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An existence result for multidimensional immiscible two-phase flows with discontinuous capillary pressure field

Clément CANCES^{*,†}, Michel PIERRE[‡]

January 31, 2011

Abstract

We consider the system of equations governing an incompressible immiscible two-phase flow within an heterogeneous porous medium made of two different rock types. Since the capillary pressure function depends on the rock type, the capillary pressure field might be discontinuous at the interface between the rocks. We prove the existence of a solution for such a flow by passing to the limit in regularizations of the problem.

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1 Introduction

The models of immiscible two-phase flows in porous media are often used to give a prediction of the motions of complex flows in subsoil, particularly in the frame of oil-engineering. So they have been widely studied, both from theoretical and numerical points of view. One of the main difficulty appearing in their study is linked to the degeneracy of the problem where one of the two phases vanishes.

Because of variations of the rock type, one has to take into account strong heterogeneities of the subsoil with respect to space in the model and to assume that the physical properties of the porous medium are even discontinuous in the case of severe variations of the rock type. It is well known that such discontinuities of the medium induce discontinuities of the fluid composition, but also discontinuous pressure fields (see [vDMdN95], [EEN98], [BDPvD03], [EEM06], [CGP09], [BLS09], [Can09]). While some mathematical analysis in the one-dimensional case has been carried out in [BDPvD03], [BLS09], [CGP09] and [Can09], ensuring the well-posedness of the problem, there is no existence result available for the solution of immiscible two-phase flows with discontinuous pressure fields in several dimensions, unless some strong assumptions are made in order to reduce the problem (see [CGP09], [EEM06]). In this paper, we propose to establish an existence result for the solution of the system of equations governing such a flow.

1.1 Presentation of the problem

For the sake of simplicity, we suppose that the porous medium, represented by a bounded open subset Ω with Lipschitz continuous boundary of \mathbb{R}^d ($d = 2$ or 3), is built of two homogeneous subdomains, represented by bounded open subsets $(\Omega_i)_{i \in \{1,2\}}$ with Lipschitz continuous boundaries such that

$$\Omega_1 \cap \Omega_2 = \emptyset, \quad \overline{\Omega_1 \cup \Omega_2} = \overline{\Omega}.$$

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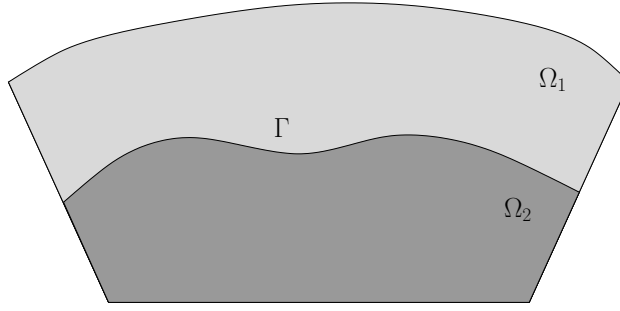


Figure 1: A example of domain Ω made of two subdomains Ω_1 and Ω_2 separated by the interface Γ

We denote by $\Gamma \subset \Omega$ the interface between the two subdomains :

$$\bar{\Gamma} = \bar{\Omega}_1 \cap \bar{\Omega}_2.$$

The porous medium Ω is supposed to be saturated by a moisture made of only two immiscible phases, the oil phase (for which the subscript o stands) and the water phase (for which the subscript w stands). One denotes by s the oil saturation, and $(1 - s)$ is thus the water saturation.

The motion of each phase in Ω_i is governed by the diphasic Darcy-Muskat laws (see e.g. [AS79]):

$$\phi_i \partial_t s - \operatorname{div} \left(\mathbf{K}_i \frac{kr_{o,i}(s)}{\mu_o} (\nabla p_o - \rho_o \mathbf{g}) \right) = 0, \quad (1)$$

$$- \phi_i \partial_t s - \operatorname{div} \left(\mathbf{K}_i \frac{kr_{w,i}(s)}{\mu_w} (\nabla p_w - \rho_w \mathbf{g}) \right) = 0, \quad (2)$$

where $\phi_i \in (0, 1)$ is the porosity of Ω_i , the symmetric definite positive matrix \mathbf{K}_i is the permeability of the rock Ω_i , $kr_{\alpha,i}$ is the relative permeability of the phase $\alpha \in \{o, w\}$ in Ω_i , $\mu_\alpha > 0$ is its viscosity, p_α its pressure, ρ_α its density and \mathbf{g} stands for the gravity. In order to simplify the problem, we suppose that there are no irreducible saturations. More precisely, we do the following assumptions on the functions $kr_{\alpha,i}$:

