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| Jean-Baptiste Gouéré. Percolation in a multiscale Boolean model. 2010. hal-00519310v2

**HAL Id: hal-00519310**

**<https://hal.science/hal-00519310v2>**

Preprint submitted on 13 Nov 2010 (v2), last revised 9 Mar 2011 (v3)

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# Percolation in a multiscale Boolean model

Jean-Baptiste Gou  r   \*

## Abstract

We consider percolation in a multiscale Boolean model. This model is defined as the union of scaled independent copies of a given Boolean model. The scale factor of the  $n^{\text{th}}$  copy is  $\rho^{-n}$ . We prove, under optimal integrability assumptions, that no percolation occurs in the multiscale Boolean model for large enough  $\rho$  if the rate of the Boolean model is below some critical value.

## 1 Introduction and statement of the main result

### 1.1 The Boolean model

Let  $d \geq 2$ . Let  $\mu$  be a finite measure on  $]0, +\infty[$ . We assume that the mass of  $\mu$  is positive. Let  $\xi$  be a Poisson point process on  $\mathbb{R}^d \times ]0, +\infty[$  whose intensity is the product of the Lebesgue measure on  $\mathbb{R}^d$  by  $\mu$ . With  $\xi$  we associate a random set  $\Sigma(\mu)$  defined as follows:

$$\Sigma(\mu) = \bigcup_{(c,r) \in \xi} B(c,r)$$

where  $B(c,r)$  is the open Euclidean ball of radius  $r$  centered at  $c$ . The random set  $\Sigma(\mu)$  is the Boolean model with parameter  $\mu$ . We shall sometimes write  $\Sigma$  to simplify the notations.

The following description may be more intuitive. Let  $\chi$  denote the projection of  $\xi$  on  $\mathbb{R}^d$ . With probability one this projection is one-to-one. We can therefore write:

$$\xi = \{(c, r(c)), c \in \chi\}.$$

Write  $\mu = m\nu$  where  $\nu$  is a probability measure. Then,  $\chi$  is a Poisson point process on  $\mathbb{R}^d$  with density  $m$ . Moreover, given  $\chi$ , the sequence  $(r(c))_{c \in \chi}$  is a sequence of independent random variable with common distribution  $\nu$ . We shall not use this point of view.

### 1.2 Percolation in the Boolean model

Let  $C$  denote the connected component of  $\Sigma$  that contains the origin. We say that  $\Sigma$  percolates if  $C$  is unbounded with positive probability. We refer to the book by Meester and Roy [9] for background on continuum percolation. Set:

$$\lambda_c(\mu) = \inf\{\lambda > 0 : \Sigma(\lambda\mu) \text{ percolates}\}.$$

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One easily check that  $\lambda_c(\mu)$  is finite as soon as  $\mu$  has a positive mass. In [4] we proved that  $\lambda_c(\mu)$  is positive if and only if:

$$\int r^d \mu(dr) < \infty.$$

The only if part had been proved earlier by Hall [7]. For all  $A, B \subset \mathbb{R}^d$ , we write  $A \leftrightarrow_\Sigma B$  if there exists a path in  $\Sigma$  from  $A$  to  $B$ . We denote by  $S(c, r)$  the Euclidean sphere of radius  $r$  centered at  $c$ :

$$S(c, r) = \{x \in \mathbb{R}^d : \|x - c\|_2 = r\}.$$

We write  $S(r)$  when  $c = 0$ .

The critical parameter  $\lambda_c(\mu)$  can also be defined as follows:

$$\lambda_c(\mu) = \sup \left\{ \lambda > 0 : P(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r)) \rightarrow 0 \text{ as } r \rightarrow \infty \right\},$$

We shall need two other critical parameters:

$$\begin{aligned} \widehat{\lambda}_c(\mu) &= \sup \left\{ \lambda > 0 : P(S(r/2) \leftrightarrow_{\Sigma(\lambda\mu)} S(r)) \rightarrow 0 \text{ as } r \rightarrow \infty \right\}, \\ \widetilde{\lambda}_c(\mu) &= \sup \left\{ \lambda > 0 : r^d P(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r)) \rightarrow 0 \text{ as } r \rightarrow \infty \right\}. \end{aligned}$$

We have (see Lemma 12) :

$$\widetilde{\lambda}_c(\mu) \leq \widehat{\lambda}_c(\mu) \leq \lambda_c(\mu). \quad (1)$$

When the support of  $\mu$  is bounded,

$$P(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r))$$

decays exponentially fast to 0 as soon as  $\lambda < \lambda_c(\mu)$  (see for example [9], Section 12.10 in [6] in the case of constant radii or the papers [10], [13], [18] and [19]). Therefore:

$$\widetilde{\lambda}_c(\mu) = \widehat{\lambda}_c(\mu) = \lambda_c(\mu) \text{ as soon as the support of } \mu \text{ is bounded.} \quad (2)$$

### Remarks.

- The threshold parameter  $\widehat{\lambda}_c(\mu)$  is positive if and only if  $\int r^d \mu(dr)$  is finite (i.e., if and only if  $\lambda_c(\mu)$  is positive). See Lemma 13.
- Using ideas of [4], we can check that  $\widetilde{\lambda}_c(\mu)$  is positive if and only if

$$x^d \int_x^\infty r^d \mu(dr) \rightarrow 0 \text{ as } x \rightarrow \infty.$$

If we only use results stated in [4], we can easily get the following weaker statements. Let  $D(\lambda\mu)$  denote the Euclidean diameter of the connected component of  $\Sigma(\lambda\mu)$  that contains the origin. Note that  $\widetilde{\lambda}_c(\mu)$  is positive if and only if there exists  $\lambda$  such that:

$$r^d P(D(\lambda\mu) \geq r) \rightarrow 0, \text{ as } r \rightarrow \infty. \quad (3)$$

