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► **To cite this version:**

Louis Jeanjean, Boyan Sirakov. Multiple solutions for an elliptic problem with quadratic growth in the gradient. 2010. hal-00519454v1

HAL Id: hal-00519454

<https://hal.science/hal-00519454v1>

Preprint submitted on 20 Sep 2010 (v1), last revised 11 Oct 2012 (v2)

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MULTIPLE SOLUTIONS FOR AN ELLIPTIC PROBLEM WITH QUADRATIC GROWTH IN THE GRADIENT

LOUIS JEANJEAN* AND BOYAN SIRAKOV†

ABSTRACT. We consider a divergence-form elliptic problem with quadratic growth in the gradient. Assuming that the zero order term has a given sign we show the existence of at least two solutions. The case where the zero order term has the opposite sign was already intensively studied and the uniqueness is the rule. Our proofs are mainly variational but also sub- and super-solutions arguments are used.

1. INTRODUCTION

Boundary value problems for elliptic equations like

$$(1.1) \quad -\operatorname{div}(a(x, u, \nabla u)) = B(x, u, \nabla u) + f(x), \quad x \in \Omega \subset \mathbb{R}^N,$$

where $-\operatorname{div}(a(x, \cdot, \nabla \cdot))$ is a Leray-Lions operator on some Sobolev space, have been one of the central problems in the theory of elliptic PDE in divergence form. This paper is a contribution to this study for the widely explored case when the nonlinear term $B(x, u, \xi)$ has “natural growth” in the unknown function, that is, grows linearly in u and quadratically in $\xi \in \mathbb{R}^N$. The model case for our study is

$$(1.2) \quad a(x, u, \xi) = A(x)\xi, \quad B(x, u, \xi) = c_0(x)u + \mu(x)|\xi|^2,$$

where A is a positive bounded matrix, $\mu \in L^\infty(\Omega)$, and c_0, f belong to suitably chosen Lebesgue spaces.

This type of problems have generated a considerable literature. Let us mention here [9, 11, 13, 16, 18, 5, 6, 7, 2, 1] as reference papers on this subject, most closely related to the problem we consider. In these works (see also the references given in them) the existence and uniqueness of a solution of the Dirichlet boundary problem for (1.1)

2000 *Mathematics Subject Classification.* 35J25, 35J62.

Key words and phrases. Elliptic equation, natural growth, quadratic growth in the gradient, non-coercive, minimax methods, sub- and super-solutions.

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is established under various conditions on a , B and f , which will be discussed below.

The novelty in our work is in that we consider (1.1) with non-coercive dependence in the unknown function u . More specifically, when reduced to (1.2), all papers quoted above assume that the coefficient c_0 is nonpositive. In this paper we address the opposite case, in which we unveil a new phenomenon. Namely, we are going to see that, *when c_0 is positive and sufficiently close to zero, the same type of existence result as in the case $c_0 = 0$ can be obtained, but the bounded solutions are not unique.*

The paper is organized as follows. The next section contains our hypotheses and main results, together with a discussion which explains their proofs and situates them with respect to previous works. The proofs themselves can be found in Sections 3–6. We conclude with some final remarks in Section 7, where we discuss possible extensions and open problems.

2. MAIN RESULT AND DISCUSSION

In this section we state our main results on the equation

$$(2.1) \quad -\Delta u - c_0(x)u = \mu(x)|\nabla u|^2 + f(x), \quad u \in H_0^1(\Omega)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a bounded domain in \mathbb{R}^N . We have taken (1.1)-(1.2) with $A(x) \equiv I$, for shortness (more general equations will be discussed in Section 7). We assume that

- (H1) $\mu \in L^\infty(\Omega)$; $c_0, f \in L^p(\Omega)$ for some $p > \frac{N}{2}$; $c_0 \not\equiv 0$ in Ω .
- (H2) Setting $\bar{\mu} := \|\mu\|_{L^\infty(\Omega)}$, we have

$$\bar{\mu}\|f\|_{L^{\frac{N}{2}}(\Omega)} < C_N^{-1},$$

where C_N is the optimal Sobolev constant (see (3.8) below).

Theorem 1. (i) *Assume that (H1)-(H2) hold. There exists a constant $\bar{c} > 0$ depending on N , Ω , $\bar{\mu}$, $\bar{\mu}\|f\|_{L^p(\Omega)}$, such that if*

$$(2.2) \quad \|c_0\|_{L^p(\Omega)} < \bar{c},$$

then (2.1) admits a bounded solution.

(ii) *Assume in addition that $\mu(x)$ is constant, $\mu(x) \equiv \mu_0 \neq 0$. Then (2.1) admits at least two bounded solutions.*

Remark 2. Note the statement (i) in this theorem reduces to the already known results (see [15], [21], and Section 7 below) when $c_0 \equiv 0$. A weaker version of (i) appeared already in [20], where equations in non-divergence form are studied. The statement (ii), which to our

knowledge is completely new, indicates that when $c_0 \not\geq 0$, even the bounded solutions are no longer expected to be unique.

Let us now give more details on some existence and uniqueness results which appeared prior to our work. Because of the very large literature we restrict to works which encompass the model case (1.2) (the reader may consult the references in the papers quoted below for various related problems). In [9, 11] Boccardo, Murat and Puel showed that the sub- and super-solution method applies to general equations with quadratic growth in the gradient, and proved existence of bounded solutions of such equations under a hypothesis of strict coercivity in u , that is, $c_0(x) \leq -\alpha_0 < 0$ in (1.2). For results on strictly coercive equations we refer also to dall'Aglio, Giachetti and Puel [13]. Next, the equation (1.1)-(1.2) with $c_0 \equiv 0$ was studied by Ferone and Murat [15, 16]. In that case it turns out that existence can be proved only under a smallness hypothesis on the source term f , as in (H2). Further existence results with weaker assumptions of regularity on the coefficients can be found in Grenon, Murat and Porretta [18]. Uniqueness results for these solutions in natural spaces associated to the problem were obtained by Barles and Murat [6], Barles, Blanc, Georgelin, and Kobylanski [5], Barles and Porretta [7]. We also refer to the recent works by Abdellaoui, dall'Aglio and Peral [2], and Abdel Hamid and Bidaut-Veron [1] for a deep study of (2.1) with $c_0 = 0$, $\mu = 1$, and $f \geq 0$. In particular they show that under (H2) the problem (2.1) has infinitely many solutions, of which only one is such that $e^u - 1 \in H_0^1(\Omega)$. For results on other classes of equations of type (1.1), with B being for instance in the form $B(x, s, \xi) = \beta(u)|\xi|^2$ for some real function β , we refer to Boccardo, Gallouët, and Murat [8], as well as to [2], [1].

