one can easily verify that $v(0) < \infty$. By Theorems 2.2 and 2.3, the stopping time θ_1^* defined by

$$\theta_1^* := \text{ess inf} \{ \theta \in T_S, v(\theta) = \phi(\theta) + u(\theta) \text{ a.s. } \}.$$

is optimal for (2.18) and the stopping time θ_2^* given by

$$\theta_2^* = \text{ess inf} \{ \theta \in T_{\theta_1^*}, u(\theta) = \phi(\theta) \text{ a.s. } \}.$$

is an optimal for $u(\theta_1^*)$. It clearly follows that (θ_1^*, θ_2^*) is the minimal double optimal stopping time for $v(S)$.

Of course, by backward induction, this result can be generalized to swing options in the d -dimensional case with $d \geq 2$.

2.3 Maximal optimal stopping times

2.3.1 Definition of $\check{\theta}(S)$, $S \in T_0$

Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family and $\{v(\theta), \theta \in T_0\}$ be the associated value function. Fix $S \in T_0$. Let us introduce the random variable $\check{\theta}(S)$ as follows

Definition 2.4. Fix $S \in T_0$.

Let \mathcal{A}_S of stopping times θ such that $\{v(\tau), \tau \in T_{[S, \theta]}\}$ is a martingale system.

We define the random variable $\check{\theta}(S)$ by

$$\check{\theta}(S) := \text{ess sup } \mathcal{A}_S.$$

Note that the definition of $\check{\theta}(S)$ does not require any assumption on ϕ .

Note that if $v(0) < \infty$, we clearly have:

$$\check{\theta}(S) = \text{ess sup} \{ \theta \in T_S, E[v(\theta)] = E[v(S)] \},$$

because in this case $\mathcal{A}_S = \text{ess sup} \{ \theta \in T_S, E[v(\theta)] = E[v(S)] \}$.

Lemma 2.5. Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family.

For each $S \in T_0$, the set \mathcal{A}_S is stable by pairwise maximization.

Also, $\check{\theta}(S)$ is a stopping time and there exists a nondecreasing sequence of stopping times (θ^n) such that $\theta^n \in \mathcal{A}_S$ for all $n \in \mathbb{N}$ and $\theta^n \uparrow \check{\theta}(S)$ a.s..

PROOF OF LEMMA 2.5:

Fix $S \in T_0$. Let $\theta_1, \theta_2 \in \mathcal{A}_S$. Let us show that $\theta_1 \vee \theta_2$ belongs to \mathcal{A}_S . Note that this property is intuitive since if $\{v(\tau)\}_{\tau \in T_{[S, \theta_1]}}$ and $\{v(\tau)\}_{\tau \in T_{[S, \theta_2]}}$ are martingale systems, then it is quite clear that $\{v(\tau)\}_{\tau \in T_{[S, \theta_1 \vee \theta_2]}}$ is a martingale system. For the sake of completeness, let us verify this property. We have clearly that a.s.

$$E[v(\theta_1 \vee \theta_2) | \mathcal{F}_S] = E[v(\theta_2) \mathbf{1}_{\theta_2 > \theta_1} | \mathcal{F}_S] + E[v(\theta_1) \mathbf{1}_{\theta_1 \geq \theta_2} | \mathcal{F}_S]. \quad (2.19)$$

Since $\theta_2 \in \mathcal{A}_S$, we have that on $\{\theta_2 > \theta_1\}$, $v(\theta_1) = E[v(\theta_2) | \mathcal{F}_{\theta_1}]$ a.s. It follows that $E[v(\theta_2) \mathbf{1}_{\theta_2 > \theta_1} | \mathcal{F}_S] = E[v(\theta_1) \mathbf{1}_{\theta_2 > \theta_1} | \mathcal{F}_S]$ a.s. This with equality (2.19) gives that $E[v(\theta_1 \vee \theta_2) | \mathcal{F}_S] = E[v(\theta_1) | \mathcal{F}_S]$ a.s.. Now, since $\theta_1 \in \mathcal{A}_S$, $E[v(\theta_1) | \mathcal{F}_S] = v(S)$ a.s.. Hence, we have shown that $E[v(\theta_1 \vee \theta_2) | \mathcal{F}_S] = v(S)$ a.s. which gives that $\theta_1 \vee \theta_2 \in \mathcal{A}_S$.

It remains to verify the second point of the lemma. Now, by classical results, there exists a nondecreasing sequence of stopping times (θ^n) such that $\theta^n \in \mathcal{A}_S$ for all $n \in \mathbb{N}$ and $\theta^n \uparrow \check{\theta}(S)$ a.s.. In particular, $\check{\theta}(S)$ is a stopping time. \square

Lemma 2.6. *Suppose $\{\phi(\theta), \theta \in T_0\}$ is such that $v(0) < \infty$. Fix $S \in T_0$. If $\theta \in T_S$ is an optimal stopping time for $v(S)$, then $\theta \in \mathcal{A}_S$.*

PROOF: By the optimality criterium (Proposition 1.4), $E[v(\theta)] = E[v(S)]$ which gives the desired result. \square

2.3.2 Characterization of $\check{\theta}(S)$ as the maximal optimal stopping time

In this subsection, $\{\phi(\theta), \theta \in T_0\}$ is supposed to be USCE and such that $v(0) < \infty$. Fix $S \in T_0$. By Theorem 2.2, we know that there exists an optimal stopping time for $v(S)$. In the sequel, we show that $\check{\theta}(S)$ is the maximal optimal stopping time. More precisely,

Theorem 2.4. (Characterization of $\check{\theta}(S)$ as the maximal optimal stopping time)
Suppose $\{\phi(\theta), \theta \in T_0\}$ is such that $v(0) < \infty$ and is USCE. Fix $S \in T_0$. Then, $\check{\theta}(S)$ is the maximal optimal stopping time for $v(S)$.

PROOF OF THEOREM 2.4: Fix $S \in T_0$.

To simplify the notation, in the following, the stopping time $\check{\theta}(S)$ will be denoted by $\check{\theta}$.

Step 1: Let us show that $\check{\theta} \in \mathcal{A}_S$.

Recall that by Lemma 2.5, there exists a nondecreasing sequence of stopping times (θ^n) such that $\theta^n \in \mathcal{A}_S$ for all $n \in \mathbb{N}$ and $\theta^n \uparrow \check{\theta}$ a.s..

For each $n \in \mathbb{N}$, since $\theta^n \in \mathcal{A}_S$, we have

$$E[v(\theta^n)] = E[v(S)].$$

Since by Proposition 2.9, the value function $\{v(\theta), \theta \in T_0\}$ is LCE, by letting n tend to ∞ in the previous equality, we derive that $E[v(\check{\theta})] = E[v(S)]$, which gives that $\check{\theta} \in \mathcal{A}_S$.

Step 2: Let us show now that $\check{\theta}$ is optimal for $v(S)$.

The optimality of $\theta_*(\check{\theta})$ for $v(\check{\theta})$ gives the equality

$$E[\phi(\theta_*(\check{\theta}))] = E[v(\check{\theta})].$$

Also, since $\check{\theta} \in \mathcal{A}_S$, we clearly have that

$$E[v(\check{\theta})] = E[v(S)].$$

The two previous equalities give that $E[\phi(\theta_*(\check{\theta}))] = E[v(S)]$. In other words, $\theta_*(\check{\theta})$ is optimal for $v(S)$. By lemma 2.6, this implies that $\theta_*(\check{\theta}) \in \mathcal{A}_S$.

Hence since $\check{\theta} = \text{ess sup } \mathcal{A}_S$, it follows that $\theta_*(\check{\theta}) \leq \check{\theta}$ a.s. Now, $\theta_*(\check{\theta}) \geq \check{\theta}$ a.s. Hence, $\theta_*(\check{\theta}) = \check{\theta}$ a.s. It clearly follows that $\check{\theta}$ is optimal for $v(S)$.