If  $E(D(\lambda\mu)^d)$  is finite then (3) holds. If (3) holds then  $E(D(\lambda\mu)^{d-\varepsilon})$  is finite for any small enough  $\varepsilon > 0$ . By Theorem 2.2 of [4] we thus get the following implications:

$$\int_0^{+\infty} r^{2d} \mu(dr) < \infty \text{ implies } \widetilde{\lambda}_c(\mu) > 0 \text{ implies } \forall \varepsilon > 0 : \int_0^{+\infty} r^{2d-\varepsilon} \mu(dr) < \infty.$$

### 1.3 A multiscale Boolean model

Let  $\rho > 1$  be a scale factor. Let  $(\Sigma_n)_{n \geq 0}$  be a sequence of independent copies of  $\Sigma(\mu)$ . In this paper, we are interested in percolation properties of the following multiscale Boolean model:

$$\Sigma^\rho(\mu) = \bigcup_{n \geq 0} \rho^{-n} \Sigma_n. \quad (4)$$

We shall sometimes write  $\Sigma^\rho$  to simplify the notations. As before, we say that  $\Sigma^\rho$  percolates if the connected component of  $\Sigma^\rho$  that contains the origin is unbounded with positive probability.

This model seems to have been first introduced as a model of failure in geophysical medias in the 80'. We refer to the paper by Molchanov, Pisarenko and Reznikova [14] for an account of those studies. For more recent results we refer to [2], [9], [10], [11], [12] and [16].

This model is related to a discrete model introduced by Mandelbrot [8]. We refer to the survey by L. Chayes [3] and, for more recent results, to [1], [15] and [17].

In [11], Menshikov, Popov and Vachkovskaia considered the case where the radii of the unscaled process  $\Sigma_0$  equal 1. They proved the following result.

**Theorem 1 ([11])** *If  $\lambda < \lambda_c(\delta_1)$  then, for all large enough  $\rho$ ,  $\Sigma^\rho(\lambda\mu)$  does not percolate.*

In [12] the same authors considered the case where the radii are random and can be unbounded. They considered the following sub-autosimilarity assumption on the measure  $\mu$ :

$$\lim_{a \rightarrow \infty} \sup_{r \geq 1/2} \frac{a^d \mu([ar, +\infty[)}{\mu([r, +\infty[)} = 0 \quad (5)$$

with the convention  $0/0 = 0$ . They proved the following result.

**Theorem 2 ([12])** *Assume that the measure  $\mu$  satisfies (5). If  $\lambda < \tilde{\lambda}_c(\mu)$  then, for all large enough  $\rho$ ,  $\Sigma^\rho(\lambda\mu)$  does not percolate.*

Note that (5) is fulfilled for any measure with bounded support. Because of (2), Theorem 2 is then a generalization of Theorem 1.

In [5] we proved the following related result in which  $\rho$  is fixed.

**Theorem 3** *Let  $\rho > 1$ . There exists  $\lambda > 0$  such that  $\Sigma^\rho(\lambda\mu)$  does not percolate if and only if:*

$$\int_{[1, +\infty[} \beta^d \ln(\beta) \mu(d\beta) < \infty. \quad (6)$$

The main result of this paper is the first item of the following theorem. The second item is easy and already contained in Theorem 3. Recall that, by Lemma 13,  $\hat{\lambda}_c(\mu)$  is positive as soon as  $\int r^d \mu(dr)$  is finite and therefore as soon as (6) holds.

**Theorem 4**

1. Assume (6). Then, for all  $\lambda < \hat{\lambda}_c(\mu)$ , there exists  $\rho(\lambda) > 1$  such that, for all  $\rho \geq \rho(\lambda)$ :

$$P(S(r/2) \leftrightarrow_{\Sigma^\rho(\lambda\mu)} S(r)) \rightarrow 0 \text{ as } r \rightarrow \infty \quad (7)$$

and therefore  $\Sigma^\rho(\lambda\mu)$  does not percolate.

2. Assume that (6) does not hold. Then, for all  $\lambda > 0$  and for all  $\rho > 1$ ,  $\Sigma^\rho(\lambda\mu)$  percolates.

Theorem 4 is a generalization of Theorem 2 and thus of Theorem 1. Indeed, by (1), one has  $\lambda < \widehat{\lambda}_c$  as soon as  $\lambda < \widetilde{\lambda}_c$ . Moreover, by the second item of Theorem 4, (6) has to be a consequence of the assumptions of Theorem 2 at least when  $\widetilde{\lambda}_c(\mu)$  is positive. For example, one can check that (5) is stronger than (6)<sup>1</sup>. Alternatively, one can check that (6) is a consequence of  $\widetilde{\lambda}_c(\mu) > 0$  (see the remarks at the end of Section 1.2).

Let us denote by  $\lambda_c(m_\infty^\rho)$  and  $\widehat{\lambda}_c(m_\infty^\rho)$  the  $\lambda_c$  and  $\widehat{\lambda}_c$  critical thresholds for the multiscale model with scale parameter  $\rho$ . Theorems 3 and 4 yield the following result:

1. If (6) holds then  $\lambda_c(m_\infty^\rho) > 0$  (and actually the proof of Theorem 3 yields  $\widehat{\lambda}_c(m_\infty^\rho) > 0$ ) for all  $\rho > 1$  and  $\widehat{\lambda}_c(m_\infty^\rho) \rightarrow \widehat{\lambda}_c(\mu) > 0$  as  $\rho \rightarrow \infty$ .
2. Otherwise,  $\widehat{\lambda}_c(m_\infty^\rho) = \lambda_c(m_\infty^\rho) = 0$  for all  $\rho > 1$ .