Very recently the second author [22] obtained existence and uniqueness results for fully nonlinear equations in non-divergence form with quadratic dependence in the gradient, in which case the adapted weak notion of solution is the viscosity one. The idea of our study originated from that paper.

In previous works on (1.1) (for example [9, 11, 13, 15, 16]) the existence of one solution is typically obtained by looking first at approximated problems for which finding a regular solution is easy, and then by passing to the limit. A key point at this step is to benefit from a priori bounds in $W^{1,p}(\Omega)$ on the sequence of approximating solutions, which are obtained under the condition $c_0 \leq 0$.

Let us now present our approach to (2.1) and provide some intuition why the case $c_0 \not\geq 0$ is different from the cases $c_0 = 0$ or $c_0 \leq -\alpha_0 < 0$.

Assume for the moment that $\mu > 0$ is a constant and c_0 and f are smooth functions. We make the (very standard) change of unknown $v = \frac{1}{\mu}(e^{\mu u} - 1)$ in (2.1) and we observe that if a solution of

$$(2.3) \quad -\Delta v - [c_0(x) + \mu f(x)]v = c_0(x)g(v) + f(x), \quad v \in H_0^1(\Omega),$$

where

$$(2.4) \quad g(s) = \begin{cases} \frac{1}{\mu}(1 + \mu s)\ln(1 + \mu s) - s & \text{if } s > -\frac{1}{\mu} \\ -s & \text{if } s \leq -\frac{1}{\mu}, \end{cases}$$

satisfies $v > -\frac{1}{\mu}$, then $u = \frac{1}{\mu}\ln(1 + \mu v)$ is a solution of (2.1).

Equation (2.3) admits a variational formulation, in other words, its solutions in $H_0^1(\Omega)$ can be represented as critical points of a functional defined on this space. Namely, critical points of

$$(2.5) \quad I(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} [c_0(x) + \mu f(x)]v^2 dx \\ - \int_{\Omega} c_0(x)G(v)dx - \int_{\Omega} f(x)v dx$$

on $H_0^1(\Omega)$ are weak solutions of (2.3). Here $G(s) = \int_0^s g(t)dt$.

If we assume that $c_0 \leq -\alpha_0 < 0$ it is easily seen that I is coercive (that is, $\lim_{\|v\| \rightarrow \infty} I(v) = +\infty$), without assumptions on the size of f , and the existence of a global minimum follows. Indeed, the third term in the definition of I dominates the second and the fourth, since $\lim_{s \rightarrow \infty} s^{-2}G(s) = +\infty$, see Lemma 6.

If $c_0 = 0$ then I becomes

$$I(v) = \frac{1}{2} \int_{\Omega} |\nabla v|^2 - \mu f(x)v^2 dx - \int_{\Omega} f(x)v dx,$$

and this functional is coercive if and only if

$$(2.6) \quad \inf_{\|v\|_2=1} \int_{\Omega} |\nabla v|^2 - \mu f(x)v^2 dx > 0,$$

which in turn holds under the condition (H2), which appeared in [15].

Now if $c_0 \not\geq 0$ the geometry of I is completely different, we have $\inf I(v) = -\infty$, so no global minimum exists. On the other hand, it can be shown that under (2.6) I takes strictly positive values on the boundary of some large ball in $H_0^1(\Omega)$ centered at the origin, provided c is sufficiently small. In view of $I(0) = 0$ this implies that I has a local minimum in some neighborhood of the origin, and consequently at least one more critical point (of saddle type) could be expected to exist.

These properties were essentially shown to be true in the classical work by Ambrosetti and Rabinowitz [4], under some hypotheses which

in particular require $G(v)$ to behave like a power v^p , with $p \in (2, 2^*)$. In our situation these hypotheses are not satisfied, $G(v)$ has slow growth, and a significant difficulty is to show that the so-called Palais-Smale sequences, or at least Cerami sequences (see Section 5) for I are bounded. To overcome this difficulty we use some ideas introduced in [19].

In the case where the function μ is non constant we cannot directly make the change of unknown $v = \frac{1}{\mu}(e^{\mu u} - 1)$ in (2.1). We find one solution by showing that there exist ordered lower and upper solutions, which we construct by solving (2.1) with μ replaced by $\pm \|\mu\|_{L^\infty(\Omega)}$.

The proof of Theorem 1 will be given in the following sections. First, in Section 3 we give some preliminaries and study the relation between the problems (2.1) and (2.3), in the case $\mu > 0$ is a constant. In Section 4 we establish some facts on the geometry of the functional $I(v)$, and show it admits a local minimum. The core of the multiplicity result is in Section 5, where we show that Cerami sequences for I are bounded. In Section 6 we finish the proof of Theorem 1.

Some Notation.

- (1) We denote by X the space $H_0^1(\Omega)$ equipped with the Poincaré norm $\|u\| = \int_\Omega |\nabla u|^2$, and by X^{-1} its dual.
- (2) For $v \in L^1(\Omega)$ we define $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$.
- (3) The norm $(\int_\Omega |u|^p dx)^{1/p}$ in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. We denote by p' the conjuguate exponent of p , namely $p' = (p - 1)/p$.
- (4) We denote by $C, D > 0$ any positive constants which are not essential in the problem and may vary from one line to another.

3. THE LINK BETWEEN PROBLEMS (2.1) AND (2.3)

We consider the problem

$$(3.1) \quad -\Delta v - [c_0(x) + \mu f(x)]v = c_0(x)g(v) + f(x), \quad v \in X$$

where g is given by (2.4).

Lemma 3. *Assume that (H1)-(H2) hold, and $\mu > 0$ is a constant. If $v \in X$ is a solution of (3.1) which satisfies*

$$v > -1/\mu + \varepsilon \quad \text{on } \Omega, \quad \text{for some } \varepsilon > 0,$$

then $u = \frac{1}{\mu} \ln(1 + \mu v)$ is a solution of (2.1).

