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A note on the wavelet deconvolution of a density from mixtures under quadrant dependence

Christophe Chesneau

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Abstract We consider the density convolution model: $Y = X + \epsilon$, where X and ϵ are independent random variables. We suppose that the density of X is a finite mixture with unknown components. We want to estimate a component of this mixture from pairwise positive quadrant dependent observations $Y_1, \ldots, Y_n$. To reach this goal, a linear wavelet estimator is developed. We measure its performance by determining an upper bound of the mean integrated squared error.

Keywords Density deconvolution · Mixture · Quadrant dependence · Wavelets · Besov balls.

2000 Mathematics Subject Classification 62G07, 62G20.

1 Motivations

We consider the following model:

$$Y_v = X_v + \epsilon_v, \qquad v \in \{1, \ldots, n\}, \quad n \in \mathbb{N}^*, \tag{1}$$

where $X_1, \ldots, X_n$ are n random variables and $\epsilon_1, \ldots, \epsilon_n$ are n identically distributed random variables. For any $v \in \{1, \ldots, n\}$, X_v and ϵ_v are independent. The density of ϵ_1, denoted g, is known and ordinary smooth (to be defined in Section 2). For any $v \in \{1, \ldots, n\}$, the density of X_v is the following finite mixture:

$$h_v(x) = \sum_{d=1}^{m} w_d(v) f_d(x), \qquad x \in [-\Omega, \Omega],$$

where $\Omega \in (0, \infty)$, $m \in \mathbb{N}^*$,

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- $(w_d(v))_{(v,d)\in\{1,\ldots,n\}\times\{1,\ldots,m\}}$ are known positive weights such that, for any $v \in \{1,\ldots,n\}$,

$$\sum_{d=1}^{m} w_d(v) = 1,$$

- $f_1,\ldots,f_m$ are m unknown densities.

We suppose that $Y_1,\ldots,Y_n$ are pairwise positive quadrant dependent (PPQD) (to be defined in Section 2). For a fixed $q \in \{1,\ldots,m\}$, we aim to estimate f_q when only $Y_1,\ldots,Y_n$ are observed.

Let us now present a brief review on (1) under various configurations. In the case where $Y_1,\ldots,Y_n$ are independent, (1) has been recently considered in Van Es et al. (2008), Lee et al. (2010) and Chesneau (2010a). Note that, in this case, if $m = 1$, $w_1(1) = \ldots = w_1(n) = 1$ and $f_q = f_1 = f$, it becomes the standard convolution density model. See e.g. Caroll and Hall (1988), Devroye (1989), Fan (1991), Pensky and Vidakovic (1999), Fan and Koo (2002), Butucea and Matias (2005), Comte et al. (2006), Delaigle and Gijbels (2006), Lacour (2006) and Butucea and Tsybakov (2008a,b). In the case where $Y_1,\ldots,Y_n$ have some kinds of dependence and $m = 1$, $w_1(1) = \ldots = w_1(n) = 1$ and $f_q = f_1 = f$, we refer to e.g. Masry (2003), Van Zanten and Zareba (2008), Comte et al. (2008) and Kulik (2008). The estimation of f_q from independent observed $X_1,\ldots,X_n$ has been investigated by e.g. Maiboroda (1996), Hall and Zhou (2003), Pokhyl'ko (2005) and Prakasa Rao (2010). The same problem with observed PPQD $X_1,\ldots,X_n$ has been considered in Chesneau (2010b). However, to the best of our knowledge, the estimation of f_q from $Y_1,\ldots,Y_n$ in the PPQD case is a new challenge.

We estimate f_q by a linear wavelet estimator. It is "similar" to the one of Pensky and Vidakovic (1999), Fan and Koo (2002) and Van Zanten and Zareba (2008). It has the originality to incorporate some technical tools on mixture and to take into account the PPQD case. We measure its performance by considering the mean integrated squared error (MISE) under the assumption that f_q belongs to a Besov ball $B^s_{p,r}(M)$ (to be defined in Section 3). We prove that it attains the rate of convergence $r_n = (\rho_n/n^{1-\theta})^{2s/(2s+2\delta+4)}$, where ρ_n depends on the weights of the mixture, θ is a "dependence factor" and δ is a parameter related to the ordinary smooth assumption on g.