Let us denote by  $D^\rho(\lambda\mu)$  the diameter of the connected component of  $\Sigma^\rho(\lambda\mu)$  that contains the origin. The following result is an easy consequence of Theorem 4 above and Theorems 2.9 and 1.2 in [5].

**Theorem 5** *Let  $s > 0$ ,  $\lambda > 0$  and  $\rho > 1$ .*

1. *If  $\int_{[1,+\infty[} \beta^{d+s} \mu(d\beta) < \infty$  and (7) holds, then  $E(D^\rho(\lambda\mu)^s) < \infty$ .*
2. *If  $\int_{[1,+\infty[} \beta^{d+s} \mu(d\beta) = \infty$  then  $E(D^\rho(\lambda\mu)^s) = \infty$ .*

## 2 Proof of Theorem 4

### 2.1 Some notations

In the whole of Section 2, we make the following assumptions:

- $\mu$  satisfies (6).
- $1 < \widehat{\lambda}_c(\mu)$ .

For all  $\eta > 0$ , we denote by  $T_\eta\mu$  the measure defined by  $T_\eta\mu(A) = \mu(A - \eta)$ . In other words, we can build  $\Sigma(T_\eta\mu)$  from  $\Sigma(\mu)$  by adding  $\eta$  to each radius.

For all  $\rho > 1$ , we denote by  $H^\rho\mu$  the measure defined by  $H^\rho\mu(A) = \rho^d\mu(\rho A)$ . With this definition,  $\rho^{-1}\Sigma(\mu)$  is a Boolean model driven by the measure  $H^\rho\mu$ . For all  $n \geq 0$ , we let:

$$m_n^\rho = \sum_{k=0}^n H^{\rho^k} \mu.$$

With this definition and the notations of (4),

$$\bigcup_{k=0}^n \rho^{-k} \Sigma_k$$

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1. From (5) one gets the existence of  $a > 1$  such that, for all  $r \geq a$ , one has  $\mu([r, +\infty[) \leq 2^{-1}a^{-d}\mu([r/a, +\infty[)$ . By induction and standard computations this yields, for all  $r \geq a$ ,  $\mu([r, +\infty[) \leq Ar^{-\ln(2)/\ln(a)-d}$ . Therefore, for a small enough  $\eta > 0$ , one has  $\int r^{d+\eta}\mu(dr) < \infty$ .

is a Boolean model driven by  $m_n^\rho$ . We also let:

$$m_\infty^\rho = \sum_{k \geq 0} H^{\rho^k} \mu.$$

So,  $\Sigma^\rho(\mu)$  is a Boolean model driven by the locally finite measure  $m_\infty^\rho$ .

Let  $p(a, \mu)$  denote the probability of existence of a path from  $S(a/2)$  to  $S(a)$  in  $\Sigma(\mu)$ :

$$p(a, \mu) = P(S(a/2) \leftrightarrow_{\Sigma(\mu)} S(a)).$$

We aim at proving that, for large enough  $\rho$ ,  $p(a, m_\infty^\rho) \rightarrow 0$  as  $a$  tends to infinity and  $\Sigma^\rho(\mu)$  does not percolate. The first item of Theorem 4 follows by applying this result to the measure  $\lambda\mu$ . Recall that the second item of Theorem 4 is contained in Theorem 3.

## 2.2 Ideas

In this subsection we first sketch the proof of the existence of  $\rho$  and  $a$  such that  $p(a, m_\infty^\rho)$  is small. This gives the main ingredients of the proof of the first item of Theorem 4. A full proof is given in Subsection 2.3. We then give the ideas of the proof of Theorems 1 and 2 by Menshikov, Popov and Vachkovskaia. Their basic strategy is similar but the implementation of the proofs are different.

### Sketch of the proof of the first item of Theorem 4

Consider a small  $\varepsilon_1 > 0$ . Fix a small  $\eta > 0$  and a large  $a$  such that (see Lemma 6):

$$p(a, T_\eta \mu) \leq \varepsilon_1/2. \quad (8)$$

For all  $n \geq 1$ , write:

$$m_n^\rho = H^\rho m_{n-1}^\rho + \mu.$$

If the event  $\{S(a/2) \leftrightarrow_{\Sigma(m_n^\rho)} S(a)\}$  occurs, then either the event  $\{S(a/2) \leftrightarrow_{T_\eta \mu} S(a)\}$  occurs (with a natural coupling between the Boolean models) either in  $\Sigma(H^\rho m_{n-1}^\rho) \cap B(a)$  one can find a component of diameter at least  $\eta$ . We use this observation through its following crude consequence (see Lemma 7):

$$p(a, m_n^\rho) \leq p(a, T_\eta \mu) + C a^d \eta^{-d} p(\eta/2, H^\rho m_{n-1}^\rho).$$

By scaling and by (8), this yields:

$$p(a, m_n^\rho) \leq \varepsilon_1/2 + C a^d \eta^{-d} p(\rho \eta/2, m_{n-1}^\rho). \quad (9)$$

But for any  $\varepsilon_2$ , any small enough  $\varepsilon_1$  and any large enough  $a$  we can find  $\tau$  such that (see Lemmas 8 and 9):

$$p(\tau a, m_{n-1}^\rho) \leq \varepsilon_2 \text{ as soon as } p(a, m_{n-1}^\rho) \leq \varepsilon_1. \quad (10)$$

An important fact is that  $\tau$  does not depends on  $n$  nor on  $\rho$ , provided  $\rho \geq \rho_0$  where  $\rho_0$  is an arbitrary constant strictly larger than 1. Here we use assumption (6) to bound error terms due to the existence of large balls.