Proof. The equation (3.1) can be rewritten, for $v > -1/\mu$,

$$(3.2) \quad -\Delta v = \frac{c_0(x)}{\mu}(1 + \mu v) \ln(1 + \mu v) + (1 + \mu v)f(x).$$

Let $v \in X$ be a solution of (3.2), we want to show that $u = \frac{1}{\mu} \ln(1 + \mu v)$ is a solution of (2.1), that is, if $\phi \in C_0^\infty(\Omega)$, then

$$(3.3) \quad \int_{\Omega} \nabla u \nabla \phi - \mu |\nabla u|^2 \phi - c_0(x) u \phi dx = \int_{\Omega} f(x) \phi dx.$$

Let $\psi = \frac{\phi}{1 + \mu v}$. Clearly $\psi \in X$ and thus it can be used to test (3.2). We get

$$(3.4) \quad \int_{\Omega} \nabla v \nabla \psi dx = \int_{\Omega} \frac{c_0(x)}{\mu} \ln(1 + \mu v) \phi dx + \int_{\Omega} f(x) \phi dx.$$

But

$$(3.5) \quad \int_{\Omega} \frac{c_0(x)}{\mu} \ln(1 + \mu v) \phi dx = \int_{\Omega} c_0(x) u \phi dx$$

and

$$\begin{aligned} \int_{\Omega} \nabla v \nabla \psi dx &= \int_{\Omega} \nabla \left(\frac{1}{\mu} (e^{\mu u} - 1) \right) \nabla \left(\frac{\phi}{1 + \mu v} \right) dx \\ &= \int_{\Omega} e^{\mu u} \nabla u \left(\frac{\nabla \phi}{1 + \mu v} - \frac{\mu \phi \nabla v}{(1 + \mu v)^2} \right) dx \\ &= \int_{\Omega} \nabla u \left(\nabla \phi - \frac{\mu \phi \nabla \left(\frac{1}{\mu} (e^{\mu u} - 1) \right)}{(1 + \mu v)} \right) dx \\ (3.6) \quad &= \int_{\Omega} \nabla u (\nabla \phi - \mu \phi \nabla u) dx = \int_{\Omega} \nabla u \nabla \phi - \mu |\nabla u|^2 \phi dx. \end{aligned}$$

Combining (3.4), (3.5) and (3.6), we see that u satisfies (3.3). \square

Next we recall the following standard fact.

Lemma 4. 1. Given $h \in L^{N/2}(\Omega)$, set

$$E_h(u) = \int_{\Omega} |\nabla u|^2 - h(x) |u|^2 dx,$$

for $u \in X$. Then

$$\|h^+\|_{\frac{N}{2}} < C_N^{-1}$$

implies that the quantity $E_h(u)$ defines a norm on X which is equivalent to the standard norm, and

$$\lambda(h, \Omega) := \inf_{\|u\|_2=1} E_h(u) > 0.$$

This last property implies that the operator $-\Delta - h$ satisfies the maximum principle in Ω , that is, if $-\Delta u - hu \geq 0$ in X^{-1} for some $u \in X$, then $u^- \in X$ yields $u^- \equiv 0$ in Ω .

2. If in addition $h \in L^p(\Omega)$ for some $p > N/2$ then $\lambda(h, \Omega)$ is attained by a function $\phi_1 \in W^{2,p}(\Omega) \cap X \subset C^\alpha(\Omega) \cap X$ ($\alpha > 0$), such that $\phi_1 > 0$ in Ω and

$$-\Delta\phi_1 - h\phi_1 = \lambda(h, \Omega)\phi_1 \quad \text{in } \Omega.$$

The function ϕ_1 is unique up to a multiplication by a constant. Moreover, $\lambda(h, \Omega)$ is continuous, that is, if $h_n \rightarrow h$ in $L^p(\Omega)$ and Ω_n is a sequence of domains which approximates Ω then $\lambda(h_n, \Omega_n) \rightarrow \lambda(h, \Omega)$.

Proof. We only sketch this proof, since the result is fairly standard. The first statement trivially follows from the Sobolev embedding and the fact that for any $v \in X$,

$$(3.7) \quad \int_{\Omega} h(x)v^2 dx \leq \|h\|_{\frac{N}{2}} \|v\|_{2^*}^2 \leq C_N \|h\|_{\frac{N}{2}} \|\nabla v\|_2^2.$$

Here $2^* = \frac{2N}{N-2}$ and

$$(3.8) \quad C_N = \min\{\|\nabla v\|_2 : v \in H_0^1(\Omega), \|v\|_{2^*} = 1\} > 0$$

is the optimal constant in Sobolev's inequality. Note C_N depends only on N ; the exact value of C_N can be found in [24]. The maximum principle is obtained by multiplying $-\Delta u - hu \geq 0$ by u^- and integrating.

For the second statement, we note that the infimum is attained since $\int_{\Omega} h(x)v^2 dx \leq \|h\|_{\frac{N}{2}+\varepsilon} \|v\|_{2^*-\delta}^2$ for some $\delta, \varepsilon > 0$, and the embedding $X \rightarrow L^s(\Omega)$ is compact for $s \in [2, 2^*)$. The function ϕ_1 exists by the theory of Lagrange multipliers, and is unique as a consequence of the maximum principle. Finally, the continuity property follows from standard elliptic estimates applied to the functions $\phi_1^{(n)}$, ϕ_1 (normalized so that $\|\phi_1^{(n)}\|_{L^\infty} = \|\phi_1\|_{L^\infty} = 1$) corresponding respectively to $\lambda(h_n, \Omega_n)$, $\lambda(h, \Omega)$, and the uniqueness of these functions. \square

The next lemma shows that Lemma 3 can be applied, provided the function c_0 is sufficiently small.

Lemma 5. *Assume that (H1)-(H2) hold and $\mu > 0$ is a constant. There exists a constant $\bar{c} > 0$ depending on $N, \Omega, \mu\|f\|_p$ such that if*

$$(3.9) \quad \|c_0\|_p < \bar{c},$$

then any solution v of (3.1) satisfies $v > -\frac{1}{2\mu}$ in Ω .

Proof. Let $v \in X$ be a solution of (3.1). Setting $w = 1 + \mu v$ in (3.1) it is easy to see that

$$(3.10) \quad -\Delta w - \mu f(x)w = c_0(x)\tilde{g}(w) \quad \text{in } \Omega,$$

and $w - 1 \in X$, $w \in W^{2,p}(\Omega) \subset L^\infty(\Omega)$, where $\tilde{g}(s) = s \ln s$ if $s > 0$ and $\tilde{g}(s) = 0$ if $s \leq 0$. The minimal value of \tilde{g} on \mathbb{R} is $-1/e$, hence

$$(3.11) \quad -\Delta w - \mu f(x)w \geq -e^{-1}c_0(x) \quad \text{in } \Omega.$$

