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Conjugacy p-separability of right-angled Artin groups and applications

Emmanuel Toinet

ABSTRACT. We prove that every subgroup of p-power index in a right-angled Artin group is conjugacy p-separable. In particular, every right-angled Artin group is conjugacy p-separable. A consequence of this result is that the outer automorphism group of a right-angled Artin group is virtually residually p-finite. Another consequence of this result is that the outer automorphism group of a right-angled Artin group is $\mathcal{K}$-residual, where $\mathcal{K}$ is the class of all outer automorphism groups of finite p-groups. We also prove that the Torelli group of a right-angled Artin group is residually torsion-free nilpotent, hence residually p-finite and bi-orderable.

1 Introduction

Let $\Gamma = (V,E)$ be a (finite) simplicial graph. The *right-angled Artin group* associated to Γ is the group G_Γ defined by the presentation:

$$G_\Gamma = < V \mid vw = wv, \forall \{v,w\} \in E >.$$

Note that, if Γ is discrete, then G_Γ is free, and if Γ is complete, then G_Γ is free abelian. The *rank* of G_Γ is by definition the number of vertices of Γ. A *special subgroup* of G_Γ is a subgroup generated by a subset W of V - it is naturally isomorphic to the right-angled Artin group $G_{\Gamma(W)}$, where $\Gamma(W)$ denotes the full subgraph of Γ generated by W. Let v be a vertex of Γ. The *star* of v, denoted by $star(v)$, is the subset of V of all vertices which are adjacent to v plus v itself. We refer to [C] for a general survey of right-angled Artin groups.

Not much is known on automorphims of right-angled Artin groups. In 1989, Servatius conjectured a generating set for $Aut(G_\Gamma)$ (see [Ser]). He proved his conjecture in certain special cases, for example when the graph is a tree. Then Laurence proved the conjecture in the general case (see [L]). Charney and Vogtmann showed that $Out(G_\Gamma)$ is virtually torsion-free and has finite virtual cohomological dimension (see [CV]). Day found a finite presentation for $Aut(G_\Gamma)$ (see [D]). Recently, Minasyan proved that $Out(G_\Gamma)$ is residually finite (see [M]). This result was obtained independently by Charney and Vogtmann (see [CV2]), using different technics. More recently, Day

showed that $Out(G_\Gamma)$ is either virtually nilpotent or contains a free group (see [D3]).

Let p be a prime number. A group G is said to be *residually p-finite* if for every $g \in G\backslash\{1\}$, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \neq 1$. Note that residually p-finite implies residually finite as well as residually nilpotent. For g, $h \in G$, the notation $g \sim h$ means that g and h are conjugate. We say that G is *conjugacy p-separable* if for all g, $h \in G$, either $g \sim h$, or there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$. Obviously, if a group is conjugacy p-separable, then it is residually p-finite. Following Ivanova (see [I]), we say that a subset S of a group G is *finitely p-separable* if for every $g \in G \setminus S$, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \notin \varphi(S)$. We say that an element g of G is *C_{fp}-separable* if its conjugacy class, $g^G = \{\alpha g \alpha^{-1} \mid \alpha \in G\}$, is finitely p-separable. Note that G is conjugacy p-separable if and only if every element of G is C_{fp}-separable. Examples of groups which are known to be conjugacy p-separable include free groups (see [LS]) and fundamental groups of oriented closed surfaces (see [P]).

Let G be a group. The *pro-p topology* on G is defined by taking the normal subgroups of p-power index in G as a basis of neighbourhoods of 1 (see [RZ]). Equipped with the pro-p topology, G becomes a topological group. Observe that G is Hausdorff if and only if it is residually p-finite. One can easily prove that a subset S of G is closed in the pro-p topology on G if and only if it is finitely p-separable.

Definition 1.1 *Let G be a group. We say that G is hereditarily conjugacy p-separable if every subgroup of p-power index in G is conjugacy p-separable.*

Hereditary conjugacy p-separability is obviously stronger than conjugacy p-separability.

Our main theorem is the following:

Theorem 1.2 *Every right-angled Artin group is hereditarily conjugacy p-separable.*

More generally, let $\mathcal{K}$ be a class of group (e.g., the class of all finite groups, the class of all finite p-groups, etc.). We say that a group G is *$\mathcal{K}$-residual* if for all $g \in G\backslash\{1\}$, there exists a homomorphism φ from G to some group of $\mathcal{K}$ such that $\varphi(g) \neq 1$. Note that if $\mathcal{K}$ is the class of all finite p-groups, one study residual p-finiteness. We say that a group G is *conjugacy $\mathcal{K}$-separable* if for all g, $h \in G$, either $g \sim h$, or there exists a homomorphism φ from G to some group of $\mathcal{K}$ such that $\varphi(g) \nsim \varphi(h)$. Note that if $\mathcal{K}$ is the class of all finite p-groups, one study conjugacy p-separability. In Section 7, we prove the following:

Theorem 1.3 *Every right-angled Artin group is conjugacy $\mathcal{K}$-separable, where $\mathcal{K}$ is the class of all torsion-free nilpotent groups.*

Let $\mathcal{P}$ be a group property (e.g., abelian, nilpotent, etc.). A group G is said to be *virtually* $\mathcal{P}$ if there exists a finite index subgroup $H < G$ such that H has Property $\mathcal{P}$. In Section 7, we prove:

Theorem 1.4 *The outer automorphism group of a right-angled Artin group is virtually residually p-finite.*

Theorem 1.5 *The outer automorphism group of a right-angled Artin group is $\mathcal{K}$-residual, where $\mathcal{K}$ is the class of all outer automorphism groups of finite p-groups.*

Let $G = G_\Gamma$ be a right-angled Artin group. Let r be the rank of G. We denote by $\mathcal{T}(G)$ the kernel of the natural homomorphism $Out(G) \to GL_r(\mathbb{Z})$. We say that $\mathcal{T}(G)$ is the *Torelli group* of G. In Section 7 we prove the following. After submitting the first version of this paper, the author learned that the following was obtained independently by Wade (see [W]), using different arguments.

Theorem 1.6 *The Torelli group of a right-angled Artin group is residually torsion-free nilpotent.*

This implies that the Torelli group of a right-angled Artin group is residually p-finite for every prime number p and bi-orderable.

Our proof of Theorem 1.2 is purely combinatorial. It is based on HNN extensions (see below for the definition of an HNN extension). The basic idea is that a right-angled Artin group of rank r ($r \geq 1$) can be written as an HNN extension of any of its special subgroups of rank r - 1. Our proof was inspired by Minasyan's paper (see [M]). In his paper, Minasyan proved that every finite index subgroup in a right-angled Artin group is conjugacy separable - that is, right-angled Artin groups are "hereditarily conjugacy separable". To this end, he introduced the centralizer condition and proved that a group is hereditarily conjugacy separable if and only if it is conjugacy separable and satisfies the centralizer condition. In Section 3, we introduce the p centralizer condition which is analog of the centralizer condition in [M], and we prove that a group is hereditarily conjugacy p-separable if and only if it is conjugacy p-separable and satisfies the p centralizer condition. In Section 4, we prove the following:

Theorem 1.7 *Every extension of a free group by a finite p-group is conjugacy p-separable.*

From a technical point of view, Theorem 1.7 is the main result of our paper. In Section 5, we deal with retractions that are key tools in the proof of our main theorem, which is the object of Section 6.

My gratefulness goes to my Ph.D. thesis advisor, Luis Paris, for his trust, time and advice.

2 HNN extensions

We start recalling the definition of an HNN extension (see [LS]). This notion of HNN extension will be of great importance in our proof, because, as we will see, a right-angled Artin group of rank r ($r \geq 1$) can be written as an HNN extension of any of its special subgroups of rank r - 1.

Let H be a group. Then by the notation:

$$< H,\, h,... \mid r,... >,$$

we mean the group defined by the presentation whose generators are the generators of H together with h,... and the relators of H together with r,...

Let H be a group. Let I be a set of indices. Let $\{K_i\}_{i\in I}$ and $\{L_i\}_{i\in I}$ be families of subgroups of H and let $\{\psi_i : K_i \to L_i\}_{i\in I}$ be a family of isomorphisms. The *HNN extension with base H, stable letters t_i ($i \in I$), and associated subgroups K_i and L_i ($i \in I$)*, is the group defined by the presentation:

$$G = < H,\, t_i\ (i \in I) \mid t_i^{-1}k_it_i = \psi_i(k_i),\, \forall\ k_i \in K_i\ (i \in I) >.$$

In particular, let H be a group, let K and L be subgroups of H and let ψ be an isomorphism from K to L. The *HNN extension of H relative to ψ* is the group defined by the presentation:

$$G = < H,\, t \mid t^{-1}kt = \psi(k),\, \forall\ k \in K >.$$

From now on, we suppose $K = L$ and $\psi = id_K$. In this case, G is called the *HNN extension of H relative to K*:

$$G = < H,\, t \mid t^{-1}kt = k,\, \forall\ k \in K >.$$

Every element of G can be written as a word $x_0t^{a_1}x_1...t^{a_n}x_n$ ($n \geq 0$, x_0,..., $x_n \in H$, a_1,..., $a_n \in \mathbb{Z} \setminus \{0\}$). Following Minasyan (see [M]), we will say that the word $x_0t^{a_1}x_1...t^{a_n}x_n$ is *reduced* if $x_0 \in H$, x_1,..., $x_{n-1} \in H \setminus K$, and $x_n \in H$. Every element of G can be represented by such a reduced word. Note that our definition of a reduced word is stronger than the definition of a reduced word in [LS].

Lemma 2.1 (Britton's Lemma) *If a word $x_0t^{a_1}x_1...t^{a_n}x_n$ is reduced with $n \geq 1$, then $x_0t^{a_1}x_1...t^{a_n}x_n \neq 1$.*

Proof: Proved in [M]. □

Lemma 2.2 *If $x_0t^{a_1}x_1...t^{a_n}x_n$ and $y_0t^{b_1}y_1...t^{b_m}y_m$ are reduced words such that $x_0t^{a_1}x_1...t^{a_n}x_n = y_0t^{b_1}y_1...t^{b_m}y_m$, then $m = n$ and $a_i = b_i$ for all $i \in$ {1,...,n}.*

Proof: Proved in [M]. □

A *cyclic permutation* of the word $t^{a_1}x_1...t^{a_n}x_n$ is a word $t^{a_k}x_k...t^{a_n}x_nt^{a_1}x_1$ $...t^{a_{k-1}}x_{k-1}$ with $k \in \{1,...,n\}$. A word $t^{a_1}x_1...t^{a_n}x_n$ is said to be *cyclically reduced* if any cyclic permutation of $t^{a_1}x_1...t^{a_n}x_n$ is reduced. Note that, if $t^{a_1}x_1...t^{a_n}x_n$ is reduced and $n \geq 2$, then $t^{a_1}x_1...t^{a_n}x_n$ is cyclically reduced if and only if $x_n \in H \setminus K$. Every element of G is conjugate to a cyclically reduced word.

Lemma 2.3 (Collins' Lemma) *If $g = t^{a_1}x_1...t^{a_n}x_n$ ($n \geq 1$) and $h = t^{b_1}y_1...t^{b_m}y_m$ ($m \geq 1$) are cyclically reduced and conjugate, then there exists a cyclic permutation h^* of h and an element $\alpha \in K$ such that $g = \alpha h^* \alpha^{-1}$.*

Proof: Proved in [M]. □

Remark: There exists a natural homomorphism $f : G \to H$, defined by $f(h) = h$ for all $h \in H$, and $f(t) = 1$.

Remark: Let P be a group and let $\varphi : H \to P$ be a homomorphism. Let Q be the HNN extension of P relative to $\varphi(K)$:

$$Q = < P, \overline{t} \mid \overline{t}^{-1}\varphi(k)\overline{t} = \varphi(k), \forall\ k \in K >.$$

Then φ induces a homomorphism $\overline{\varphi} : G \to Q$, defined by $\overline{\varphi}(h) = \varphi(h)$ for all $h \in H$, and $\overline{\varphi}(t) = \overline{t}$.

Lemma 2.4 *With the notations of the previous remark, $ker(\overline{\varphi})$ is the normal closure of $ker(\varphi)$ in G.*

Proof: Proved in [M]. □

The following simple observation is the key in the proof of our main theorem. Let G be a right-angled Artin group of rank r ($r \geq 1$). Let H be a special subgroup of G of rank r - 1. In other words, there exists a partition of V: $V = W \cup \{t\}$ such that $H = < W >$. Then G can be written as the HNN extension of H relative to the special subgroup $K = C(t) = < star(t) >$ of H:

$$G = < H, t \mid t^{-1}kt = k, \forall\ k \in K >.$$

3 Hereditary conjugacy p-separability and p centralizer condition

We start with an observation that the reader has to keep in mind, because it will be used repeatedly in the rest of the paper: if H and K are two normal subgroups of p-power index in a group G, then $H \cap K$ is a normal subgroup of p-power index in G.

If G is a group, H is a subgroup of G, and $g \in G$, then we set $C_H(g) = \{h \in H \mid gh = hg\}$.

Definition 3.1 *We say that G satisfies the p centralizer condition (pCC) if, for every normal subgroup H of p-power index in G, and for all $g \in G$, there exists a normal subgroup K of p-power index in G such that $K < H$, and:*

$$C_{G/K}(\varphi(g)) \subset \varphi(C_G(g)H),$$

where $\varphi : G \to G/K$ denotes the canonical projection.

We shall show that a group G is hereditarily conjugacy p-separable if and only if it is conjugacy p-separable and satisfies the p centralizer condition (see Proposition 3.6). For technical reasons, we have to introduce the following definitions:

Definition 3.2 *Let G be a group, H be a subgroup of G, and $g \in G$. We say that the pair (H, g) satisfies the p centralizer condition in G (pCC_G) if, for every normal subgroup K of p-power index in G, there exists a normal subgroup L of p-power index in G such that $L < K$, and:*

$$C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K),$$

where $\varphi : G \to G/L$ denotes the canonical projection. We say that H satisfies the p centralizer condition in G (pCC_G) if the pair (H, g) satisfies the p centralizer condition in G for all $g \in G$.