The paper is organized as follows. Assumptions on (1) and some notations are introduced in Section 2. Section 3 briefly describes the wavelet basis and the Besov balls. The linear wavelet estimator is presented in Section 4. The upper bound result is set in Section 5. Section 6 is devoted to the proofs.

2 Assumptions and notations

Assumption on $f_1,\ldots,f_m$. For any $d \in \{1,\ldots,m\}$, we assume that the support of f_d is $[-\Omega,\Omega]$.

Assumptions on $Y_1, \ldots, Y_n$. Recall that $Y_1, \ldots, Y_n$ are PPQD i.e. for any $(v, \ell) \in \{1, \ldots, n\}^2$ with $v \neq \ell$ and any $(x, y) \in \mathbb{R}^2$,

$$\mathbb{P}(Y_v > x, Y_\ell > y) \geq \mathbb{P}(Y_v > x)\mathbb{P}(Y_\ell > y).$$

See Lehmann (1966) and Newman (1980).
Moreover, we assume that there exist positive real numbers $b_0, \ldots, b_{n-1}$ such that

1. for any $(v, \ell) \in \{1, \ldots, n\}^2$,

$$\mathbb{C}(Y_v, Y_\ell) = b_{|v-\ell|}, \tag{2}$$

2. there exist two constants, $C > 0$ and $\theta \in [0, 1)$, satisfying

$$\sum_{u=0}^{n-1} b_u \leq C n^\theta. \tag{3}$$

Assumptions on g. We define the Fourier transform of a function h by

$$\mathcal{F}(h)(x) = \int_{-\infty}^{\infty} h(y) e^{-ixy} dy, \qquad x \in \mathbb{R},$$

whenever this integral exists. The notation $\overline{\cdot}$ will be used for the complex conjugate.
We assume that there exist two constants, $c_* > 0$ and $\delta > 1$, such that

$$|\,\mathcal{F}(g)(x)| \geq \frac{c_*}{(1 + x^2)^{\delta/2}}, \qquad x \in \mathbb{R}. \tag{4}$$

This assumption controls the decay of the Fourier coefficients of g, and thus the smoothness of g.

Assumptions on the weights. Set

$$\Gamma_n = \left(\frac{1}{n} \sum_{v=1}^{n} w_k(v) w_\ell(v) \right)_{(k,\ell) \in \{1, \ldots, m\}^2}.$$

We suppose that $\det(\Gamma_n) > 0$. For the considered q and any $v \in \{1, \ldots, n\}$, we set

$$a_q(v) = \frac{1}{\det(\Gamma_n)} \sum_{k=1}^{m} (-1)^{k+q} \gamma_{q,k}^n w_k(v), \tag{5}$$

where $\gamma_{q,k}^n$ denotes the determinant of the minor (q, k) of the matrix Γ_n. Then, for any $k \in \{1, \ldots, m\}$,

$$\frac{1}{n} \sum_{v=1}^{n} a_q(v) w_k(v) = \begin{cases} 1 & \text{if } k = q, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

and

$$(a_q(1), \ldots, a_q(n)) = \underset{(b_1,\ldots,b_n)\in\mathbb{R}^n}{\operatorname{argmin}} \frac{1}{n}\sum_{v=1}^{n} b_v^2.$$

Technical details can be found in Maiboroda (1996).
We set

$$\rho_n = \frac{1}{n}\sum_{v=1}^{n} a_q^2(v) \tag{7}$$

and, for technical reasons, we suppose that $\rho_n < n^{1-\theta}$ where θ refers to (3).