Let $\bar{\phi}_1 > 0$ denote the function given by the previous lemma for $h = \mu f$ and a slightly larger domain $\bar{\Omega} \supset \Omega$ taken so that the corresponding $\lambda_1(\mu f, \bar{\Omega})$ is still strictly positive (we extend all coefficients of the equation as zero outside Ω). We set $z = w/\bar{\phi}_1$, $w = z\bar{\phi}_1$ in (3.11) and use the equation satisfied by $\bar{\phi}_1$, to conclude that

$$(3.12) \quad -\Delta z - 2\frac{\nabla \bar{\phi}_1}{\bar{\phi}_1} \cdot \nabla z + \lambda_1(\mu f, \bar{\Omega})z \geq -(e\bar{\phi}_1)^{-1}c_0(x) \quad \text{in } \Omega.$$

We can now apply Theorem 8.16 of [17] (see also the remarks immediately preceding and following this theorem and its proof) and conclude that

$$(3.13) \quad \sup_{\Omega}(w^-) \leq C \sup_{\Omega}(z^-) \leq C\|c_0\|_p$$

for some constant C which depends on $N, \Omega, \|\bar{\phi}_1\|_{W^{1,p}(\Omega)}$, and on a positive lower bound for $\bar{\phi}_1$ on Ω . By the previous lemma and by the elliptic estimates the last two quantities depend only on N, Ω and $\mu\|f\|_p$. This finishes the proof of Lemma 5. \square

4. ON THE GEOMETRY OF THE FUNCTIONAL $I(v)$

We associate to equation (3.1) the functional $I : X \rightarrow \mathbb{R}$ defined by

$$I(v) = \frac{1}{2}E_{c_0+\mu f}(v) - \int_{\Omega} c_0(x)G(v)dx - \int_{\Omega} f(x)vdx.$$

Under our assumptions it is standard to show that $I \in C^1(X, \mathbb{R})$. By (H2) and Lemma 4, if

$$(4.1) \quad \|c_0\|_{\frac{N}{2}} \leq \varepsilon_0/2, \quad \text{with} \quad \varepsilon_0 := C_N^{-1} - \bar{\mu}\|f\|_{\frac{N}{2}} > 0.$$

then $E_{c_0+\mu f}(\cdot)$ is equivalent to the standard norm on X . We shall assume in the rest of the paper that (4.1) holds.

Recall $G(s) = \int_0^s g(t)dt$ and define $H(s) = \frac{1}{2}g(s)s - G(s)$. In the following lemma we gather some simple and useful properties of g, G and H .

Lemma 6.

- (i) g is continuous on \mathbb{R} , $g > 0$ on $\mathbb{R} \setminus \{0\}$, $G \geq 0$ on \mathbb{R}^+ and $G \leq 0$ on \mathbb{R}^- .
- (ii) for any $r \in (1, 2)$ there exists $C = C(r, \mu) > 0$ such that we have $|g(s)| \leq C|s|^r$ for any $s \in \mathbb{R}$.
- (iii) $g(s)/s \rightarrow 0$ as $s \rightarrow 0$.

- (iv) $g(s)/s \rightarrow +\infty$ and $G(s)/s^2 \rightarrow +\infty$ as $s \rightarrow +\infty$.
- (v) The function H satisfies $H(s) \leq (s/t)H(t)$, for $0 \leq s \leq t$.
- (vi) The function H is bounded on \mathbb{R}^- .

Proof. We have $g(0) = 0$ and, for $s > -1/\mu$, $g'(s) = \ln(1 + \mu s)$. Thus $g'(0) = 0$, $g(s) > 0$ if $s \neq 0$. Now direct calculations show that

$$g(s) \leq \ln(1 + \mu s) s \quad \text{if } s \geq 0,$$

and $g(s) \leq |s|$ if $s \leq 0$. Hence (i), (ii) and (iii) hold. By the definition of g , (iv) clearly holds. Also $H(0) = 0$ and we get, for $s \geq 0$,

$$H'(s) = \frac{1}{2}[g'(s)s - g(s)] = \frac{1}{2}\left[s - \frac{1}{\mu}\ln(1 + \mu s)\right].$$

Thus $H''(s) = 2\mu s(1 + \mu s)^{-1}$, so $H''(s) \geq 0$ for $s \geq 0$. From the convexity of H , we deduce that, if $0 < s \leq t$,

$$H(s) \leq \frac{s}{t}H(t) + \left(1 - \frac{s}{t}\right)H(0) = \frac{s}{t}H(t),$$

which proves (v). Finally, we trivially check that

$$H(s) = -G\left(-\frac{1}{\mu}\right) - \frac{1}{2\mu^2}$$

is constant for $s \leq -1/\mu$, which implies (vi). The lemma is proved. \square

The next lemma concerns the geometrical structure of I . We are going to denote with $B(0, \rho)$ the ball in X with radius ρ .

Lemma 7. *Assume that (H1)-(H2) hold. There exist constants $\alpha = \alpha(N, \Omega, \bar{\mu}, \varepsilon_0) > 0$, $\beta > 0$ and $\rho > 0$ such that if $0 < \|c_0\|_{\frac{N}{2}} \leq \alpha$ then*

- (i) $I(v) \geq \beta$ for $\|u\| = \rho$.
- (ii) $\inf_{v \in B(0, \rho)} I(v) \leq 0$, and $\inf_{v \in B(0, \rho)} I(v) < 0$ if $f \not\equiv 0$.
- (iii) There exists $v_0 \in X$ such that $\|v_0\| > \rho$ and $I(v_0) \leq 0$.

Proof. Let $r > 1$, close to 1, satisfy $(r+1)p' < \frac{2N}{N-2}$. We can choose such r since $p > \frac{N}{2}$. By Lemma 6 we have

$$(4.2) \quad |G(s)| \leq C|s|^{r+1}, \quad \text{for all } s \in \mathbb{R}.$$

Using (4.2), we get, for any $v \in X$,

$$(4.3) \quad \int_{\Omega} c_0(x)G(v)dx \leq C\|c_0\|_p\|v\|_{(r+1)p'}^{r+1} \leq C\|c_0\|_p\|v\|^{r+1},$$

where we used the Hölder and Sobolev inequalities. Also

$$\int_{\Omega} f(x)v(x)dx \leq \|f\|_{\frac{N}{2}}\|v\|_{\frac{N}{N-2}} \leq D\|f\|_{\frac{N}{2}}\|v\| \leq D(\bar{\mu}C_N)^{-1}\|v\|,$$

for some $D = D(N, \Omega) > 0$, by (H2). We then get, for any $v \in X$, because of (4.1),

$$(4.4) \quad I(v) \geq \frac{\varepsilon_0}{4} \|v\|^2 - D(\mu C_N)^{-1} \|v\| - C \|c_0\|_p \|v\|^{r+1}.$$

We fix first $\rho > 0$ sufficiently large so that if $\|v\| = \rho$

$$\frac{\varepsilon_0}{4} \|v\|^2 - D(\mu C_N)^{-1} \|v\| \geq \frac{\varepsilon_0}{8} \rho,$$

and then $\|c_0\|_p$ small enough to ensure that $I(v) \geq \frac{\varepsilon_0}{16} \rho$, for any $v \in X$ with $\|v\| = \rho$. This proves (i).

