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Percolation in a multiscale Boolean model

Jean-Baptiste Gouéré *

Abstract

We consider percolation in a multiscale Boolean model. This model is defined as the union of scaled independent copies of a given Boolean model. The scale factor of the n^{th} copy is ρ^{-n} . We prove, under optimal integrability assumptions, that no percolation occurs in the multiscale Boolean model for large enough ρ if the rate of the Boolean model is below some critical value.

1 Introduction and statement of the main result

1.1 The Boolean model

Let $d \geq 2$. Let μ be a finite measure on $]0, +\infty[$. We assume that the mass of μ is positive. Let ξ be a Poisson point process on $\mathbb{R}^d \times]0, +\infty[$ whose intensity is the product of the Lebesgue measure on \mathbb{R}^d by μ . With ξ we associate a random set $\Sigma(\mu)$ defined as follows:

$$\Sigma(\mu) = \bigcup_{(c,r) \in \xi} B(c,r)$$

where $B(c,r)$ is the open Euclidean ball of radius r centered at c . The random set $\Sigma(\mu)$ is the Boolean model with parameter μ . We shall sometimes write Σ to simplify the notations.

The following description may be more intuitive. Let χ denote the projection of ξ on \mathbb{R}^d . With probability one this projection is one-to-one. We can therefore write:

$$\xi = \{(c, r(c)), c \in \chi\}.$$

Write $\mu = m\nu$ where ν is a probability measure. Then, χ is a Poisson point process on \mathbb{R}^d with density m . Moreover, given χ , the sequence $(r(c))_{c \in \chi}$ is a sequence of independent random variable with common distribution ν . We shall not use this point of view.

1.2 Percolation in the Boolean model

Let C denote the connected component of Σ that contains the origin. We say that Σ percolates if C is unbounded with positive probability. We refer to the book by Meester and Roy [9] for background on continuum percolation. Set:

$$\lambda_c(\mu) = \inf\{\lambda > 0 : \Sigma(\lambda\mu) \text{ percolates}\}.$$

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One easily check that $\lambda_c(\mu)$ is finite as soon as μ has a positive mass. In [4] we proved that $\lambda_c(\mu)$ is positive if and only if:

$$\int r^d \mu(dr) < \infty.$$

The only if part had been proved earlier by Hall [7]. For all $A, B \subset \mathbb{R}^d$, we write $A \leftrightarrow_{\Sigma} B$ if there exists a path in Σ from A to B . We denote by $S(c, r)$ the Euclidean sphere of radius r centered at c :

$$S(c, r) = \{x \in \mathbb{R}^d : \|x - c\|_2 = r\}.$$

We write $S(r)$ when $c = 0$.

The critical parameter $\lambda_c(\mu)$ can also be defined as follows:

$$\lambda_c(\mu) = \sup \{ \lambda > 0 : P(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r)) \rightarrow 0 \text{ as } r \rightarrow \infty \},$$

We shall need two other critical parameters:

$$\begin{aligned} \widehat{\lambda}_c(\mu) &= \sup \{ \lambda > 0 : P(S(r/2) \leftrightarrow_{\Sigma(\lambda\mu)} S(r)) \rightarrow 0 \text{ as } r \rightarrow \infty \}, \\ \widetilde{\lambda}_c(\mu) &= \sup \{ \lambda > 0 : r^d P(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r)) \rightarrow 0 \text{ as } r \rightarrow \infty \}. \end{aligned}$$

We have (see Lemma 12) :

$$\widetilde{\lambda}_c(\mu) \leq \widehat{\lambda}_c(\mu) \leq \lambda_c(\mu). \quad (1)$$

When the support of μ is bounded,

$$P(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r))$$

decays exponentially fast to 0 as soon as $\lambda < \lambda_c(\mu)$ (see for example [9], Section 12.10 in [6] in the case of constant radii or the papers [10], [13], [18] and [19]). Therefore:

$$\widetilde{\lambda}_c(\mu) = \widehat{\lambda}_c(\mu) = \lambda_c(\mu) \text{ as soon as the support of } \mu \text{ is bounded.} \quad (2)$$

Remarks.

- The treshold parameter $\widehat{\lambda}_c(\mu)$ is positive if and only if $\int r^d \mu(dr)$ is finite (i.e., if and only if $\lambda_c(\mu)$ is positive). See Lemma 13.
- Using ideas of [4], we can check that $\widetilde{\lambda}_c(\mu)$ is positive if and only if

$$x^d \int_x^\infty r^d \mu(dr) \rightarrow 0 \text{ as } x \rightarrow \infty.$$

If we only use results stated in [4], we can easily get the following weaker statements. Let $D(\lambda\mu)$ denote the Euclidean diameter of the connected component of $\Sigma(\lambda\mu)$ that contains the origin. Note that $\widetilde{\lambda}_c(\mu)$ is positive if and only if there exists λ such that:

$$r^d P(D(\lambda\mu) \geq r) \rightarrow 0, \text{ as } r \rightarrow \infty. \quad (3)$$