Assumption 1 For $i \in \{1, 2\}$,

- $kr_{o,i} \in C^1$ is (strictly) increasing on $[0, 1]$ with $kr_{o,i}(0) = 0$ and $kr_{o,i}(1) = 1$;
- $kr_{w,i} \in C^1$ is (strictly) decreasing on $[0, 1]$ with $kr_{w,i}(0) = 1$ and $kr_{w,i}(1) = 0$.

The difference between the phase pressures, so called *capillary pressure*, is given by the following simplified law

$$p_o - p_w = \pi_i(s). \quad (3)$$

We do the following reasonable assumption on the capillary pressure functions.

Assumption 2 For $i \in \{1, 2\}$, the function π_i belongs to $C^1((0, 1); \mathbb{R}) \cap L^1((0, 1); \mathbb{R})$, and are (strictly) increasing.

Note that the functions π_i are not supposed to be bounded near 0 and 1, but, thanks to the monotony of π_i , we can define

$$\bar{\mathbb{R}} \ni \pi_i(0) = \lim_{s \rightarrow 0^+} \pi_i(s), \quad \bar{\mathbb{R}} \ni \pi_i(1) = \lim_{s \rightarrow 1^-} \pi_i(s).$$

Remark 1.1 The most classical choices for the capillary pressure functions are the so-called Van Genuchten and Brooks-Corey capillary pressure functions, respectively defined by (we suppose here that the water phase is the wetting phase)

$$\pi_{VG}(s) = A \left((1 - s)^{-\frac{\nu}{\nu-1}} - 1 \right)^{\frac{1}{\nu}}, \quad \pi_{BC}(s) = B + C(1 - s)^{\frac{1}{\lambda}},$$

where $A > 0$, $B \geq 0$, $C > 0$, $\nu > 2$, and $\lambda > 1$ are parameters depending on the rock type. These choices of capillary pressure functions satisfy Assumption 2.

It has been stressed in [ALV84] that the natural topology for the phase pressures in Ω_i is governed by the quantity

$$\iint \left(\frac{kr_{o,i}(s)}{\mu_o} (\nabla p_o)^2 + \frac{kr_{w,i}(s)}{\mu_w} (\nabla p_w)^2 \right) dxdt. \quad (4)$$

In particular, assume that $s(x, t) = 0$ for some $x \in \Omega_i$, then it follows from Assumption 1 that $kr_{o,i}(s(x, t)) = 0$. As a consequence, the control of the quantity (4) provides no information on $p_o(x, t)$. But because of the relation (3), then the oil-pressure $p_o(x, t)$ can not exceed the threshold

value $p(x, t) + \pi_i(0)$, otherwise the oil-phase should be present. Hence, $p_o(x, t)$ should be defined in a multivalued way, i.e.

$$s(x, t) = 0 \Leftrightarrow p_o(x, t) = [-\infty, p_w(x, t) + \pi_i(0)]. \quad (5)$$

Similarly, on has

$$s(x, t) = 1 \Leftrightarrow p_w(x, t) = [-\infty, p_o(x, t) - \pi_i(1)]. \quad (6)$$

We deduce from (5) and (6) that the capillary pressure function π_i has to be extended into a monotone graph $\tilde{\pi}_i$, defined by

$$\tilde{\pi}_i(s) = \begin{cases} \pi_i(s) & \text{if } s \in (0, 1) \\ [-\infty, \pi_i(0)] & \text{if } s = 0 \\ [\pi_i(1), +\infty] & \text{if } s = 1. \end{cases}$$