Step 3: Let us show that $\check{\theta}$ is the maximal optimal stopping time for $v(S)$.

By lemma 2.6, we have that each θ which is optimal for $v(S)$ belongs to \mathcal{A}_S and hence is smaller than $\check{\theta}$ (since $\check{\theta} = \text{ess sup } \mathcal{A}_S$). This gives step 3.

Hence, the proof of Theorem 2.4 is ended. \square

3 Links between v , v^+ and ϕ

Recall that we have shown that for all $S \in T_0$, $v(S) = \phi(S) \vee v^+(S)$ a.s. (see Proposition 1.5). One can wonder if it possible to have conditions which ensure that $v(S) = \phi(S)$ a.s. or $v(S) = v^+(S)$. In this section, we will first give some strict supermartingale conditions on v which ensure the equality between v and ϕ (locally).

Second, we will give some properties of v and v^+ which ensure the equality between v and v^+ (for some stopping times which will be specified).

3.1 When does the value function coincide with the reward?

We will now give some strict martingale conditions on v which ensure the equality between $v(\theta)$ and $\phi(\theta)$ for a given stopping time θ .

Let us first note that for $A \in \mathcal{F}_S$, the restriction of $\check{\theta}(S)$ to A can be characterized as follows:

Lemma 3.7. (*localization lemma*) *Suppose the reward $\{\phi(\theta), \theta \in T_0\}$ is an admissible family. Fix $\theta \in T_S$. Let $A \in \mathcal{F}_S$. Let $\mathcal{A}_S(A)$ be the set of r.v. $\theta \mathbf{1}_A$ with $\theta \in T_S$ such that $\{v(\tau)\}_{\tau \in T_{[S, \theta]}}$ is a martingale system on A . We have*

$$\check{\theta}(S) \mathbf{1}_A := \text{ess sup } \mathcal{A}_S(A) \text{ a.s.} \quad (3.1)$$

Note that in the case where $v(0) < \infty$, equality (3.1) can be written as follows:

$$\check{\theta}(S) \mathbf{1}_A = \text{ess sup} \{ \theta \mathbf{1}_A, \theta \in T_S \text{ and } E[v(\theta) | \mathcal{F}_S] \mathbf{1}_A = v(S) \mathbf{1}_A \text{ a.s.} \} \text{ a.s.}$$

PROOF : For the sake of completeness, we give the proof of this lemma. To simplify notation, let us denote $\check{\theta}(S)$ by $\check{\theta}$ and $\mathcal{A}_S(A)$ by $\mathcal{A}(A)$. Also let us introduce the following r.v. $\tilde{\theta} := \text{ess sup} \mathcal{A}(A)$.

Note first that the inequality $\check{\theta} \mathbf{1}_A \leq \tilde{\theta}$ a.s. is clear.

It remains to show the other inequality.

By similar arguments to those used in the proof of Lemma 2.5, the set $\mathcal{A}(A)$ is stable by pairwise maximization. Hence, there exists a nondecreasing sequence of stopping times (θ^n) in T_S such that $\theta^n \mathbf{1}_A \in \mathcal{A}(A)$ for all $n \in \mathbb{N}$ and $\theta^n \mathbf{1}_A \uparrow \tilde{\theta}$. For each $n \in \mathbb{N}$, since $\theta^n \in \mathcal{A}(A)$, the family $\{v(\tau)\}_{\tau \in T_{[S, \theta^n]}}$ is a martingale system on A . Put $\bar{\theta}^n = \theta^n \mathbf{1}_A + S \mathbf{1}_{A^c}$. Since $A \in \mathcal{F}_S$, we clearly have that $\bar{\theta}^n$ is a stopping time and that $\{v(\tau)\}_{\tau \in T_{[S, \bar{\theta}^n]}}$ is a martingale system.

Hence, $\bar{\theta}^n \leq \check{\theta}$ a.s. (by definition of $\check{\theta}$), which gives $\theta^n \mathbf{1}_A \leq \check{\theta} \mathbf{1}_A$ a.s. for each n .

By letting n tend to ∞ , we derive that $\tilde{\theta} \leq \check{\theta} \mathbf{1}_A$ a.s.. Since the other inequality has already been shown, it follows that $\tilde{\theta} = \check{\theta} \mathbf{1}_A$ a.s. and the proof is ended.

Let us give another possible proof. Note first that $\tilde{\theta}$ corresponds to the $\check{\theta}$ associated with the value function $\{V(\theta), \theta \in T_S\}$ associated with the reward $\{\phi(\theta) \mathbf{1}_A, \theta \in T_S\}$. Since $v(\theta) \mathbf{1}_A = V(\theta)$ a.s. for each $\theta \in T_S$, it follows that $\tilde{\theta} = \check{\theta} \mathbf{1}_A$ a.s.. \square

Let \mathcal{X} be the set of all real valued random variables. We introduce the following relation on \mathcal{X} :

Definition 3.5. (*Relation \prec*)

Fix $X, X' \in \mathcal{X}$ and $A \in \mathcal{F}$.

We say that $X \prec X'$ on A if $X \leq X'$ a.s. on A and if $P(\{X \neq X'\} \cap A) \neq 0$.

Remark 3.15. Clearly, we define the relation $X \succ X'$ on A in the same way.

Also, to simplify notation, the relation \leq a.s. will be denoted by \preceq and \geq a.s. by \succeq .

Define $T'_0 := \{\theta \in T_0, \theta < T \text{ a.s.}\}$. Fix $\theta \in T'_0$ and $A \in \mathcal{F}_\theta$.

Definition 3.6. Let $\{u(\theta), \theta \in T_0\}$ be a supermartingale system.

u is said to be a martingale system on the right at θ on A if there exists $\theta' \in T_0$ with $\theta \prec \theta'$ on A such that $\{u(\tau)\}_{\tau \in T_{[\theta, \theta']}}$ is a martingale system on A .

This notion is of course local in time (and in space because of A). Note that if $u(0) < \infty$, this property is equivalent to the following one:

$$\exists \theta' \in T_0 \text{ s.t. } \theta \prec \theta' \text{ on } A, E[u(\theta')\mathbf{1}_A] = E[u(\theta)\mathbf{1}_A]. \quad (3.2)$$

Definition 3.7. Let $\{u(\theta), \theta \in T_0\}$ be a supermartingale system.

u is said to be a strict supermartingale system on the right at θ on A if it is not a martingale system on the right at θ on A , that is if for each $\theta' \in T_0$ with $\theta \prec \theta'$ on A , $\{u(\tau)\}_{\tau \in T_{[\theta, \theta']}}$ is not a martingale system on A .

If $u(0) < \infty$, this property can be written as follows:

$$\forall \theta' \in T_0 \text{ s.t. } \theta \prec \theta' \text{ on } A, E[u(\theta')\mathbf{1}_A] < E[u(\theta)\mathbf{1}_A]. \quad (3.3)$$

By using the localization lemma (Lemma 3.7), one can easily derive the following lemma:

Lemma 3.8. Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family. Let $\theta \in T_0$ and $A \in \mathcal{F}_\theta$ be such that $\theta \prec T$ on A .

- v is a martingale on the right at θ on A if and only if $\theta \prec \check{\theta}(\theta)$ on A .
- v is a strict supermartingale on the right at θ on A if and only if $\theta = \check{\theta}(\theta)$ a.s. on A .

We now state the following Theorem which gives a sufficient condition which ensures locally the equality between v and ϕ .

Theorem 3.5. Suppose $\{\phi(\theta), \theta \in T_0\}$ is USCE and such that $v(0) < \infty$. Let $\theta \in T_0$ and $A \in \mathcal{F}_\theta$ be such that $\theta \prec T$ on A .