If $E(D(\lambda\mu)^d)$ is finite then (3) holds. If (3) holds then $E(D(\lambda\mu)^{d-\varepsilon})$ is finite for any small enough $\varepsilon > 0$. By Theorem 2.2 of [4] we thus get the following implications:

$$\int_0^{+\infty} r^{2d} \mu(dr) < \infty \text{ implies } \widetilde{\lambda}_c(\mu) > 0 \text{ implies } \forall \varepsilon > 0 : \int_0^{+\infty} r^{2d-\varepsilon} \mu(dr) < \infty.$$

1.3 A multiscale Boolean model

Let $\rho > 1$ be a scale factor. Let $(\Sigma_n)_{n \geq 0}$ be a sequence of independent copies of $\Sigma(\mu)$. In this paper, we are interested in percolation properties of the following multiscale Boolean model:

$$\Sigma^\rho(\mu) = \bigcup_{n \geq 0} \rho^{-n} \Sigma_n. \quad (4)$$

We shall sometimes write Σ^ρ to simplify the notations. As before, we say that Σ^ρ percolates if the connected component of Σ^ρ that contains the origin is unbounded with positive probability.

This model seems to have been first introduced as a model of failure in geophysical medias in the 80'. We refer to the paper by Molchanov, Pisarenko and Reznikova [14] for an account of those studies. For more recent results we refer to [2], [9], [10], [11], [12] and [16].

This model is related to a discrete model introduced by Mandelbrot [8]. We refer to the survey by L. Chayes [3] and, for more recent results, to [1], [15] and [17].

In [11], Menshikov, Popov and Vachkovskaia considered the case where the radii of the unscaled process Σ_0 equal 1. They proved the following result.

Theorem 1 ([11]) *If $\lambda < \lambda_c(\delta_1)$ then, for all large enough ρ , $\Sigma^\rho(\lambda\mu)$ does not percolate.*

In [12] the same authors considered the case where the radii are random and can be unbounded. They considered the following sub-autosimilarity assumption on the measure μ :

$$\lim_{a \rightarrow \infty} \sup_{r \geq 1/2} \frac{a^d \mu([ar, +\infty[)}{\mu([r, +\infty[)} = 0 \quad (5)$$

with the convention $0/0 = 0$. They proved the following result.

Theorem 2 ([12]) *Assume that the measure μ satisfies (5). If $\lambda < \tilde{\lambda}_c(\mu)$ then, for all large enough ρ , $\Sigma^\rho(\lambda\mu)$ does not percolate.*

Note that (5) is fulfilled for any measure with bounded support. Because of (2), Theorem 2 is then a generalization of Theorem 1.

In [5] we proved the following related result in which ρ is fixed.

Theorem 3 *Let $\rho > 1$. There exists $\lambda > 0$ such that $\Sigma^\rho(\lambda\mu)$ does not percolate if and only if:*

$$\int_{[1, +\infty[} \beta^d \ln(\beta) \mu(d\beta) < \infty. \quad (6)$$

The main result of this paper is the first item of the following theorem. The second item is easy and already contained in Theorem 3. Recall that, by Lemma 13, $\widehat{\lambda}_c(\mu)$ is positive as soon as $\int r^d \mu(dr)$ is finite and therefore as soon as (6) holds.

Theorem 4

1. Assume (6). Then, for all $\lambda < \widehat{\lambda}_c(\mu)$, there exists $\rho(\lambda) > 1$ such that, for all $\rho \geq \rho(\lambda)$:

$$P(S(r/2) \leftrightarrow_{\Sigma^\rho(\lambda\mu)} S(r)) \rightarrow 0 \text{ as } r \rightarrow \infty \quad (7)$$

and therefore $\Sigma^\rho(\lambda\mu)$ does not percolate.

2. Assume that (6) does not hold. Then, for all $\lambda > 0$ and for all $\rho > 1$, $\Sigma^\rho(\lambda\mu)$ percolates.

Theorem 4 is a generalization of Theorem 2 and thus of Theorem 1. Indeed, by (1), one has $\lambda < \widehat{\lambda}_c$ as soon as $\lambda < \widetilde{\lambda}_c$. Moreover, by the second item of Theorem 4, (6) has to be a consequence of the assumptions of Theorem 2 at least when $\widetilde{\lambda}_c(\mu)$ is positive. For example, one can check that (5) is stronger than (6)¹. Alternatively, one can check that (6) is a consequence of $\widetilde{\lambda}_c(\mu) > 0$ (see the remarks at the end of Section 1.2).

Let us denote by $\lambda_c(m_\infty^\rho)$ and $\widehat{\lambda}_c(m_\infty^\rho)$ the λ_c and $\widehat{\lambda}_c$ critical thresholds for the multiscale model with scale parameter ρ . Theorems 3 and 4 yield the following result:

1. If (6) holds then $\lambda_c(m_\infty^\rho) > 0$ (and actually the proof of Theorem 3 yields $\widehat{\lambda}_c(m_\infty^\rho) > 0$) for all $\rho > 1$ and $\widehat{\lambda}_c(m_\infty^\rho) \rightarrow \widehat{\lambda}_c(\mu) > 0$ as $\rho \rightarrow \infty$.
2. Otherwise, $\widehat{\lambda}_c(m_\infty^\rho) = \lambda_c(m_\infty^\rho) = 0$ for all $\rho > 1$.