3 Wavelets and Besov balls

Wavelet basis. Let $N \in \mathbb{N}^*$, ϕ be a father wavelet of a multiresolution analysis on $\mathbb{R}$ and ψ be the associated mother wavelet. Assume that
- $\operatorname{supp}(\phi) = \operatorname{supp}(\psi) = [1-N, N]$,
- $\int_{1-N}^{N} \phi(x)dx = 1$,
- for any $v \in \{0, \ldots, N-1\}$, $\int_{1-N}^{N} x^v \psi(x)dx = 0$,
- ϕ and ψ are of class $\mathcal{C}^{\upsilon}$, $\upsilon > 2+\delta$, where δ is the one in (4).

(For instance, the Daubechies wavelets dbN with an appropriate N satisfy these assumptions).
Set

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k), \qquad \psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k).$$

Then, with an appropriate treatments at the boundaries, there exists an integer τ and a set of consecutive integers Λ_j with a length proportional to 2^j such that, for any integer $\ell \geq \tau$, the collection

$$\mathcal{B} = \{\phi_{\ell,k}(.),\ k \in \Lambda_\ell;\ \psi_{j,k}(.);\ j \in \mathbb{N} - \{0, \ldots, \ell-1\},\ k \in \Lambda_j\},$$

is an orthonormal basis of $\mathbb{L}^2([-\Omega, \Omega]) = \{h : [-\Omega, \Omega] \to \mathbb{R};\ \int_{-\Omega}^{\Omega} h^2(x)dx < \infty\}$.We refer to Cohen et al. (1993).
For any integer $\ell \geq \tau$, any $h \in \mathbb{L}^2([-\Omega, \Omega])$ can be expanded on $\mathcal{B}$ as

$$h(x) = \sum_{k\in\Lambda_\ell} \alpha_{\ell,k}\phi_{\ell,k}(x) + \sum_{j=\ell}^{\infty}\sum_{k\in\Lambda_j} \beta_{j,k}\psi_{j,k}(x), \quad x \in [-\Omega, \Omega],$$

where $\alpha_{j,k}$ and $\beta_{j,k}$ are the wavelet coefficients of h defined by

$$\alpha_{j,k} = \int_{-\Omega}^{\Omega} h(x)\phi_{j,k}(x)dx, \qquad \beta_{j,k} = \int_{-\Omega}^{\Omega} h(x)\psi_{j,k}(x)dx. \tag{8}$$

Besov balls. Let $M > 0$, $s > 0$, $p \geq 1$ and $r \geq 1$. A function h belongs to $B^s_{p,r}(M)$ if and only if there exists a constant $M^* > 0$ (depending on M) such that the associated wavelet coefficients (8) satisfy

$$\left(\sum_{j=\tau-1}^{\infty} \left(2^{j(s+1/2-1/p)} \left(\sum_{k \in \Lambda_j} |\beta_{j,k}|^p \right)^{1/p} \right)^r \right)^{1/r} \leq M^*.$$

We set $\beta_{\tau-1,k} = \alpha_{\tau,k}$. In this expression, s is a smoothness parameter and p and r are norm parameters. Besov balls contain the Hölder and Sobolev balls. See Meyer (1992).

4 Linear estimator

Assuming that $f_q \in B^s_{p,r}(M)$ with $p \geq 2$, we define the linear estimator $\widehat{f}$ by

$$\widehat{f}(x) = \sum_{k \in \Lambda_{j_0}} \widehat{\alpha}_{j_0,k} \phi_{j_0,k}(x), \qquad x \in [-\Omega, \Omega], \tag{9}$$

where

$$\widehat{\alpha}_{j_0,k} = \frac{1}{2\pi n} \sum_{v=1}^{n} a_q(v) \int_{-\infty}^{\infty} \frac{\overline{\mathcal{F}(\phi_{j,k})}(x)}{\mathcal{F}(g)(x)} e^{-ixY_v} dx, \tag{10}$$

$a_q(1), \ldots, a_q(n)$ are defined by (5), j_0 is the integer such that

$$\frac{1}{2} \left(\frac{n^{1-\theta}}{\rho_n} \right)^{1/(2s+2\delta+4)} < 2^{j_0} \leq \left(\frac{n^{1-\theta}}{\rho_n} \right)^{1/(2s+2\delta+4)},$$

ρ_n is defined by (7) and θ is the one in (3).