If the value function $\{v(\theta), \theta \in T_0\}$ is a strict supermartingale on the right at θ on A , then $v(\theta) = \phi(\theta)$ a.s. on A .

PROOF : The second point of Lemma 3.8 with the optimality criterium and the fact that $\check{\theta}(\theta)$ is an optimal stopping time give the desired result. \square

Remark 3.16. Note that if the reward is given by a continuous adapted process (ϕ_t) , this theorem leads to a well known result stated for example in Appendix D of Karatzas and Shreve ((1998) (see also Proposition 3.16 for a slight generalization).

3.2 Martingale properties of v and v^+ and when do v and v^+ coincide?

We will now study some links between v and v^+ .

We introduce the following definition:

Definition 3.8. (Localization) Let $\{h(\theta), \theta \in T_0\}$ be an admissible family.

Let S, S' be two stopping times and let $A \in \mathcal{F}_S$. Suppose $S < S'$ a.s. on A .

An admissible family h restricted to $[S, S' [$, that is $\{h(\theta)\}_{\theta \in T_{[S, S' [}}$, is said to be a martingale system on A if for each $\theta, \theta' \in T_0$ such that $S \leq \theta, \theta' < S'$ a.s. on A ,

$$E[h(\theta') | \mathcal{F}_\theta] = h(\theta) \text{ a.s. on } A \cap \{\theta \leq \theta'\}. \quad (3.4)$$

We can define similarly the notion of martingale system $\{h(\tau)\}_{\tau \in T_{[S, S']}}$ on A (resp. for $T_{]S, S']}$). Note that $T_{]S, S']}$ on A can be empty.

Note now the following property: if $\{v(\tau), \tau \in T_{[0, \theta]}\}$ is a martingale, then by Lemma 1.2, for each $\tau \in T_0$,

$$v(\tau) = E[v(\theta) | \mathcal{F}_\tau] \leq v^+(\tau) \text{ a.s. on } \{\theta > \tau\},$$

and hence $v(\tau) = v^+(\tau)$ a.s. on $\{\theta > \tau\}$. A generalization of this property is given in the following proposition.

Proposition 3.10. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family.*

Fix $S, S' \in T_0$ such that $S \leq S'$ a.s.. Suppose $\{v(\tau)\}_{\tau \in T_{[S, S'']}}$ is a martingale system. Then, for each $\tau \in T_0$, $v(\tau) = v^+(\tau)$ a.s. on $\{S \leq \tau < S'\}$.

Also, $\{v^+(\tau)\}_{\tau \in T_{]S, S']}}$ is a martingale system on $\{S < S'\}$.

PROOF : Fix $\tau \in T_0$.

Since $\{v(\tau)\}_{\tau \in T_{[S, S'']}}$ is a martingale system, we have a.s.

$$v(\tau) \mathbf{1}_{\{S \leq \tau < S'\}} = E[v(S') | \mathcal{F}_\tau] \mathbf{1}_{\{S \leq \tau < S'\}} \leq v^+(\tau) \mathbf{1}_{\{S \leq \tau < S'\}},$$

where the inequality follows from Lemma 1.2. Hence, $v(\tau) = v^+(\tau)$ a.s. on $\{S \leq \tau < S'\}$, which gives the first assertion.

It follows that for each $\tau \in T_0$ such that $S \leq \tau < S'$ a.s. on $\{S < S'\}$,

$$E[v^+(\tau) | \mathcal{F}_S] = E[v(\tau) | \mathcal{F}_S] = v(S) = v^+(S) \text{ a.s. on } \{S < S'\}. \quad (3.5)$$

since $\{S < S'\}$ belongs to \mathcal{F}_S and $v(\tau) = v^+(\tau)$ a.s. on $\{S < S'\}$. Note that here, the assumption that $S \leq \tau < S'$ a.s. on $\{S < S'\}$ is important because the set $\{S \leq \tau < S'\}$ does not necessarily belong to \mathcal{F}_S . Hence, $\{v^+(\tau)\}_{\tau \in T_{]S, S']}}$ is a martingale system on $\{S < S'\}$. \square

From this proposition, we derive the following corollary:

Corollary 3.2. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family.*

For each $\tau \in T_0$, we have $v(\tau) = v^+(\tau)$ a.s. on $\{S \leq \tau < \check{\theta}(S)\}$.

Also, the family $\{v^+(\tau)\}_{\tau \in T_{]S, \check{\theta}(S)}}$ is a martingale system (on $\{S < \check{\theta}(S)\}$).

PROOF : Fix $\tau \in T_0$.

Recall that there exists a nondecreasing sequence of stopping times $(\theta^n) \in T_S$ such that for all $n \in \mathbb{N}$, $\{v(\tau)\}_{\tau \in T_{[S, \theta^n]}}$ is a martingale system and $\theta^n \uparrow \check{\theta}(S)$ a.s..

Note now that $\mathbf{1}_{\{S \leq \tau < \check{\theta}(S)\}} = \lim_{n \rightarrow \infty} \uparrow \mathbf{1}_{\{S \leq \tau < \theta^n\}}$ a.s.. For each n , $\{v(\tau)\}_{\tau \in T_{[S, \theta^n]}}$ is a martingale system. Hence, by Proposition 3.10, the equality

$$v(\tau) \mathbf{1}_{\{S \leq \tau < \theta^n\}} = v^+(\tau) \mathbf{1}_{\{S \leq \tau < \theta^n\}}$$

holds a.s.. By letting n tend to ∞ , we derive that $v(\tau) \mathbf{1}_{\{S \leq \tau < \check{\theta}(S)\}} = v^+(\tau) \mathbf{1}_{\{S \leq \tau < \check{\theta}(S)\}}$ a.s. which gives the first assertion.

The second assertion follows. \square

Note now that for each $S' \in T_0$, if $v(0) < \infty$, if $v(0) = v^+(0)$ and if $\{v^+(\tau)\}_{\tau \in T_{[0, S'']}}$ is a martingale system, then $v(\tau) = v^+(\tau)$ a.s. for each $\tau \in T_{[0, S']}$. Indeed, the supermartingale property of v and the martingale property of v^+ give that $E[v(\tau) - v^+(\tau)] \leq v(0) - v^+(0) = 0$ and since $v(\tau) - v^+(\tau) \geq 0$ a.s., we derive that $v(\tau) = v^+(\tau)$ a.s. More generally, the following property holds.

Proposition 3.11. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family such that $v(0) < \infty$. Fix $S, S' \in T_0$ such that $S \leq S'$ a.s. Suppose $\{v^+(\tau)\}_{\tau \in T_{[S, S'']}}$ is a martingale system. Then, the two following properties hold:*

- *If $\tau \in T_0$ is such that $S < \tau \leq S'$ a.s. on $\{S < S'\}$, then $v(\tau) = v^+(\tau)$ a.s. on $\{S < S'\}$. Also, $\{v(\tau)\}_{\tau \in T_{[S, S'']}}$ is a martingale system on $\{S < S'\}$.*
- *Put $A := \{v^+(S) = v(S)\}$. If $\tau \in T_0$ is such that $S \leq \tau \leq S'$ a.s. on A , then $v(\tau) = v^+(\tau)$ a.s. on A . Also, the family $\{v(\tau)\}_{\tau \in T_{[S, S'']}}$ is a martingale system on A .*

PROOF :

Let us prove the first property. Let $\tau \in T_0$ be such that $\{S < \tau \leq S'\}$ a.s. on $\{S < S'\}$. We have:

$$v^+(S) = E[v^+(\tau)|\mathcal{F}_S] \leq E[v(\tau)|\mathcal{F}_S] \leq v^+(S) \text{ a.s. on } \{S < S'\}.$$

where the last inequality follows from Lemma 1.2. Hence, all the inequalities are equalities. Since $v(0) < \infty$, we derive that $v(\tau) = v^+(\tau)$ a.s. on $\{S < S'\}$ and the first point follows. Let us prove the second point. We have a.s.