Let us denote by $D^\rho(\lambda\mu)$ the diameter of the connected component of $\Sigma^\rho(\lambda\mu)$ that contains the origin. The following result is an easy consequence of Theorem 4 above and Theorems 2.9 and 1.2 in [5].

Theorem 5 *Let $s > 0$, $\lambda > 0$ and $\rho > 1$.*

1. *If $\int_{[1,+\infty[} \beta^{d+s} \mu(d\beta) < \infty$ and (7) holds, then $E(D^\rho(\lambda\mu)^s) < \infty$.*
2. *If $\int_{[1,+\infty[} \beta^{d+s} \mu(d\beta) = \infty$ then $E(D^\rho(\lambda\mu)^s) = \infty$.*

2 Proof of Theorem 4

2.1 Some notations

In the whole of Section 2, we make the following assumptions:

- μ satisfies (6).
- $1 < \widehat{\lambda}_c(\mu)$.

For all $\eta > 0$, we denote by $T_\eta\mu$ the measure defined by $T_\eta\mu(A) = \mu(A - \eta)$. In other words, we can build $\Sigma(T_\eta\mu)$ from $\Sigma(\mu)$ by adding η to each radius.

For all $\rho > 1$, we denote by $H^\rho\mu$ the measure defined by $H^\rho\mu(A) = \rho^d \mu(\rho A)$. With this definition, $\rho^{-1}\Sigma(\mu)$ is a Boolean model driven by the measure $H^\rho\mu$. For all $n \geq 0$, we let:

$$m_n^\rho = \sum_{k=0}^n H^{\rho^k} \mu.$$

With this definition and the notations of (4),

$$\bigcup_{k=0}^n \rho^{-k} \Sigma_k$$

1. From (5) one gets the existence of $a > 1$ such that, for all $r \geq a$, one has $\mu([r, +\infty[) \leq 2^{-1} a^{-d} \mu([r/a, +\infty[)$. By induction and standard computations this yields, for all $r \geq a$, $\mu([r, +\infty[) \leq A r^{-\ln(2)/\ln(a)-d}$. Therefore, for a small enough $\eta > 0$, one has $\int r^{d+\eta} \mu(dr) < \infty$.

is a Boolean model driven by m_n^ρ . We also let:

$$m_\infty^\rho = \sum_{k \geq 0} H^{\rho^k} \mu.$$

So, $\Sigma^\rho(\mu)$ is a Boolean model driven by the locally finite measure m_∞^ρ .

Let $p(a, \mu)$ denote the probability of existence of a path from $S(a/2)$ to $S(a)$ in $\Sigma(\mu)$:

$$p(a, \mu) = P(S(a/2) \leftrightarrow_{\Sigma(\mu)} S(a)).$$

We aim at proving that, for large enough ρ , $p(a, m_\infty^\rho) \rightarrow 0$ as a tends to infinity and $\Sigma^\rho(\mu)$ does not percolate. The first item of Theorem 4 follows by applying this result to the measure $\lambda\mu$. Recall that the second item of Theorem 4 is contained in Theorem 3.

2.2 Ideas

In this subsection we first sketch the proof of the existence of ρ and a such that $p(a, m_\infty^\rho)$ is small. This gives the main ingredients of the proof of the first item of Theorem 4. A full proof is given in Subsection 2.3. We then give the ideas of the proof of Theorems 1 and 2 by Menshikov, Popov and Vachkovskaia. Their basic strategy is similar but the implementation of the proofs are different.

Sketch of the proof of the first item of Theorem 4

Consider a small $\varepsilon_1 > 0$. Fix a small $\eta > 0$ and a large a such that (see Lemma 6):

$$p(a, T_\eta \mu) \leq \varepsilon_1/2. \quad (8)$$

For all $n \geq 1$, write:

$$m_n^\rho = H^\rho m_{n-1}^\rho + \mu.$$

If the event $\{S(a/2) \leftrightarrow_{\Sigma(m_n^\rho)} S(a)\}$ occurs, then either the event $\{S(a/2) \leftrightarrow_{T_\eta \mu} S(a)\}$ occurs (with a natural coupling between the Boolean models) either in $\Sigma(H^\rho m_{n-1}^\rho) \cap B(a)$ one can find a component of diameter at least η . We use this observation through its following crude consequence (see Lemma 7):

$$p(a, m_n^\rho) \leq p(a, T_\eta \mu) + C a^d \eta^{-d} p(\eta/2, H^\rho m_{n-1}^\rho).$$

By scaling and by (8), this yields:

$$p(a, m_n^\rho) \leq \varepsilon_1/2 + C a^d \eta^{-d} p(\rho\eta/2, m_{n-1}^\rho). \quad (9)$$

But for any ε_2 , any small enough ε_1 and any large enough a we can find τ such that (see Lemmas 8 and 9):

$$p(\tau a, m_{n-1}^\rho) \leq \varepsilon_2 \text{ as soon as } p(a, m_{n-1}^\rho) \leq \varepsilon_1. \quad (10)$$

An important fact is that τ does not depends on n nor on ρ , provided $\rho \geq \rho_0$ where ρ_0 is an arbitrary constant strictly larger than 1. Here we use assumption (6) to bound error terms due to the existence of large balls.