The capillary pressure graph $\tilde{\pi}_i$ admits a continuous inverse, denoted by π_i^{-1} , mapping \mathbb{R} to $[0, 1]$, that, thanks to Assumption 2, satisfies

$$\pi_i^{-1} \in L^1(\mathbb{R}_-), \quad (\pi_i^{-1} - 1) \in L^1(\mathbb{R}_+). \quad (7)$$

Let us now focus on the transmission conditions at the interface Γ . On one hand, because of mass balance of each phase, both phase fluxes have to be continuous, i.e. for $\alpha \in \{o, w\}$, one has

$$\sum_{i \in \{1, 2\}} \left(\mathbf{K}_i \frac{kr_{\alpha, i}(s)}{\mu_\alpha} (\nabla p_\alpha - \rho_\alpha \mathbf{g}) \right) \cdot \mathbf{n}_i = 0, \quad (8)$$

where \mathbf{n}_i denotes the outward normal to $\partial\Omega_i$. On the other hand, following [EEM06], we prescribe the continuity of the pressure of the mobile phases :

$$kr_{\alpha, 1}(s_1) (p_{\alpha, 1} - p_{\alpha, 2})^+ - kr_{\alpha, 2}(s_2) (p_{\alpha, 2} - p_{\alpha, 1})^+ = 0, \quad (9)$$

where $s_i, p_{\alpha, i}$ denote the traces on Γ from Ω_i of s, p_α respectively. The relation (9) claims that either the pressure of the phase α is continuous through the interface Γ , or the phase α is missing at the side of the interface where its pressure is the larger. Using the multivalued formalism introduced in (5) and (6), the relation (9) is equivalent to

$$p_{\alpha, 1} \cap p_{\alpha, 2} \neq \emptyset. \quad (10)$$

In order to close the system, we impose a no-flux boundary condition for each phase on $\partial\Omega$

$$\left(\mathbf{K}_i \frac{kr_{\alpha, i}(s)}{\mu_\alpha} (\nabla p_\alpha - \rho_\alpha \mathbf{g}) \right) \cdot \mathbf{n} = 0, \quad (11)$$

where \mathbf{n} denote the outward normal to Ω , and an initial condition

$$s_0(x) \in L^\infty(\Omega, [0, 1]). \quad (12)$$

It is worth noticing that due to the choice of the boundary condition (11), the pressure are only determined up to a constant.

The purpose of this paper is to show that, after suitable reformulation, the problem (1)–(3), (8), (10)–(12) admits a solution.

1.2 Reformulation of the problem

Classical computations (see e.g. [CJ86], [AKM90], [Arb92]) allow to rewrite the equations (1)–(3) under the form

$$\phi_i \partial_t s - \operatorname{div} \left(\mathbf{K}_i \left(\frac{kr_{o, i}(s)}{\mu_o} (\nabla P - \rho_o \mathbf{g}) + \nabla \varphi_i(s) \right) \right) = 0, \quad (13)$$

$$- \operatorname{div} \left(\mathbf{K}_i (M_i(s) \nabla P - \zeta_i(s) \mathbf{g}) \right) = 0, \quad (14)$$

with

$$M_i(s) = \frac{kr_{o, i}(s)}{\mu_o} + \frac{kr_{w, i}(s)}{\mu_w}, \quad \varphi_i(s) = \int_0^s \frac{kr_{o, i}(a) kr_{w, i}(a)}{kr_{o, i}(a) \mu_w + kr_{w, i}(a) \mu_o} \pi'_i(a) da, \\ \zeta_i(s) = \frac{kr_{o, i}(s)}{\mu_o} \rho_o + \frac{kr_{w, i}(s)}{\mu_w} \rho_w$$

and

$$P = p_w + \int_0^{\pi_i(s)} \frac{kr_{o, i}(\pi_i^{-1}(a))}{kr_{o, i}(\pi_i^{-1}(a)) + \frac{\mu_o}{\mu_w} kr_{w, i}(\pi_i^{-1}(a))} da = p_w + \lambda_{w, i}(\pi_i(s)) \quad (15)$$

$$= p_o - \int_0^{\pi_i(s)} \frac{kr_{w, i}(\pi_i^{-1}(a))}{\frac{\mu_w}{\mu_o} kr_{o, i}(\pi_i^{-1}(a)) + kr_{w, i}(\pi_i^{-1}(a))} da = p_o + \lambda_{o, i}(\pi_i(s)). \quad (16)$$