If G is a group, H is a subgroup of G, and $g \in G$, then we set: $g^H = \{\alpha g \alpha^{-1} \mid \alpha \in H\}$. In order to prove Proposition 3.6, we need the following statements, which are analogs of some statements obtained in [M] (Lemma 3.4, Corollary 3.5 and Lemma 3.7, respectively):

Lemma 3.3 *Let G be a group, H be a subgroup of G, and $g \in G$. Suppose that the pair (G, g) satisfies pCC_G, and that g^G is finitely p-separable in G. If $C_G(g)H$ is finitely p-separable in G, then g^H is also finitely p-separable in G.*

Proof: Let $h \in G$ such that $h \notin g^H$. If $h \notin g^G$, then, since g^G is finitely p-separable in G, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(h) \notin \varphi(g^G)$. In particular, $\varphi(h) \notin \varphi(g^H)$. Thus we can assume that $h \in g^G$. Let $\alpha \in G$ such that $h = \alpha g \alpha^{-1}$. Suppose that $C_G(g) \cap \alpha^{-1}H \neq \emptyset$. Let $k \in C_G(g) \cap \alpha^{-1}H$. We obtain: $\alpha k \in H$ and $h = \alpha g \alpha^{-1} = \alpha k g (\alpha k)^{-1} \in g^H$ - a contradiction. Thus $C_G(g) \cap \alpha^{-1}H = \emptyset$, i.e. $\alpha^{-1} \notin C_G(g)H$. As $C_G(g)H$ is finitely p-separable in G, there exists a normal subgroup K of p-power index in G such that $\alpha^{-1} \notin C_G(g)HK$. Now the condition pCC_G implies that there exists a normal subgroup L of p-power index in G such that $L < K$, and:

$$C_{G/L}(\varphi(g)) \subset \varphi(C_G(g)K),$$

where $\varphi : G \to G/L$ denotes the canonical projection. We have: $\varphi(h) \notin \varphi(g^H)$. Indeed, if there exists $\beta \in H$ such that $\varphi(h) = \varphi(\beta g \beta^{-1})$, then $\varphi(g) = \varphi(\beta^{-1} h \beta)$, and $\varphi(\alpha^{-1}\beta)\varphi(g) = \varphi(\alpha^{-1}\beta)\varphi(\beta^{-1}h\beta) = \varphi(\alpha^{-1}h\alpha)\varphi(\alpha^{-1}\beta) = \varphi(g)\varphi(\alpha^{-1}\beta)$ i.e. $\varphi(\alpha^{-1}\beta) \in C_{G/L}(\varphi(g))$. But then $\varphi(\alpha^{-1}) \in C_{G/L}(\varphi(g))\varphi(H) \subset \varphi(C_G(g)KH)$. We obtain: $\alpha^{-1} \in C_G(g)HKL = C_G(g)HK$ (because $L < K$) - a contradiction. □

Corollary 3.4 *Let G be a conjugacy p-separable group satisfying pCC, and H be a subgroup of G such that $C_G(h)H$ is finitely p-separable in G for all $h \in H$. Then H is conjugacy p-separable. Moreover, for all $h \in H$, h^H is finitely p-separable in G.*

Proof: Let $h \in H$. Since G satisfies pCC, the pair (G, h) satisfies pCC_G. Since G is conjugacy p-separable, h^G is finitely p-separable in G. Lemma 3.3 now implies that h^H is finitely p-separable in G. Therefore h^H is finitely p-separable in H. □

Lemma 3.5 *Let G be a group, H be a subgroup of G, and $g \in G$. Let K be a normal subgroup of p-power index in G. If $g^{H \cap K}$ is finitely p-separable in G, then there exists a normal subgroup L of p-power index in G such that $L < K$, and:*

$$C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K),$$

where $\varphi : G \to G/L$ denotes the canonical projection.

Proof: Note that $H \cap K$ is of finite index n in H. Actually, $H \cap K$ is of p-power index in H (because $\frac{H}{H \cap K} \simeq \frac{KH}{K} < \frac{G}{K}$). However, we do not need this fact. There exists $\alpha_1, \ldots, \alpha_n \in H$ such that $H = \sqcup_{i=1}^n \alpha_i (H \cap K)$. Up to renumbering, we can assume that there exists $l \in \{0, \ldots, n\}$ such that $\alpha_i^{-1} g \alpha_i \in g^{H \cap K}$ for all $i \in \{1, \ldots, l\}$ and $\alpha_i^{-1} g \alpha_i \notin g^{H \cap K}$ for all $i \in \{l+1, \ldots, n\}$. By the assumptions, there exists a normal subgroup L of p-power index in G such that $\alpha_i^{-1} g \alpha_i \notin g^{H \cap K} L$ for all $i \in \{l+1, \ldots, n\}$. Up to replacing L by $L \cap K$,

we can assume that $L < K$. Let $\varphi : G \to G/L$ be the canonical projection. Let $\overline{h} \in C_{\varphi(H)}(\varphi(g))$. There exists $h \in H$ such that $\overline{h} = \varphi(h)$. There exists $i \in \{1,...,n\}$ and $k \in H \cap K$ such that $h = \alpha_i k$. We have : $\varphi(h^{-1}gh) = \varphi(h)^{-1}\varphi(g)\varphi(h) = \varphi(g)$. Thus $h^{-1}gh \in gL$. But then $\alpha_i^{-1}g\alpha_i = kh^{-1}ghk^{-1} \in kgLk^{-1} = kgk^{-1}L \subset g^{H\cap K}L$. Therefore $i \leq l$. Then there exists $\beta \in H \cap K$ such that: $\alpha_i^{-1}g\alpha_i = \beta g \beta^{-1}$. This is to say that $\alpha_i\beta \in C_H(g)$, and then $h = \alpha_i k = (\alpha_i\beta)(\beta^{-1}k) \in C_H(g)(H \cap K) \subset C_H(g)K$. We have shown that $C_{\varphi(H)}(\varphi(g)) \subset \varphi(C_H(g)K)$. □

We are now ready to prove:

Proposition 3.6 *A group is hereditarily conjugacy p-separable if and only if it is conjugacy p-separable and satisfies pCC.*

Proof: Suppose that G is conjugacy p-separable and satisfies pCC. Let H be a subgroup of p-power index in G. Thus H is closed in the pro-p topology on G (because $G \setminus H = \cup\{gH \mid g \notin H\}$). Let $h \in H$. The set $C_G(h)H$ is a finite union of left cosets modulo H and thus is closed in the pro-p topology on G. Corollary 3.4 now implies that H is conjugacy p-separable. Therefore G is hereditarily conjugacy p-separable. Suppose now that G is hereditarily conjugacy p-separable. In particular, G is conjugacy p-separable. We shall show that G satisfies pCC. Let $g \in G$. Let K be a normal subgroup of p-power index in G. We set $H = K < g >$. We have $K < H$. Thus H is a subgroup of p-power index in G. Therefore it is conjugacy p-separable. And $g^{H\cap K} = g^H \subset H$. As g^H is closed in the pro-p topology on H, it is closed in the pro-p topology on G. The result now follows from lemma 3.5. □

4 Extensions of free groups by finite p-groups are conjugacy p-separable

We start with an observation that the reader has to keep in mind because it will be used repeatedly in the proof of Theorem 4.2: if $\varphi : G \to H$ is a homomorphism from a group G to a group H, whose kernel is torsion-free, then the restriction of φ to any finite subgroup of G is injective.

We need the following lemma:

Lemma 4.1 *Let $G = G_1 * ... * G_n$ be a free product of n conjugacy p-separable groups $G_1,..., G_n$. Let g, $h \in G \setminus \{1\}$ be two non-trivial elements of finite order in G such that $g \nsim h$. There exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$.*

Proof: Since g is of finite order in G, there exists $i \in \{1,...,n\}$ such that g is conjugate to an element of finite order in G_i. Thus we may assume that g belongs to G_i. Similarly, we may assume that there exists j in $\{1,...,n\}$ such that h belongs to G_j. Suppose that $i \neq j$. Let $\varphi : G_i \to P$ be a homomorphism from G_i onto a finite p-group P such that $\varphi(g) \neq 1$. Let $\widetilde{\varphi} : G \to P$ be the natural homomorphism extending φ. Then $\widetilde{\varphi}(g) \nsim \widetilde{\varphi}(h)$. Suppose that $i = j$. Then g and h are not conjugate in G_i - otherwise they would be conjugate in G. Since G_i is conjugacy p-separable, there exists a homomorphism $\varphi : G_i \to P$ from G_i onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$. Let $\widetilde{\varphi} : G \to P$ be defined as above. We have $\widetilde{\varphi}(g) \nsim \widetilde{\varphi}(h)$. $\square$

Recall that a *subnormal subgroup* of a group G is a subgroup H of G such that there exists a finite sequence of subgroups of G:

$$H = H_0 < H_1 < ... < H_n = G,$$

such that H_i is normal in H_{i+1} for all $i \in \{0,...,n\text{-}1\}$.

In Section 4, by a graph, we mean a unoriented graph, possibly with loops or multiple edges.

Recall that a *graph of groups* is a connected graph $\Gamma = (V, E)$, together with a function $\mathcal{G}$ which assigns:

- to each vertex $v \in V$, a group G_v,
- and to each edge $e = \{v,w\} \in E$, a group G_e together with two injective homomorphisms $\alpha_e : G_e \to G_v$ and $\beta_e : G_e \to G_w$ - we are not assuming that $v \neq w$ -,

(see [Se], see also [Dy]). The groups G_v ($v \in V$) are called the *vertex groups* of Γ, the groups G_e ($e \in E$) are called the *edge groups* of Γ. The monomorphisms α_e and β_e ($e \in E$) are called the *edge monomorphisms*. The images of the edge groups under the edge monomorphisms are called the *edge subgroups*.

Choose disjoint presentations $G_v = < X_v \mid R_v >$ for the vertex groups of Γ. Choose a maximal tree T in Γ. Assign a direction to each edge of Γ. Let $\{t_e \mid e \in E\}$ be a set in one-to-one correspondence with the set of edges of Γ, and disjoint from the X_v, $v \in V$. The *fundamental group* of the above graph of groups Γ is the group G_Γ defined by the presentation whose generators are:

$$X_v \ (v \in V),$$
$$t_e \ (e \in E)$$

(called vertex and edge generators respectively) and whose relations are:

$$R_v \ (v \in V),$$
$$t_e = 1 \ (e \in T),$$
$$t_e \alpha_e(g_e) t_e^{-1} = \beta_e(g_e), \ \forall \ g_e \in G_e \ (e \in E).$$

(called vertex, tree, and edge relations respectively). One can prove that this is well-defined - that is, independent of our choice of T, etc. Note that it suffices to write the edge relations for g_e in a set of generators for G_e.

<u>Convention</u>: The groups G_v $(v \in V)$ and G_e $(e \in E)$ will be regarded as subgroups of G_Γ.

Let $\{\Gamma_i\}_{i \in I}$ be a collection of connected and pairwise disjoint subgraphs of Γ. We may define a graph of groups Γ^* from Γ *by contracting* Γ_i *to a point for all* $i \in I$, as follows. The graph Γ^* is obtained from Γ by contracting Γ_i to a point p_i for all $i \in I$. The function $\mathcal{G}^*$ is obtained from $\mathcal{G}$ by using the fundamental group of Γ_i for the vertex group at p_i, and by composing the edge monomorphisms of Γ by the natural inclusions of the vertex groups of Γ_i into the fundamental group of Γ_i, if necessary. Clearly, G_Γ is isomorphic to the fundamental group G_{Γ^*} of Γ^*.

If $\pi : G_\Gamma \to H$ is a homomorphism from G_Γ to a group H, such that the restriction of π to each edge subgroup of Γ is injective, then we may define a graph of groups Γ' from Γ *by composing with* π, as follows. The vertex set of Γ' is V, and the edge set of Γ' is E. The vertex groups of Γ' are the groups $G'_v = \pi(G_v)$ $(v \in V)$, and the edge groups of Γ' are the groups $G'_e = G_e$ $(e \in E)$. The edge monomorphisms are the monomorphisms $\alpha'_e = \pi \circ \alpha_e$ and $\beta'_e = \pi \circ \beta_e$ $(e \in E)$. Present G_Γ and $G_{\Gamma'}$ using the same symbols for edge generators and with the same choice of maximal tree. There exist two homomorphisms, $\pi_V : G_\Gamma \to G_{\Gamma'}$ and $\pi_E : G_{\Gamma'} \to H$ such that the diagram:

$$\begin{tikzcd} G_\Gamma \arrow[r, "\pi"] \arrow[d, "\pi_V"'] & H \\ G_{\Gamma'} \arrow[ur, "\pi_E"'] & \end{tikzcd}$$

commutes, and that the restriction of π_E to each vertex group of $G_{\Gamma'}$ is injective. The homomorphism π_V is given by:

$$(\pi_V)_{|G_v} = \pi_{|G_v}, \ \forall \ v \in V,$$
$$\pi_V(t_e) = t_e, \ \forall \ e \in E.$$

And the homomorphism π_E is given by:

$$(\pi_E)_{|G'_v} = (id_H)_{|G'_v}, \ \forall \ v \in V,$$
$$\pi_E(t_e) = \pi(t_e), \ \forall \ e \in E.$$

In [Dy], Dyer proved that every extension of a free group by a finite group is conjugacy separable. The following theorem is the analog of Dyer's theorem for conjugacy p-separability. From a technical point of view, it is the main result of our paper.

Theorem 4.2 *Every extension of a free group by a finite p-group is conjugacy p-separable.*

Proof: Our proof was inspired by Dyer's one (see [Dy]). Let G be an extension of a free group by a finite p-group. In other words, there exists a short exact sequence:

$$1 \longrightarrow F \longrightarrow G \xrightarrow{\pi} P \longrightarrow 1 \ ,$$

where F is a free group, and P is a finite p-group. Let $g \in G$. Let $h \in G$ such that $g \nsim h$.

<u>Step 1</u>: We show that we may assume that G satisfies a short exact sequence:

$$1 \longrightarrow F \longrightarrow G \xrightarrow{\pi} C_{p^n} \longrightarrow 1 \ ,$$

where F is a free group, $n \geq 1$, C_{p^n} denotes the cyclic group of order p^n, and $\pi(g) = \pi(h)$.

Since G is an extension of a free group by a finite p-group, G is residually p-finite by [G], Lemma 1.5. Therefore, if $g = 1$, then $g^G = \{1\}$ is finitely p-separable in G. On the other hand, if g is of infinite order in G, then g^G is finitely p-separable in G by [I], Proposition 5. Therefore we may assume that $g \neq 1$ and that g is of finite order in G. Similarly, we may assume that $h \neq 1$ and that h is of finite order in G. If $\pi(g)$ and $\pi(h)$ are not conjugate in P, we are done. Thus, up to replacing h by a conjugate of itself, we may assume that $\pi(g) = \pi(h)$. Since $ker(\pi) = F$ is torsion-free, g and h have the same order p^n ($n \in \mathbb{N}^*$). Let H be the subgroup of G generated by g and F. Note that H is a subgroup of p-power index in G, and that g and h belong to H. As $\frac{G}{F} = P$ is nilpotent, H is subnormal in G. Thus we may replace G by H, by [I], Proposition 4[1], so as to assume that G satisfies the short exact sequence:

$$1 \longrightarrow F \longrightarrow G \xrightarrow{\pi} C_{p^n} \longrightarrow 1 \ .$$

[1]Strictly speaking, it follows from the proof of [I], Proposition 4, that, if there exists a homomorphism $\varphi : H \to P$ from H onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$, then there exists a homomorphism $\psi : G \to Q$ from G onto a finite p-group Q such that $\psi(g) \nsim \psi(h)$. The exact statement of [I], Proposition 4, is slightly different.

Now, G is the fundamental group of a graph of groups Γ, whose vertex groups are all finite groups, by [S]. As $\pi_{|G_v}$ is injective for all $v \in V$, G_v is isomorphic to a subgroup of C_{p^n} for all $v \in V$. From now on, the groups G_v ($v \in V$) will be regarded as subgroups of C_{p^n}.

Step 2: We show that we may assume that all edge groups are non-trivial, that if two different vertices are connected by an edge, then the corresponding edge group is a proper subgroup of C_{p^n}, and that g and h belong to two different vertex groups.

First, we show that we may assume that all edge groups are non-trivial. Indeed, Let Γ_0 be the subgraph of Γ whose vertices are all the vertices of Γ, and whose edges are the edges of Γ for which the edge group is non-trivial. Let Γ_1,..., Γ_r be the connected components of Γ_0. Let Γ^* be the graph of groups obtained from Γ by contracting Γ_i to a point for all $i \in \{1,...,r\}$. Let T be a maximal tree of Γ^*. Then G is isomorphic to the fundamental group G^* of Γ^*. Observe that G^* is the free product of the free group on $\{t_e \mid e \in E \setminus T\}$ and the fundamental groups of the Γ_i ($i \in \{1,...,r\}$). Thus, it suffices to consider the case where $\Gamma = \Gamma_i$ ($i \in \{1,...,r\}$), by Lemma 4.1. Since each Γ_i ($i \in \{1,...,r\}$) is a graph of groups whose edge groups are all non-trivial, the first part of the assertion is proved.