$$v^+(\tau)\mathbf{1}_{\{S \leq \tau \leq S'\} \cap A} = v^+(S)\mathbf{1}_{\{S=\tau\} \cap A} + v^+(\tau)\mathbf{1}_{\{S < \tau \leq S'\} \cap A}.$$

By the first point, it follows that a.s.

$$v^+(\tau)\mathbf{1}_{\{S \leq \tau \leq S'\} \cap A} = v(S)\mathbf{1}_{\{S=\tau\} \cap A} + v(\tau)\mathbf{1}_{\{S < \tau \leq S'\} \cap A} = v(\tau)\mathbf{1}_{\{S \leq \tau \leq S'\} \cap A}.$$

By noticing that $A \in \mathcal{F}_S$, one can easily derive the martingale property of $\{v(\tau)\}_{\tau \in T_{[S, S'']}}$ on A , which gives the second point. \square

3.3 Definition of $\check{\theta}^+(S)$ and comparison between $\check{\theta}(S)$ and $\check{\theta}^+(S)$

Fix $S \in T_0$. Recall that we have defined the stopping time $\check{\theta}(S)$ associated with the value function v (Definition 2.4). Let us now define similarly the random variable $\check{\theta}^+(S)$ associated with v^+ as follows

Definition 3.9. *Let \mathcal{A}_S^+ be the set of stopping times θ such that $\{v^+(\tau)\}_{\tau \in [S, \theta]}$ is a martingale system. We define the random variable $\check{\theta}^+(S)$ by*

$$\check{\theta}^+(S) := \text{ess sup } \mathcal{A}_S^+.$$

By similar arguments to those of the proof of Lemma 2.5, the set \mathcal{A}_S^+ is stable by pairwise maximization. Hence, there exists an optimizing nondecreasing sequence of stopping times (θ^n) in T_S which are in \mathcal{A}_S^+ such that $\check{\theta}^+(S) = \lim_{n \rightarrow \infty} \uparrow \theta^n$ a.s. and hence $\check{\theta}^+(S)$ is a stopping time.

Note that $\check{\theta}^+(S)$ clearly satisfies the same localization property as $\check{\theta}(S)$ (see Lemma 3.7). More precisely,

Lemma 3.9. *Suppose the reward $\{\phi(\theta), \theta \in T_0\}$ is an admissible family. Fix $\theta \in T_S$. Let $A \in \mathcal{F}_S$. Let $\mathcal{A}_S^+(A)$ be the set of r.v. $\theta\mathbf{1}_A$ with $\theta \in T_S$ such that $\{v^+(\tau)\}_{\tau \in T_{[S, \theta]}}$ is a martingale system on A . We have*

$$\check{\theta}^+(S)\mathbf{1}_A := \text{ess sup } \mathcal{A}_S^+(A) \text{ a.s..}$$

We state the following theorem:

Theorem 3.6. *Suppose the reward $\{\phi(\theta), \theta \in T_0\}$ is an admissible family such that $v(0) < \infty$. Then, for each $S \in T_0$, we have:*

•

$$\check{\theta}^+(S) \leq \check{\theta}(S) \text{ a.s. on } \{v(S) = v^+(S)\}.$$

- Suppose $\check{\theta}(S)$ is “accessible” on a set $A \in \mathcal{F}_S$, that is there exists a nondecreasing sequence of stopping times $(\theta_n)_{n \in \mathbb{N}}$ such that $\theta_n < \theta$ a.s. on A for each $n \in \mathbb{N}$ and such that $\theta_n \uparrow \theta$ on A .

Then,

$$\check{\theta}^+(S) \geq \check{\theta}(S) \text{ a.s. on } A.$$

- $\check{\theta}(S) = S \leq \check{\theta}^+(S)$ a.s. on $\{v^+(S) < v(S)\}$.

PROOF:

Step 1: Let us prove the first point. Put $A := \{v(S) = v^+(S)\}$. Let $\theta \in \mathcal{A}_S^+(A)$. Hence, $\{v^+(\tau)\}_{\tau \in T_{[S, \theta]}}$ is a martingale system on A . By the second point of Proposition 3.11, $\{v(\tau)\}_{\tau \in T_{[S, \theta]}}$ is a martingale system on A . It follows that $\mathcal{A}_S^+(A) \subset \mathcal{A}_S(A)$. By taking the essential supremum and by using localization properties of $\check{\theta}^+(S)$ and $\check{\theta}(S)$ (see Lemma 3.9 and Lemma 3.7), this clearly implies that $\check{\theta}^+(S)\mathbf{1}_A \leq \check{\theta}(S)\mathbf{1}_A$ a.s.

Step 2: Let us now prove the second point.

The proof of this property is clearly based on Corollary 3.2 and the following lemma:

Lemma 3.10. *Let $\theta \in T_S$ be a stopping time supposed to be “accessible” on A , where $A \in \mathcal{F}_S$. We have*

$$\text{ess sup } T_{[S, \theta]} = \theta, \text{ a.s. on } A \cap \{S < \theta\}. \quad (3.6)$$

In particular, if θ is predictable, then the previous equality is satisfied a.s. on $\{S < \theta\}$.

Suppose this lemma has been already proven. Recall that $\{v^+(\tau)\}_{\tau \in T_{[S, \check{\theta}(S)]}}$ is a martingale system (a.s. on $\{S < \check{\theta}(S)\}$) (by Corollary 3.2). It follows that, a.s. on $\{S < \check{\theta}(S)\}$,

$$\mathcal{A}_S^+ \supset T_{[S, \check{\theta}(S)]},$$

which gives that $\check{\theta}^+(S) = \text{ess sup } \mathcal{A}_S^+ \geq \text{ess sup } T_{[S, \check{\theta}(S)]}$ a.s..

Now, by Lemma 3.10, $\text{ess sup } T_{[S, \check{\theta}(S)]} = \check{\theta}(S)$ a.s. on $A \cap \{S < \check{\theta}(S)\}$.

This clearly gives that $\check{\theta}^+(S) \geq \check{\theta}(S)$ a.s. on A , which ends step 2.

The third point of the Theorem is clear. \square

PROOF OF LEMMA 3.10:

We clearly have $\text{ess sup } T_{[0, \theta]} \leq \theta$ a.s..

Suppose θ is accessible on A , where $A \in \mathcal{F}_S$. By definition, there exists a nondecreasing sequence of stopping times $(\theta_n)_{n \in \mathbb{N}}$ with $\theta_n < \theta$ a.s. on A for each $n \in \mathbb{N}$ and such that $\theta_n \uparrow \theta$ on A . Hence, $\text{ess sup } T_{[S, \theta]} \geq \theta_n$ a.s. on $A \cap \{S < \theta\}$. By letting n tend to ∞ , it gives that $\text{ess sup } T_{[S, \theta]} \geq \theta$ a.s. on $A \cap \{S < \theta\}$ and the desired result follows.

The second point follows clearly since a predictable stopping time is by definition an accessible stopping time on Ω . \square

Corollary 3.3. *Suppose $\check{\theta}(S)$ is predictable, then $\check{\theta}^+(S) = \check{\theta}(S)$ a.s. on $\{v(S) = v^+(S)\}$.*

Note that if $\check{\theta}(S)$ is not predictable, it is possible that $P(\{\check{\theta}^+(S) < \check{\theta}(S)\} \cap \{v(S) = v^+(S)\}) > 0$ as in the following example.

EXAMPLE:

Let ν be a random variable on a probability space which takes its values in $[0, T]$ and such that

$P(\nu > t) < 1$ for each $t \in]0, T]$. Let us consider the (completed) filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$ defined by $\mathcal{F}_t := \mathcal{F}_{t^+}^0$ where $\mathcal{F}_t^0 = \sigma(\nu \wedge t)$. Note that the stopping times relative to this filtration are given by $\nu \wedge t$ for $0 \leq t \leq T$. Let us consider the reward associated with the adapted process $\phi_t := \mathbf{1}_{[0, \nu]}(t)$ for $t \in [0, T]$.