The integer j_0 is chosen to minimize the MISE of $\widehat{f}$ (see the proof of Theorem 1 below).

5 Upper bounds

Theorem 1 *Consider* (1) *under the assumptions of Section* 2. *Suppose that* $f_q \in B^s_{p,r}(M)$ *with* $s > 0$, $p \geq 2$ *and* $r \geq 1$. *Let* $\widehat{f}$ *be* (9). *Then there exists a constant* $C > 0$ *such that*

$$\mathbb{E}\left(\int_{-\Omega}^{\Omega} \left(\widehat{f}(x) - f_q(x) \right)^2 dx \right) \leq C \left(\frac{\rho_n}{n^{1-\theta}} \right)^{2s/(2s+2\delta+4)}.$$

The proof of Theorem 1 uses a moment inequality on (10) and a suitable decomposition of the MISE.

Note that the rate of convergence of $\widehat{f}$ in the PPQD case is greater than the one obtained in the independent case i.e. $(\rho_n/n)^{2s/(2s+2\delta+1)}$ (see (Chesneau 2010a, Theorem 1)).

Due to the definition of j_0, $\widehat{f}$ is not adaptive with respect to s. Adaptivity can perhaps be achieved by using another wavelet estimator as the hard thresholding one. This approach works when $Y_1, \ldots, Y_n$ are independent (see (Chesneau 2010a, Theorem 2)), but the proof of this fact uses technical probability inequalities (Bernstein, Rosenthal, ...) and it is not immediately clear how to extend this to the PPQD case.

6 Proofs

In this section, we consider (1) under the assumptions of Section 2. Moreover, C denotes any constant that does not depend on j, k and n. Its value may change from one term to another and may depends on ϕ.

Lemma 1 *For any $k \in \Lambda_{j_0}$, set*

$$h_{j_0,k}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{\mathcal{F}(\phi_{j_0,k})}(x)}{\mathcal{F}(g)(x)} e^{-ixy} dx, \qquad y \in \mathbb{R}.$$

Then there exists a constant $C > 0$ such that

$$\sup_{y \in \mathbb{R}} |h'_{j_0,k}(y)| \leq C 2^{j_0(\delta+3/2)}.$$

Proof of Lemma 1. With the aid of differentiation under the integral sign, we have

$$h'_{j_0,k}(y) = -i \frac{1}{2\pi} \int_{-\infty}^{\infty} x \frac{\overline{\mathcal{F}(\phi_{j_0,k})}(x)}{\mathcal{F}(g)(x)} e^{-ixy} dx, \qquad y \in \mathbb{R}.$$

Now note that, since $\phi \in \mathcal{C}^{\upsilon}$, there exists a constant $C > 0$ such that $|\mathcal{F}(\phi)(x)| \leq C(1+|x|)^{-\upsilon}$, $x \in \mathbb{R}$ (see Meyer (1992)). Therefore, using $\upsilon > 2+\delta$, we have

$$\begin{aligned} &\int_{-\infty}^{\infty} |x|(1+x^2)^{\delta/2} |\mathcal{F}(\phi)(x)| \, dx \\ &\leq C \int_{-\infty}^{\infty} |x|(1+x^2)^{\delta/2}(1+|x|)^{-\upsilon} dx < \infty. \end{aligned} \tag{11}$$