We choose  $\varepsilon_2$  such that:

$$Ca^d \eta^{-d} \varepsilon_2 = \varepsilon_1/2.$$

We set  $\rho = 2\tau a/\eta$ . Then, (9) and (10) can be rewritten as follows:

$$p(a, m_n^\rho) \leq \varepsilon_1/2 + Ca^d \eta^{-d} p(\tau a, m_{n-1}^\rho) \quad (11)$$

$$Ca^d \eta^{-d} p(\tau a, m_{n-1}^\rho) \leq \varepsilon_1/2 \text{ as soon as } p(a, m_{n-1}^\rho) \leq \varepsilon_1. \quad (12)$$

As moreover (8) implies  $p(a, m_0^\rho) \leq \varepsilon_1$  we get, by induction and then sending  $n$  to infinity (see Lemma 10):

$$p(a, m_\infty^\rho) \leq \varepsilon_1.$$

The convergence of  $p(a, m_\infty^\rho)$  to 0 is then extracted from the above result for a small enough  $\varepsilon_2$  and from arguments behind (10) applied to  $m_\infty^\rho$  and other  $\varepsilon$ .

### Sketch of the proofs of Theorems 1 and 2 by Menshikov, Popov and Vachkovskaia

Let us quickly describe the ideas of the proofs of Menshikov, Popov and Vachkovskaia. Those ideas are used in their papers [11] and [12] through a discretization of space ; we describe them in a slightly more geometric way. For simplicity we only consider two scales:  $\rho^{-1}\Sigma_1$  and  $\Sigma_0$ . For simplicity, we also assume that the radius is one in the unscaled model ( $\mu = \lambda\delta_1$ ). We assume that the scale factor  $\rho$  is large enough. Assume that  $C$  is a connected component of  $\rho^{-1}\Sigma_1 \cup \Sigma_0$  whose diameter is at least  $\alpha$  (it can be much larger) for a small enough constant  $\alpha > 0$ . Then,  $C$  is included in the union of the following kind of sets:

1. connected components of  $\rho^{-1}\Sigma_1$  whose diameter is at least  $\alpha$  ;
2. balls of  $\Sigma_0$  enlarged by  $\alpha$  (same centers but the radii are  $1 + \alpha$  instead of 1).

Then, they show that the union of all those sets is stochastically dominated by a Boolean model similar to  $\Sigma_0$  but with radii enlarged by a factor  $\alpha$  and with density of centers  $1 + \alpha'$  times the corresponding density for  $\Sigma_0$  for a suitable  $\alpha' > 0$ . This part uses  $\lambda < \widehat{\lambda}_c$ . In some sense, one can therefore control percolation in the union of two models by percolation in one model. Iterating the argument with some care in the constants  $\alpha$  and  $\alpha'$ , one sees that – for large enough  $\rho$  – one can control percolation in the multiscale model by percolation in a subcritical model. This yields the result.

### 2.3 Proof of Theorem 4

As  $1 < \widehat{\lambda}_c(\mu)$ , we know that  $p(a, \mu)$  tends to 0 as  $a$  tends to infinity. We need the following slightly stronger consequence.

**Lemma 6** *There exists  $\eta > 0$  such that  $p(a, T_\eta \mu)$  tends to 0.*

**Proof.** Let  $\varepsilon > 0$  and  $x > 0$ . We have:

$$\begin{aligned} H^{1+\varepsilon} T_{\varepsilon^2} \mu([x, +\infty[) &= (1 + \varepsilon)^d T_{\varepsilon^2} \mu([x(1 + \varepsilon), +\infty[) \\ &= (1 + \varepsilon)^d \mu([x(1 + \varepsilon) - \varepsilon^2, +\infty[) \\ &\leq \kappa(\varepsilon)(1 + \varepsilon)^d \mu([x, +\infty[) \end{aligned} \quad (13)$$

where

$$\kappa(\varepsilon) = \frac{\mu([0, +\infty[)}{\mu([\varepsilon, +\infty[)}.$$

The inequality is proven as follows. If  $x \geq \varepsilon$ , then  $[x(1 + \varepsilon) - \varepsilon^2, +\infty[ \subset [x, +\infty[$  and the result follows from  $\kappa(\varepsilon) \geq 1$ . If, on the contrary,  $x < \varepsilon$ , then the left hand side is bounded above by  $(1 + \varepsilon)^d \mu([0, +\infty[)$  which is itself bounded above by the right hand side.

Note that  $\kappa(\varepsilon)(1 + \varepsilon)^d$  tends to 1 as  $\varepsilon$  tends to 0. Let us say that a measure  $\nu$  is subcritical if  $\widehat{\lambda}_c(\nu) > 1$ . As  $\mu$  is subcritical, we get that  $\kappa(\varepsilon)(1 + \varepsilon)^d \mu$  is subcritical for small enough  $\varepsilon$ . We fix such an  $\varepsilon$ . By (13) we can couple a Boolean model driven by  $H^{1+\varepsilon}T_{\varepsilon^2}\mu$  and a Boolean model driven by  $\kappa(\varepsilon)(1 + \varepsilon)^d \mu$  in such a way that the first one is contained in the second one. Therefore the first one is subcritical. By scaling, a Boolean model driven by  $T_{\varepsilon^2}\mu$  is then subcritical. We take  $\eta = \varepsilon^2$ .  $\square$

**Lemma 7** *Let  $\nu_1$  and  $\nu_2$  be two finite measures on  $]0, +\infty[$ . One has, for all  $\eta > 0$  and  $a \geq 4\eta$ :*

$$p(a, \nu_1 + \nu_2) \leq p(a, T_\eta \nu_1) + C_1 a^d \eta^{-d} p(\eta/2, \nu_2)$$

where  $C_1 = C_1(d) > 0$  depends only on the dimension  $d$ .