Note that the associated value function is given by $v_t = \mathbf{1}_{[0, \nu]}(t)$ for $t \in [0, T]$.

Also, v^+ is given by $v_t^+ = \mathbf{1}_{[0, \nu]}(t)$ for $t \in [0, T]$.

Fix the initial time $S = 0$. The maximal optimal stopping time is clearly given by $\check{\theta}(0) = \nu$.

However, $\check{\theta}^+(0) = 0$. Indeed, for each $t \in]0, T]$,

$$E[v_{\nu \wedge t}^+] = P(\nu \wedge t < \nu) = P(t < \nu) < 1 = v_0^+.$$

It follows that for each $t > 0$, $(v_{s \wedge \nu}^+)_{0 \leq s \leq t}$ is not a martingale which implies that $\check{\theta}^+(0) = 0$. Note also that $T_{[0, \check{\theta}(0)]} = T_{[0, \nu]} = \{0\}$.

3.4 Some applications of the Doob-Meyer decomposition

Suppose that the reward $\{\phi(\theta), \theta \in T_0\}$ is such that the associated family $\{v^+(S), S \in T_0\}$ is of class \mathcal{D} (which is satisfied if for example $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$ or if $\{\phi(\theta), \theta \in T_0\}$ is uniformly integrable). Note that since the reward $\{\phi(\theta), \theta \in T_0\}$ is not supposed to be RCE, the associated value function $\{v(\theta), \theta \in T_0\}$ is not necessarily RCE. However, the family $\{v^+(S), S \in T_0\}$ is RCE (see Proposition 1.6). Since it is a supermartingale, by classical results (see Appendix A2 Proposition A.20), there exists a RCLL supermartingale process denote by (v_t^+) such that for each $\theta \in T_0$, $v_\theta^+ = v^+(\theta)$ a.s.. Since it is a supermartingale of class \mathcal{D} , by the Doob-Meyer decomposition, there exists a uniformly integrable RCLL martingale (M_t) and a predictable RC nonincreasing process (A_t) with $A_0 = 0$ and $E[A_T] < \infty$ such that for each $t \in [0, T]$,

$$v_t^+ = M_t - A_t \text{ a.s.} \quad (3.7)$$

We have the following characterization of $\check{\theta}^+(S)$ which is rather intuitive:

Proposition 3.12. *Suppose that $\{\phi(\theta), \theta \in T_0\}$ is an admissible family such that $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$. For almost every ω , we have:*

$$\check{\theta}^+(S)(\omega) = \inf\{t \geq S(\omega), A_t(\omega) > A_S(\omega)\} \wedge T,$$

where (A_t) is the non decreasing predictable RCLL process which appears in the Doob-Meyer decomposition of (v_t^+) .

PROOF : To simplify notation, $\check{\theta}^+(S)$ will be denoted $\check{\theta}^+$.

Let us introduce the following stopping time defined for each ω by

$$\bar{\theta}(\omega) = \inf\{t \geq S(\omega), A_t(\omega) > A_S(\omega)\} \wedge T.$$

Step 1: Let us show that $\check{\theta}^+ \leq \bar{\theta}$ a.s.

Recall that there exists a nondecreasing sequence of stopping times (θ^n) in T_S which are such that $\{v^+(\tau)\}_{\tau \in [S, \theta^n]}$ is a martingale system and such that $\check{\theta}^+ = \lim_{n \rightarrow \infty} \uparrow \theta^n$ a.s. Hence, we have that for each $n \in \mathbb{N}$, $A_{\theta^n} = A_S$ a.s.. By definition of $\bar{\theta}$, this gives that for each $n \in \mathbb{N}$, $\theta^n \leq \bar{\theta}$ a.s.. By letting n tend to ∞ , it follows that $\check{\theta}^+ \leq \bar{\theta}$ a.s. which ends step 1.

Step 2: Let us show the other inequality that is $\check{\theta}^+ \geq \bar{\theta}$ a.s.

Let T_1 be the time of the first jump of the process (A_t) . By convention, for each ω , if there is no jump between $S(\omega)$ and T , $T_1(\omega)$ is taken to be equal to T . Since (A_t) is predictable, T_1 is a predictable stopping time.

Recall that (A_t) is a RCLL non decreasing process. Without loss of generality, we can suppose that for each $\omega \in \Omega$, the map $t \mapsto A_t(\omega)$ is RCLL. Hence, we have that

- either the map $t \mapsto A_t(\omega)$ is constant on $[S(\omega), T_1(\omega)[$ and hence, in this case, $\bar{\theta}(\omega) = T_1(\omega)$,
- otherwise, $\bar{\theta}(\omega) < T_1(\omega)$ and the map $t \mapsto A_t(\omega)$ is continuous at $\bar{\theta}(\omega)$, constant on $[S(\omega), \bar{\theta}(\omega)]$ and starts to increase at $\bar{\theta}(\omega)$.

Now, since T_1 is a predictable stopping time, there exists a non decreasing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $S \leq \tau_n < T_1$ a.s. for each $n \in \mathbb{N}$ and such that $\tau_n \uparrow T_1$ a.s. For each n , define the stopping time

$$\bar{\theta}_n := \bar{\theta} \wedge \tau_n.$$

Note that $\bar{\theta} = \lim_{n \rightarrow \infty} \uparrow \bar{\theta}_n$. It is clear that for each n , $A_{\bar{\theta}_n} = A_S$ a.s., which yields the equality $v_{\bar{\theta}_n}^+ = M_{\bar{\theta}_n} - A_S$ a.s. In other terms, $\{v^+(\tau)\}_{\tau \in [S, \bar{\theta}_n]}$ is a martingale system. Hence, for each n ,

$$\bar{\theta}^n \leq \check{\theta}^+ \text{ a.s.}$$

By letting n tend to ∞ , we derive that $\bar{\theta} = \lim_{n \rightarrow \infty} \bar{\theta}^n \leq \check{\theta}^+$ a.s., which makes step 2 ended. At last, step 1 with step 2 give the desired result. \square

Remark 3.17. Fix $S \in T_0$. Suppose that $\check{\theta}(S)$ is predictable. By Corollary 3.3, $\check{\theta}^+(S) = \check{\theta}(S)$ a.s. on $\{v(S) = v^+(S)\}$. This with Proposition 3.12 gives

$$\check{\theta}(S) = \inf\{t \geq S, A_t > A_S\} \wedge T \text{ a.s..} \quad (3.8)$$

In other words, without any regularity assumption on the reward family, we have a characterization of an ‘‘upper bound’’ of optimal stopping times as a hitting time.

Also, in the case where the reward family is supposed to be USCE, this gives a characterization of the maximal optimal stopping time.

Note also that in the case of Brownian filtration, since all the stopping times are predictable, equality (3.8) holds for any $S \in T_0$.

Case of a RCE reward

We will now consider the particular case where the reward family $\{\phi(\theta), \theta \in T_0\}$ is supposed to be RCE and such that $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$.

In this case, $v(S) = v^+(S)$ a.s. for each $S \in T_0$. Let (v_t) be the RCLL adapted process which aggregates the value function. Note that $(v_t) = (v_t^+)$. With this notation, the Doob-Meyer decomposition (3.7) can be written

$$v_t = M_t - A_t \text{ a.s..}$$

In the following, we give a few properties concerning the non decreasing process (A_t) of the Doob-Meyer decomposition of (v_t) .