Now, we show that we may assume that if two different vertices are connected by an edge, then the corresponding edge group is a proper subgroup of C_{p^n}. Indeed, let Γ_0 be the subgraph of Γ whose vertices are all the vertices of Γ, and whose edges are the edges of Γ for which the edge group is isomorphic to C_{p^n}. Let Γ_1,..., Γ_r be the connected components of Γ_0. Choose a maximal tree T_i in Γ_i, for all $i \in \{1,...,r\}$. Let Γ^* be the graph of groups obtained from Γ by contracting T_i to a point for all $i \in \{1,...,r\}$. Then G is isomorphic to the fundamental group G^* of Γ^*. Note that a vertex group of Γ^* is either a vertex group of Γ, or the fundamental group of T_i, for some $i \in \{1,...,r\}$, in which case it is isomorphic to C_{p^n} (because each T_i ($i \in \{1,...,r\}$) is a tree of groups whose vertex and edge groups are all equal to C_{p^n}). Thus, we may replace Γ by Γ^*, so that the second part of the assertion is proved.

Since g is of finite order in G, there exists a vertex v of Γ, an element g_0 of finite order in the vertex group G_v of v, and an element α of G such that $g = \alpha g_0 \alpha^{-1}$. Similarly, there exists a vertex w of Γ, an element h_0 of finite order in the vertex group G_w of w, and an element β of G such that $h = \beta h_0 \beta^{-1}$. As C_{p^n} is abelian, we have: $\pi(g_0) = \pi(h_0)$. Thus, up to replacing g by g_0 and h by h_0, we may assume that g belongs to G_v, and h belongs to G_w. Since $\pi_{|G_v}$ is injective, and $\pi(g) = \pi(h)$, we have $v \neq w$.

Step 3: We show that we may assume that Γ has exactly two vertices, and that all edges join these two vertices.

Indeed, choose a maximal tree T in Γ. There is a path P in T joining v to w. Choose an edge e on this path. Then $T \setminus \{e\}$ is the disjoint union of two trees, T_v and T_w - say $v \in T_v$ and $w \in T_w$. Let Γ_v be the full subgraph of Γ generated by the vertices of T_v, and Γ_w be the full subgraph of Γ generated by the vertices of T_w. Let Γ^* be the graph of groups obtained from Γ by contracting Γ_v to a point v^* and Γ_w to a point w^*. Observe that Γ^* has exactly two vertices and that all edges join these two vertices. The vertex groups of Γ^* are the fundamental groups of Γ_v and Γ_w, respectively. The edge groups of Γ^* are non-trivial proper subgroups of C_{p^n}. And G is isomorphic to the fundamental group G^* of Γ^*. Now, since the restriction of π to each edge subgroup of Γ^* is injective, we may define a graph of groups Γ' from Γ^* by composing with π, as described above. Denote by G' the fundamental group of Γ'. There exist two homomorphisms $\pi_V : G \to G'$, and $\pi_E : G' \to C_{p^n}$ such that the diagram:

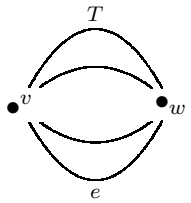
$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\pi_V"'] & C_{p^n} \\ G' \arrow[ur, "\pi_E"'] & \end{tikzcd}$$

commutes, and that the restriction of π_E to each vertex group of Γ' is injective. Consequently, $ker(\pi_E)$ is free by [Se], II, 2.6., Lemma 8.

Set $g' = \pi_V(g)$, and $h' = \pi_V(h)$. As g' and h' have order p^n, the vertex groups of Γ' are equal to C_{p^n}. The edge groups of Γ' are non-trivial proper subgroups of C_{p^n}. Observe that g' and h' belong to two different vertex groups, and that g' (resp. h') is not conjugate to an element of one of the edge groups. Let e be an edge of Γ'. Then g' and h' are not conjugate in $G'_v *_{G'_e} G'_w$, by [MKS], Theorem 4.6. (ii). Observe that G' is an HNN extension (in the general sense) of $G'_v *_{G'_e} G'_w$ with stable letters t_a ($a \in E \setminus \{e\}$), and associated subgroups $\alpha'_a(G'_a)$, $\beta'_a(G'_a)$ ($a \in E \setminus \{e\}$). Therefore g' and h' are not conjugate in G' (see [Dy2], Theorem 3). Thus, we may replace Γ by Γ', G by G', g by g', and h by h', so as to assume that Γ has two vertices and that all edges join these two vertices.

<u>Step 4</u>: We show that we may assume that Γ has at most two edges.

Suppose that Γ has more than two edges. Choose a maximal tree T in Γ - that is, an edge of Γ. Present $G_v = < g \mid g^{p^n} = 1 >$, $G_w = < h \mid h^{p^n} = 1 >$, and G as described above. Choose an edge $e \in E \setminus T$.



The edge relations corresponding to e can be reduced to the following:

$$t_e \alpha_e(g_e) t_e^{-1} = \beta_e(g_e),$$

where g_e is a generator of G_e. Let p^s be the order of G_e ($s \in \{1,...,n\text{-}1\}$). Then $\alpha_e(g_e)$ generates a subgroup of order p^s of G_v. But there exists a unique subgroup of order p^s in G_v; it is cyclic, generated by g^{p^r}, where $r = n - s$. Thus, up to replacing g_e by the preimage of g^{p^r} under α_e, we may assume that $\alpha_e(g_e) = g^{p^r}$. There exists $k \in \mathbb{N}$, such that p and k are coprime, and that $\beta_e(g_e) = h^{kp^r}$. Therefore the edge relation corresponding to e can be written:

$$t_e g^{p^r} t_e^{-1} = h^{kp^r},$$

where $r \in \{1,...,n\text{-}1\}$, $k \in \mathbb{N}$, and p and k are coprime. Now, since $\pi : G \to C_{p^n}$ satisfies $\pi(g) = \pi(h)$, we have: $\pi(g)^{p^r} = \pi(h)^{kp^r} = \pi(g)^{kp^r}$, and then $\pi(g)^{(k-1)p^r} = 1$ (in C_{p^n}). As $\pi(g)$ has order p^n in C_{p^n}, we deduce that p^{n-r} divides $k - 1$. There exists $a \in \mathbb{Z}$ such that $k = ap^{n-r} + 1$. We conclude that the edge relation corresponding to e can be written:

$$t_e g^{p^r} t_e^{-1} = h^{p^r},$$

where $r \in \{1,...,n\text{-}1\}$.

Let H be the normal subgroup of G generated by the elements:

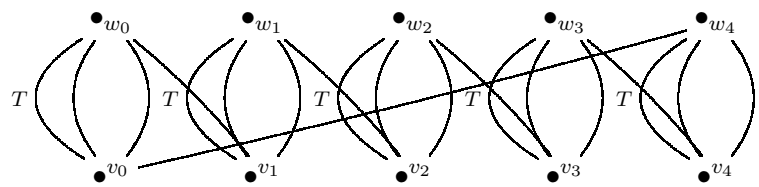
$$g,\ h,\ t_a\ (a \in E \setminus \{e\}),\ t_e^p.$$

Then H has index p in G, and g and h belong to H. Thus we may replace G by H by [I], Proposition 4. Let G_0 be the fundamental group of the graph of groups $\Gamma \setminus \{e\}$. Set $G_0 = < X_0 \mid R_0 >$, where the presentation is as fundamental group of the graph of groups $\Gamma \setminus \{e\}$. Set $G_i = t_e^i G_0 t_e^{-i} = < X_i \mid R_i >$, for all $i \in \{1,...,p\text{-}1\}$. Clearly $\{1,t_e,...,t_e^{p-1}\}$ is a Schreier transversal for H in G. The Reidemeister-Schreier method yields the presentation:

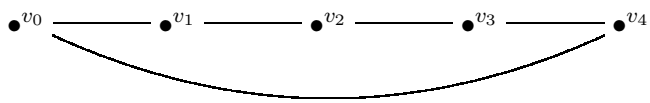
$$H = < X_0,\ X_1,...,\ X_{p-1},\ u \mid R_0,\ R_1,...,\ R_{p-1},\ g_1^{p^r} = h_0^{p^r},\ g_2^{p^r} = h_1^{p^r},...,\ g_{p-1}^{p^r} = h_{p-2}^{p^r},\ ug_0^{p^r}u^{-1} = h_{p-1}^{p^r} >,$$

where $u = t_e^p$, $g_i = t_e^i g t_e^{-i}$ ($i \in \{0,...,p\text{-}1\}$), and $h_j = t_e^j h t_e^{-j}$ ($j \in \{0,...,p\text{-}1\}$). Replace g by g_0, and h by h_1. Observe that H is the fundamental group of a graph of groups $\widetilde{\Gamma}$, as follows. The graph $\widetilde{\Gamma}$ has $2p$ vertices, say v_0, w_0, v_1, w_1,..., v_{p-1}, w_{p-1}, and $p|E|$ edges. Let $\widetilde{\Gamma}_i$ be the full subgraph of $\widetilde{\Gamma}$ generated by $\{v_i, w_i\}$ for all $i \in \{0,...,p\text{-}1\}$. Then $\widetilde{\Gamma}_i$ is isomorphic to $\Gamma \setminus \{e\}$. There is one edge joining w_0 to v_1, one edge joining w_1 to v_2,..., one edge joining w_{p-2} to v_{p-1}, and one edge joining v_0 to w_{p-1}, and the edge groups

associated to these egdes are isomorphic to G_e. Note that g belongs to the vertex group of v_0 and h belongs to the vertex group of w_1.



Let Γ^* be the graph of groups obtained from $\widetilde{\Gamma}$ by contracting $\widetilde{\Gamma}_i$ to a point for all $i \in \{1,...,p\text{-}1\}$. Then G is isomorphic to the fundamental group of Γ^*. The graph Γ^* has p vertices, say $v_0,..., v_{p-1}$. There is one edge joining v_0 to v_1, one edge joining v_1 to $v_2,...$, one edge joining v_{p-2} to v_{p-1}, and one edge joining v_0 to v_{p-1}, and the edge groups associated to these edges are all isomorphic to G_e. Note that g belongs to the vertex group of v_0 and h belongs to the vertex group of v_1.



Let T be the maximal tree $T = v_0v_1...v_{p-2}v_{p-1}$. Then $T \setminus \{v_0v_1\}$ is the disjoint union of two trees : v_0 and $v_1v_2...v_{p-2}v_{p-1}$. Set $\Gamma_1^* = v_0$ and $\Gamma_2^* = v_1v_2...v_{p-2}v_{p-1}$. Let Λ be the graph of groups obtained from Γ^* by contracting Γ_i^* to a point for all $i \in \{1,2\}$. Let Λ' be the graph of groups obtained from Λ by composing with π. As in Step 3, we may replace Γ by Λ', so as to assume that Γ has two vertices and two edges joining these two vertices.

<u>End of the proof</u>: Present $G_v = < g \mid g^{p^n} = 1 >$, $G_w = < h \mid h^{p^n} = 1 >$, and G as described above. There are two cases:

<u>Case 1</u>: Γ has one edge.

In this case, G is an amalgamated product of two finite abelian p-groups. Since G is residually p-finite, G is conjugacy p-separable by [I], Theorem 2. Thus, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$.

<u>Case 2</u>: Γ has two edges.

We have:

$$G = < g,\, h,\, t \mid g^{p^n} = 1,\, h^{p^n} = 1,\, g^{p^r} = h^{p^r},\, tg^{p^s}t^{-1} = h^{p^s} >,$$

where $r \in \{1,...,n\text{-}1\}$, $s \in \{1,...,n\text{-}1\}$. Let:

$$A = C_{p^n} \times \underbrace{C_{p^s} \times ... \times C_{p^s}}_{p^r\text{-}1} \times C_{p^r}$$

Set $m = p^r + 1$. Present each factor of this product in the natural way, using generators x_1,..., x_m respectively. Let α be the automorphism of A defined by:

$$\begin{gathered}\alpha(x_1) = x_1x_2x_m \\ \alpha(x_i) = x_{i+1},\ \forall\ i \in \{2,...,m\text{-}2\} \\ \alpha(x_{m-1}) = (x_2...x_{m-1})^{-1} \\ \alpha(x_m) = x_m\end{gathered}$$

It is easily seen that α has order m - 1 = p^r. We have:

$$\begin{gathered}\alpha^0(x_1) = x_1, \\ \alpha^1(x_1) = x_1x_2x_m, \\ \alpha^2(x_1) = x_1x_2x_3x_m^2, \\ ... \\ \alpha^{m-2}(x_1) = x_1x_2x_3...x_{m-1}x_m^{m-2}.\end{gathered}$$

Let $B = A \rtimes < \alpha >$ be the semidirect product of A by $< \alpha >$. Note that B is a finite p-group. Let $\varphi : G \to B$ be the homomorphism defined by:

$$\begin{gathered}\varphi(g) = x_1, \\ \varphi(h) = x_1x_m, \\ \varphi(t) = \alpha.\end{gathered}$$

Observe that the conjugacy class of $\varphi(g)$ in B is $\varphi(g)^B = \{\alpha^k(x_1) \mid k \in \{0,...,m\text{-}2\}\}$. Thus, $\varphi(g)$ and $\varphi(h)$ are not conjugate in B. □

Corollary 4.3 *Let P be a finite p-group. Let A be a subgroup of P. Let Q be the HNN extension of P relative to A:*

$$Q = < P,\ t \mid t^{-1}at = a,\ \forall\ a \in A >.$$

Then Q is hereditarily conjugacy p-separable.

Proof: Let R be a subgroup of p-power index in Q. Actually, we do not need the fact that R is of p-power index. Let $f : Q \to P$ be the natural homomorphism. We have $ker(f) \cap P = \{1\}$. Therefore $ker(f)$ is free by [KS], Theorem 6. That is, Q is an extension of a free group by a finite p-group. Thus R is itself an extension of a free group by a finite p-group. Therefore R is conjugacy p-separable by Theorem 4.2. □

5 Retractions

Definition 5.1 *Let G be a group and H be a subgroup of G. We say that H is a retract of G if there exists a homomorphism $\rho_H : G \to G$ such that $\rho_H(G) = H$ and $\rho_H(h) = h$ for all $h \in H$. The homomorphism ρ_H is called a retraction of G onto H.*

Remark: If G is a right-angled Artin group, and H is a special subgroup of G, then H is a retract of G. A retraction of G onto H is given by:

$$\rho_H(v) = \begin{cases} v & \text{if } v \in W \\ 1 & \text{if } v \in V \setminus W \end{cases}$$

In what follows (Lemma 5.2 to Lemma 5.12), we shall prove several results on retractions that will allow us later to control the growth of some intersections of subgroups of a right-angled Artin group in homomorphic images of it (see Lemma 6.3).

Lemma 5.2 *Let G be a group and H be a subgroup of G. Suppose that H is a retract of G. Let ρ_H be a retraction of G onto H. Let N be a normal subgroup of G such that $\rho_H(N) \subset N$. Then ρ_H induces a retraction $\rho_{\overline{H}} : G/N \to G/N$ of G/N onto the canonical image $\overline{H}$ of H in G/N, defined by: $\rho_{\overline{H}}(gN) = \rho_H(g)N$ for all $gN \in G/N$.*

Proof : Proved in [M]. □

Remark: Let G be a group and let H and H' be two subgroups of G. Suppose that H and H' are retracts of G and that the corresponding retractions, ρ_H and $\rho_{H'}$, commute. Then $\rho_H(H') = \rho_{H'}(H) = H \cap H'$. Moreover $H \cap H'$ is a retract of G. A retraction of G onto $H \cap H'$ is given by $\rho_{H \cap H'} = \rho_H \circ \rho_{H'} = \rho_{H'} \circ \rho_H$.