Next, note that $I(0) = 0$, so $\inf_{v \in B(0, \rho)} I(v) \leq 0$. If $f \not\equiv 0$, take a function $v \in C_0^\infty(\Omega)$, such that $\int_\Omega f(x)v dx > 0$ and consider the map $t \rightarrow I(tv)$ for $t > 0$. We have

$$(4.5) \quad \begin{aligned} I(tv) &= \frac{t^2}{2} \|v\|^2 - \int_\Omega c_0(x) G(tv) dx - t \int_\Omega f(x)v dx \\ &= t^2 \left[\frac{1}{2} \|v\|^2 - \int_\Omega c_0(x) \frac{G(tv)}{t^2 v^2} v^2 dx - \frac{1}{t} \int_\Omega f(x)v dx \right]. \end{aligned}$$

By Lemma 6 we have $G(s)/s^2 \rightarrow 0$ as $s \rightarrow 0$, thus

$$\int_\Omega c_0(x) \frac{G(tv)}{t^2 v^2} |v|^2 dx \rightarrow 0$$

as $t \rightarrow 0$, since $v \in C_0^\infty(\Omega)$. Then (4.5) implies $I(tv) < 0$ for $t > 0$ small enough. This proves (ii).

Finally, to prove (iii) we consider again the map $t \rightarrow I(tv)$, $t > 0$, and take $v \in C_0^\infty(\Omega)$ with $v \geq 0$, $c_0 v \not\equiv 0$. Then since by Lemma 6 $G(s)/s^2 \rightarrow +\infty$ as $s \rightarrow +\infty$, we now have

$$\int_\Omega c_0(x) \frac{G(tv)}{t^2 v^2} |v|^2 dx \rightarrow +\infty,$$

so $I(tv) \rightarrow -\infty$ as $t \rightarrow +\infty$. This of course implies (iii). \square

In view of Lemma 7 it can be expected that for $\|c_0\|_p$ sufficiently small I has two critical points, one of which is a local minimum, while the other is of saddle type.

Lemma 8. *Assume that (H1)-(H2) are satisfied, and that $\|c_0\|_p$ is sufficiently small to ensure (4.1) and Lemma 7 hold. Then the functional I possesses a critical point $v \in B(0, \rho)$, with $I(v) \leq 0$, which is a local minimum.*

Proof. By Lemma 7 (i) and (ii) there are $\rho, \beta > 0$ such that

$$m := \inf_{v \in B(0, \rho)} I(v) \leq 0 \quad \text{and} \quad I(v) \geq \beta > 0 \quad \text{if} \quad \|v\| = \rho.$$

Let $(v_n) \subset B(0, \rho) \subset X$ be a sequence such that $I(v_n) \rightarrow m$. Since $(v_n) \subset X$ is bounded we have, up to a subsequence, $v_n \rightharpoonup v$ weakly in X , for some $v \in X$. Now, by standard properties of the weak convergence and $f \in L^{N/2}(\Omega) \subset X^{-1}$,

$$\|v\|^2 \leq \liminf_{n \rightarrow \infty} \|v_n\|^2 \quad \text{and} \quad \int_{\Omega} f(x)v_n dx \rightarrow \int_{\Omega} f(x)v dx.$$

Also, since $v_n \rightarrow v$ in $L^q(\Omega)$ for $1 \leq q < \frac{2N}{N-2}$ and $c_0 \in L^p(\Omega)$ we readily obtain, using (4.2), that

$$\int_{\Omega} c_0(x)G(v_n) dx \rightarrow \int_{\Omega} c_0(x)G(v) dx.$$

We deduce that $v \in B(0, \rho)$ and

$$I(v) \leq \liminf_{n \rightarrow \infty} I(v_n) = m = \inf_{v \in B(0, \rho)} I(v).$$

Thus v is a local minimum of I and, by standard arguments, a critical point of I . \square

Now we define the mountain pass level

$$\hat{c} = \inf_{g \in \Gamma} \max_{t \in [0, 1]} I(g(t))$$

where

$$\Gamma = \{g \in C([0, 1], X) : g(0) = 0, g(1) = v_0\},$$

with $v_0 \in X$ given by Lemma 7 (iii). We shall prove that I possesses a critical point at the mountain pass level, that is, there exists $v \in X$ such that $I(v) = \hat{c}$ and $I'(v) = 0$. Since $\hat{c} > 0$ (by Lemma 7 (i)), this critical point will be different from the local minimum given by Lemma 8.

It is a standard fact that any C^1 -functional having a mountain pass geometry admits a Cerami sequence at the mountain pass level (see for example [14]). In other words, there exists a sequence $(v_n) \subset X$ such that

$$I(v_n) \rightarrow \hat{c} \quad \text{and} \quad (1 + \|v_n\|)I'(v_n) \rightarrow 0.$$

If we manage to show that $(v_n) \subset X$ admits a convergent subsequence, its limit is the desired critical point. A first essential step in the proof of this fact is showing that (v_n) is bounded.

5. BOUNDEDNESS OF THE CERAMI SEQUENCES

The following lemma is the key point in the proof of Theorem 1.

Lemma 9. *Assume that (H1)-(H2) are satisfied, and that $\|c_0\|_p$ is sufficiently small to ensure (4.1) and Lemma 7 hold. Then the Cerami sequences for I at any level $d \in \mathbb{R}^+$ are bounded.*

Proof. Let $(v_n) \subset X$ be a Cerami sequence for I at a level $d \in \mathbb{R}^+$. Assume for contradiction that $\|v_n\| \rightarrow \infty$ and set

$$w_n = \frac{v_n}{\|v_n\|}.$$

Since $(w_n) \subset X$ is bounded we have $w_n \rightharpoonup w$ in X and $w_n \rightarrow w$ in $L^q(\Omega)$, for $1 \leq q < \frac{2N}{N-2}$ (up to a subsequence). We write $w = w^+ - w^-$. We shall distinguish the two cases $c_0 w^+ \equiv 0$ and $c_0 w^+ \not\equiv 0$, and prove they are both impossible.