We choose ε_2 such that:

$$Ca^d \eta^{-d} \varepsilon_2 = \varepsilon_1/2.$$

We set $\rho = 2\tau a/\eta$. Then, (9) and (10) can be rewritten as follows:

$$p(a, m_n^\rho) \leq \varepsilon_1/2 + Ca^d \eta^{-d} p(\tau a, m_{n-1}^\rho) \quad (11)$$

$$Ca^d \eta^{-d} p(\tau a, m_{n-1}^\rho) \leq \varepsilon_1/2 \text{ as soon as } p(a, m_{n-1}^\rho) \leq \varepsilon_1. \quad (12)$$

As moreover (8) implies $p(a, m_0^\rho) \leq \varepsilon_1$ we get, by induction and then sending n to infinity (see Lemma 10):

$$p(a, m_\infty^\rho) \leq \varepsilon_1.$$

The convergence of $p(a, m_\infty^\rho)$ to 0 is then extracted from the above result for a small enough ε_2 and from arguments behind (10) applied to m_∞^ρ and other ε .

Sketch of the proofs of Theorems 1 and 2 by Menshikov, Popov and Vachkovskaia

Let us quickly describe the ideas of the proofs of Menshikov, Popov and Vachkovskaia. Those ideas are used in their papers [11] and [12] through a discretization of space ; we describe them in a slightly more geometric way. For simplicity we only consider two scales: $\rho^{-1}\Sigma_1$ and Σ_0 . For simplicity, we also assume that the radius is one in the unscaled model ($\mu = \lambda\delta_1$). We assume that the scale factor ρ is large enough. Assume that C is a connected component of $\rho^{-1}\Sigma_1 \cup \Sigma_0$ whose diameter is at least α (it can be much larger) for a small enough constant $\alpha > 0$. Then, C is included in the union of the following kind of sets:

1. connected components of $\rho^{-1}\Sigma_1$ whose diameter is at least α ;
2. balls of Σ_0 enlarged by α (same centers but the radii are $1 + \alpha$ instead of 1).

Then, they show that the union of all those sets is stochastically dominated by a Boolean model similar to Σ_0 but with radii enlarged by a factor α and with density of centers $1 + \alpha'$ times the corresponding density for Σ_0 for a suitable $\alpha' > 0$. This part uses $\lambda < \widehat{\lambda}_c$. In some sense, one can therefore control percolation in the union of two models by percolation in one model. Iterating the argument with some care in the constants α and α' , one sees that – for large enough ρ – one can control percolation in the multiscale model by percolation in a subcritical model. This yields the result.

2.3 Proof of Theorem 4

As $1 < \widehat{\lambda}_c(\mu)$, we know that $p(a, \mu)$ tends to 0 as a tends to infinity. We need the following slightly stronger consequence.

Lemma 6 *There exists $\eta > 0$ such that $p(a, T_\eta \mu)$ tends to 0.*

Proof. Let $\varepsilon > 0$ and $x > 0$. We have:

$$\begin{aligned} H^{1+\varepsilon} T_{\varepsilon^2} \mu([x, +\infty[) &= (1 + \varepsilon)^d T_{\varepsilon^2} \mu([x(1 + \varepsilon), +\infty[) \\ &= (1 + \varepsilon)^d \mu([x(1 + \varepsilon) - \varepsilon^2, +\infty[) \\ &\leq \kappa(\varepsilon)(1 + \varepsilon)^d \mu([x, +\infty[) \end{aligned} \quad (13)$$

where

$$\kappa(\varepsilon) = \frac{\mu(]0, +\infty[)}{\mu([\varepsilon, +\infty[)}.$$

The inequality is proven as follows. If $x \geq \varepsilon$, then $[x(1 + \varepsilon) - \varepsilon^2, +\infty[\subset [x, +\infty[$ and the result follows from $\kappa(\varepsilon) \geq 1$. If, on the contrary, $x < \varepsilon$, then the left hand side is bounded above by $(1 + \varepsilon)^d \mu(]0, +\infty[)$ which is itself bounded above by the right hand side.

Note that $\kappa(\varepsilon)(1 + \varepsilon)^d$ tends to 1 as ε tends to 0. Let us say that a measure ν is subcritical if $\widehat{\lambda}_c(\nu) > 1$. As μ is subcritical, we get that $\kappa(\varepsilon)(1 + \varepsilon)^d \mu$ is subcritical for small enough ε . We fix such an ε . By (13) we can couple a Boolean model driven by $H^{1+\varepsilon}T_{\varepsilon^2}\mu$ and a Boolean model driven by $\kappa(\varepsilon)(1 + \varepsilon)^d \mu$ in such a way that the first one is contained in the second one. Therefore the first one is subcritical. By scaling, a Boolean model driven by $T_{\varepsilon^2}\mu$ is then subcritical. We take $\eta = \varepsilon^2$. \square

Lemma 7 *Let ν_1 and ν_2 be two finite measures on $]0, +\infty[$. One has, for all $\eta > 0$ and $a \geq 4\eta$:*

$$p(a, \nu_1 + \nu_2) \leq p(a, T_\eta \nu_1) + C_1 a^d \eta^{-d} p(\eta/2, \nu_2)$$

where $C_1 = C_1(d) > 0$ depends only on the dimension d .