Proposition 5.3 *Let G be a group and $H_1,\ldots, H_n$ be n subgroups of G. Suppose that $H_1,\ldots, H_n$ are retracts of G and that the corresponding retractions pairwise commute. Then, for every normal subgroup K of p-power index in G, there exists a normal subgroup N of p-power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$ for all $i \in \{1,\ldots,n\}$. Consequently, for every $i \in \{1,\ldots,n\}$, the retraction ρ_{H_i} induces a retraction $\rho_{\overline{H_i}}$ of G/N onto the canonical image $\overline{H_i}$ of H_i in G/N.*

Proof: Proved in [M] - see Remark 4.4. □

Lemma 5.4 *Let G be a group and let H and H' be two subgroups of G. Suppose that H and H' are retracts of G and that the corresponding retractions, ρ_H and $\rho_{H'}$, commute. Let N be a normal subgroup of G and assume that $\rho_H(N) \subset N$ and $\rho_{H'}(N) \subset N$. Then, if $\varphi : G \to G/N$ denotes the canonical projection, $\varphi(H \cap H') = \varphi(H) \cap \varphi(H')$.*

Proof: Proved in [M]. □

The next statement is analog of Lemma 4.6 in [M]:

Corollary 5.5 *Let G be a group and H_1,..., H_n be n subgroups of G. Suppose that H_1,..., H_n are retracts of G and that the corresponding retractions ρ_{H_1},..., ρ_{H_n} pairwise commute. Then, for every normal subgroup K of p-power index in G, there exists a normal subgroup N of p-power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$, for all $i \in \{1,...,n\}$. Moreover, if $\varphi : G \to G/N$ denotes the canonical projection, then $\varphi(\bigcap_{i=1}^n H_i) = \bigcap_{i=1}^n \varphi(H_i)$.*

Proof: By Proposition 5.3, there exists a normal subgroup N of p-power index in G such that $N < K$ and $\rho_{H_i}(N) \subset N$ for all $i \in \{1,...,n\}$. We denote by $\varphi : G \to G/N$ the canonical projection. We argue by induction on $k \in \{1,...,n\}$ to prove that $\varphi(\bigcap_{i=1}^k H_i) = \bigcap_{i=1}^k \varphi(H_i)$. If $k = 1$, then the result is trivial. Thus we can assume that $k \geq 2$ and that the result has been proved for k - 1. We set $H' = \bigcap_{i=1}^{k-1} H_i$. By the above remark, H' is a retract of G. A retraction of G onto H' is given by $\rho_{H'} = \rho_{H_1} \circ ... \circ \rho_{H_{k-1}}$. We have:

$$\begin{aligned}\rho_{H'}(N) &= \rho_{H_1}(...(\rho_{H_{k-2}}(\rho_{H_{k-1}}(N)))) \\ &\subset \rho_{H_1}(...(\rho_{H_{k-2}}(N))) \\ &\subset ... \\ &\subset \rho_{H_1}(N) \\ &\subset N.\end{aligned}$$

The retractions $\rho_{H'}$ and ρ_{H_k} commute, so we can apply Lemma 5.4 to conclude that $\varphi(H' \cap H_k) = \varphi(H') \cap \varphi(H_k)$. By the induction hypothesis, $\varphi(H')$ $= \bigcap_{i=1}^{k-1} \varphi(H_i)$. Finally $\varphi(\bigcap_{i=1}^k H_i) = \bigcap_{i=1}^k \varphi(H_i)$. □

In the following lemmas, G is a group, and A and B are two subgroups of G. We assume that A and B are retracts of G and that the corresponding retractions, ρ_A and ρ_B, commute.

Lemma 5.6 *Let x, $y \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$ ($\in AxB$) and $\beta = \rho_A(\rho_B(y)y^{-1})y\rho_B(y^{-1})$ ($\in AyB$). Are equivalent:*

1. *$y \in AxB$,*
2. *$\beta \in \alpha^{A \cap B}$.*

Proof: Proved in [M]. □

Lemma 5.7 *Let $x \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$ ($\in AxB$) and $\gamma = \rho_A(\rho_B(x)x^{-1})$ ($\in A$). Then we have:*

$$A \cap xBx^{-1} = \gamma^{-1} C_{A \cap B}(\alpha)\gamma.$$

Proof: Proved in [M]. $\square$

The next five statements are analogs of some statements in [M] (Lemma 5.3, Corollary 5.4, Lemma 5.5, Lemma 5.6 and Lemma 5.7, respectively):

Lemma 5.8 *Let $x \in G$. We set: $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$ ($\in AxB$). If $\alpha^{A\cap B}$ is finitely p-separable in G, then AxB is also finitely p-separable in G.*

Proof: Let $y \in G$ such that $y \notin AxB$. We set $\beta = \rho_A(\rho_B(y)y^{-1})y\rho_B(y^{-1})$. By Lemma 5.6, we have $\beta \notin \alpha^{A\cap B}$. Since $\alpha^{A\cap B}$ is finitely p-separable in G, there exists a normal subgroup K of p-power index in G such that, if $\psi : G \to G/K$ denotes the canonical projection, we have: $\psi(\beta) \notin \psi(\alpha^{A\cap B}) = \psi(\alpha)^{\psi(A\cap B)}$. By Corollary 5.5, there exists a normal subgroup N of p-power index in G such that $N < K$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$ and, if $\varphi : G \to G/N$ denotes the canonical projection, then: $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. Assume that $\varphi(\beta) \in \varphi(\alpha)^{\varphi(A\cap B)}$. There exists $g \in A \cap B$ such that $\varphi(\beta) = \varphi(g)\varphi(\alpha)\varphi(g)^{-1}$. Then $\beta \in g\alpha g^{-1}N$. Since $N < K$, we obtain $\beta \in g\alpha g^{-1}K$. But this contradicts the fact that $\psi(\beta) \notin \psi(\alpha)^{\psi(A\cap B)}$. Therefore we have: $\varphi(\beta) \notin \varphi(\alpha)^{\varphi(A\cap B)}$ i.e. $\varphi(\beta) \notin \varphi(\alpha)^{\varphi(A)\cap\varphi(B)}$. We set $\overline{A} = \varphi(A)$ and $\overline{B} = \varphi(B)$. By Lemma 5.2, ρ_A induces a retraction $\rho_{\overline{A}}$ of G/N onto $\overline{A}$ and ρ_B induces a retraction $\rho_{\overline{B}}$ of G/N onto $\overline{B}$. We set: $\overline{x} = \varphi(x)$ and $\overline{y} = \varphi(y)$. We have: $\varphi(\alpha) = \rho_{\overline{A}}(\rho_{\overline{B}}(\overline{x})\overline{x}^{-1})\overline{x}\rho_{\overline{B}}(\overline{x}^{-1})$ and $\varphi(\beta) = \rho_{\overline{A}}(\rho_{\overline{B}}(\overline{y})\overline{y}^{-1})\overline{y}\rho_{\overline{B}}(\overline{y}^{-1})$. By Lemma 5.6, we have $\overline{y} \notin \overline{A}\overline{x}\overline{B}$ i.e. $\varphi(y) \notin \varphi(AxB)$. $\square$

Corollary 5.9 *Let G be a group and A and B be two subgroups of G. Suppose that G is residually p-finite. If A and B are retracts of G, such that the corresponding retractions commute, then AB is finitely p-separable in G.*

Proof: We apply Lemma 5.8 to $x = 1$. $\square$

Lemma 5.10 *Let G be a group and A be a subgroup of G. Suppose that G is residually p-finite and that A is a retract of G. Then if a subset S of A is closed in the pro-p topology on A, it is also closed in the pro-p topology on G.*

Proof: We denote by $\overline{S}$ the closure of S in G - equipped with the pro-p topology. We shall show that $\overline{S} \subset S$. By Corollary 5.9, A is closed in G. Therefore $\overline{S} \subset A$. Let $a \in G\backslash S$. We can assume that $a \in A$. There exists a homomorphism φ from A onto a finite p-group P such that $\varphi(a) \notin \varphi(S)$. We set: $\psi = \varphi \circ \rho_A$. We have: $\psi(a) = \varphi(a) \notin \varphi(S) = \psi(S)$. Then $a \notin \overline{S}$.$\square$

Lemma 5.11 *Let $x \in G$. We set $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$. Suppose that the pair ($A\cap B$,α) satisfies the p centralizer condition in G. Then, for every normal subgroup K of p-power index in G, there exists a normal*

subgroup N of p-power index in G such that $N < K$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$ and, if $\varphi : G \to G/N$ denotes the canonical projection, $\varphi(A) \cap \varphi(xBx^{-1}) \subset \varphi(A \cap xBx^{-1})\varphi(K)$.

Proof: Let K be a normal subgroup of p-power index in G. We set $\gamma = \rho_A(\rho_B(x)x^{-1}) \in A$. By Lemma 5.7, we have: $A \cap xBx^{-1} = \gamma^{-1}C_{A\cap B}(\alpha)\gamma$. Since the pair $(A \cap B, \alpha)$ satisfies pCC_G, there exists a normal subgroup L of p-power index in G such that $L < K$ and, if $\psi : G \to G/L$ denotes the canonical projection, $C_{\psi(A\cap B)}(\psi(\alpha)) \subset \psi(C_{A\cap B}(\alpha)K)$. This is equivalent to $\psi^{-1}(C_{\psi(A\cap B)}(\psi(\alpha))) \subset C_{A\cap B}(\alpha)K$. Indeed let $g \in \psi^{-1}(C_{\psi(A\cap B)}(\psi(\alpha)))$. We have $\psi(g) \in C_{\psi(A\cap B)}(\psi(\alpha)) \subset \psi(C_{A\cap B}(\alpha)K)$. Then $g \in C_{A\cap B}(\alpha)KL \subset C_{A\cap B}(\alpha)K$ (because $L < K$). By corollary 5.5, there exists a normal subgroup N of p-power index in G such that $N < L$, $\rho_A(N) \subset N$, $\rho_B(N) \subset N$ and, if $\varphi : G \to G/N$ denotes the canonical projection, $\varphi(A \cap B) = \varphi(A) \cap \varphi(B)$. We set $\overline{A} = \varphi(A)$, $\overline{B} = \varphi(B)$. By Lemma 5.2, ρ_A induces a retraction $\rho_{\overline{A}}$ of G/N onto $\overline{A}$ and ρ_B induces a retraction $\rho_{\overline{B}}$ of G/N onto $\overline{B}$. Obviously $\rho_{\overline{A}}$ and $\rho_{\overline{B}}$ commute. We set $\overline{x} = \varphi(x)$, $\overline{\alpha} = \rho_{\overline{A}}(\rho_{\overline{B}}(\overline{x})\overline{x}^{-1})\overline{x}\rho_{\overline{B}}(\overline{x}^{-1})$ $(\in G/N)$ and $\overline{\gamma} = \rho_{\overline{A}}(\rho_{\overline{B}}(\overline{x})\overline{x}^{-1})$ $(\in \overline{A})$. Observe that $\overline{\alpha} = \varphi(\alpha)$ and $\overline{\gamma} = \varphi(\gamma)$. Then, by Lemma 5.7, we have: $\overline{A} \cap \overline{x}\overline{B}\overline{x}^{-1} = \overline{\gamma}^{-1}C_{\overline{A}\cap\overline{B}}(\overline{\alpha})\overline{\gamma}$. But $\overline{A} \cap \overline{B} = \varphi(A \cap B)$. We have:

$$\varphi^{-1}(\overline{A} \cap \overline{x}\overline{B}\overline{x}^{-1}) = \varphi^{-1}(\overline{\gamma}^{-1}C_{\varphi(A\cap B)}(\overline{\alpha})\overline{\gamma}) = \gamma^{-1}\varphi^{-1}(C_{\varphi(A\cap B)}(\overline{\alpha}))\gamma.$$

We have:

$$\varphi^{-1}(C_{\varphi(A\cap B)}(\overline{\alpha})) \subset \psi^{-1}(C_{\psi(A\cap B)}(\psi(\alpha))).$$

Indeed let $g \in \varphi^{-1}(C_{\varphi(A\cap B)}(\varphi(\alpha)))$. We have $\varphi(g) \in \varphi(A \cap B)$ i.e. $g \in (A \cap B)N$, which implies $g \in (A \cap B)L$ i.e. $\psi(g) \in \psi(A \cap B)$; and $\varphi(g)\varphi(\alpha) = \varphi(\alpha)\varphi(g)$ i.e. $g\alpha g^{-1}\alpha^{-1} \in N$, which implies $g\alpha g^{-1}\alpha^{-1} \in L$ i.e. $\psi(g)\psi(\alpha) = \psi(\alpha)\psi(g)$. Then:

$$\varphi^{-1}(C_{\varphi(A\cap B)}(\overline{\alpha})) \subset C_{A\cap B}(\alpha)K.$$

Therefore:

$$\varphi^{-1}(\overline{A} \cap \overline{x}\overline{B}\overline{x}^{-1}) \subset \gamma^{-1}C_{A\cap B}(\alpha)\gamma K = (A \cap xBx^{-1})K.$$

We conclude that:

$$\varphi(A) \cap \varphi(xBx^{-1}) \subset \varphi(A \cap xBx^{-1})\varphi(K).$$

□

Lemma 5.12 *Let x, $y \in G$. We set $C = xBx^{-1}$ $(< G)$ and $\alpha = \rho_A(\rho_B(x)x^{-1})x\rho_B(x^{-1})$. If $\alpha^{A\cap B}$ and $y^{A\cap C}$ are finitely p-separable in G and if the pair $(A \cap B, \alpha)$ satisfies pCC_G, then $C_A(y)C$ is finitely p-separable in G.*

Proof: Let $z \in G$ such that $z \notin C_A(y)C$. Suppose first that $z \notin AC$. Since $\alpha^{A\cap B}$ is finitely p-separable in G, AxB is finitely p-separable in G by Lemma 5.8. Therefore $AC = AxBx^{-1}$ is also finitely p-separable in G. Consequently there exists a normal subgroup N of p-power index in G such that $z \notin ACN$. We obviously have $z \notin C_A(y)CN$. Thus we can assume that $z \in AC$. There exist $a \in A$, $c \in C$ such that $z = ac$. Since $z \notin C_A(y)C$, $a^{-1}ya \notin y^{A\cap C}$. Indeed, if there exist $g \in A \cap C$ such that $a^{-1}ya = gyg^{-1}$, then $(ag)^{-1}y(ag) = y$ i.e. $ag \in C_A(y)$. We obtain $a \in C_A(y)C$, and then $z \in C_A(y)C$ - a contradiction. Now $y^{A\cap C}$ is finitely p-separable in G. Then there exists a normal subgroup K of p-power index in G such that $a^{-1}ya \notin y^{A\cap C}K$. By Lemma 5.11, there exists a normal subgroup N of p-power index in G such that $N < K$ and, if $\varphi : G \to G/N$ denotes the canonical projection, $\varphi(A) \cap \varphi(C) \subset \varphi(A \cap C)\varphi(K)$. For a subset S of G, we set $\overline{S} = \varphi(S)$. For an element g of G, we set $\overline{g} = \varphi(g)$. We have: $\overline{y}^{\overline{A}\cap\overline{C}} \subset \overline{y}^{\overline{A\cap C}.\overline{K}}$. Note that $\overline{K} \lhd G/N$. Then $\overline{y}^{\overline{A}\cap\overline{C}} \subset \overline{y}^{\overline{A\cap C}}\overline{K}$. Observe that $\overline{a}^{-1}\overline{y}\,\overline{a} \notin \overline{y}^{\overline{A\cap C}}\overline{K}$ - otherwise we would have $a^{-1}ya \in y^{A\cap C}KN$, and then $a^{-1}ya \in y^{A\cap C}K$ (because $N < K$). We deduce that $\overline{a}^{-1}\overline{y}\,\overline{a} \notin \overline{y}^{\overline{A}\cap\overline{C}}$. Now it suffices to show that $\varphi(z) \notin \varphi(C_A(y)C)$. Suppose the contrary. There exist $a' \in C_A(y)$, $c' \in C$ such that $\varphi(z) = \varphi(a'c')$. Then $\varphi(ac) = \varphi(a'c')$. Thus $\varphi(a'^{-1}a) = \varphi(c'c^{-1})$. We set $\overline{g} = \varphi(a'^{-1}a) = \varphi(c'c^{-1})$ $(\in \overline{A} \cap \overline{C})$. We have: $\varphi(z) = \varphi(a')\overline{g}\varphi(c)$ and $\overline{a} = \varphi(z)\varphi(c)^{-1} = \varphi(a')\overline{g}$. Then $\overline{a}^{-1}\overline{y}\,\overline{a} = \overline{g}^{-1}\varphi(a'^{-1}ya')\overline{g} = \overline{g}^{-1}\varphi(y)\overline{g} = \overline{g}^{-1}\overline{y}\,\overline{g} \in \overline{y}^{\overline{A}\cap\overline{C}}$ - a contradiction. We have shown that $C_A(y)C$ is finitely p-separable in G. $\square$

6 Proof of the main theorem

We turn now to the proof that right-angled Artin groups are hereditarily conjugacy p-separable. We need the following theorem, which is due to Duchamp and Krob (see [DK2]).