First we assume that $c_0 w^+ = 0$, and define the sequence $(z_n) \subset X$ by $z_n = t_n v_n$ with $t_n \in [0, 1]$ satisfying

$$(5.1) \quad I(z_n) = \max_{t \in [0, 1]} I(tv_n)$$

(if t_n defined by (5.1) is not unique we choose its smallest possible value). Let us show that

$$(5.2) \quad \lim_{n \rightarrow \infty} I(z_n) = +\infty.$$

Seeking a contradiction we assume that for some $M < \infty$

$$(5.3) \quad \liminf_{n \rightarrow \infty} I(z_n) \leq M,$$

and we define $(k_n) \subset X$ by

$$k_n = \frac{\sqrt{4M}}{\|v_n\|} v_n = \sqrt{4M} w_n.$$

Then $k_n \rightharpoonup k := \sqrt{4M} w$ in X and $k_n \rightarrow k$ in $L^q(\Omega)$ for any $1 \leq q < \frac{2N}{N-2}$. Thus, as in the proof of Lemma 8, we have

$$(5.4) \quad \int_{\Omega} c_0(x) G(k_n) dx \rightarrow \int_{\Omega} c_0(x) G(k) dx.$$

Now, recall that $G(s) \leq 0$ for $s \leq 0$, see Lemma 6. Since we have assumed $c_0(x) = 0$ if $k(x) > 0$, we obtain

$$(5.5) \quad \int_{\Omega} c_0(x) G(k) dx \leq 0.$$

Also, since $f \in L^{N/2}(\Omega) \subset X^{-1}$

$$(5.6) \quad \left| \int_{\Omega} f(x) k_n dx \right| \leq \sqrt{4M} \|f\|_{X^{-1}} \|w_n\| \leq \sqrt{4M} \|f\|_{X^{-1}}.$$

Combining (5.4), (5.5) and (5.6) it follows that

$$(5.7) \quad \begin{aligned} I(k_n) &= 2M - \int_{\Omega} c_0(x) G(k_n) dx - \int_{\Omega} f(x) k_n dx \\ &\geq 2M - \sqrt{4M} \|f\|_{X^{-1}} + o(1). \end{aligned}$$

Thus, taking $M > 0$ larger if necessary, we can assume that

$$(5.8) \quad I(k_n) \geq (3/2)M$$

for all sufficiently large $n \in \mathbb{N}$. Since k_n and z_n lay on the same ray in X for all $n \in \mathbb{N}$, we see by the definition of z_n that (5.8) contradicts (5.3) (note $\sqrt{4M}/\|v_n\| < 1$ since $\|v_n\| \rightarrow \infty$). Thus (5.2) holds.

We remark that $I(v_n) \rightarrow d$ and $I(z_n) \rightarrow \infty$ imply that $t_n \in (0, 1)$. Hence by the definition of z_n we have that $\langle I'(z_n), z_n \rangle = 0$, for all $n \in \mathbb{N}$. Thus, with H defined as in Lemma 6,

$$(5.9) \quad \begin{aligned} I(z_n) &= I(z_n) - \frac{1}{2} \langle I'(z_n), z_n \rangle \\ &= \int_{\Omega} c_0(x)H(z_n)dx - \frac{1}{2} \int_{\Omega} f(x)z_n dx. \end{aligned}$$

Combining (5.2) and (5.9) we see that

$$(5.10) \quad \frac{1}{2} \int_{\Omega} f(x)z_n dx = -M(n) + \int_{\Omega} c_0(x)H(z_n)dx$$

where $M(n)$ is a quantity such that $M(n) \rightarrow +\infty$. To show that $c_0 w^+ = 0$ does not occur we shall prove that (5.10) is impossible.

Observe that, for $n \in \mathbb{N}$ large enough,

$$(5.11) \quad \begin{aligned} d + 1 &\geq I(v_n) = I(v_n) - \frac{1}{2} \langle I'(v_n), v_n \rangle + o(1) \\ &= \int_{\Omega} c_0(x)H(v_n)dx - \frac{1}{2} \int_{\Omega} f(x)v_n dx + o(1) \end{aligned}$$

(note that $\langle I'(v_n), v_n \rangle \rightarrow 0$, since (v_n) is a Cerami sequence). Thus, for some $D > 0$,

$$\int_{\Omega} c_0(x)H(v_n)dx \leq D + \frac{1}{2} \int_{\Omega} f(x)v_n dx = D + \frac{1}{2t_n} \int_{\Omega} f(x)z_n dx$$

or equivalently, using (5.10)

$$(5.12) \quad \int_{\Omega} c_0(x)H(v_n)dx \leq D - \frac{M(n)}{t_n} + \frac{1}{t_n} \int_{\Omega} c_0(x)H(z_n)dx.$$

Now we decompose Ω into $\Omega = \Omega_n^+ \cup \Omega_n^-$ with

$$\Omega_n^+ = \{x \in \Omega : z_n(x) \geq 0\} \quad \text{and} \quad \Omega_n^- = \Omega \setminus \Omega_n^+.$$

On Ω_n^+ we have, by Lemma 6 (v) and $c_0 \geq 0$, that

$$\int_{\Omega_n^+} c_0(x)H(z_n)dx \leq t_n \int_{\Omega_n^+} c_0(x)H(v_n)dx.$$

On Ω_n^- we have, by Lemma 6 (vi) and $|\Omega| < \infty$, that for some $D > 0$

$$\int_{\Omega_n^-} c_0(x)H(z_n)dx \leq D.$$

Then it follows from (5.12) that

$$\int_{\Omega_n^-} c_0(x)H(v_n)dx \leq D - \frac{M(n)}{t_n} + \frac{D}{t_n}.$$

Letting $n \rightarrow \infty$ and using $t_n \in [0, 1]$ we see that

$$\int_{\Omega_n^-} c_0(x)H(v_n)dx \rightarrow -\infty$$

which is impossible since, by Lemma 6 (vi), H is bounded on \mathbb{R}^- and $|\Omega| < \infty$. At this point we have shown that $c_0w^+ = 0$ is impossible.