**Proof.** Let  $(x_i)_{i \in I}$  be a family of points such that :

- The balls  $B(x_i, \eta/4)$ ,  $i \leq I$ , cover  $B(a)$ .
- There are at most  $C_1 a^d \eta^{-d}$  points in the family where  $C_1 = C_1(d)$  depends only on the dimension  $d$ .

We couple the different Boolean model as follows. Let  $\Sigma(\nu_1)$  be a Boolean model driven by  $\nu_1$ . Let  $\Sigma(\nu_2)$  be a Boolean model driven by  $\nu_2$ . Assume that  $\Sigma(\nu_1)$  and  $\Sigma(\nu_2)$  are independent. Then  $\Sigma(\nu_1) \cup \Sigma(\nu_2)$  is a Boolean model driven by  $\nu_1 + \nu_2$ . We set  $\Sigma(\nu_1 + \nu_2) = \Sigma(\nu_1) \cup \Sigma(\nu_2)$ . We also consider  $\Sigma(T_\eta \nu_1)$ , the Boolean model obtained by adding  $\eta$  to the radius of each ball of  $\Sigma(\nu_1)$ . Thus  $\Sigma(T_\eta \nu_1)$  is driven by  $T_\eta \nu_1$ .

Let us prove the following property:

$$\{S(a/2) \leftrightarrow_{\Sigma(\nu_1 + \nu_2)} S(a)\} \subset \{S(a/2) \leftrightarrow_{\Sigma(T_\eta \nu_1)} S(a)\} \cup \bigcup_{i \in I} \{S(x_i, \eta/4) \leftrightarrow_{\Sigma(\nu_2)} S(x_i, \eta/2)\}. \quad (14)$$

Assume that  $\Sigma(\nu_1 + \nu_2) = \Sigma(\nu_1) \cup \Sigma(\nu_2)$  connects  $S(a/2)$  with  $S(a)$ . Recall  $a \geq 4\eta$ . If the diameter of all connected components of  $\Sigma(\nu_2) \cap B(a)$  are less or equal to  $\eta$ , then  $\Sigma(T_\eta \nu_1)$  connects  $S(a/2)$  with  $S(a)$ . Otherwise, let  $C$  be a connected component of  $\Sigma(\nu_2) \cap B(a)$  with diameter at least  $\eta$ . Let  $x, y$  be two points of  $C$  such that  $\|x - y\| > \eta$ . The point  $x$  belongs to a ball  $B(x_i, \eta/4)$ . As  $y$  does not belong to  $B(x_i, \eta/2)$ , the component  $C$  connects  $S(x_i, \eta/4)$  to  $S(x_i, \eta/2)$ . Therefore,  $\Sigma(\nu_2)$  connects  $S(x_i, \eta/4)$  to  $S(x_i, \eta/2)$ . We have proven (14). The lemma follows.  $\square$

The following lemma is essentially the first item of Proposition 3.1 in [4]. For the sake of completeness we nevertheless provide a proof.

**Lemma 8** *Let  $\nu$  be a finite measure on  $]0, +\infty[$ . There exists a constant  $C_2 = C_2(d) > 0$  such that, for all  $a > 0$ :*

$$p(10a, \nu) \leq C_2 p(a, \nu)^2 + C_2 \int_{[a, +\infty[} r^d \nu(dr).$$



**Proof.** Let  $K$  be a finite subset of  $S(5)$  such that  $K + B(1/2)$  covers  $S(5)$ . Let  $L$  be a finite subset of  $S(10)$  such that  $L + B(1/2)$  covers  $S(10)$ . Let  $A$  be the following event: there exists a random ball  $B(c, r)$  of  $\Sigma(\nu)$  such that  $r \geq a$  and  $B(c, r) \cap B(10a)$  is non empty. We have:

$$\{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\} \setminus A \subset \{S(5a) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(10a)\}$$

where, in the last event, we ask for a path using only balls of  $\Sigma(\nu)$  of radius at most  $a$ . Let us prove the following:

$$\begin{aligned} & \{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\} \setminus A \\ & \subset \bigcup_{k \in K, l \in L} \{S(ak, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(ak, a)\} \cap \{S(al, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(al, a)\}. \end{aligned} \quad (15)$$

Assume that the event on the left hand side occurs. Then, by the previous remark, there exists a path from a point  $x \in S(5a)$  to a point  $y \in S(10a)$  that is contained in balls of  $\Sigma(\nu)$  of radius at most  $a$ . As  $Ka + B(a/2)$  covers  $S(5a)$ , there exists  $k \in K$  such that  $x$  belongs to  $B(ka, a/2)$ . Using the previous path, one gets that the event

$$\{S(ak, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(ak, a)\}$$

occurs. By a similar arguments involving  $y$  we get (15).

Observe that, for all  $k \in K$  and  $l \in L$ , the events

$$\{S(ak, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(ak, a)\} \text{ and } \{S(al, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(al, a)\}$$

are independent. Indeed, the first one depends only on balls with centers in  $B(ak, 2a)$ , the second one depends only on balls with centers in  $B(al, 2a)$ , and  $\|ak - al\| \geq 5a$ . Using this independence, stationarity and (15), we then get:

$$P(\{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\}) \leq CP(S(a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(a))^2 + P(A)$$

where  $C$  is the product of the cardinality of  $K$  by the cardinality of  $L$ . The probability  $P(A)$  is bounded above by standard computations.  $\square$

From the previous lemma, we deduce the following result.