Theorem 6.1 *Right-angled Artin groups are residually p-finite.*

This theorem can also be proved using HNN extensions (see [Lo]).

Basically, Proposition 6.2 establishes the main result. Proposition 6.2.1 and Proposition 6.2.2 will be proved simultaneously by induction on the rank of G.

Proposition 6.2 *Let G be a right-angled Artin group.*

1. *Every special subgroup S of G satisfies the p centralizer condition in G (pCC_G).*

2. *For all $g \in G$ and for all special subgroup S of G, g^S is finitely p-separable in G.*

From now on, we assume that G is a right-angled Artin group of rank r ($r \geq 1$), and that H is a special subgroup of G of rank r - 1. Thus, G can be written as an HNN extension of H relative to the special subgroup $K = C(t) = < star(t) >$ of H:

$$G = < H, t \mid t^{-1}kt = k, \forall\ k \in K >.$$

Recall that H is a retract of G. A retraction of G onto H is given by:

$$\rho_H(v) = \begin{cases} v & \text{if } v \in W \\ 1 & \text{if } v \in V \setminus W \end{cases}$$

We also assume that:

- every special subgroup S of H satisfies the p centralizer condition in H (pCC_H),
- for all $h \in H$ and for all special subgroup S of H, h^S is finitely p-separable in H.

The next results (Lemma 6.3 to Lemma 6.13) are preliminaries to the proof of Proposition 6.2.

In general, if A and B are subgroups of a group G, the image of the intersection of A and B under a homomorphism $\varphi : G \to H$ do not coincide with the intersection of the images of A and B in H. However, the p centralizer condition and the above results on retractions will allow us to obtain the following lemma, which will be used to apply Minasyan's criterion for conjugacy in HNN extensions (see Lemma 6.4).

Lemma 6.3 *Let be given A_0, a conjugate of a special subgroup of H, A_1,..., A_n, n special subgroups of H and α, x_0,..., x_n, y_1,..., y_n, 2(n+1) elements of H. Then, for every normal subgroup L of p-power index in H, there exists a normal subgroup N of p-power index in H such that $N < L$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then:*

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n} \overline{x_i}\overline{A_i}\overline{y_i} \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_i A_i y_i)L)$$

where $\overline{A_i} = \varphi(A_i)$ ($i \in \{0,...,n\}$), $\overline{\alpha} = \varphi(\alpha)$, $\overline{x_j} = \varphi(x_j)$ ($j \in \{0,...,n\}$), $\overline{y_k} = \varphi(y_k)$ ($k \in \{1,...,n\}$).

Proof: Let L be a subgroup of p-power index in H. We argue by induction on n. Strictly speaking, the basis of our induction is n = 0 but we will need the case n = 1. By the assumptions, there exist a special subgroup A of H and an element β of H such that $A_0 = \beta A \beta^{-1}$.
<u>n = 0</u>: We set $x = \beta^{-1}x_0\beta$. The pair (A,x) satisfies pCC_H by the assumptions. There exists a normal subgroup N of p-power index in H such

that $N < L$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then $C_{\varphi(A)}(\varphi(x)) \subset \varphi(C_A(x)L)$. But $C_{A_0}(x_0) = \beta C_A(x)\beta^{-1}$. We deduce that: $\varphi(\alpha)C_{\varphi(A_0)}(\varphi(x_0)) \subset \varphi((\alpha C_{A_0}(x_0))L)$.
<u>n = 1</u>: There are two cases:
<u>Case 1</u>: $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 = \emptyset$.
This is equivalent to say that: $x_1 \notin \alpha C_{A_0}(x_0) y_1^{-1} A_1$. We set $B = (y_1\beta)^{-1}A_1$ $y_1\beta$, so that we have: $x_1 \notin \alpha\beta(C_A(x)B)\beta^{-1}y_1^{-1}$. Now the intersection of conjugates of two special subgroups of H is a conjugate of a special subgroup of H (see [M], Lemma 6.5). Then $A \cap A_1$ is a conjugate of a special subgroup C of H. There exists $\gamma \in H$ such that $A \cap A_1 = \gamma C \gamma^{-1}$. Therefore if $h \in H$, $h^{A\cap A_1} = \gamma(\gamma^{-1}h\gamma)^C\gamma^{-1}$. Now $(\gamma^{-1}h\gamma)^C$ is finitely p-separable in H by the assumptions. We deduce that $h^{A\cap A_1}$ is finitely p-separable in H. With the same argument, $h^{A\cap B}$ is finitely p-separable in H. Now the pair $(A \cap A_1, h)$ satisfies pCC_H by the assumptions. We deduce that $C_A(x)B$ is finitely p-separable in H by Lemma 5.12. This implies that $\alpha C_{A_0}(x_0)y_1^{-1}A_1$ is finitely p-separable in H. There exists a normal subgroup M of p-power index in H such that $x_1 \notin \alpha C_{A_0}(x_0)y_1^{-1}A_1M$. Up to replacing M by $M \cap L$, we can assume that $M < L$. Now the pair (A_0, x_0) satisfies pCC_H by the assumptions. There exists a normal subgroup N of p-power index in H such that $N < M$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then $C_{\varphi(A_0)}(\varphi(x_0)) \subset \varphi(C_{A_0}(x_0)M)$, or, equivalently, $\varphi^{-1}(C_{\varphi(A_0)}(\varphi(x_0))) \subset C_{A_0}(x_0)M$. Then $\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0})\overline{y_1}^{-1}\overline{A_1}) \subset \alpha\varphi^{-1}(C_{\overline{A_0}}(\overline{x_0}))y_1^{-1}A_1 \subset \alpha C_{A_0}(x_0)y_1^{-1}A_1M$. Therefore: $x_1 \notin \varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0})$ $\overline{y_1}^{-1}\overline{A_1})$. Finally: $\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\overline{A_1}\overline{y_1} = \emptyset$.
<u>Case 2</u>: $\alpha C_{A_0}(x_0) \cap x_1 A_1 y_1 \neq \emptyset$.
Remark: If G is a group and H, K are two subgroups of G such that $aH \cap bKc \neq \emptyset$ - where a, b, $c \in G$ -, then for all $g \in aH \cap bKc$, we have $aH \cap bKc = g(H \cap c^{-1}Kc)$.
Choose $g \in \alpha C_{A_0}(x_0) \cap x_1A_1y_1$. Then we have: $\alpha C_{A_0}(x_0) \cap x_1A_1y_1 = g(C_{A_0}(x_0) \cap y_1^{-1}A_1y_1)$. We set $D = A_0 \cap y_1^{-1}A_1y_1$. Then $\alpha C_{A_0}(x_0) \cap x_1A_1y_1$ $= gC_D(x_0)$. Now, D is a conjugate of a special subgroup E of H by [M], lemma 6.5. There exists $\delta \in H$ such that $D = \delta E \delta^{-1}$. As above, the pair (D, x_0) satisfies pCC_H. There exists a normal subgroup M of p-power index in H such that $M < L$ and, if $\psi : H \to H/M$ denotes the canonical projection, we have: $C_{\psi(D)}(\psi(x_0)) \subset \psi(C_D(x_0)L)$. Now by Lemma 5.11, there exists a normal subgroup N of p-power index in H such that $N < M$ and, if φ $: H \to H/N$ denotes the canonical projection, then $\varphi(A) \cap \varphi((y_1\beta)^{-1}A_1y_1\beta)$ $\subset \varphi(A \cap (y_1\beta)^{-1}A_1y_1\beta)\varphi(M)$. Therefore:

$$\begin{gathered}\overline{A_0} \cap \overline{y_1}^{-1}\overline{A_1}\overline{y_1} = \varphi(\beta A\beta^{-1}) \cap \varphi(y_1^{-1}A_1y_1) = \\ \varphi(\beta)(\varphi(A) \cap \varphi((y_1\beta)^{-1}A_1y_1\beta))\varphi(\beta^{-1}) \subset \\ \varphi(\beta)(\varphi(A \cap (y_1\beta)^{-1}A_1y_1\beta)\varphi(M))\varphi(\beta^{-1}) = \varphi(A_0 \cap y_1^{-1}A_1y_1)\varphi(M) = \\ \varphi(D)\varphi(M) \ (*).\end{gathered}$$

For $S \subset H$, we set $\overline{S} = \varphi(S)$ and for $h \in H$, we set $\overline{h} = \varphi(h)$. We have $\overline{g} \in \overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\overline{A_1}\overline{y_1}$. Therefore $\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\overline{A_1}\overline{y_1} = \overline{g}(C_{\overline{A_0}}(\overline{x_0}) \cap \overline{y_1}^{-1}\overline{A_1}\overline{y_1})$. Considering $(*)$, we obtain:

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\overline{A_1}\overline{y_1} = \overline{g}C_{\overline{A_0} \cap \overline{y_1}^{-1}\overline{A_1}\overline{y_1}}(\overline{x_0}) \subset \overline{g}C_{\varphi(D)\varphi(M)}(\overline{x_0}).$$

Recall that $N < M$. Then $\psi : H \to H/M$ induces a homomorphism $\widetilde{\psi} : H/N \to H/M$ such that $\psi = \widetilde{\psi} \circ \varphi$. Note that $\widetilde{\psi}(\varphi(D)\varphi(M)) = \psi(D)$. Let $z \in C_{\varphi(D)\varphi(M)}(\overline{x_0})$. Then:

$$\widetilde{\psi}(z) \in C_{\psi(D)}(\psi(x_0)) \subset \psi(C_D(x_0)L) = \widetilde{\psi}(\varphi(C_D(x_0)L)).$$

Therefore $z \in \varphi(C_D(x_0)L)ker(\widetilde{\psi}) = \varphi(C_D(x_0)L)$ because $ker(\widetilde{\psi}) = \varphi(M) < \varphi(L)$. We deduce that $C_{\varphi(D)\varphi(M)}(\overline{x_0}) \subset \varphi(C_D(x_0)L)$. We conclude that

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \overline{x_1}\overline{A_1}\overline{y_1} \subset \overline{g}\varphi(C_D(x_0)L) = \varphi(gC_D(x_0)L) = \varphi((\alpha C_{A_0}(x_0) \cap x_1A_1y_1)L).$$

<u>Inductive step</u>: Suppose that $n \geq 1$ and that the result has been proved for n - 1.
Note that if $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = \emptyset$, then by the induction hypothesis, there exists a normal subgroup N of p-power index in H such that, if $\varphi : H \to H/N$ denotes the canonical projection, then

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n-1} \overline{x_i}\overline{A_i}\overline{y_i} \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i)L) = \emptyset.$$

Obviously:

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n} \overline{x_i}\overline{A_i}\overline{y_i} = \emptyset \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_iA_iy_i)L)$$

Thus we can assume that $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i \neq \emptyset$. Therefore there exists $g \in H$ such that $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = g(C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_iy_i)$. We set $F = A_0 \cap \bigcap_{i=1}^{n-1} y_i^{-1}A_iy_i$ - F is a conjugate of a special subgroup of H by [M], Lemma 6.5. We have: $\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i = gC_F(x_0)$. Now, by the case n = 1, there exists a normal subgroup M of p-power index of H such that $M < L$ and, if $\psi : H \to H/M$ denotes the canonical projection, then:

$$\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_nA_ny_n) \subset \psi((gC_F(x_0) \cap x_nA_ny_n)L).$$

This is equivalent to:

$$\psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_nA_ny_n)) \subset (gC_F(x_0) \cap x_nA_ny_n)L.$$

On the other hand, by the induction hypothesis, there exists a normal subgroup N of p-power index in H such that $N < M$ and, if $\varphi : H \to H/N$ denotes the canonical projection, then:

$$\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n-1} \overline{x_i}\overline{A_i}\overline{y_i} \subset \varphi((\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_iA_iy_i)M)$$

or, equivalently:

$$\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n-1} \overline{x_i}\overline{A_i}\overline{y_i}) \subset (\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i)M$$

Thus we have:

$$\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n} \overline{x_i}\overline{A_i}\overline{y_i}) =$$
$$\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n-1} \overline{x_i}\overline{A_i}\overline{y_i}) \cap \varphi^{-1}(\overline{x_n}\overline{A_n}\overline{y_n}) \subset$$
$$(\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n-1} x_i A_i y_i)M \cap x_n A_n y_n N = gC_F(x_0)M \cap x_n A_n y_n N.$$

Recall that $N < M$. Thus we have:

$$\varphi^{-1}(\overline{\alpha}C_{\overline{A_0}}(\overline{x_0}) \cap \bigcap_{i=1}^{n} \overline{x_i}\overline{A_i}\overline{y_i}) \subset gC_F(x_0)M \cap x_n A_n y_n M \subset$$
$$\psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0))) \cap \psi^{-1}(\psi(x_n A_n y_n)) =$$
$$\psi^{-1}(\psi(g)C_{\psi(F)}(\psi(x_0)) \cap \psi(x_n A_n y_n)) \subset (gC_F(x_0) \cap x_n A_n y_n)L =$$
$$(\alpha C_{A_0}(x_0) \cap \bigcap_{i=1}^{n} x_i A_i y_i)L.$$

□

We need the following criterion for conjugacy in HNN extensions:

Lemma 6.4 *Let $G = < H,\ t \mid t^{-1}kt = k,\ \forall\ k \in K >$ be an HNN extension. Let S be a subgroup of H. Let $g = x_0 t^{a_1} x_1 ... t^{a_n} x_n$ ($n \geq 1$) and $h = y_0 t^{b_1} y_1 ... t^{b_m} y_m$ be elements of G in reduced form. Then $h \in g^S$ if and only if all of the following conditions hold:*

1. *$m = n$ and $a_i = b_i$, for all $i \in \{1,...,n\}$,*
2. *$y_0 ... y_n \in (x_0 ... x_n)^S$,*
3. *if $\alpha \in S$ satisfies $y_0 ... y_n = \alpha x_0 ... x_n \alpha^{-1}$, then:*

$$\alpha C_S(x_0 ... x_n) \cap y_0 K x_0^{-1} \cap (y_0 y_1) K (x_0 x_1)^{-1} \cap ... \cap$$
$$(y_0 ... y_{n-1}) K (y_0 ... y_{n-1})^{-1} \neq \emptyset.$$

Proof: Proved in [M]. □

Lemma 6.5 *Let S be a special subgroup of H. Let $g \in G \setminus H$. Let $h \in G \setminus g^S$. There exists a normal subgroup L of p-power index in H such that, if $\varphi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$ and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have $\overline{\varphi}(h) \notin \overline{\varphi}(g)^{\overline{\varphi}(S)}$.*

Proof: Write $g = x_0 t^{a_1} x_1 ... t^{a_n} x_n$ and $h = y_0 t^{b_1} y_1 ... t^{b_m} y_m$ in reduced forms. We have $n \geq 1$ - as $g \notin H$.

Step 1: We assume that the first condition in Minasyan's criterion (see Lemma 6.4) is not satisfied by g and h.