Proof. Let $(x_i)_{i \in I}$ be a family of points such that :

- The balls $B(x_i, \eta/4)$, $i \leq I$, cover $B(a)$.
- There are at most $C_1 a^d \eta^{-d}$ points in the family where $C_1 = C_1(d)$ depends only on the dimension d .

We couple the different Boolean model as follows. Let $\Sigma(\nu_1)$ be a Boolean model driven by ν_1 . Let $\Sigma(\nu_2)$ be a Boolean model driven by ν_2 . Assume that $\Sigma(\nu_1)$ and $\Sigma(\nu_2)$ are independent. Then $\Sigma(\nu_1) \cup \Sigma(\nu_2)$ is a Boolean model driven by $\nu_1 + \nu_2$. We set $\Sigma(\nu_1 + \nu_2) = \Sigma(\nu_1) \cup \Sigma(\nu_2)$. We also consider $\Sigma(T_\eta \nu_1)$, the Boolean model obtained by adding η to the radius of each ball of $\Sigma(\nu_1)$. Thus $\Sigma(T_\eta \nu_1)$ is driven by $T_\eta \nu_1$.

Let us prove the following property:

$$\{S(a/2) \leftrightarrow_{\Sigma(\nu_1 + \nu_2)} S(a)\} \subset \{S(a/2) \leftrightarrow_{\Sigma(T_\eta \nu_1)} S(a)\} \cup \bigcup_{i \in I} \{S(x_i, \eta/4) \leftrightarrow_{\Sigma(\nu_2)} S(x_i, \eta/2)\}. \quad (14)$$

Assume that $\Sigma(\nu_1 + \nu_2) = \Sigma(\nu_1) \cup \Sigma(\nu_2)$ connects $S(a/2)$ with $S(a)$. Recall $a \geq 4\eta$. If the diameter of all connected components of $\Sigma(\nu_2) \cap B(a)$ are less or equal to η , then $\Sigma(T_\eta \nu_1)$ connects $S(a/2)$ with $S(a)$. Otherwise, let C be a connected component of $\Sigma(\nu_2) \cap B(a)$ with diameter at least η . Let x, y be two points of C such that $\|x - y\| > \eta$. The point x belongs to a ball $B(x_i, \eta/4)$. As y does not belong to $B(x_i, \eta/2)$, the component C connects $S(x_i, \eta/4)$ to $S(x_i, \eta/2)$. Therefore, $\Sigma(\nu_2)$ connects $S(x_i, \eta/4)$ to $S(x_i, \eta/2)$. We have proven (14). The lemma follows. \square

The following lemma is essentially the first item of Proposition 3.1 in [4]. For the sake of completeness we nevertheless provide a proof.

Lemma 8 *Let ν be a finite measure on $]0, +\infty[$. There exists a constant $C_2 = C_2(d) > 0$ such that, for all $a > 0$:*

$$p(10a, \nu) \leq C_2 p(a, \nu)^2 + C_2 \int_{[a, +\infty[} r^d \nu(dr).$$

Proof. Let K be a finite subset of $S(5)$ such that $K + B(1/2)$ covers $S(5)$. Let L be a finite subset of $S(10)$ such that $L + B(1/2)$ covers $S(10)$. Let A be the following event: there exists a random ball $B(c, r)$ of $\Sigma(\nu)$ such that $r \geq a$ and $B(c, r) \cap B(10a)$ is non empty. We have:

$$\{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\} \setminus A \subset \{S(5a) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(10a)\}$$

where, in the last event, we ask for a path using only balls of $\Sigma(\nu)$ of radius at most a . Let us prove the following:

$$\begin{aligned} & \{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\} \setminus A \\ & \subset \bigcup_{k \in K, l \in L} \{S(ak, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(ak, a)\} \cap \{S(al, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(al, a)\}. \end{aligned} \quad (15)$$

Assume that the event on the left hand side occurs. Then, by the previous remark, there exists a path from a point $x \in S(5a)$ to a point $y \in S(10a)$ that is contained in balls of $\Sigma(\nu)$ of radius at most a . As $Ka + B(a/2)$ covers $S(5a)$, there exists $k \in K$ such that x belongs to $B(ka, a/2)$. Using the previous path, one gets that the event

$$\{S(ak, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(ak, a)\}$$

occurs. By a similar arguments involving y we get (15).

Observe that, for all $k \in K$ and $l \in L$, the events

$$\{S(ak, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(ak, a)\} \text{ and } \{S(al, a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(al, a)\}$$

are independent. Indeed, the first one depends only on balls with centers in $B(ak, 2a)$, the second one depends only on balls with centers in $B(al, 2a)$, and $\|ak - al\| \geq 5a$. Using this independence, stationarity and (15), we then get:

$$P(\{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\}) \leq CP(S(a/2) \leftrightarrow_{\Sigma(\nu)}^{\leq a} S(a))^2 + P(A)$$

where C is the product of the cardinality of K by the cardinality of L . The probability $P(A)$ is bounded above by standard computations. \square

From the previous lemma, we deduce the following result.