**Lemma 9** *Let  $\varepsilon > 0$ . There exists  $C_3 = C_3(d) > 0$ ,  $a_0 = a_0(d, \mu)$  and  $k_0 = k_0(d, \mu, \varepsilon)$  such that, for all  $N$ , all  $\rho \geq 2$  and all  $a \geq a_0$ : if  $p(a, m_N^\rho) \leq C_3$  then for all  $k \geq k_0$ ,  $p(a10^k, m_N^\rho) \leq \varepsilon$ .*

**Proof.** For all  $\rho \geq 2$  and all  $a \geq 1$  we have:

$$\begin{aligned} \int_{[a, +\infty[} r^d m_\infty^\rho(dr) &= \sum_{k \geq 0} \rho^{kd} \int_{]0, +\infty[} 1_{[a, +\infty[}(r\rho^{-k})(r\rho^{-k})^d \mu(dr) \\ &= \int_{]0, +\infty[} \sum_{k \geq 0} 1_{[a, +\infty[}(r\rho^{-k}) r^d \mu(dr) \\ &= \int_{[a, +\infty[} (\lfloor \ln(r/a) \ln(\rho) \rfloor + 1) r^d \mu(dr) \\ &\leq \int_{[a, +\infty[} (\ln(r) \ln(2)^{-1} + 1) r^d \mu(dr). \end{aligned}$$

Let  $C_2$  be the constant given by Lemma 8. By (6) we can chose  $a_0 = a_0(d, \mu) \geq 1$  such that

$$C_2^2 \int_{[a_0, +\infty[} (\ln(r) \ln(2)^{-1} + 1) r^d \mu(dr) \leq \frac{1}{4}. \quad (16)$$

Let  $C_3 = (2C_2)^{-1}$ . Let  $N$ ,  $\rho$  and  $a$  be as in the statement of the lemma. From Lemma 8 we get:

$$C_2 p(10a, m_N^\rho) \leq (C_2 p(a, m_N^\rho))^2 + C_2^2 \int_{[a, +\infty[} r^d m_N^\rho(dr) \quad (17)$$

$$\leq (C_2 p(a, m_N^\rho))^2 + C_2^2 \int_{[a, +\infty[} (\ln(r) \ln(2)^{-1} + 1) r^d \mu(dr) \quad (18)$$

Let  $(u_k)$  be a sequence defined by  $u_0 = 1/2$  and, for all  $k \geq 0$ :

$$u_{k+1} = u_k^2 + C_2^2 \int_{[a_0 10^k, +\infty[} (\ln(r) \ln(2)^{-1} + 1) r^d \mu(dr). \quad (19)$$

Note that the sequence  $(u_k)$  only depends on  $d$  and  $\mu$ .

Assume that  $p(a, m_N^\rho) \leq C_3$ . We then have  $C_2 p(a, m_N^\rho) \leq u_0$ . Using  $a \geq a_0$  and (18), we then get  $C_2 p(a 10^k, m_N^\rho) \leq u_k$  for all  $k$ . Therefore, it suffices to show that the sequence  $(u_k)$  tends to 0.

Using (19), (16) and  $u_0 = 1/2$  we get  $0 \leq u_k \leq 1/2$  for all  $k$ . Therefore,  $0 \leq \limsup u_k \leq 1/2$ . By (19) and by the convergence of the integrale we also get  $\limsup u_k \leq (\limsup u_k)^2$ . As a consequence,  $\limsup u_k = 0$  and the lemma is proven.  $\square$

**Lemma 10** *For all  $a > 0$  and  $\rho > 1$  the following convergence holds:*

$$p(a, m_\infty^\rho) = \lim_{N \rightarrow \infty} p(a, m_N^\rho).$$

**Proof.** The sequence of events

$$A_N = \{S(a/2) \leftrightarrow_{\Sigma(m_N^\rho)} S(a)\}$$

is increasing (we use the natural coupling between our Boolean models). Therefore, it suffices to show that the union of the previous events is

$$A = \{S(a/2) \leftrightarrow_{\Sigma(m_\infty^\rho)} S(a)\}.$$

If  $A$  occurs, then there is a path from  $S(a/2)$  to  $S(a)$  that is contained in  $\Sigma(m_\infty^\rho)$ . By a compactness argument, this path is included in a finite union of ball of  $\Sigma(m_\infty^\rho)$ . Therefore, there exists  $N$  such that the path is included in  $\Sigma(m_N^\rho)$  and  $A_N$  occurs. This proves  $A \subset \cup A_N$ . The other inclusion is straightforward.  $\square$

**Proof of the second item of Theorem 4.** By Lemma 6, we can fix  $\eta_1 > 0$  such that  $p(a, T_{10\eta_1} \mu)$  tends to 0 as  $a$  tends to  $\infty$ . Let  $C_1$  be given by Lemma 7. Let  $a_0$  and  $C_3$  be as given by Lemma 9. Fix  $a_1 \geq \max(40\eta_1, a_0, 1)$  such that  $p(a, T_{10\eta_1} \mu) \leq C_3/2$  for all  $a \geq a_1$ . Let  $k_0$  be given by Lemma 9 with the choice:

$$\varepsilon = C_1^{-1} (10a_1)^{-d} \eta_1^d C_3/2.$$

Therefore, for all  $\rho \geq 2$ , all  $N$ , all  $a \in [a_1, 10a_1]$  and all  $\eta \in [\eta_1, 10\eta_1]$ :

$$C_1 a^d \eta^{-d} p(a 10^k, m_N^\rho) \leq \frac{C_3}{2} \text{ for all } k \geq k_0 \text{ as soon as } p(a, m_N^\rho) \leq C_3.$$