The special subgroup K is closed in the pro-p topology on H (see [Lo]). Thus there exists a normal subgroup L of p-power index in H such that:

$$\forall\ i \in \{1,...,n\text{-}1\},\ x_i \notin KL\ (*),$$
$$\forall\ j \in \{1,...,m\text{-}1\},\ y_j \notin KL\ (**).$$

We denote by $\varphi : H \to P = H/L$ the canonical projection. If Q denotes the HNN extension of P relative to $\varphi(K)$:

$$Q = < P, \overline{t} \mid \overline{t}^{-1}\varphi(k)\overline{t} = \varphi(k), \forall\ k \in K >,$$

and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ - with $\overline{\varphi}_{|_H} = \varphi$ and $\overline{\varphi}(t) = \overline{t}$ -, then $\overline{\varphi}(g) = \overline{x_0}\overline{t}^{a_1}\overline{x_1}...\overline{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h) = \overline{y_0}\overline{t}^{b_1}\overline{y_1}...\overline{t}^{b_m}\overline{y_m}$ are reduced products in Q by $(*)$ and $(**)$ - where $\overline{x_i} = \overline{\varphi}(x_i)$ $(i \in \{0,...,n\})$ and $\overline{y_j} = \overline{\varphi}(y_j)$ $(j \in \{0,...,m\})$. But then the first condition in Minasyan's criterion will not hold for $\overline{\varphi}(g)$ and $\overline{\varphi}(h)$.

Conclusion of Step 1: We can assume that $m = n$ and $a_i = b_i$ for all $i \in \{1,...,n\}$.

Step 2: We assume that the second condition in Minasyan's criterion is not satisfied by g and h. We set $x = x_0...x_n$ and $y = y_0...y_n$. Thus $y \notin x^S$.

By the assumptions, x^S is finitely p-separable in H. Therefore there exists a homomorphism φ from H onto a finite p-group P such that $\varphi(y) \notin \varphi(x)^{\varphi(S)}$. Denote by Q the HNN extension of P relative to $\varphi(K)$, and by $\overline{\varphi} : G \to Q$ the homomorphism induced by φ. Now let $f : Q \to P$ be the natural homomorphism. We have:

$$f(\overline{\varphi}(g)) = f(\overline{x_0}\overline{t}^{a_1}\overline{x_1}...\overline{t}^{a_n}\overline{x_n}) = \overline{x_0}...\overline{x_n} = \varphi(x),$$
$$f(\overline{\varphi}(h)) = f(\overline{y_0}\overline{t}^{a_1}\overline{y_1}...\overline{t}^{a_n}\overline{y_n}) = \overline{y_0}...\overline{y_n} = \varphi(y).$$

Since $\varphi(y) \notin \varphi(x)^{\varphi(S)}$, we see that $\overline{\varphi}(h) \notin \overline{\varphi}(g)^{\overline{\varphi}(S)}$.

Conclusion of Step 2: We can assume that $y \in x^S$. There exists $\alpha \in S$ such that $y = \alpha x \alpha^{-1}$.

End of the proof: Considering Minasyan's criterion, since $h \notin g^S$, we must have:

$$\alpha C_S(x_0...x_n) \cap y_0 K x_0^{-1} \cap (y_0 y_1)K(x_0 x_1)^{-1} \cap ... \cap (y_0...y_{n-1})K(x_0...x_{n-1})^{-1} = \emptyset.$$

Since K is closed in the pro-p topology on H (see [Lo]), there exists a normal subgroup L of p-power index in H such that:

$$\forall\ i \in \{1,...,n\text{-}1\},\ x_i \notin KL\ (*),$$
$$\forall\ j \in \{1,...,m\text{-}1\},\ y_j \notin KL\ (**).$$

Now by Lemma 6.3, there exists a normal subgroup N of p-power index in H such that $N < L$ and, if $\varphi : H \to P = H/N$ denotes the canonical projection, then:

$$\overline{\alpha}C_{\overline{S}}(\overline{x}) \cap \overline{y_0}\overline{K}\overline{x_0}^{-1} \cap \overline{y_0}\,\overline{y_1}\overline{K}(\overline{x_0}\,\overline{x_1})^{-1} \cap ... \cap \overline{y_0}...\overline{y_{n-1}}\overline{K}(\overline{x_0}...\overline{x_{n-1}})^{-1} \subset$$
$$\varphi((\alpha C_S(x) \cap y_0Kx_0^{-1} \cap y_0y_1K(x_0x_1)^{-1} \cap ... \cap y_0...y_{n-1}K(x_0...x_{n-1})^{-1})L) =$$
$$\emptyset \ (***).$$

where $\overline{S} = \varphi(S)$, $\overline{\alpha} = \varphi(\alpha)$, $\overline{x} = \varphi(x)$, $\overline{x_i} = \varphi(x_i)$ ($i \in \{0,...,n\}$), $\overline{y_j} = \varphi(y_j)$ ($j \in \{0,...,n\}$). Let Q be the HNN extension of P relative to $\varphi(K)$ and let $\overline{\varphi} : G \to Q$ be the homomorphism induced by φ. Then, by $(*)$ and $(**)$, $\overline{\varphi}(g) = \overline{x_0}\overline{t}^{a_1}\overline{x_1}...\overline{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h) = \overline{y_0}\overline{t}^{a_1}\overline{y_1}...\overline{t}^{a_n}\overline{y_n}$ are reduced elements of Q. So, in view of $(***)$, we have $\overline{\varphi}(h) \notin \overline{\varphi}(g)^{\overline{\varphi}(S)}$. □

Lemma 6.6 *Let $g_0 = t^{a_1}x_1...t^{a_n}x_n$ ($n \geq 1$) and $h_0 = t^{b_1}y_1...t^{b_m}y_m$ be cyclically reduced elements of G. Let $h_1,..., h_k$ be elements of G. If $h_i \notin g_0^K$ for all $i \in \{1,...,k\}$, then there exists a normal subgroup L of p-power index in H such that, if $\varphi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$ and $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have:*

1. *$\overline{\varphi}(g_0) = \overline{t}^{a_1}\overline{x_1}...\overline{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h_0) = \overline{t}^{b_1}\overline{y_1}...\overline{t}^{b_m}\overline{y_m}$ are cyclically reduced in Q - where $\overline{x_i} = \overline{\varphi}(x_i)$ ($i \in \{1,...,n\}$) and $\overline{y_j} = \overline{\varphi}(y_j)$ ($j \in \{1,...n\}$).*
2. *$\overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\overline{\varphi}(K)}$ for all $i \in \{1,...,k\}$.*

Proof: Since K is closed in the pro-p topology on H (see [Lo]), there exists a normal subgroup L_0 of p-power index in H such that:

$$\forall\ i \in \{1,...,n\text{-}1\},\ x_i \notin KL_0\ (*),$$
$$\forall\ j \in \{1,...,m\text{-}1\},\ y_j \notin KL_0\ (**).$$

Let $i \in \{1,...,k\}$. Since $h_i \notin g_0^K$, there exists a normal subgroup L_i of p-power index in H such that, if $\varphi_i : H \to P_i = H/L_i$ denotes the canonical projection, if Q_i denotes the HNN extension of P_i relative to $\varphi_i(K)$ and if $\overline{\varphi_i} : G \to Q_i$ denotes the homomorphism induced by φ_i, we have $\overline{\varphi_i}(h_i) \notin \overline{\varphi_i}(g_0)^{\overline{\varphi_i}(K)}$ - by Lemma 6.5. We set $L = L_0 \cap L_1... \cap L_k$. We note that L is a normal subgroup of p-power index in H. Let $\varphi : H \to P = H/L$ be the canonical projection, let Q be the HNN extension of P relative to $\varphi(K)$ and let $\overline{\varphi} : G \to Q$ be the homomorphism induced by φ. Then since $L < L_0$, $\overline{\varphi}(g_0) = \overline{t}^{a_1}\overline{x_1}...\overline{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h_0) = \overline{t}^{b_1}\overline{y_1}...\overline{t}^{b_m}\overline{y_m}$ are cyclically reduced in Q by $(*)$ and $(**)$ - where $\overline{x_i} = \overline{\varphi}(x_i)$ ($i \in \{1,...,n\}$) and $\overline{y_j} = \overline{\varphi}(y_j)$ ($j \in \{1,...,m\}$). Moreover since $L < L_i$ for all $i \in \{1,...,k\}$, we have $\overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\overline{\varphi}(K)}$ for all $i \in \{1,...,k\}$. □

Lemma 6.7 *Let $G = < H, t \mid t^{-1}kt = k, \forall\, k \in K >$ be an HNN extension. Let S be a subgroup of H. Let $g = x_0 t^{a_1} x_1 ... t^{a_n} x_n$ be an element of G in reduced form ($n \geq 1$). Then:*

$$C_S(g) = \\ C_S(x_0...x_n) \cap x_0 K x_0^{-1} \cap (x_0x_1)K(x_0x_1)^{-1} \cap ... \cap (x_0...x_{n-1})K(x_0...x_{n-1})^{-1}.$$

Proof: Proved in [M]. □

Lemma 6.8 *Let S be a special subgroup of H. Let L be a normal subgroup of p-power index in G and let $g = x_0 t^{a_1} x_1 ... t^{a_n} x_n$ be an element of G in reduced form and not contained in H. Then there exists a normal subgroup N of p-power index of H such that if $\varphi : H \to P = H/N$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$ and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, then:*

1. *$C_{\overline{\varphi}(S)}(\overline{\varphi}(g)) \subset \overline{\varphi}(C_S(g)L)$,*
2. *$ker(\varphi) = N < H \cap L$,*
3. *$ker(\overline{\varphi}) < L$.*

Proof: We have $n \geq 1$ - as $g \notin H$.
As above, K is closed in the pro-p topology on H (see [Lo]). Therefore there exists a normal subgroup M of p-power index in H such that:

$$\forall\, i \in \{1,...,n\text{-}1\},\ x_i \notin MK\ (*).$$

We set $L' = H \cap L$. Note that L' is a normal subgroup of p-power index in H. Thus, up to replacing M by $M \cap L'$, we can assume that $M < L'$. We set $x = x_0...x_n$. We have:

$$C_S(g) = \\ C_S(x) \cap x_0 K x_0^{-1} \cap (x_0x_1)K(x_0x_1)^{-1} \cap ... \cap (x_0...x_{n-1})K(x_0...x_{n-1})^{-1},$$

by Lemma 6.7. We denote by I the intersection in the right-hand side. By Lemma 6.3, there exists a normal subgroup N of p-power index in H such that $N < M$ and if $\varphi : H \to P = H/N$ denotes the canonical projection, we have:

$$C_{\overline{S}}(\overline{x}) \cap \overline{x_0}\overline{K}\overline{x_0}^{-1} \cap \overline{x_0}\ \overline{x_1}\overline{K}(\overline{x_0}\ \overline{x_1})^{-1} \cap ... \cap \overline{x_0}...\overline{x_{n-1}}\overline{K}(\overline{x_0}...\overline{x_{n-1}})^{-1} \subset \\ \varphi(IM)$$

where $\overline{S} = \varphi(S)$, $\overline{x} = \varphi(x)$, $\overline{x_i} = \varphi(x_i)$ ($i \in \{0,...,n\}$). We denote by J the intersection in the left-hand side. Let Q be the HNN extension of P relative to $\varphi(K)$, let $\overline{\varphi} : G \to Q$ be the homomorphism induced by φ. Then $\overline{x_0}\overline{t}^{a_1}\overline{x_1}...\overline{t}^{a_n}\overline{x_n}$ is a reduced form of $\overline{\varphi}(g)$ in Q by $(*)$. But then $C_{\overline{\varphi}(S)}(\overline{\varphi}(g))$ $= J$ - by Lemma 6.7. Now $\varphi(M) < \varphi(L') = \overline{\varphi}(L') < \overline{\varphi}(L)$. Therefore:

$$C_{\overline{\varphi}(S)}(\overline{\varphi}(g)) = J \subset \varphi(IM) = \varphi(I)\varphi(M) \subset \overline{\varphi}(I)\overline{\varphi}(L) = \overline{\varphi}(C_S(g))\overline{\varphi}(L) = \overline{\varphi}(C_S(g)L).$$

Finally we remark that $ker(\varphi) = N < M < L' = H \cap L < L$. Since $ker(\overline{\varphi})$ is the normal closure of $ker(\varphi)$ in G, we conclude that $ker(\overline{\varphi}) < L$ (because L is normal in G). □

A *prefix* of $t^{a_1}x_1...t^{a_n}x_n$ is an element of G of the form $t^{a_1}x_1...t^{a_k}x_k$ for some $k \in \{0,...,n\}$. We need the following result:

Proposition 6.9 *Let $G = < H, t \mid t^{-1}kt = k, \forall\ k \in K >$ be an HNN extension. Let $g = t^{a_1}x_1...t^{a_n}x_n$ be a cyclically reduced element of G ($n \geq 1$). Let $\{p_1,...,p_{n+1}\}$ be the set of all prefixes of g - we are not assuming that $p_1,..., p_{n+1}$ are ordered. There are two cases:*

1. *if $x_n \in K$, then $n = 1$ and $C_G(g) = < t > C_K(g)$.*
2. *if $x_n \in H \setminus K$, let $\{p_1,...,p_m\}$ be the set of prefixes of g satisfying $p_i^{-1}gp_i \in g^K$ ($m \in \{0,...,n+1\}$). For each $i \in \{1,...,m\}$, we choose $\alpha_i \in K$ such that $p_i^{-1}gp_i = \alpha_i^{-1}g\alpha_i$. We set $S = \{\alpha_i p_i^{-1} \mid i \in \{1,...,m\}\}$. Then $C_G(g) = C_K(g) < g > S$.*

Proof: Proved in [M]. □

Lemma 6.10 *Let L be a normal subgroup of p-power index in G. Let $g_0 = t^{a_1}x_1...t^{a_n}x_n$ ($n \geq 1$) be a cyclically reduced element of G. There exists a normal subgroup N of p-power index in H such that, if $\varphi : H \to P = H/N$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$ and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have:*

1. *$C_Q(\overline{\varphi}(g_0)) \subset \overline{\varphi}(C_G(g_0)L)$,*
2. *$ker(\varphi) = N < H \cap L$,*
3. *$ker(\overline{\varphi}) < L$.*

Proof: Let $\{p_1,...,p_{n+1}\}$ be the set of all prefixes of g_0. Renumbering $p_1,..., p_{n+1}$, if necessary, we can assume that there exists $m \in \{0,...,n\}$ such that $p_i^{-1}g_0p_i \in g_0^K$ for all $i \in \{1,...,m\}$ and $p_i^{-1}g_0p_i \notin g_0^K$ for all $i \in \{m+1,...,n+1\}$. For each $i \in \{1,...,m\}$, we choose $\alpha_i \in K$ such that $p_i^{-1}g_0p_i = \alpha_i^{-1}g_0\alpha_i$. We set $S = \{\alpha_i p_i^{-1} \mid i \in \{1,...,m\}\}$. We set $h_i = p_i^{-1}g_0p_i$ for all $i \in \{m+1,...,n+1\}$. By Lemma 6.6, there exists a normal subgroup N_1 of p-power index in H such that, if $\varphi_1 : H \to P_1 = H/N_1$ denotes the canonical projection, if Q_1 denotes the HNN extension of P_1 relative to $\varphi_1(K)$, and if $\overline{\varphi_1} : G \to Q_1$ denotes the homomorphism induced by φ_1, then $\varphi_1(g_0)$ is cyclically reduced in Q_1, and $\overline{\varphi_1}(h_i) \notin \overline{\varphi_1}(g_0)^{\overline{\varphi_1}(K)}$ for

all $i \in \{m+1,...,n+1\}$. On the other hand, by Lemma 6.8, there exists a normal subgroup N_2 of p-power index in H such that, if $\varphi_2 : H \to P_2 = H/N_2$ denotes the canonical projection, if Q_2 denotes the HNN extension of P_2 relative to $\varphi_2(K)$ and if $\overline{\varphi_2} : G \to Q_2$ denotes the homomorphism induced by φ_2, we have: $C_{\overline{\varphi_2}(K)}(\overline{\varphi_2}(g_0)) \subset \overline{\varphi_2}(C_K(g_0)L)$, $ker(\varphi_2) < H \cap L$ and $ker(\overline{\varphi_2}) < L$. We set $N = N_1 \cap N_2$. Note that N is a normal subgroup of p-power index in H. Let $\varphi : H \to P = H/N$ be the canonical projection, let Q be the HNN extension of P relative to $\varphi(K)$ and let $\overline{\varphi} : G \to Q$ be the homomorphism induced by φ. Since $N < N_1$, $\overline{\varphi}(g_0)$ is cyclically reduced in Q and $\overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\overline{\varphi}(K)}$ for all $i \in \{m+1,...,n+1\}$. On the other hand, since $N < N_2$, we have:

$$\overline{\varphi}^{-1}(C_{\overline{\varphi}(K)}(\overline{\varphi}(g_0))) \subset \overline{\varphi_2}^{-1}(C_{\overline{\varphi_2}(K)}(\overline{\varphi_2}(g_0))) \subset C_K(g_0)L \ (*).$$