Since for each $S \in T_0$, $\check{\theta}(S) = \check{\theta}^+(S)$ a.s., by Proposition 3.12, we have:

Proposition 3.13. Suppose $\{\phi(\theta), \theta \in T_0\}$ is a RCE admissible family such that $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$. Fix $S \in T_0$. For almost every ω , we have:

$$\check{\theta}(S)(\omega) = \inf\{t \geq S(\omega), A_t(\omega) > A_S(\omega)\} \wedge T,$$

where (A_t) is the non decreasing predictable RCLL process which appears in the Doob-Meyer decomposition of (v_t) .

Furthermore, if $\{\phi(\theta), \theta \in T_0\}$ is supposed to be left-USCE, then $\check{\theta}(S)$ is the maximal optimal stopping time.

Remark 3.18. Note that the second assertion of this proposition is well known in the discrete time case (see Neveu (1975)) and in the case of a continuous reward (ϕ_t) (see equality (D.35) in Karatzas and Shreve (1998)).

Also, Theorem 3.5 leads to the following property:

Proposition 3.14. Suppose $\{\phi(\theta), \theta \in T_0\}$ is a RCE and left-USCE admissible family and that $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$. Let $\theta \in T_0$. For almost every ω s.t. $\theta(\omega) < T$, we have:

if for each $t' > \theta(\omega)$, $A_{t'}(\omega) > A_\theta(\omega)$, then $v(\theta)(\omega) = \phi(\theta)(\omega)$.

Remark 3.19. This property can also be written as follows. For almost every ω such that $\theta(\omega) < T$, we have: if $v(\theta)(\omega) \neq \phi(\theta)(\omega)$, then the nondecreasing function $s \mapsto A_s(\omega)$ is locally constant on the right of $\theta(\omega)$, that is there exists $t'(\omega) > \theta(\omega)$, such that $A_{t'}(\omega) = A_\theta(\omega)$.

PROOF OF PROPOSITION 3.14: Let us introduce the following set:

$$A = \left\{ \omega \in \Omega \text{ s.t. } \theta(\omega) < T \text{ and } \left(\forall t' \in]\theta(\omega), T], A_{t'}(\omega) > A_\theta(\omega) \text{ and } v(\theta)(\omega) > \phi(\theta)(\omega) \right) \right\}.$$

Without loss of generality, we can suppose for each ω , the function $t \mapsto A_t(\omega)$ is RC. Then, one can easily see that for each $p \in \mathbb{N}^*$,

$$A = \bigcap_{n \geq p} \left\{ A_{(\theta + \frac{1}{n}) \wedge T} > A_\theta \text{ and } v(\theta) > \phi(\theta) \right\} \cap \{\theta < T\},$$

which implies that $A \in \mathcal{F}_{\theta + \frac{1}{p}}$. Hence,

$$A \in \bigcap_{p \geq 1} \mathcal{F}_{(\theta + \frac{1}{p}) \wedge T} = \mathcal{F}_\theta,$$

by the right-continuity of the filtration (\mathcal{F}_t) . Suppose now that $P(A) > 0$.

The definition of A gives clearly that for each $\theta' \in T_0$ with $\mathbf{1}_A \theta \prec \mathbf{1}_A \theta'$, we have $\mathbf{1}_A A_{\theta'} \succ \mathbf{1}_A A_\theta$. Now, one can easily show this inequality is equivalent to

$$E[v(\theta') | \mathcal{F}_\theta] \prec v(\theta) \quad \text{a.s. on } A.$$

Hence, v is a strict supermartingale on the right at θ on A , which gives $v(\theta) = \phi(\theta)$ a.s. on A by Theorem 3.5. This provides the expected contradiction. Therefore, $P(A) = 0$. \square

In the following, we state the continuity property of the non decreasing process (A_t) (well known in the case of a continuous reward process).

Proposition 3.15. Suppose $\{\phi(\theta), \theta \in T_0\}$ is a RCE and left-USCE admissible family and that $E[\text{ess sup}_{\theta \in T_0} \phi(\theta)] < \infty$. Then, the nondecreasing process (A_t) is continuous.

PROOF: Recall that the value function (v_t) is a RCLL supermartingale which is uniformly integrable and LCE (see Proposition 2.9). Then, the result follows from a result of the general theory of processes (see Proposition A.21 in the Appendix). \square

Furthermore, in case of a RCLL adapted reward process (ϕ_t) , we have

Proposition 3.16. Suppose now that the reward ϕ is given by a RCLL adapted process (ϕ_t) which is left-USCE and such that $E[\text{ess sup}_{t \in [0, T]} \phi_t] < \infty$.

Then, for almost every ω , the nondecreasing continuous function $t \mapsto A_t(\omega)$ is “flat” away from the set $\mathcal{H}(\omega) := \{t \in [0, T], v_t(\omega) = \phi_t(\omega)\}$ i.e. $\int_0^T \mathbf{1}_{v_t > \phi_t} dA_t = 0$ a.s.

Remark 3.20. *In the case of a continuous reward (ϕ_t) , this proposition corresponds to Theorem D13 in Karatzas and Shreve ((1998)).*

PROOF: The proof is based on Proposition 3.14 and on classical analytic arguments. Note that these analytic arguments are the same as those used in the proof of Theorem D13 in Karatzas and Shreve ((1998)).

Without loss of generality, we can suppose for each ω , the maps $t \mapsto v_t(\omega)$, $t \mapsto \phi_t(\omega)$ are right-continuous and $t \mapsto A_t(\omega)$ is continuous.

Let us denote by $\mathcal{J}(\omega)$ the set on which the nondecreasing function $t \mapsto A_t(\omega)$ is “flat”:

$$\mathcal{J}(\omega) := \{t \in]0, T[, \exists \varepsilon > 0 \text{ with } A_{t-\varepsilon}(\omega) = A_{t+\varepsilon}(\omega)\}$$

The set $\mathcal{J}(\omega)$ is clearly open and hence can be written as a countable union of disjoint intervals: $\mathcal{J}(\omega) = \cup_i]\alpha_i(\omega), \beta_i(\omega)[$. We consider

$$\hat{\mathcal{J}}(\omega) = \cup_i]\alpha_i(\omega), \beta_i(\omega)[= \{t \in [0, T[, \exists \varepsilon > 0 \text{ with } A_t(\omega) = A_{t+\varepsilon}(\omega)\}.$$

The nondecreasing function $t \mapsto A_t(\omega)$ is “flat” on $\hat{\mathcal{J}}(\omega)$ in the sense that $\int_0^T \mathbf{1}_{\hat{\mathcal{J}}(\omega)} dA_t(\omega) = \sum_i (A_{\beta_i(\omega)} - A_{\alpha_i(\omega)}) = 0$. We will show that for almost every ω ,

$$\mathcal{H}^c(\omega) \subset \hat{\mathcal{J}}(\omega),$$

which will clearly give the desired result.