Using (4), the equality $|\mathcal{F}(\phi_{j_0,k})(x)| = 2^{-j_0/2}\left|\mathcal{F}(\phi)(x/2^{j_0})\right|$, the change of variables $u = x/2^{j_0}$ and (11), we obtain

$$\begin{aligned}
\sup_{y\in\mathbb{R}}|h'_{j_0,k}(y)| &\leq \frac{1}{2\pi}\int_{-\infty}^{\infty}|x|\frac{|\mathcal{F}(\phi_{j_0,k})(x)|}{|\mathcal{F}(g)(x)|}dx \\
&\leq \frac{1}{2\pi c_*}2^{-j_0/2}\int_{-\infty}^{\infty}|x|(1+x^2)^{\delta/2}\left|\mathcal{F}(\phi)(x/2^{j_0})\right|dx \\
&= \frac{1}{2\pi c_*}2^{-j_0/2}\int_{-\infty}^{\infty}|2^{j_0}u|(1+2^{2j_0}u^2)^{\delta/2}\left|\mathcal{F}(\phi)(u)\right|2^{j_0}du \\
&\leq \frac{1}{2\pi c_*}2^{j_0(\delta+3/2)}\int_{-\infty}^{\infty}|u|(1+u^2)^{\delta/2}\left|\mathcal{F}(\phi)(u)\right|du \\
&= C2^{j_0(\delta+3/2)}.
\end{aligned}$$

The proof of Lemma 1 is complete.

□

Proposition 1 *For any $k \in \Lambda_{j_0}$, let $\alpha_{j_0,k}$ be the wavelet coefficient* (8) *of f_q and $\widehat{\alpha}_{j_0,k}$ be* (10)*. Then there exists a constant $C > 0$ such that*

$$\mathbb{E}\left(\left|\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}\right|^2\right) \leq C2^{(2\delta+3)j_0}\frac{\rho_n}{n^{1-\theta}}.$$

Proof of Proposition 1. Since X_v and ϵ_v are independent, for any $x \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}\left(e^{-ixY_v}\right) &= \mathbb{E}\left(e^{-ixX_v}\right)\mathbb{E}\left(e^{-ix\epsilon_v}\right) = \mathcal{F}(h_v)(x)\mathcal{F}(g)(x) \\
&= \sum_{d=1}^{m} w_d(v)\mathcal{F}(f_d)(x)\mathcal{F}(g)(x). \qquad (12)
\end{aligned}$$

It follows from the Fubini theorem, (12), (6) and the Parseval-Plancherel theorem that

$$\begin{aligned}
\mathbb{E}\left(\widehat{\alpha}_{j_0,k}\right) &= \frac{1}{2\pi n}\sum_{v=1}^{n}a_q(v)\int_{-\infty}^{\infty}\frac{\overline{\mathcal{F}(\phi_{j_0,k})}(x)}{\mathcal{F}(g)(x)}\mathbb{E}\left(e^{-ixY_v}\right)dx \\
&= \frac{1}{2\pi n}\sum_{v=1}^{n}a_q(v)\int_{-\infty}^{\infty}\frac{\overline{\mathcal{F}(\phi_{j_0,k})}(x)}{\mathcal{F}(g)(x)}\sum_{d=1}^{m}w_d(v)\mathcal{F}(f_d)(x)\mathcal{F}(g)(x)dx \\
&= \sum_{d=1}^{m}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\mathcal{F}(\phi_{j_0,k})}(x)\mathcal{F}(f_d)(x)dx\right)\frac{1}{n}\sum_{v=1}^{n}a_q(v)w_d(v) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{\mathcal{F}(\phi_{j_0,k})}(x)\mathcal{F}(f_q)(x)dx = \int_{-\Omega}^{\Omega}\phi_{j_0,k}(x)f_q(x)dx = \alpha_{j_0,k}.
\end{aligned}$$

Therefore

$$\mathbb{E}\left(\left|\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}\right|^2\right) = \mathbb{V}\left(\widehat{\alpha}_{j_0,k}\right). \qquad (13)$$

For any $k \in \Lambda_{j_0}$, set

$$h_{j_0,k}(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\overline{\mathcal{F}(\phi_{j_0,k})}(x)}{\mathcal{F}(g)(x)} e^{-ixy} dx, \qquad y \in \mathbb{R}.$$