Fix  $k \geq k_0$ ,  $a \in [a_1, 10a_1]$  and  $\eta \in [\eta_1, 10\eta_1]$ . Set:

$$\rho = 2a 10^k \eta^{-1}.$$

Note  $\rho \geq 8 \geq 2$  as  $a \geq a_1 \geq 40\eta_1 \geq 4\eta$ . By Lemma 7 we have, for all  $N$ :

$$\begin{aligned} p(a, m_{N+1}^\rho) &\leq p(a, T_\eta \mu) + C_1 a^d \eta^{-d} p(\eta/2, m_{N+1}^\rho - \mu) \\ &= p(a, T_\eta \mu) + C_1 a^d \eta^{-d} p(\eta/2, H^\rho m_N^\rho). \end{aligned}$$

By definition of  $a_1$ , by  $a_1 \leq a$ , by  $\eta \leq 10\eta_1$ , by scaling and by definition of  $\rho$  we get, for all  $N$ :

$$\begin{aligned} p(a, m_{N+1}^\rho) &\leq \frac{C_3}{2} + C_1 a^d \eta^{-d} p(\rho\eta/2, m_N^\rho) \\ &= \frac{C_3}{2} + C_1 a^d \eta^{-d} p(a 10^k, m_N^\rho). \end{aligned}$$

Combining this inequality with the property defining  $k_0$ , we get that  $p(a, m_N^\rho) \leq C_3$  implies  $p(a, m_{N+1}^\rho) \leq C_3$ . As  $p(a, m_0^\rho) = p(a, \mu) \leq p(a, T_{10\eta_1} \mu) \leq C_3/2$  we get  $p(a, m_N^\rho) \leq C_3$  for all integer  $N$ .

Let  $\varepsilon > 0$ . Using again Lemma 9 we get the existence of an integer  $k'_0$  such that  $p(a 10^{k'}, m_N^\rho) \leq \varepsilon$  for all  $k' \geq k'_0$  as soon as  $p(a, m_N^\rho) \leq C_3$ . But we have proven the latter property. Therefore  $p(a 10^{k'}, m_N^\rho) \leq \varepsilon$  for all  $N$  and all  $k' \geq k'_0$ . By Lemma 10, we get  $p(a 10^{k'}, m_\infty^\rho) \leq \varepsilon$  for all  $k' \geq k'_0$ . Using the freedom on the choice of  $k \geq k_0$  and  $\eta \in [\eta_1, 10\eta_1]$ , we get that the previous result holds for all  $\rho \geq 2a 10^{k_0-1} \eta_1^{-1}$  and then for all  $\rho \geq 2a_1 10^{k_0} \eta_1^{-1}$ . Moreover, using the freedom on the choice of  $a \in [a_1, 10a_1]$  and  $k' \geq k'_0$ , we get:

$$p(r, m_\infty^\rho) \leq \varepsilon \text{ for all } r \geq a_1 10^{k'_0} \text{ and all } \rho \geq 2a_1 10^{k_0} \eta_1^{-1}.$$

Therefore,  $p(r, m_\infty^\rho)$  tends to 0 as  $r$  tends to infinity. As a consequence,  $\Sigma^\rho(\mu)$  does not percolate for any  $\rho \geq 2a_1 10^{k_0} \eta_1^{-1}$ .  $\square$

### 3 Proof of Theorem 5

**Lemma 11** *Let  $s > 0$  and  $\rho > 1$ . The following assumptions are equivalent:*

1.  $\int_{]0, +\infty[} r^{d+s} \mu(dr) < \infty$ .
2.  $\int_{[1, +\infty[} r^{d+s} m_\infty^\rho(dr) < \infty$ .

**Proof.** We have:

$$\begin{aligned} \int_{[1, +\infty[} r^{d+s} m_\infty^\rho(dr) &= \sum_{k \geq 0} \rho^{kd} \int_{]0, +\infty[} 1_{[1, +\infty[}(r \rho^{-k}) (r \rho^{-k})^{d+s} \mu(dr) \\ &= \int_{[1, +\infty[} \sum_{k \geq 0} 1_{[1, +\infty[}(r \rho^{-k}) \rho^{-ks} r^{d+s} \mu(dr). \end{aligned}$$

Therefore:

$$\int_{[1,+\infty[} r^{d+s} \mu(dr) \leq \int_{[1,+\infty[} r^{d+s} m_\infty^\rho(dr) \leq \frac{1}{1-\rho^{-s}} \int_{[1,+\infty[} r^{d+s} \mu(dr).$$

This yields the result.  $\square$

**Proof of the first item of Theorem 5.** By the discussion at the beginning of Section 1.5 in [5],  $\Sigma^\rho(\lambda\mu)$  is driven by a Poisson point process whose intensity is the product of the Lebesgue measure by the locally finite measure  $\lambda m_\infty^\rho$ . Let us check the three items of Theorem 2.9 in [5] with  $\rho = 10$  ( $\rho$  is not use in the same way in [5]). We refer to Section 2.1 of [5] for definitions.