Lemma 9 *Let $\varepsilon > 0$. There exists $C_3 = C_3(d) > 0$, $a_0 = a_0(d, \mu)$ and $k_0 = k_0(d, \mu, \varepsilon)$ such that, for all N , all $\rho \geq 2$ and all $a \geq a_0$: if $p(a, m_N^\rho) \leq C_3$ then for all $k \geq k_0$, $p(a10^k, m_N^\rho) \leq \varepsilon$.*

Proof. For all $\rho \geq 2$ and all $a \geq 1$ we have:

$$\begin{aligned} \int_{[a, +\infty[} r^d m_\infty^\rho(dr) &= \sum_{k \geq 0} \rho^{kd} \int_{]0, +\infty[} 1_{[a, +\infty[}(r\rho^{-k})(r\rho^{-k})^d \mu(dr) \\ &= \int_{]0, +\infty[} \sum_{k \geq 0} 1_{[a, +\infty[}(r\rho^{-k}) r^d \mu(dr) \\ &= \int_{[a, +\infty[} (\lfloor \ln(r/a) \ln(\rho) \rfloor + 1) r^d \mu(dr) \\ &\leq \int_{[a, +\infty[} (\ln(r) \ln(2)^{-1} + 1) r^d \mu(dr). \end{aligned}$$

Let C_2 be the constant given by Lemma 8. By (6) we can chose $a_0 = a_0(d, \mu) \geq 1$ such that

$$C_2^2 \int_{[a_0, +\infty[} (\ln(r) \ln(2)^{-1} + 1) r^d \mu(dr) \leq \frac{1}{4}. \quad (16)$$

Let $C_3 = (2C_2)^{-1}$. Let N , ρ and a be as in the statement of the lemma. From Lemma 8 we get:

$$C_2 p(10a, m_N^\rho) \leq (C_2 p(a, m_N^\rho))^2 + C_2^2 \int_{[a, +\infty[} r^d m_N^\rho(dr) \quad (17)$$

$$\leq (C_2 p(a, m_N^\rho))^2 + C_2^2 \int_{[a, +\infty[} (\ln(r) \ln(2)^{-1} + 1) r^d \mu(dr) \quad (18)$$

Let (u_k) be a sequence defined by $u_0 = 1/2$ and, for all $k \geq 0$:

$$u_{k+1} = u_k^2 + C_2^2 \int_{[a_0 10^k, +\infty[} (\ln(r) \ln(2)^{-1} + 1) r^d \mu(dr). \quad (19)$$

Note that the sequence (u_k) only depends on d and μ .

Assume that $p(a, m_N^\rho) \leq C_3$. We then have $C_2 p(a, m_N^\rho) \leq u_0$. Using $a \geq a_0$ and (18), we then get $C_2 p(a 10^k, m_N^\rho) \leq u_k$ for all k . Therefore, it suffices to show that the sequence (u_k) tends to 0.

Using (19), (16) and $u_0 = 1/2$ we get $0 \leq u_k \leq 1/2$ for all k . Therefore, $0 \leq \limsup u_k \leq 1/2$. By (19) and by the convergence of the integrale we also get $\limsup u_k \leq (\limsup u_k)^2$. As a consequence, $\limsup u_k = 0$ and the lemma is proven. \square

Lemma 10 *For all $a > 0$ and $\rho > 1$ the following convergence holds:*

$$p(a, m_\infty^\rho) = \lim_{N \rightarrow \infty} p(a, m_N^\rho).$$

Proof. The sequence of events

$$A_N = \{S(a/2) \leftrightarrow_{\Sigma(m_N^\rho)} S(a)\}$$

is increasing (we use the natural coupling between our Boolean models). Therefore, it suffices to show that the union of the previous events is

$$A = \{S(a/2) \leftrightarrow_{\Sigma(m_\infty^\rho)} S(a)\}.$$

If A occurs, then there is a path from $S(a/2)$ to $S(a)$ that is contained in $\Sigma(m_\infty^\rho)$. By a compactness argument, this path is included in a finite union of ball of $\Sigma(m_\infty^\rho)$. Therefore, there exists N such that the path is included in $\Sigma(m_N^\rho)$ and A_N occurs. This proves $A \subset \cup A_N$. The other inclusion is straightforward. \square

Proof of the second item of Theorem 4. By Lemma 6, we can fix $\eta_1 > 0$ such that $p(a, T_{10\eta_1} \mu)$ tends to 0 as a tends to ∞ . Let C_1 be given by Lemma 7. Let a_0 and C_3 be as given by Lemma 9. Fix $a_1 \geq \max(40\eta_1, a_0, 1)$ such that $p(a, T_{10\eta_1} \mu) \leq C_3/2$ for all $a \geq a_1$. Let k_0 be given by Lemma 9 with the choice:

$$\varepsilon = C_1^{-1} (10a_1)^{-d} \eta_1^d C_3/2.$$

Therefore, for all $\rho \geq 2$, all N , all $a \in [a_1, 10a_1]$ and all $\eta \in [\eta_1, 10\eta_1]$:

$$C_1 a^d \eta^{-d} p(a10^k, m_N^\rho) \leq \frac{C_3}{2} \text{ for all } k \geq k_0 \text{ as soon as } p(a, m_N^\rho) \leq C_3.$$