We now assume that $c_0w^+ \neq 0$ and we show that this property also leads to a contradiction. Since $(v_n) \subset X$ is a Cerami sequence we have $\langle I'(v_n), v_n \rangle \rightarrow 0$. Thus

$$\|v_n\|^2 - \int_{\Omega} c_0(x)g(v_n)v_n dx - \int_{\Omega} f(x)v_n dx \rightarrow 0.$$

Dividing by $\|v_n\|^2$ we get

$$\|w_n\|^2 - \int_{\Omega} c_0(x) \frac{g(v_n)}{\|v_n\|} w_n dx \rightarrow 0,$$

and since $\|w_n\| = 1$ we have

$$(5.13) \quad \int_{\Omega} c_0(x) \frac{g(v_n)}{\|v_n\|} w_n dx = \int_{\Omega} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \rightarrow 1.$$

Let

$$\Omega^+ = \{x \in \Omega : c_0(x)w(x) > 0\} \neq \emptyset.$$

We also define

$$\Omega_n^+ = \{x \in \Omega : v_n(x) \geq 0\} \quad \text{and} \quad \Omega_n^- = \Omega \setminus \Omega_n^+.$$

Now since $g(s)/s \rightarrow +\infty$ as $s \rightarrow +\infty$ and $w_n \rightarrow w > 0$ a.e. on Ω^+ it follows that

$$c_0 \frac{g(v_n)}{v_n} w_n^2 \rightarrow +\infty \quad \text{a.e. on } \Omega^+.$$

Thus, taking into account that $|\Omega^+| > 0$, we deduce that

$$(5.14) \quad \lim_{n \rightarrow +\infty} \int_{\Omega^+} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \rightarrow +\infty.$$

On the other hand we have

$$(5.15) \quad \begin{aligned} \int_{\Omega^+} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx &= \int_{\Omega} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \\ &\quad - \int_{(\Omega \setminus \Omega^+) \cap \Omega_n^+} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \\ &\quad - \int_{(\Omega \setminus \Omega^+) \cap \Omega_n^-} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx. \end{aligned}$$

But, for all $n \in \mathbb{N}$, since g is non negative,

$$(5.16) \quad \int_{(\Omega \setminus \Omega^+) \cap \Omega_n^+} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \geq 0.$$

Also, since $g(s)/s$ is bounded for $s \leq 0$ we have, for some $D > 0$,

$$(5.17) \quad \left| \int_{(\Omega \setminus \Omega^+) \cap \Omega_n^-} c_0(x) \frac{g(v_n)}{v_n} w_n^2 dx \right| \leq D \int_{\Omega} c_0(x) w_n^2 dx \\ \leq D \|c_0\|_{\frac{N}{2}} \|w_n\|^2 \leq D \|c_0\|_{\frac{N}{2}}.$$

Now combining (5.13)-(5.17) we get a contradiction. This shows that $c_0 w^+ \neq 0$ is impossible and ends the proof of the lemma. \square

Lemma 10. *Under the hypotheses of Lemma 9 any Cerami sequence for I at a level $d \in \mathbb{R}^+$ admits a convergent subsequence.*

Proof. Let $(v_n) \subset X$ be a Cerami sequence for I at a level $d \in \mathbb{R}^+$. Since by Lemma 9 this sequence is bounded, by passing to a subsequence we can assume that $v_n \rightharpoonup v$ in X and $v_n \rightarrow v \in L^q(\Omega)$, with $1 \leq q < \frac{2N}{N-2}$. The condition $I'(v_n) \rightarrow 0$ in X^{-1} is just

$$-\Delta v_n - [c_0(x) + \mu f(x)]v_n - c_0(x)g(v_n) - f(x) \rightarrow 0 \quad \text{in } X^{-1}.$$

Because $v_n \rightarrow v$ in $L^q(\Omega)$, for $1 \leq q < \frac{2N}{N-2}$ and $c_0 \in L^p(\Omega)$ for some $p > \frac{N}{2}$ we readily have that $c_0(x)g(v_n) \rightarrow c_0(x)g(v)$ in X^{-1} . Thus

$$(5.18) \quad -\Delta v_n - [c_0(x) + \mu f(x)]v_n \rightarrow c_0(x)g(v) + f(x) \quad \text{in } X^{-1}.$$

Now let $L : X \rightarrow X^{-1}$ be defined by

$$(Lu)v = \int_{\Omega} \nabla u \nabla v - [c_0(x) + \mu f(x)]uv dx.$$

The operator L is invertible by (4.1), so we can deduce from (5.18) that $v_n \rightarrow L^{-1}[c_0(x)g(v) + f(x)]$ in X . Consequently, by the uniqueness of the limit, $v_n \rightarrow v$ in X . \square

6. PROOF OF THEOREM 1

With the results from the previous section at hand, we are ready to prove the second statement in Theorem 1. We assume that $\|c_0\|_p$ is sufficiently small for the conclusions of Lemmas 4–10 to hold.

Proof of Theorem 1 (ii). Let first $\mu_0 > 0$. By Lemma 8 we have the existence of a first critical point as a local minima of I and by Lemmas 9 and 10 we obtain a second critical point at the mountain pass level $\hat{c} > 0$.

Next, if $\mu_0 < 0$ we replace u by $-u$, which is equivalent to replacing μ_0 by $-\mu_0$ and f by $-f$. Theorem 1 (ii) is proved. \square

When $\mu \in L^\infty(\Omega)$ is not constant we can no longer make the change of unknown $v = \frac{1}{\mu} \ln(e^{\mu u} - 1)$ and derive equation (3.1). However, we can make this change in the “extremal” equations, obtained by replacing $\mu(x)$ by $\bar{\mu}$ and by $-\bar{\mu}$, which will then give us upper and lower solutions to the problem, which we can prove to be ordered. Thus to obtain the existence of one solution we may appeal to a theorem stating the existence of a solution between ordered upper and lower solutions. Such results abound in the theory of elliptic PDE, see for instance [3], [12], and the references in these works. We are going to use Theorem 3.1 from [10], which is particularly adapted to our setting.

We rewrite equation (2.1) as

$$(6.1) \quad -\Delta u = F(x, u, \nabla u), \quad u \in X$$

where $F(x, u, \nabla u) = c_0(x)u + \mu(x)|\nabla u|^2 + f(x)$. We recall, see Definition 3.1 of [10], that a function $\alpha \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a lower solution of (6.1) if

$$-\Delta \alpha \leq F(x, \alpha(x), \nabla \alpha(x)) \quad \text{in } \Omega \quad \text{and} \quad \alpha \leq 0 \quad \text{on } \partial\Omega.$$

A function $\beta \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$ is an upper solution of (6.1) if

$$-\Delta \beta \geq F(x, \beta(x), \nabla \beta(x)) \quad \text{in } \Omega \quad \text{and} \quad \beta \geq 0 \quad \text{on } \partial\Omega.$$

The function F obviously satisfies the hypothesis (1.5) from [10], since

$$|F(x, s, \xi)| \leq (1 + \bar{\mu} + |u|)(|c_0(x)| + |f(x)| + |\xi|^2).$$

We have the following regularity result.