There are two cases:
<u>Case 1</u>: $x_n \in K$. Then $n = 1$, $C_G(g_0) = < t > C_K(g_0)$ and $C_Q(\overline{\varphi}(g_0)) = < \overline{t} > C_{\varphi(K)}(\overline{\varphi}(g_0))$ - by Proposition 6.9. Now $(*)$ implies:

$$C_Q(\overline{\varphi}(g_0)) \subset < \overline{\varphi}(t) > \overline{\varphi}(C_K(g_0)L) = \overline{\varphi}(< t > C_K(g_0)L) = \overline{\varphi}(C_G(g_0)L).$$

<u>Case 2</u>: $x_n \in H \setminus K$. If $i \in \{1,...,m\}$, $\overline{\varphi}(p_i)^{-1}\overline{\varphi}(g_0)\overline{\varphi}(p_i) = \overline{\varphi}(p_i^{-1}g_0p_i) \in \overline{\varphi}(g_0)^{\varphi(K)}$ - because $p_i^{-1}g_0p_i \in g_0^K$-, whereas if $i \in \{m+1,...,n+1\}$, $\overline{\varphi}(p_i)^{-1}$ $\overline{\varphi}(g_0)\overline{\varphi}(p_i) = \overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\varphi(K)}$. Therefore $\{\overline{\varphi}(p_1),...,\overline{\varphi}(p_m)\}$ is the set of all prefixes of $\overline{\varphi}(g_0)$ satisfying $\overline{\varphi}(p_i)^{-1}\overline{\varphi}(g_0)\overline{\varphi}(p_i) \in \overline{\varphi}(g_0)^{\varphi(K)}$. Now $C_G(g_0)$ $= C_K(g_0) < g_0 > S$ and $C_Q(\overline{\varphi}(g_0)) = C_{\varphi(K)}(\overline{\varphi}(g_0)) < \overline{\varphi}(g_0) > \overline{S}$ where $\overline{S}$ $= \overline{\varphi}(S) = \{\overline{\varphi}(\alpha_i)\overline{\varphi}(p_i)^{-1} \mid i \in \{1,...,m\}\}$ - by Proposition 6.9. We deduce that:

$$C_Q(\overline{\varphi}(g_0)) \subset \overline{\varphi}(C_K(g_0)L) < \overline{\varphi}((g_0) > \overline{\varphi}(S) = \overline{\varphi}(C_K(g_0)L < g_0 > S) = \overline{\varphi}(C_G(g_0)L).$$

□

Proposition 6.11 *Let G be a right-angled Artin group of rank r ($r \geq 1$). Let $g \in G$. If $g \neq 1$, then there exists a special subgroup H of rank r-1 of G such that $g \notin H^G$.*

Proof: Proved in [M]. □

Lemma 6.12 *Every special subgroup S of G satisfies the p centralizer condition in G, pCC_G.*

Proof: Let $g \in G$. Let L be a normal subgroup of p-power index in G. There are two cases:
<u>Case 1</u>: $S \neq G$.
Let H be a special subgroup of rank r - 1 of G such that $S < H$. Then G can be written as an HNN extension of H, relative to a special subgroup K of H:

$$G = < H, t \mid t^{-1}kt = k, \forall\ k \in K >.$$

We set $L' = H \cap L$. We note that L' is a normal subgroup of p-power index in H. There are two cases:
<u>Subcase 1</u>: $g \in H$.
By the assumptions, the pair (S,g) satisfies the p centralizer condition in H (pCC_H). There exists a normal subgroup M of p-power index in H such that $M < L'$ and, if $\psi : H \to P = H/M$ denotes the canonical projection, we have:

$$C_{\psi(S)}(\psi(g)) \subset \psi(C_S(g)L')\ (*).$$

We denote by $f : G \to H$ the natural homomorphism. We note that $f^{-1}(M)$ is a normal subgroup of p-power index in G (because $f^{-1}(M)$ is the kernel of the homomorphism $\psi \circ f$). Therefore, $N = L \cap f^{-1}(M)$ is a normal subgroup of p-power index in G. Moreover $N < L$ and $f(N) < M$. We denote by $\varphi : G \to Q = G/N$ the canonical projection. We observe that $ker(\psi) = M$, $ker(\varphi) = N$, $M < f^{-1}(M) \cap L \cap H = N \cap H$ and $N \cap H \subset f(N) < M$. Therefore $M = N \cap H$. Thus we can assume that $P < Q$ and $\varphi_{|_H} = \psi$. But then $\psi(L') = \varphi(L') \subset \varphi(L)$. Recall that $g \in H$ and $S < H$. Thus considering $(*)$, we obtain:

$$C_{\varphi(S)}(\varphi(g)) = C_{\psi(S)}(\psi(g)) \subset \psi(C_S(g))\psi(L') \subset \varphi(C_S(g))\varphi(L) = \varphi(C_S(g)L).$$

<u>Subcase 2</u>: $g \in G \setminus H$.
Write $g = x_0 t^{a_1} x_1 ... t^{a_n} x_n$ in a reduced form ($n \geq 1$). Then, by Lemma 6.8, there exists a normal subgroup M of p-power index in H such that, if $\psi : H \to P = H/M$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$ and if $\overline{\psi} : G \to Q$ denotes the homomorphism induced by ψ, then: $C_{\overline{\psi}(S)}(\overline{\psi}(g)) \subset \overline{\psi}(C_S(g)L)$, $ker(\psi) < H \cap L$, and $ker(\overline{\psi}) < L$. We note that $\overline{\psi}(S) \cap \overline{\psi}(L) = \psi(S) \cap \overline{\psi}(L) < P$ is finite. Since Q is residually p-finite (see [Lo], Lemma 2.8), $\overline{\psi}(g)^{\overline{\psi}(S) \cap \overline{\psi}(L)}$ is finitely p-separable in Q. Therefore, by Lemma 3.5, there exists a normal subgroup N of p-power index in Q such that $N < \overline{\psi}(L)$ and, if $\chi : Q \to R = Q/N$ denotes the canonical projection, then:

$$C_{\chi(\overline{\psi}(S))}(\chi(\overline{\psi}(g))) \subset \chi(C_{\overline{\psi}(S)}(\overline{\psi}(g))\overline{\psi}(L)).$$

We set $\varphi = \chi \circ \overline{\psi} : G \to R$. We have: $ker(\varphi) = \overline{\psi}^{-1}(ker(\chi)) = \overline{\psi}^{-1}(N) \subset \overline{\psi}^{-1}(\overline{\psi}(L)) = Lker(\overline{\psi})$. Now $ker(\overline{\psi}) < L$. Then $ker(\varphi) < L$. And:

$$C_{\varphi(S)}(\varphi(g)) = C_{\chi(\overline{\psi}(S))}(\chi(\overline{\psi}(g))) \subset \chi(C_{\overline{\psi}(S)}(\overline{\psi}(g))\overline{\psi}(L)) \subset \chi(\overline{\psi}(C_S(g)L)\overline{\psi}(L)) = \varphi(C_S(g)L).$$

Case 2: $S = G$.
If $g = 1$, then the result is trivial. Thus we can assume that $g \neq 1$. Then, by Proposition 6.11, there exists a special subgroup of rank r - 1 H of G such that $g \notin H^G$. As usual, G can be written as an HNN extension of H relative to a special subgroup K of H:

$$G = < H, t \mid t^{-1}kt = k, \forall\ k \in K >.$$

Let $g_0 = t^{a_1}x_1...t^{a_n}x_n$ be a cyclically reduced element in G conjugate to g. Choose $\alpha \in G$ such that $g = \alpha g_0 \alpha^{-1}$. Note that $g \notin H^G$ implies that $n \geq 1$. By Lemma 6.10, there exists a normal subgroup M of p-power index in H such that, if $\psi : H \to P = H/M$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\psi(K)$ and if $\overline{\psi} : G \to Q$ denotes the homomorphism induced by ψ, then: $C_Q(\overline{\psi}(g_0)) \subset \overline{\psi}(C_G(g_0)L)$, $ker(\psi) < H \cap L$ and $ker(\overline{\psi}) < L$. Now Q is hereditarily conjugacy p-separable by Corollary 4.3. Then Q satisfies the p centralizer condition by Proposition 3.6. There exists a normal subgroup N of p-power index in Q such that $N < \overline{\psi}(L)$ and if $\chi : Q \to R = Q/N$ denotes the canonical projection, we have:

$$C_R(\chi(\overline{\psi}(g_0))) \subset \chi(C_Q(\overline{\psi}(g_0))\overline{\psi}(L)).$$

We set $\varphi = \chi \circ \overline{\psi} : G \to R$. As above, we have $ker(\varphi) = \overline{\psi}^{-1}(ker(\chi)) = \overline{\psi}^{-1}(N) \subset \overline{\psi}^{-1}(\overline{\psi}(L)) = Lker(\overline{\psi})$. Now $ker(\overline{\psi}) < L$. Then $ker(\varphi) < L$. And:

$$C_R(\varphi(g_0)) = C_{\varphi(G)}(\varphi(g_0)) = C_{\chi(\overline{\psi}(G))}(\chi(\overline{\psi}(g_0))) \subset \chi(C_{\overline{\psi}(G)}(\overline{\psi}(g_0))\overline{\psi}(L)) \subset \chi(\overline{\psi}(C_G(g_0)L)\overline{\psi}(L)) = \varphi(C_G(g_0)L).$$

Finally:

$$\varphi(\alpha)C_R(\varphi(g_0))\varphi(\alpha)^{-1} \subset \varphi(\alpha)\varphi(C_G(g_0)L)\varphi(\alpha)^{-1}.$$

That is,

$$C_R(\varphi(g)) \subset \varphi(C_G(g)L).$$

□

Lemma 6.13 *For every $g \in G$ and for every special subgroup S of G, g^S is finitely p-separable in G.*

Proof: There are two cases:
Case 1: $S \neq G$.
Let H be a special subgroup of rank r - 1 of G such that $S < H$. As usual, G can be written as an HNN extension of H relative to a special subgroup K of H:

$$G = < H, t \mid t^{-1}kt = k, \forall\ k \in K >.$$

Let $g \in G$. There are two cases:
<u>Subcase 1</u>: $g \in H$.
Then g^S is finitely p-separable in H by the assumptions. Since G is residually p-finite by Theorem 6.1, g^S is finitely p-separable in G by Lemma 5.10.
<u>Subcase 2</u>: $g \in G \setminus H$.
Let $h \in G \setminus g^S$. By Lemma 6.5, there exists a normal subgroup L of p-power index in H such that, if $\varphi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$ and if $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have $\overline{\varphi}(h) \notin \overline{\varphi}(g)^{\overline{\varphi}(S)}$. Now $\overline{\varphi}(S) = \varphi(S) < P$ is finite and Q is residually p-finite (see [Lo], Lemma 2.8). Then there exists a homomorphism $\chi : Q \to R$ from Q onto a finite p-group R such that $\chi(\overline{\varphi}(h)) \notin \chi(\overline{\varphi}(g)^{\overline{\varphi}(S)})$. We set $\psi = \chi \circ \overline{\varphi} : G \to R$. It is clear that ψ suits.
<u>Case 2</u>: $S = G$.
Let $g \in G$.
If $g = 1$, then, since G is residually p-finite by Theorem 6.1, $g^G = \{1\}$ is finitely p-separable in G. Thus we can assume that $g \neq 1$. Then, by Proposition 6.11, there exits a special subgroup of rank r - 1 H of G such that $g \notin H^G$. As usual, G can be written as an HNN extension of H relative to a special subgroup K of H:

$$G = < H, t \mid t^{-1}kt = k, \forall\ k \in K >.$$

Let $h \in G \setminus g^G$. Let $g_0 = t^{a_1}x_1...t^{a_n}x_n$ and $h_0 = t^{b_1}y_1...t^{b_m}y_m$ be cyclically reduced elements of G conjugate to g and h respectively. Note that $g \notin H^G$ implies that $n \geq 1$. There are two cases:
<u>Subcase 1</u>: $h_0 \in H$. Then, by Lemma 6.6, there exists a normal subgroup L of p-power index in H such that, if $\varphi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$ and $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, we have: $\overline{\varphi}(g_0) = \overline{t}^{a_1}\overline{x_1}...\overline{t}^{a_n}\overline{x_n}$ is cyclically reduced in Q - where $\overline{x_i} = \overline{\varphi}(x_i)$ ($i \in \{1,...,n\}$). Since $n \geq 1$, we have: $\overline{\varphi}(g_0) \notin P^Q = \overline{\varphi}(H^G)$. Therefore $\overline{\varphi}(g_0) \notin \overline{\varphi}(h_0)^Q = \overline{\varphi}(h_0^G) \subset \overline{\varphi}(H^G)$. Now Q is conjugacy p-separable by Corollary 4.3. Thus there exists a homomorphism χ from Q onto a finite p-group R such that $\chi(\overline{\varphi}(g_0)) \notin \chi(\overline{\varphi}(h_0))^R$. We set $\psi = \chi \circ \overline{\varphi} : G \to R$. It is clear that ψ suits.
<u>Subcase 2</u>: $h_0 \in G \setminus H$. Let $\{h_1,...,h_m\}$ be the set of all cyclic permutations of h_0. Then, since $h \notin g^G$, we have: $h_i \notin g_0^G$ for all $i \in \{1,...,m\}$. Therefore, by Lemma 6.6, there exists a normal subgroup L of p-power index in H such that, if $\varphi : H \to P = H/L$ denotes the canonical projection, if Q denotes the HNN extension of P relative to $\varphi(K)$ and $\overline{\varphi} : G \to Q$ denotes the homomorphism induced by φ, then: $\overline{\varphi}(g_0) = \overline{t}^{a_1}\overline{x_1}...\overline{t}^{a_n}\overline{x_n}$ and $\overline{\varphi}(h_0) = \overline{t}^{b_1}\overline{y_1}...\overline{t}^{b_m}\overline{y_m}$ are cyclically reduced in Q - where $\overline{x_i} = \overline{\varphi}(x_i)$ ($i \in \{1,...,n\}$) and $\overline{y_j} = \overline{\varphi}(y_j)$ ($j \in \{1,...n\}$) - and $\overline{\varphi}(h_i) \notin \overline{\varphi}(g_0)^{\overline{\varphi}(K)}$ for all $i \in \{1,...,m\}$. Consequently, by Lemma 2.3, $\overline{\varphi}(g_0) \notin \overline{\varphi}(h_0)^Q$. Now Q is conjugacy p-separable by Corollary 4.3. Then there exists a homomorphism χ from Q onto a finite p-group R

such that: $\chi(\overline{\varphi}(g_0)) \notin \chi(\overline{\varphi}(h_0))^R$. Therefore $\chi(\overline{\varphi}(g)) \notin \chi(\overline{\varphi}(h))^R$. We set $\psi = \chi \circ \overline{\varphi} : G \to R$. It is clear that ψ suits. □

Proof of Proposition 6.2: We argue by induction on the rank r of G. If $r = 0$, then the result is trivial. Thus we can assume that $r \geq 1$ and that the result has been proved for 1,..., r - 1. Now, Proposition 6.2.1 follows from Lemma 6.12, and Proposition 6.2.2 follows from Lemma 6.13. □

We are now ready to prove:

Theorem 6.14 *Every right-angled Artin group is hereditarily conjugacy p-separable.*

Proof: Let G be a right-angled Artin group. Let $g \in G$. Then g^G is finitely p-separable in G by Proposition 6.2.1. We deduce that G is conjugacy p-separable. On the other hand, G satisfies the p centralizer condition by Proposition 6.2.2. We conclude that G is hereditarily conjugacy p-separable by Proposition 3.6. □

7 Applications

The first application that we mention is an application of our main result to separability properties of right-angled Artin groups.