Fix $k \geq k_0$, $a \in [a_1, 10a_1]$ and $\eta \in [\eta_1, 10\eta_1]$. Set:

$$\rho = 2a10^k \eta^{-1}.$$

Note $\rho \geq 8 \geq 2$ as $a \geq a_1 \geq 40\eta_1 \geq 4\eta$. By Lemma 7 we have, for all N :

$$\begin{aligned} p(a, m_{N+1}^\rho) &\leq p(a, T_\eta \mu) + C_1 a^d \eta^{-d} p(\eta/2, m_{N+1}^\rho - \mu) \\ &= p(a, T_\eta \mu) + C_1 a^d \eta^{-d} p(\eta/2, H^\rho m_N^\rho). \end{aligned}$$

By definition of a_1 , by $a_1 \leq a$, by $\eta \leq 10\eta_1$, by scaling and by definition of ρ we get, for all N :

$$\begin{aligned} p(a, m_{N+1}^\rho) &\leq \frac{C_3}{2} + C_1 a^d \eta^{-d} p(\rho\eta/2, m_N^\rho) \\ &= \frac{C_3}{2} + C_1 a^d \eta^{-d} p(a10^k, m_N^\rho). \end{aligned}$$

Combining this inequality with the property defining k_0 , we get that $p(a, m_N^\rho) \leq C_3$ implies $p(a, m_{N+1}^\rho) \leq C_3$. As $p(a, m_0^\rho) = p(a, \mu) \leq p(a, T_{10\eta_1} \mu) \leq C_3/2$ we get $p(a, m_N^\rho) \leq C_3$ for all integer N .

Let $\varepsilon > 0$. Using again Lemma 9 we get the existence of an integer k'_0 such that $p(a10^{k'}, m_N^\rho) \leq \varepsilon$ for all $k' \geq k'_0$ as soon as $p(a, m_N^\rho) \leq C_3$. But we have proven the latter property. Therefore $p(a10^{k'}, m_N^\rho) \leq \varepsilon$ for all N and all $k' \geq k'_0$. By Lemma 10, we get $p(a10^{k'}, m_\infty^\rho) \leq \varepsilon$ for all $k' \geq k'_0$. Using the freedom on the choice of $k \geq k_0$ and $\eta \in [\eta_1, 10\eta_1]$, we get that the previous result holds for all $\rho \geq 2a10^{k_0-1}\eta_1^{-1}$ and then for all $\rho \geq 2a_110^{k_0}\eta_1^{-1}$. Moreover, using the freedom on the choice of $a \in [a_1, 10a_1]$ and $k' \geq k'_0$, we get:

$$p(r, m_\infty^\rho) \leq \varepsilon \text{ for all } r \geq a_110^{k'_0} \text{ and all } \rho \geq 2a_110^{k_0}\eta_1^{-1}.$$

Therefore, $p(r, m_\infty^\rho)$ tends to 0 as r tends to infinity. As a consequence, $\Sigma^\rho(\mu)$ does not percolate for any $\rho \geq 2a_110^{k_0}\eta_1^{-1}$. \square

3 Proof of Theorem 5

Lemma 11 *Let $s > 0$ and $\rho > 1$. The following assumptions are equivalent:*

1. $\int_{]0, +\infty[} r^{d+s} \mu(dr) < \infty$.
2. $\int_{]1, +\infty[} r^{d+s} m_\infty^\rho(dr) < \infty$.

Proof. We have:

$$\begin{aligned} \int_{]1, +\infty[} r^{d+s} m_\infty^\rho(dr) &= \sum_{k \geq 0} \rho^{kd} \int_{]0, +\infty[} 1_{]1, +\infty[}(r\rho^{-k})(r\rho^{-k})^{d+s} \mu(dr) \\ &= \int_{]1, +\infty[} \sum_{k \geq 0} 1_{]1, +\infty[}(r\rho^{-k}) \rho^{-ks} r^{d+s} \mu(dr). \end{aligned}$$

Therefore:

$$\int_{[1,+\infty[} r^{d+s} \mu(dr) \leq \int_{[1,+\infty[} r^{d+s} m_\infty^\rho(dr) \leq \frac{1}{1-\rho^{-s}} \int_{[1,+\infty[} r^{d+s} \mu(dr).$$

This yields the result. \square

Proof of the first item of Theorem 5. By the discussion at the beginning of Section 1.5 in [5], $\Sigma^\rho(\lambda\mu)$ is driven by a Poisson point process whose intensity is the product of the Lebesgue measure by the locally finite measure λm_∞^ρ . Let us check the three items of Theorem 2.9 in [5] with $\rho = 10$ (ρ is not use in the same way in [5]). We refer to Section 2.1 of [5] for definitions.

1. The first item is fulfilled thanks to (7)
2. For all $\beta > 0$ and all $x \in \mathbb{R}^d$, the event $G(x, 0, \beta)$ only depends on balls $B(c, r) \in \Sigma^\rho(\lambda\mu)$ such that c belongs to $B(x, 3\beta)$. By the independance property of Poisson point processes, we then get that $G(0, 0, \beta)$ and $G(x, 0, \beta)$ are independent whenever $\|x\| \geq 10\beta$. Therefore $I(10, 0, \beta) = 0$ and the second item of Theorem 2.9 is fulfilled.
3. The third item (note that μ in [5] is m_∞^ρ in this paper) is fulfilled thanks to Lemma 11

Theorem 2.9 in [5] yields the result. \square

Proof of the second item of Theorem 5. If $\int r^d \mu(dr)$ is infinite then, $\Sigma(\lambda\mu)$ percolates for all $\lambda > 0$ (see the dicussion of Section 1.2). Therefore $\Sigma^\rho(\lambda\mu)$ percolates for all $\rho > 1$ and $\lambda > 0$. Therefore $D^\rho(\lambda\mu) = \infty$ with positive probability for all $\rho > 1$ and $\lambda > 0$.