1. The first item is fulfilled thanks to (7)
2. For all  $\beta > 0$  and all  $x \in \mathbb{R}^d$ , the event  $G(x, 0, \beta)$  only depends on balls  $B(c, r) \in \Sigma^\rho(\lambda\mu)$  such that  $c$  belongs to  $B(x, 3\beta)$ . By the independance property of Poisson point processes, we then get that  $G(0, 0, \beta)$  and  $G(x, 0, \beta)$  are independent whenever  $\|x\| \geq 10\beta$ . Therefore  $I(10, 0, \beta) = 0$  and the second item of Theorem 2.9 is fulfilled.
3. The third item (note that  $\mu$  in [5] is  $m_\infty^\rho$  in this paper) is fulfilled thanks to Lemma 11

Theorem 2.9 in [5] yields the result.  $\square$

**Proof of the second item of Theorem 5.** If  $\int r^d \mu(dr)$  is infinite then,  $\Sigma(\lambda\mu)$  percolates for all  $\lambda > 0$  (see the dicussion of Section 1.2). Therefore  $\Sigma^\rho(\lambda\mu)$  percolates for all  $\rho > 1$  and  $\lambda > 0$ . Therefore  $D^\rho(\lambda\mu) = \infty$  with positive probability for all  $\rho > 1$  and  $\lambda > 0$ .

Now, assume that  $\int r^d \mu(dr)$  is finite. Then, by the discussion at the beginning of Section 1.5 in [5],  $\Sigma^\rho(\lambda\mu)$  is driven by a Poisson point process whose intensity is the product of the Lebesgue measure by the locally finite measure  $\lambda m_\infty^\rho$ . We can therefore apply Theorem 1.2 in [5]. By Lemma 11, assumption (A3) of Theorem 1.2 in [5] is not fulfilled (note that  $\mu$  in [5] is  $m_\infty^\rho$  in this paper). Theorem 1.2 in [5] then yields the result.  $\square$

## A Critical parameters

**Lemma 12**

$$\tilde{\lambda}_c(\mu) \leq \hat{\lambda}_c(\mu) \leq \lambda_c(\mu).$$

**Proof.** The second inequality is a consequence of the following inclusion:

$$\{\{0\} \leftrightarrow_\Sigma S(r)\} \subset \{S(r/2) \leftrightarrow_\Sigma S(r)\}.$$

The first inequality can be proven as follows. Let  $r \geq 1$ . By the FKG inequality, we get:

$$\begin{aligned} P(\{0\} \leftrightarrow_\Sigma S(r)) &\geq P(B(0, 1) \subset \Sigma \text{ and } S(1) \leftrightarrow_\Sigma S(r)) \\ &\geq CP(S(1) \leftrightarrow_\Sigma S(r)) \end{aligned}$$

where  $C = P(B(0, 1) \subset \Sigma) > 0$  does not depend on  $r$ . For all large enough  $r$ , we can cover  $S(2r)$  by at most  $C'r^d$  balls  $B(x_i, 1)$  where  $C'$  only depends on the dimension  $d$ . If

there is a path in  $\Sigma$  from  $S(2r)$  to  $S(4r)$ , then there exists  $i$  and a path in  $\Sigma$  from  $S(x_i, 1)$  to  $S(x_i, r)$ . (Consider the ball  $B(x_i, 1)$  that contains the initial point of the path.) By stationarity and by the previous inequality we thus get:

$$\begin{aligned} P(S(2r) \leftrightarrow_{\Sigma} S(4r)) &\leq C' r^d P(S(1) \leftrightarrow_{\Sigma} S(r)) \\ &\leq C' C^{-1} r^d P(\{0\} \leftrightarrow_{\Sigma} S(r)). \end{aligned}$$

The first inequality stated in the lemma follows.  $\square$

**Lemma 13** *The threshold parameter  $\widehat{\lambda}_c(\mu)$  is positive if and only if  $\int r^d \mu(dr)$  is finite.*

**Proof.** If  $\widehat{\lambda}_c(\mu)$  is positive, then there exists  $\lambda > 0$  such that  $\Sigma(\lambda\mu)$  does not percolate. By Theorem 2.1 of [4] this implies that  $\int r^d \mu(dr)$  is finite.

Let us assume now that  $\int r^d \mu(dr)$  is finite. We need to prove the existence of  $\lambda > 0$  such that  $p(a, \lambda\mu)$  tends to 0. This is proven, as an intermediate result, in the proof of Theorem 1.1 in [5]. As the result is an easy consequence of Lemma 8, we find it more convenient to provide a proof here. Let  $C_2$  be the constant given by Lemma 8. For all  $a > 0$  and  $\lambda > 0$  we have:

$$C_2 p(10a, \lambda\mu) \leq (C_2 p(a, \lambda\mu))^2 + \lambda C_2^2 \int_{[a, +\infty[} r^d \mu(dr). \quad (20)$$

For all  $0 < a \leq 1$  we have, by standard computations:

$$C_2 p(a, \lambda\mu) \leq C_2 P(\text{a ball of } \Sigma(\lambda\mu) \text{ touches } B(a)) \leq C_2 v_d \lambda \int_{]0, +\infty[} (1+r)^d \mu(dr)$$

where  $v_d$  is the volume of the unit Euclidean ball. As  $\int r^d \mu(dr)$  is finite, we can therefore fix  $\lambda > 0$  such that:

$$\lambda C_2^2 \int_{[a, +\infty[} r^d \mu(dr) \leq 1/4 \text{ for all } a > 0 \text{ and } C_2 p(a, \lambda\mu) \leq 1/2 \text{ for all } 0 < a \leq 1. \quad (21)$$

By (20), (21) and by induction we get  $C_2 p(a, \lambda\mu) \leq 1/2$  for all  $a > 0$ . Therefore, we have  $0 \leq \limsup C_2 p(a, \lambda\mu) \leq 1/2$ . But (20) also yields the inequality  $\limsup C_2 p(a, \lambda\mu) \leq (\limsup C_2 p(a, \lambda\mu))^2$ . As a consequence we must have  $\limsup C_2 p(a, \lambda\mu) = 0$  and then  $p(a, \lambda\mu) \rightarrow 0$ .  $\square$

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