Then

$$\widehat{\alpha}_{j_0,k} = \frac{1}{n} \sum_{v=1}^{n} a_q(v) h_{j_0,k}(Y_v).$$

We have

$$\begin{aligned} \mathbb{V}\left(\widehat{\alpha}_{j_0,k}\right) &= \frac{1}{n^2} \sum_{v=1}^{n} \sum_{\ell=1}^{n} a_q(v) a_q(\ell) \mathbb{C}\left(h_{j_0,k}(Y_v), h_{j_0,k}(Y_\ell)\right) \\ &\leq \frac{1}{n^2} \sum_{v=1}^{n} \sum_{\ell=1}^{n} |a_q(v)||a_q(\ell)||\mathbb{C}\left(h_{j_0,k}(Y_v), h_{j_0,k}(Y_\ell)\right)|. \end{aligned} \tag{14}$$

We need the following lemma proved by (Newman 1980, Lemma 3).

Lemma 2 (Newman (1980)) *Let X and Y be two quadrant positive dependent random variables and $h : \mathbb{R} \to \mathbb{C}$ such that $\sup_{x\in\mathbb{R}} |h'(x)| < \infty$. Then*

$$|\mathbb{C}\left(h(X), h(Y)\right)| \leq \left(\sup_{x\in\mathbb{R}} |h'(x)|\right)^2 \mathbb{C}(X, Y).$$

Since $Y_1, \ldots, Y_n$ are PPQD, it follows from Lemma 2 that, for any $(v, \ell) \in \{1, \ldots, n\}^2$ with $v \neq \ell$,

$$|\mathbb{C}\left(h_{j_0,k}(Y_v), h_{j_0,k}(Y_\ell)\right)| \leq \left(\sup_{y\in\mathbb{R}} |h'_{j_0,k}(y)|\right)^2 \mathbb{C}(Y_v, Y_\ell). \tag{15}$$

Using (15), (2) and Lemma 1, we obtain

$$\begin{aligned} &\frac{1}{n^2} \sum_{v=1}^{n} \sum_{\ell=1}^{n} |a_q(v)||a_q(\ell)||\mathbb{C}\left(h_{j_0,k}(Y_v), h_{j_0,k}(Y_\ell)\right)| \\ &\leq \frac{1}{n^2} \left(\sup_{y\in\mathbb{R}} |h'_{j_0,k}(y)|\right)^2 \sum_{v=1}^{n} \sum_{\ell=1}^{n} |a_q(v)||a_q(\ell)| b_{|v-\ell|} \\ &\leq C \frac{1}{n^2} 2^{j_0(2\delta+3)} \sum_{v=1}^{n} \sum_{\ell=1}^{n} |a_q(v)||a_q(\ell)| b_{|v-\ell|}. \end{aligned} \tag{16}$$

We have

$$\begin{aligned}\sum_{v=1}^{n}\sum_{\ell=1}^{n}|a_q(v)||a_q(\ell)|b_{|v-\ell|} &= b_0 n\rho_n + 2\sum_{v=2}^{n}\sum_{\ell=1}^{v-1}|a_q(v)||a_q(\ell)|b_{v-\ell}\\ &\le b_0 n\rho_n + \sum_{v=2}^{n}\sum_{\ell=1}^{v-1}\left(a_q^2(v)+a_q^2(\ell)\right)b_{v-\ell}\\ &= b_0 n\rho_n + \sum_{v=2}^{n}\sum_{u=1}^{v-1}\left(a_q^2(v)+a_q^2(v-u)\right)b_u\\ &= b_0 n\rho_n + \sum_{v=2}^{n}a_q^2(v)\sum_{u=1}^{v-1}b_u + \sum_{v=2}^{n}\sum_{u=1}^{v-1}a_q^2(v-u)b_u.\end{aligned}$$