Now, assume that $\int r^d \mu(dr)$ is finite. Then, by the discussion at the beginning of Section 1.5 in [5], $\Sigma^\rho(\lambda\mu)$ is driven by a Poisson point process whose intensity is the product of the Lebesgue measure by the locally finite measure λm_∞^ρ . We can therefore apply Theorem 1.2 in [5]. By Lemma 11, assumption (A3) of Theorem 1.2 in [5] is not fulfilled (note that μ in [5] is m_∞^ρ in this paper). Theorem 1.2 in [5] then yields the result. \square

A Critical parameters

Lemma 12

$$\tilde{\lambda}_c(\mu) \leq \hat{\lambda}_c(\mu) \leq \lambda_c(\mu).$$

Proof. The second inequality is a consequence of the following inclusion:

$$\{\{0\} \leftrightarrow_\Sigma S(r)\} \subset \{S(r/2) \leftrightarrow_\Sigma S(r)\}.$$

The first inequality can be proven as follows. Let $r \geq 1$. By the FKG inequality, we get:

$$\begin{aligned} P(\{0\} \leftrightarrow_\Sigma S(r)) &\geq P(B(0, 1) \subset \Sigma \text{ and } S(1) \leftrightarrow_\Sigma S(r)) \\ &\geq CP(S(1) \leftrightarrow_\Sigma S(r)) \end{aligned}$$

where $C = P(B(0, 1) \subset \Sigma) > 0$ does not depend on r . For all large enough r , we can cover $S(2r)$ by at most $C'r^d$ balls $B(x_i, 1)$ where C' only depends on the dimension d . If

there is a path in Σ from $S(2r)$ to $S(4r)$, then there exists i and a path in Σ from $S(x_i, 1)$ to $S(x_i, r)$. (Consider the ball $B(x_i, 1)$ that contains the initial point of the path.) By stationarity and by the previous inequality we thus get:

$$\begin{aligned} P(S(2r) \leftrightarrow_{\Sigma} S(4r)) &\leq C' r^d P(S(1) \leftrightarrow_{\Sigma} S(r)) \\ &\leq C' C^{-1} r^d P(\{0\} \leftrightarrow_{\Sigma} S(r)). \end{aligned}$$

The first inequality stated in the lemma follows. \square

Lemma 13 *The threshold parameter $\widehat{\lambda}_c(\mu)$ is positive if and only if $\int r^d \mu(dr)$ is finite.*

Proof. If $\widehat{\lambda}_c(\mu)$ is positive, then there exists $\lambda > 0$ such that $\Sigma(\lambda\mu)$ does not percolate. By Theorem 2.1 of [4] this implies that $\int r^d \mu(dr)$ is finite.

Let us assume now that $\int r^d \mu(dr)$ is finite. We need to prove the existence of $\lambda > 0$ such that $p(a, \lambda\mu)$ tends to 0. This is proven, as an intermediate result, in the proof of Theorem 1.1 in [5]. As the result is an easy consequence of Lemma 8, we find it more convenient to provide a proof here. Let C_2 be the constant given by Lemma 8. For all $a > 0$ and $\lambda > 0$ we have:

$$C_2 p(10a, \lambda\mu) \leq (C_2 p(a, \lambda\mu))^2 + \lambda C_2^2 \int_{[a, +\infty[} r^d \mu(dr). \quad (20)$$

For all $0 < a \leq 1$ we have, by standard computations:

$$C_2 p(a, \lambda\mu) \leq C_2 P(\text{a ball of } \Sigma(\lambda\mu) \text{ touches } B(a)) \leq C_2 v_d \lambda \int_{]0, +\infty[} (1+r)^d \mu(dr)$$

where v_d is the volume of the unit Euclidean ball. As $\int r^d \mu(dr)$ is finite, we can therefore fix $\lambda > 0$ such that:

$$\lambda C_2^2 \int_{[a, +\infty[} r^d \mu(dr) \leq 1/4 \text{ for all } a > 0 \text{ and } C_2 p(a, \lambda\mu) \leq 1/2 \text{ for all } 0 < a \leq 1. \quad (21)$$

By (20), (21) and by induction we get $C_2 p(a, \lambda\mu) \leq 1/2$ for all $a > 0$. Therefore, we have $0 \leq \limsup C_2 p(a, \lambda\mu) \leq 1/2$. But (20) also yields the inequality $\limsup C_2 p(a, \lambda\mu) \leq (\limsup C_2 p(a, \lambda\mu))^2$. As a consequence we must have $\limsup C_2 p(a, \lambda\mu) = 0$ and then $p(a, \lambda\mu) \rightarrow 0$. \square

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