Let us denote by \mathbb{Q} the set of rationals. By Proposition 3.14 (or Remark 3.19) applied to constant stopping times $\theta := t$, where $t \in \mathbb{Q} \cap [0, T[$, it follows that for a.e. ω ,

$$\{t \in \mathbb{Q} \cap [0, T[\text{ s.t. } v_t(\omega) > \phi_t(\omega)\} \subset \hat{\mathcal{J}}(\omega). \quad (3.9)$$

Let us now show that the desired inclusion

$$\mathcal{H}^c(\omega) = \{t \in [0, T[\text{ s.t. } v_t(\omega) > \phi_t(\omega)\} \subset \hat{\mathcal{J}}(\omega)$$

holds for a.e. ω . Fix ω such that (3.9) holds and fix $t \in \mathcal{H}^c(\omega)$. Since $v_t(\omega) > \phi_t(\omega)$ and since the maps $t \mapsto v_t(\omega)$ and $t \mapsto \phi_t(\omega)$ are right-continuous, there exists a non increasing sequence of rationals $t_n(\omega) \in \mathbb{Q} \cap [0, T[$ such that $t = \lim_{n \rightarrow \infty} \downarrow t_n(\omega)$ with $v_{t_n(\omega)}(\omega) > \phi_{t_n(\omega)}(\omega)$ for each n . Using the above inclusion (3.9), the equality $\hat{\mathcal{J}}(\omega) = \cup_i]\alpha_i(\omega), \beta_i(\omega)[$ and the fact that $t = \lim_{n \rightarrow \infty} \downarrow t_n(\omega)$, we derive that there exist i and n_0 (which both depend on ω) such that for each $n \geq n_0$, $t_n(\omega) \in]\alpha_i(\omega), \beta_i(\omega)[$. It follows that the limit $t \in]\alpha_i(\omega), \beta_i(\omega)[$, which gives that $t \in \hat{\mathcal{J}}(\omega)$. Hence, the inclusion $\mathcal{H}^c(\omega) \subset \hat{\mathcal{J}}(\omega)$ is proven, which ends the proof. \square

Remark 3.21. *Note that in the paper on multiple stopping time problem (see Kobylanski et al. (2010) and Kobylanski et al. (2010)), the value function can be written as the value function of a one stopping time problem with a “new reward” family $\{\phi(\theta), \theta \in T_0\}$ which is RCE but which a priori cannot be aggregated by a RCLL adapted process. Hence, Proposition 3.16 cannot be applied. However, Propositions 3.13, 3.14 and 3.15 do hold true.*

4 The particular case of a reward process

In this section, we give an application of the previous results to a particular case where the reward is given by a progressive process $(\phi_t, 0 \leq t \leq T)$ which satisfies some regularity properties precised below. In this case, the value function can be aggregated. Also, the $(1 - \lambda)$ -optimal stopping times as well as the minimal optimal stopping time can be written as hitting times of processes.

4.1 Case of a reward process (ϕ_t)

Let $(\phi_t, 0 \leq t \leq T)$ be a progressive process. Note that the family $\{\phi(\theta), \theta \in T_0\}$ defined by $\phi(\theta) = \phi_\theta$ is admissible. Suppose that $v(0) < \infty$.

Recall first that since $\{v^+(\theta), \theta \in T_0\}$ is a supermartingale system which is RCE, there exists a RCLL process $(v_t^+, 0 \leq t \leq T)$ that *aggregates* the family $\{v^+(\theta), \theta \in T_0\}$ that is such that

$$v^+(\theta) = v_\theta^+ \text{ a.s. for all } \theta \in T_0. \quad (4.1)$$

Define the process (v_t) by

$$v_t := \phi_t \vee v_t^+. \quad (4.2)$$

By Proposition 1.5, we clearly have

Proposition 4.17. *Suppose that the reward is given by a progressive process (ϕ_t) such that the associated value function satisfies $v(0) < \infty$. Then, the adapted process (v_t) defined by (4.2) aggregates the value function family $\{v(S), S \in T_0\}$, that is for all $S \in T_0$, $v(S) = v_S$ a.s..*

Moreover, if the family $\{\phi_\theta, \theta \in T_0\}$ is right-USC (resp. right-USCE), then the first (resp. second) point of Theorem 2.1 can be applied, that is for each $\lambda \in]0, 1[$, $\theta^\lambda(S)$ is $(1 - \lambda)$ -optimal for v_S (resp. for $E[v_S]$).

We now state that under some additional assumptions, for each $\lambda \in]0, 1[$, $\theta^\lambda(S)$ can be written as a hitting time of processes.

Definition 4.10. *A process $(\phi_t, 0 \leq t \leq T)$ is said to be right upper-semicontinuous (right-USC) if for almost every ω , the function $t \mapsto \phi_t(\omega)$ is right upper-semicontinuous, that is for each $t \in [0, T]$,*

$$\phi_t(\omega) \geq \limsup_{s \rightarrow t^+} \phi_s(\omega).$$

For each $\lambda \in]0, 1[$ let us define the following stopping time for each ω by:

$$\tau^\lambda(S)(\omega) := \inf\{t \geq S(\omega), \lambda v_t^+(\omega) \leq \phi_t(\omega)\}.$$

Proposition 4.18. *Suppose that $(\phi_t, 0 \leq t \leq T)$ is a right-USC progressive process with $v(0) < \infty$. Fix $S \in T_0$. Then, for each $\lambda \in]0, 1[$,*

$$\tau^\lambda(S) = \theta^\lambda(S) \text{ a.s..}$$

In particular, $\tau^\lambda(S)$ is a $(1 - \lambda)$ -optimal stopping time for v_S , that is $\lambda v_S \leq E[\phi_{\tau^\lambda(S)}] | \mathcal{F}_S$ a.s.

Remark 4.22. *Note that the condition $v(0) < \infty$ is satisfied for example if $E[\text{ess sup}_{0 \leq t \leq T} \phi_t] < \infty$.*

PROOF: To simplify the notation, let us denote $\theta^\lambda(S)$ by θ^λ and $\tau^\lambda(S)$ by τ^λ .

By the first point of Lemma 2.3, for almost every $\omega \in \Omega$,

$$\lambda v_{\theta^\lambda(\omega)}(\omega) \leq \phi_{\theta^\lambda(\omega)}(\omega) \text{ a.s. ,}$$

which implies that $\tau^\lambda(\omega) \leq \theta^\lambda(\omega)$ by definition of $\tau^\lambda(\omega)$.

Let us show the other inequality. Suppose first we have shown that for each $\lambda \in]0, 1[$, the stopping time τ^λ satisfies

$$\lambda v_{\tau^\lambda} \leq \phi_{\tau^\lambda} \text{ a.s..} \quad (4.3)$$

Hence, $\tau^\lambda \in \mathbb{T}_S = \{\theta \in T_S, \lambda v^+(\theta) \leq \phi(\theta) \text{ a.s.}\}$ which implies that $\theta^\lambda = \text{ess inf } \mathbb{T}_S \leq \tau^\lambda$ a.s.. Consequently, the desired equality $\tau^\lambda = \theta^\lambda$ a.s. follows.

It remains to show inequality (4.3). The proof is done by fixing $\omega \in \Omega$ such that the function $t \mapsto \phi_t(\omega)$ is right upper-semicontinuous and the function $t \mapsto v_t^+(\omega)$ is right continuous. Then, the same arguments of analysis as those in the proof of Lemma 2.3 (case a) hold except that the minimizing sequence (θ^n) in \mathbb{T}_S for θ^λ is replaced by a non increasing sequence of reals (t^n) (which depend of ω) in $[S(\omega), T]$ such that $\tau^\lambda(\omega) = \lim_{n \rightarrow \infty} \downarrow t^n$ and such that for each n , $\lambda v_{t^n}^+(\omega) \leq \phi_{t^n}(\omega)$. \square

Suppose that $\{\phi_\theta, \theta \in T_0\}$ is right and left-USCE. Then, Theorems 2.2 and 2.3 can be applied. Moreover, we have:

Proposition 4.19. *Suppose that $(\phi_t, 0 \leq t \leq T)$ is a right-USC progressive process with $v(0) < \infty$ and that it is left-USCE. Then, for all $S \in T_0$, the stopping time $\tau_*(S)$ defined by*

$$\tau_*(S) = \inf\{t \geq S, v_t = \phi_t\} = \inf\{t \geq S, v_t^+ \leq \phi_t\}$$

is the minimal optimal stopping time for v_S . In other words, $\tau_*(S) = \theta_*(S)$ a.s.

PROOF: Again, let us denote $\theta_*(S)$ by θ_* , $\tau_*(S)$ by τ_* , $\theta^\lambda(S)$ by θ^λ and $\tau^\lambda(S)$ by τ^λ .