Using (3), we obtain

$$\sum_{v=2}^{n}a_q^2(v)\sum_{u=1}^{v-1}b_u \le n\rho_n\left(\sum_{u=0}^{n-1}b_u\right) \le C\rho_n n^{\theta+1}$$

and

$$\sum_{v=2}^{n}\sum_{u=1}^{v-1}a_q^2(v-u)b_u = \sum_{u=1}^{n-1}b_u\sum_{v=u+1}^{n}a_q^2(v-u) \le n\rho_n\left(\sum_{u=0}^{n-1}b_u\right) \le C\rho_n n^{\theta+1}.$$

Hence

$$\sum_{v=1}^{n}\sum_{\ell=1}^{n}|a_q(v)||a_q(\ell)|b_{|v-\ell|} \le C\rho_n n^{\theta+1}. \tag{17}$$

Putting (13), (14), (16) and (17) together, we obtain

$$\mathbb{E}\left(|\widehat{\alpha}_{j_0,k}-\alpha_{j_0,k}|^2\right) \le C2^{(2\delta+3)j_0}\frac{\rho_n}{n^{1-\theta}}.$$

This ends the proof of Proposition 1.

□

Proof of Theorem 1. We expand the function f_q on $\mathcal{B}$ as

$$f_q(x) = \sum_{k\in\Lambda_{j_0}}\alpha_{j_0,k}\phi_{j_0,k}(x) + \sum_{j=j_0}^{\infty}\sum_{k\in\Lambda_j}\beta_{j,k}\psi_{j,k}(x), \quad x\in[-\Omega,\Omega],$$

where

$$\alpha_{j_0,k} = \int_{-\Omega}^{\Omega}f_q(x)\phi_{j_0,k}(x)dx, \qquad \beta_{j,k} = \int_{-\Omega}^{\Omega}f_q(x)\psi_{j,k}(x)dx.$$

We have, for any $x \in [-\Omega, \Omega]$,

$$\widehat{f}(x) - f_q(x) = \sum_{k \in \Lambda_{j_0}} \left(\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}\right) \phi_{j_0,k}(x) - \sum_{j=j_0}^{\infty} \sum_{k \in \Lambda_j} \beta_{j,k} \psi_{j,k}(x).$$

Since $\mathcal{B}$ is an orthonormal basis of $\mathbb{L}^2([-\Omega, \Omega])$, we have

$$\mathbb{E}\left(\int_{-\Omega}^{\Omega} \left(\widehat{f}(x) - f_q(x)\right)^2 dx\right) =$$

$$\sum_{k \in \Lambda_{j_0}} \mathbb{E}\left(|\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^2\right) + \sum_{j=j_0}^{\infty} \sum_{k \in \Lambda_j} \beta_{j,k}^2.$$

Using Proposition 1 and the definition of j_0, we obtain

$$\sum_{k \in \Lambda_{j_0}} \mathbb{E}\left(|\widehat{\alpha}_{j_0,k} - \alpha_{j_0,k}|^2\right) \leq C 2^{j_0} 2^{(2\delta+3)j_0} \frac{\rho_n}{n^{1-\theta}} \leq C \left(\frac{\rho_n}{n^{1-\theta}}\right)^{2s/(2s+2\delta+4)}.$$

Since $p \geq 2$, we have $B^s_{p,r}(M) \subseteq B^s_{2,\infty}(M)$. Hence

$$\sum_{j=j_0}^{\infty} \sum_{k \in \Lambda_j} \beta_{j,k}^2 \leq C 2^{-2j_0 s} \leq C \left(\frac{\rho_n}{n^{1-\theta}}\right)^{2s/(2s+2\delta+4)}.$$

Therefore

$$\mathbb{E}\left(\int_{-\Omega}^{\Omega} \left(\widehat{f}(x) - f_q(x)\right)^2 dx\right) \leq C \left(\frac{\rho_n}{n^{1-\theta}}\right)^{2s/(2s+2\delta+4)}.$$

The proof of Theorem 1 is complete.

□

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