Recall that a group is said to be conjugacy $\mathcal{K}$-separable, where $\mathcal{K}$ is a class of groups, if for all g, $h \in G$, either $g \sim h$, or there exists a homomorphism φ from G to some group of $\mathcal{K}$ such that $\varphi(g) \nsim \varphi(h)$.

For a group G, we denote by $(C^n(G))_{n\geq 1}$ the lower central series of G. Recall that $(C^n(G))_{n\geq 1}$ is defined inductively by $C^1(G) = G$, and $C^{n+1}(G) = [G, C^n(G)]$ for all $n \geq 1$.

Theorem 7.1 *Every right-angled Artin group is conjugacy $\mathcal{K}$-separable, where $\mathcal{K}$ is the class of all torsion-free nilpotent groups.*

Proof: Let G be a right-angled Artin group. Let g, $h \in G$ such that $g \nsim h$. Let p be a prime number. Then G is conjugacy p-separable by Theorem 6.14. Thus, there exists a homomorphism φ from G onto a finite p-group P such that $\varphi(g) \nsim \varphi(h)$. Now, P is nilpotent. Therefore, there exists $n \geq 1$ such that $C^n(P) = \{1\}$. Let $\pi : G \to \frac{G}{C^n(G)}$ be the canonical projection. It follows from [DK2] that $\frac{G}{C^n(G)}$ is a torsion-free nilpotent group. Since $\varphi(C^n(G)) < C^n(P) = \{1\}$, φ induces a homomorphism $\widetilde{\varphi} : \frac{G}{C^n(G)} \to P$ such that $\varphi = \widetilde{\varphi} \circ \pi$. As $\varphi(g) \nsim \varphi(h)$, we have $\pi(g) \nsim \pi(h)$. □

We turn now to applications of our main result to residual properties of outer automorphism groups of right-angled Artin groups.

Let G be a group. An automorphism φ of G is said to be *conjugating* if for every $g \in G$, $\varphi(g) \sim g$. We say that G has *Property A* if every conjugating automorphism of G is inner. Minasyan proved (see [M]):

Proposition 7.2 *Right-angled Artin groups have Property A.*

Let G be a group. We denote by $\mathcal{I}_p(G)$ the kernel of the natural homomorphism $Out(G) \to GL(H_1(G, \mathbb{F}_p))$ (where $\mathbb{F}_p$ denotes the finite field with p elements). Paris proved (see [P]):

Theorem 7.3 *Let G be a finitely generated group. If G is conjugacy p-separable and has Property A, then $\mathcal{I}_p(G)$ is residually p-finite.*

Recall that a group G is said to be virtually $\mathcal{P}$, where $\mathcal{P}$ is a group property, if there exists a finite index subgroup $H < G$ such that H has Property $\mathcal{P}$. We are now ready to prove:

Theorem 7.4 *The outer automorphism group of a right-angled Artin group is virtually residually p-finite.*

Proof: This follows immediately from Theorem 6.14, Proposition 7.2 and Theorem 7.3. □

Recall that a group G is $\mathcal{K}$-residual, where $\mathcal{K}$ is a class of groups, if for all $g \in G \backslash \{1\}$, there exists a homomorphism φ from G to some group of $\mathcal{K}$ such that $\varphi(g) \neq 1$. Myasnikov proved (see [My]):

Theorem 7.5 *Let G be a finitely generated group. If G is conjugacy p-separable and has property A, then $Out(G)$ is $\mathcal{K}$-residual, where $\mathcal{K}$ is the class of all outer automorphism groups of finite p-groups.*

We are now ready to prove:

Theorem 7.6 *The outer automorphism group of a right-angled Artin group is $\mathcal{K}$-residual, where $\mathcal{K}$ is the class of all outer automorphism groups of finite p-groups.*

Proof: This follows immediately from Theorem 6.14, Proposition 7.2 and Theorem 7.5. □

The next application was suggested to the author by Ruth Charney and Luis Paris.

Let $G = G_\Gamma$ be a right-angled Artin group. Let r be the rank of G. We denote by $T(G)$ the kernel of the natural homomorphism $Aut(G) \to GL_r(\mathbb{Z})$ and by $\mathcal{T}(G)$ the kernel of the natural homomorphism $Out(G) \to GL_r(\mathbb{Z})$. We call $\mathcal{T}(G)$ the Torelli group of G. Note that $\mathcal{T}(G) = T(G)/Inn(G)$. In [D2], Day proved that $T(G)$ is finitely generated. Therefore $\mathcal{T}(G)$ is finitely generated. Now, we prove the following:

Theorem 7.7 *The Torelli group of a right-angled Artin group is residually torsion-free nilpotent.*

In order to prove Theorem 7.7, we have to introduce the notion of separating $\mathbb{Z}$-linear central filtration.

Recall that a *central filtration* on a group G is a sequence $(G_n)_{n\geq 1}$ of subgroups of G satisfying the conditions:

$$G_1 = G,$$
$$G_n > G_{n+1},$$
$$[G_m, G_n] < G_{m+n} \text{ for all } m,\, n \geq 1.$$

Let $\mathcal{F} = (G_n)_{n\geq 1}$ be a central filtration. Then the mapping $G \times G \to G$, $(x, y) \mapsto xyx^{-1}y^{-1}$ induces on:

$$\mathcal{L}_{\mathcal{F}}(G) = \bigoplus_{n\geq 1} \frac{G_n}{G_{n+1}}$$

a Lie bracket which makes $\mathcal{L}_{\mathcal{F}}(G)$ into a graded Lie $\mathbb{Z}$-algebra.

We say that $(G_n)_{n\geq 1}$ is a *separating filtration* if $\cap_{n\geq 1} G_n = \{1\}$. We say that $(G_n)_{n\geq 1}$ is $\mathbb{Z}$-*linear* if for all $n \geq 1$, the $\mathbb{Z}$-module $\frac{G_n}{G_{n+1}}$ is free of finite rank.

For a group G, we denote by $(C^n_{\mathbb{Z}}(G))_{n\geq 1}$ the sequence of subgroups of G defined inductively by $C^1_{\mathbb{Z}}(G) = G$, and $[G, C^n_{\mathbb{Z}}(G)] < C^{n+1}_{\mathbb{Z}}(G)$ and $\frac{C^{n+1}_{\mathbb{Z}}(G)}{[G, C^n_{\mathbb{Z}}(G)]}$ is the torsion subgroup of $\frac{C^n_{\mathbb{Z}}(G)}{[G, C^n_{\mathbb{Z}}(G)]}$ for all $n \geq 1$.

Proposition 7.8 *For all m, $n \geq 1$, $[C^m_{\mathbb{Z}}(G), C^n_{\mathbb{Z}}(G)] < C^{m+n}_{\mathbb{Z}}(G)$.*

Proof: Proved in [BL] (see Proposition 7.2). □

Thus, $(C^n_{\mathbb{Z}}(G))_{n\geq 1}$ is a central filtration on G. We denote by $\mathcal{L}_{\mathbb{Z}}(G)$ the corresponding graded Lie $\mathbb{Z}$-algebra.

For a group G, we denote by $\mathcal{A}b(G)$ the abelianization $\frac{G}{[G,G]}$ of G, and by $Z(G)$ the center of G. For a Lie algebra $\mathfrak{g}$, we denote by $Z(\mathfrak{g})$ the center of $\mathfrak{g}$.

Let G be a group. For $n \geq 1$, we denote by A_n the kernel of the natural homomorphism $Aut(G) \to Aut(\frac{G}{C^{n+1}_{\mathbb{Z}}(G)})$. Let $\pi : Aut(G) \to Out(G)$ be the canonical projection. For $n \geq 1$, we set $B_n = \pi(G_n)$.

Theorem 7.9 *If $\mathcal{A}b(G)$ is finitely generated, and $Z(\mathbb{F}_p \otimes \mathcal{L}_{\mathbb{Z}}(G)) = \{0\}$ for every prime number p, then $(B_n)_{n\geq 1}$ is a $\mathbb{Z}$-linear central filtration on B_1. Furthermore, $(B_n)_{n\geq 1}$ is separating if and only if G satisfies the condition:*

$(IN(G))$: For every $\varphi \in Aut(G)$, if φ induces an inner automorphism of $\frac{G}{C^n_{\mathbb{Z}}(G)}$ for all $n \geq 1$, then φ is inner.

Proof: Proved in [BL] (see Corollary 9.9). □

From now on, we assume that $G = G_\Gamma$ is a right-angled Artin group of rank r ($r \geq 1$). We shall show that G satisfies the conditions of Theorem 7.9. Since B_1 is precisely the Torelli group of G, Theorem 7.7 will then result from the following:

Theorem 7.10 *Let B be a group. Suppose that B admits a separating $\mathbb{Z}$-linear central filtration, $(B_n)_{n\geq 1}$. Then B is residually torsion-free nilpotent.*

Proof: Proved in [BL] (see Theorem 6.1). □

We need to introduce the following notations. Let K be a commutative ring. We denote by M_Γ the monoid defined by the presentation:

$$M_\Gamma = < V \mid vw = wv, \forall \{v,w\} \in E >,$$

by A_Γ the associative K-algebra of the monoid M_Γ, and by L_Γ the Lie K-algebra defined by the presentation:

$$L_\Gamma = < V \mid [v;w] = 0, \forall \{v,w\} \in E >.$$

The following theorem is due to Duchamp and Krob (see [DK]):

Theorem 7.11 *The K-module L_Γ is free.*

Thus, by the Poincaré-Birkhoff-Witt theorem, L_Γ can be regarded as a Lie subalgebra of its enveloping algebra, for which Duchamp and Krob established the following (see [DK]):

Theorem 7.12 *The enveloping algebra of L_Γ is isomorphic to A_Γ.*

Furthermore, in [DK2], Duchamp and Krob proved the following theorem, which generalizes a well-known theorem of Magnus (see [MKS]):

Theorem 7.13 *Suppose that $K = \mathbb{Z}$. The graded Lie $\mathbb{Z}$-algebra associated to the lower central series of G is isomorphic to L_Γ.*

Let $Z = \cap_{v \in V} star(v)$. Then $Z(G)$ is the special subgroup of G generated by Z. Let H be the special subgroup of G generated by $V \setminus Z$. We have: $G = H \times Z$. The proof of Theorem 7.7 will use the following:

Lemma 7.14 *Suppose that $Z(G_\Gamma) = \{1\}$. Then $Z(L_\Gamma) = \{0\}$.*

Proof: Let $g \in Z(L_\Gamma)$. Suppose that $g \neq 0$. Let $v \in V$. We have $[g; v] = 0$ (in L_Γ). Now, L_Γ can be regarded as a Lie subalgebra of A_Γ by Theorem 7.11 and Theorem 7.12. Thus, we have $gv = vg$ (in A_Γ). Therefore g belongs to the subalgebra of A_Γ generated by $star(v)$ (see [KR]). Since v is arbitrary, this leads to a contradiction with our assumption. □

Remark: In the above lemma, K is arbitrary.

From now on, we assume that $K = \mathbb{Z}$. Recall that $(C^n(G))_{n \geq 1}$ denotes the lower central series of G. We are now ready to prove:

Theorem 7.15 *The Torelli group of a right-angled Artin group is residually torsion-free nilpotent.*

Proof: Let $Z = \cap_{v \in V} star(v)$. Then $Z(G)$ is the special subgroup of G generated by Z. Let H be the special subgroup of G generated by $V \setminus Z$. We have: $G = H \times Z(G)$. Note that $Z(H) = \{1\}$. First, we show that $\mathcal{T}(G) = \mathcal{T}(H)$. Let $\varphi : T(G) \to T(H)$ be the homomorphism defined by:

$$\varphi(\alpha)(h, k) = (\alpha(h), k),$$

for all $\alpha \in \mathcal{T}(G)$, $h \in H$, $k \in Z(G)$. Clearly, φ is well-defined and injective. We shall show that φ is surjective. Let $\beta \in \mathcal{T}(G)$. For $g \in G$, we set $\beta(g) = (\beta_1(g), \beta_2(g))$, where $\beta_1(g) \in H$ and $\beta_2(g) \in Z(G)$. Let $h \in H$. We denote by $\overline{h}$ the canonical image of h in $\mathcal{A}b(H)$. Note that the canonical image of h in $\mathcal{A}b(G) = \mathcal{A}b(H) \times Z(G)$ is $(\overline{h},1)$. Since $\beta \in T(G)$, we have: $(\overline{h},1) = (\overline{\beta_1(h)},\beta_2(h))$, and then $\beta_2(h) = 1$. Let $k \in Z(G)$. Since $\beta(k)$ lies in the center of G, we have $\beta_1(k) = 1$. Note that the canonical image of k in $\mathcal{A}b(G)$ is $(1,k)$. As $\beta \in T(G)$, we have $\beta_2(k) = k$. Finally, we have:

$$\beta(h, k) = (\beta_1(h), k),$$

for all $h \in H$ and $k \in Z(G)$. Applying the same argument to β^{-1}, we obtain that the restriction α of β_1 to H is an automorphism of H. Therefore $\beta = \varphi(\alpha)$. We have shown that $T(G) = T(H)$. It is easily seen that this implies that $\mathcal{T}(G) = \mathcal{T}(H)$. Thus, up to replacing G by H, we can assume that $Z(G) = \{1\}$. On the other hand, it follows from [DK2] that for all $n \geq 1$, there exists $d_n \in \mathbb{N}$ such that:

$$\frac{C^n(G)}{C^{n+1}(G)} \simeq \mathbb{Z}^{d_n}.$$

Now, for all $n \geq 1$, $C^n(G) < C^n_{\mathbb{Z}}(G)$, and $\frac{C^n_{\mathbb{Z}}(G)}{C^n(G)}$ is the torsion subgroup of $\frac{G}{C^n(G)}$ by [BL], Proposition 7.2. It follows that $C^n_{\mathbb{Z}}(G) = C^n(G)$ for all $n \geq 1$, and that $\mathcal{L}_{\mathbb{Z}}(G) = L_\Gamma$ by Theorem 7.13. Since $Z(G) = \{1\}$, we have $Z(\mathbb{F}_p \otimes L_\Gamma) = \{0\}$ for every prime number p - by Lemma 7.14. We deduce that $(B_n)_{n\geq 1}$ is a $\mathbb{Z}$-linear central filtration on $\mathcal{T}(G)$ by Theorem 7.9. Now, let $\varphi \in Aut(G)$ such that φ induces an inner automorphism on $\frac{G}{C^n(G)}$ for all $n \geq 1$. Let $g \in G$. Suppose that $\varphi(g)$ and g are not conjugate in G. Then it follows from the proof of Theorem 7.1 that there exists $n \geq 1$ such that the canonical images of $\varphi(g)$ and g in $\frac{G}{C^n(G)}$ are not conjugate in $\frac{G}{C^n(G)}$ - contradicting our assumption. Thus φ is conjugating. Therefore φ is inner by Proposition 7.2. We deduce that $(B_n)_{n\geq 1}$ is separating by Theorem 7.9. We conclude that $\mathcal{T}(G)$ is residually torsion-free nilpotent by Theorem 7.10. □

Corollary 7.16 *The Torelli group of a right-angled Artin group is residually p-finite for every prime number p.*

Proof: This follows immediately from Theorem 7.15 and Theorem 2.1.(i) in [G]. □

Corollary 7.17 *The Torelli group of a right-angled Artin group is bi-orderable.*

Proof: This follows immediately from Theorem 7.15 and [R] (see also [MR], Theorem 2.4.8.). □

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