First, we clearly have that $\lim_{\lambda \uparrow 1} \uparrow \tau^\lambda \leq \tau_*$ a.s.. Also,

$$\lim_{\lambda \uparrow 1} \uparrow \tau^\lambda = \lim_{\lambda \uparrow 1} \uparrow \theta^\lambda = \theta_* \text{ a.s.,}$$

and hence, $\theta_* \leq \tau_*$ a.s..

Furthermore, for almost every ω , since $v_{\theta_*(\omega)}(\omega) = \phi_{\theta_*(\omega)}(\omega)$, it follows that $\tau_*(\omega) \leq \theta_*(\omega)$ by definition of $\tau_*(\omega)$.

Thus, we have proven that $\theta_* = \tau_*$ a.s.. \square

Remark 4.23. *Note that in the paper on multiple stopping time problem of Kobylanski et al. (2010), the value function can be written as the value function of a one stopping time problem with a “new reward” family $\{\phi(\theta), \theta \in T_0\}$ which is RCE and LCE. In that paper, under some specific assumptions, the reward family is shown to be aggregated by a progressive process which is right-USC because it is the limit of a non increasing sequence of RC processes (see Theorem 4.1 and Corollary 4.1). The results of Propositions 4.18 and 4.19 applied to that case exactly correspond to Theorem 4.2 and Proposition 4.3 in Kobylanski et al. (2010).*

4.2 Some examples

We are given a RCLL adapted process $(X_t)_{0 \leq t \leq T}$ valued in \mathbb{R}^d which is supposed to be LC that is left continuous along stopping times.

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}^+$ be an upper-semicontinuous function such that

$$E[\text{ess sup}_{t \in [0, T]} f(X_t)] < \infty. \quad (4.4)$$

For instance, the function $f = \mathbf{1}_F$, where F is a closed subset of \mathbb{R}^d , satisfies these conditions.

Then, the process (ϕ_t) given by

$$\phi_t = f(X_t), \quad 0 \leq t \leq T,$$

is USCE.

Also, note that, since the sum and the product of positive upper-semicontinuous functions are upper-semicontinuous, one can easily build number of USCE rewards.

For example, let us consider the process (ϕ_t^1) defined by

$$\phi_t^1 = f^1(X_t) \mathbf{1}_{\{X_t \in F^1\}} + f^2(X_t) \mathbf{1}_{\{X_t \in F^2\}}, \quad 0 \leq t \leq T,$$

where f^1, f^2 are upper-semicontinuous positive functions satisfying (4.4) and where F^1 and F^2 are closed subsets of \mathbb{R}^d and let us consider for $d = 1$ the process (ϕ_t^2) defined by

$$\phi_t^2 = f \left(\int_0^t X_s ds \right) \mathbf{1}_{\{X_t^* \in [a, b]\}}, \quad 0 \leq t \leq T,$$

where $X_t^* := \sup_{0 \leq s \leq t} X_s$ and f is an upper-semicontinuous positive function satisfying (4.4).

Note that (ϕ_t^1) and (ϕ_t^2) are USCE by the above remarks and the fact that the processes (X_t^*) and $(\int_0^t X_s ds)$ are RCLL and LC.

This opens a way to a large range of applications to all kinds of exotic American options.

A

A.1 Essential supremum

We recall the following theorem on the essential supremum of a family of random variables (see for example Neveu (1975)).

Theorem A.7. (Essential supremum) *Let F be a non empty family of measurable real valued functions $f : \Omega \rightarrow \overline{\mathbb{R}}$ defined on a probability space (Ω, \mathcal{F}, P) . Then there exists a measurable function $g : \Omega \rightarrow \overline{\mathbb{R}}$ such that*

1. for all $f \in F$, $f \leq g$ a.s.,
2. if h is a measurable function satisfying $f \leq h$ a.s. for all $f \in F$, then $g \leq h$ a.s..

Moreover there exists a countable subset G^* of F such that $g = \sup_{f \in G^*} f$.

This function g , which is unique a.s., is called the essential supremum of F and is denoted $\text{ess sup } F$.

A.2 Two results of the general theory of processes

Recall the following classical aggregation result.

Proposition A.20. *Let $\{h(S), S \in T_0\}$ be a supermartingale admissible system which is RCE with $h(0) < \infty$. Then, there exists a RCLL adapted process (h_t) which aggregates the family $\{h(S), S \in T_0\}$, that is $h_S = h(S)$ a.s. for each $S \in T_0$. In particular, the family $\{h(\theta), \theta \in T_0\}$ is RC (that is right continuous along stopping times).*

PROOF: For completeness, we give the short proof based on a well known result of the general theory of processes. Let us consider the process $(h(t))_{0 \leq t \leq T}$. It is a supermartingale and the function $t \mapsto E(h(t))$ is right continuous. By a classical result (see Theorem 3.13 in Karatzas and Shreve (1994)), there exists a RCLL supermartingale $(h_t)_{0 \leq t \leq T}$ which is a modification of the process $(h(t))_{0 \leq t \leq T}$, that is such that for each $t \in [0, T]$, the equality $h_t = h(t)$ holds a.s. Then, it is clear that for each dyadic stopping time $S \in T_0$, $h_S = h(S)$ a.s.. This implies that $E[h_S] = E[h(S)]$. Since the process $(h_t)_{0 \leq t \leq T}$ is RCLL and since the family $\{h(S), S \in T_0\}$ is right continuous in expectation, this equality still holds for any stopping time $S \in T_0$. Then, it remains to show this implies that $h_S = h(S)$ a.s. This is classical. Let $A \in \mathcal{F}_S$ and define $S_A = S \mathbf{1}_A + T \mathbf{1}_{A^c}$. Since S_A is a stopping time, $E[h_{S_A}] = E[h(S_A)]$. Since $h_T = h(T)$ a.s.,

it gives that $E[h_S \mathbf{1}_A] = E[h(S) \mathbf{1}_A]$, which gives the desired result. The RC property clearly follows. \square

This Proposition with the supermartingale and RCE properties of v^+ gives the following property:

Corollary A.4. *Let $\{\phi(\theta), \theta \in T_0\}$ be an admissible family such that $v(0) < \infty$. Then, there exists a RCLL adapted process (v_t^+) which aggregates the family $\{v^+(S), S \in T_0\}$, that is $v_S^+ = v^+(S)$ a.s. for each $S \in T_0$.*

In particular, the family $\{v^+(S), S \in T_0\}$ is RC.

Recall now a result of the general theory of processes (Dellacherie and Meyer (1980) Th 10. Chap VII).

Proposition A.21. *Let (u_t) be a RCLL supermartingale of class \mathcal{D} and LCE. Then, the non decreasing predictable process (A_t) of the Doob-Meyer decomposition of (u_t) is continuous.*

PROOF: For the sake of completeness, we give here the proof which is based on a classical result of the general theory of processes. Note first that since (u_t) is LCE, it follows that (A_t) is also LCE.

Let θ be a predictable stopping time of T_0 . There exists a sequence of stopping times (θ^n) announcing θ that is a nondecreasing sequence of stopping times (θ^n) in T_0 such that $\theta_n < \theta$ a.s. on $\theta > 0$ and $\theta_n = 0$ otherwise and such that $\theta = \lim_{n \rightarrow \infty} \uparrow \theta^n$.

Since (A_t) is LCE, $E[A_\theta] = \lim_{n \rightarrow \infty} E[A_{\theta_n}] = E[A_{\theta-}]$, which gives that $A_\theta = A_{\theta-}$ a.s.

Hence, the predictable processes (A_t) and (A_{t-}) satisfy that for each predictable stopping time $\theta \in T_0$, $A_\theta = A_{\theta-}$ a.s.

By a classical result of the general theory of processes based on a section theorem (see Dellacherie and Meyer (1977) Th 86. Chap IV), it follows that the predictable processes (A_t) and (A_{t-}) are indistinguishable. Hence, the process (A_t) is continuous. \square

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