Lemma 11. *Assume that $\mu \in L^\infty(\Omega)$ and that c_0 and f belong to $L^p(\Omega)$ for a $p > \frac{N}{2}$. Then any solution of (3.1) belongs to $W^{2,q}(\Omega)$ for any $1 \leq q < p$. In particular, any solution of (3.1) and thus of (6.1) is bounded, by the Sobolev embeddings.*

Proof. Let $v \in X$ be a solution of (3.1). First we claim that v belongs to $L^q(\Omega)$, for each $q < \infty$. For this we write (3.1) as

$$-\Delta v = [c_0(x) + \mu f(x)]v + c_0(x)g(v) + f(x) := g(x, v)$$

and we use Lemma B.3 of [23]. According to this result in order to prove the claim it suffices to check that

$$|g(x, v)| \leq a(x)(1 + |v|)$$

with a function $a \in L^{\frac{N}{2}}(\Omega)$. But this amounts to showing that

$$c_0(x) + \mu(x)f(x) + c_0(x)\frac{g(v)}{v} \in L^{\frac{N}{2}}(\Omega).$$

By assumption c_0 and μf belong to $L^{\frac{N}{2}}(\Omega)$. This is also the case for $c_0 g(v)/v$ because of the slow growth of $g(s)/s$ as $|s| \rightarrow \infty$, see Lemma 6. Specifically, for any $r \in (0, 1)$ there exists a $D > 0$ such that

$$\left| \frac{g(s)}{s} \right| \leq D|s|^r, \quad \text{for any } s \in \mathbb{R}.$$

Thus, since $c_0 \in L^p(\Omega)$, for some $p > \frac{N}{2}$, by taking $r > 0$ sufficiently small (satisfying $r < \frac{4p-2N}{p(N-2)}$) we see by using the Hölder inequality that $c_0 g(v)/v \in L^{\frac{N}{2}}(\Omega)$. Finally, again by Hölder inequalities $g(x, v) \in L^q(\Omega)$, so the statement of the Lemma follows from standard regularity results. \square

Proof of Theorem 1 (i). By Theorem 1 (ii), which we already proved, there exists a bounded solution $\bar{u} \in X$ of

$$(6.2) \quad -\Delta u - c_0(x)u = \bar{\mu}|\nabla u|^2 + f^+(x), \quad u \in X$$

which is then clearly an upper solution of (2.1). In addition, \bar{u} is positive in Ω , by the maximum principle (Lemma 4). Similarly, there is a negative bounded solution \underline{u} (replace u by $-u$) of

$$(6.3) \quad -\Delta u - c_0(x)u = -\bar{\mu}|\nabla u|^2 - f^-(x), \quad u \in X$$

which is a lower solution of (2.1). So we are in a position to apply Theorem 3.1 from [10], which yields a solution $u \in X$ of (2.1), with $\underline{u} \leq u \leq \bar{u}$. This ends the proof of Theorem 1. \square

7. FINAL REMARKS

Since this is the first paper on non-coercive problems with natural growth in the divergence-form setting, we have chosen to concentrate on a very particular case of (1.1), which is sufficient to highlight the differences with the coercive case, and to exhibit the new phenomena which may arise. However the hypotheses we made on (1.1) in order to prove Theorem 1 can be generalized in various ways. For instance, if $f \geq 0$ we see that, by Lemma 4, any solution v of (3.1) satisfies $v \geq 0$ on Ω , provided $\|c_0 + \mu f\|_{N/2} < C_N^{-1}$. Then we do not need any more Lemma 5 and, inspecting the proofs of the remaining lemmas, one can see that requiring that c_0 belongs to $L^p(\Omega)$ for a $p > \frac{N}{2}$, and that $f \in L^{\frac{N}{2}}(\Omega)$ suffices to get the conclusion of Theorem 1. In this case our solutions of (3.1) and thus of (2.1) are not necessarily bounded. One may in general ask whether it is possible to consider coefficients c_0, f , which are less regular, thus obtaining solutions which are less regular, like for instance in [18].

Next, the conclusion of Theorem 1 holds, with straightforward adaptations, if instead of (2.1), we consider equations of the form

$$(7.1) \quad -\operatorname{div}(A(x)\nabla u) = H(x, u, \nabla u), \quad u \in X,$$

with, say, $A \in L^\infty(\Omega)^{N \times N}$, $A \geq \lambda I$ for some $\lambda > 0$, and, for a $\beta > 0$

$$(7.2) \quad |H(x, s, \xi)| \leq \beta|\xi|^2 + c_0(x)|s| + f(x).$$

Here c_0 and f belong to $L^p(\Omega)$ for some $p > N/2$. To see this let us first consider the special case

$$(7.3) \quad H(x, s, \xi) = \mu \langle A(x)\xi, \xi \rangle + c_0(x)s + f(x).$$

where $\mu > 0$ is a constant. For (7.1)-(7.3) the change of unknown $v = \frac{1}{\mu} \ln(e^{\mu u} - 1)$ is still possible and the equation corresponding to (3.1) is

$$-\operatorname{div}(A(x)\nabla v) - [c_0(x) + \mu f(x)]v = c_0(x)g(v) + f(x).$$

This equation is variational and can be treated just like (3.1). In particular, assuming that $\|c_0 + \mu f\|_p$ is small enough we obtain the existence of two solutions. For general H satisfying (7.2) the existence of at least one solution is proved using Theorem 3.1 of [10] and the fact that if $\bar{u} \in X$ is a solution of (7.1)-(7.3) with f replaced by $|f|$ and $\mu = \beta/\lambda$, then $\bar{u} \geq 0$ is an upper solution and $-\bar{u} \leq 0$ a lower solution of (7.1)-(7.2).

Let us summarize our remarks on the importance of the change of variables $u = \frac{1}{\mu} \ln(1 + \mu v)$ which we made. If the operators $\operatorname{div}(a)$ and B in (1.1) can be appropriately bounded above and below by quantities such that this change can be made in the corresponding ‘‘extremal’’ equations, leading to new equations for which our critical point method can be applied, then we obtain an upper and a lower solution for the initial problem, and hence a solution to this problem. However, for the multiplicity result we do need to be able to make this change of variables in the initial problem. For instance, we do not know whether (2.1) with non-constant $\mu(x)$ such that $0 < \mu_1 \leq \mu(x) \leq \mu_2$ has two bounded solutions.

On the other hand, many previous papers on coercive problems with natural growth allow considerably more general equations (1.1) into which this change of unknown function cannot directly be made. The point is that it is possible to mimic the change of variables, by testing the weak formulation of (1.1) with suitably chosen functions, which somehow take account of the change of unknown (see for instance Remark 2.10 in [16] for more details). It is certainly a very interesting open problem whether a multiplicity result can be proved for more general non-coercive problems with natural growth.

Acknowledgements: The first author thanks C. De Coster for fruitful discussions.

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