The function φ'_i is given by

$$\varphi'_i(s) = \frac{kr_{o, i}(s) kr_{w, i}(s)}{kr_{o, i}(s) \mu_w + kr_{w, i}(s) \mu_o} \pi'_i(s),$$

where $\pi'_i(s)$ can eventually tend to $+\infty$ as s tends to 0 or 1, but simultaneously, the ratio $\frac{kr_{o, i}(s) kr_{w, i}(s)}{kr_{o, i}(s) \mu_w + kr_{w, i}(s) \mu_o}$ tends to 0. By the following assumption, we assume that the product remains bounded.

Assumption 3 The functions φ_i are Lipschitz continuous on $[0, 1]$.

Thanks to Assumption 1 and since π_i is supposed to be increasing, the functions φ_i defined above are such that φ_i^{-1} are continuous functions on $[\varphi_i(0), \varphi_i(1)]$. We define the quantity

$$\alpha_M = \min_{i \in \{1,2\}} \left(\min_{s \in [0,1]} M_i(s) \right),$$

it is then easy to check that $\alpha_M > 0$.

We now focus on the transmission conditions. The conservation of the oil-phase at the interface Γ can be written

$$\sum_{i \in \{1,2\}} \mathbf{K}_i \left(\frac{kr_{o,i}(s)}{\mu_o} (\nabla P - \rho_o \mathbf{g}) + \nabla \varphi_i(s) \right) \cdot \mathbf{n}_i = 0, \quad (17)$$

while the conservation of the total flux at the interface yields

$$\sum_{i \in \{1,2\}} \left(\mathbf{K}_i (M_i(s) \nabla P - \zeta_i(s) \mathbf{g}) \right) \cdot \mathbf{n}_i = 0. \quad (18)$$

The relation (10) on the interface Γ implies that

$$\text{there exists } \pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \text{ s.t. } P_1 - \lambda_{w,1}(\pi) = P_2 - \lambda_{w,2}(\pi). \quad (19)$$

Remark 1.2 1. In (19), requiring that $P_1 - \lambda_{w,1}(\pi) = P_2 - \lambda_{w,2}(\pi)$ corresponds to imposing the continuity (in the multivalued sense (10)) of the water pressure. By adding π to this relation, we also recover the continuity of the oil-pressure (in the same weak sense (10)), so that the relation (19) contains the the continuity of both pressures.

2. Following [CGP09] and [BLS09], in the one-dimensional case, the relation (19) can be reduced to

$$\tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \neq \emptyset.$$

It has been proved in [CGP09], [Can09] that this relation ensures the existence and the uniqueness of the solution to the problem.

The no-flux boundary condition for each phase on $\partial\Omega \cap \partial\Omega_i$ is replaced by

$$\mathbf{K}_i \left(\frac{kr_{o,i}(s)}{\mu_o} (\nabla P - \rho_o \mathbf{g}) + \nabla \varphi_i(s) \right) \cdot \mathbf{n} = 0, \quad (20)$$

$$\left(\mathbf{K}_i (M_i(s) \nabla P - \zeta_i(s) \mathbf{g}) \right) \cdot \mathbf{n} = 0. \quad (21)$$

All along the paper, for any $f \in \{\phi, \mathbf{K}, \dots\}$, we denote by $x \mapsto f(s, x)$ the piecewise constant function equal to $f_i(s)$ if $x \in \Omega_i$. For $T > 0$, then we denote by Q_T (resp. $Q_{i,T}$) the cylinder $\Omega \times (0, T)$ (resp. $\Omega_i \times (0, T)$).

Definition 1.1 (weak solution) A couple (s, P) is said to be a weak solution to the problem (12)-(14), (17)-(21) in the cylinder Q_T if it fulfills the following points:

1. $s \in L^\infty(Q_T, [0, 1])$, $\phi \partial_t s \in L^2((0, T); (H^1(\Omega))')$ and $\varphi_i(s) \in L^2((0, T); H^1(\Omega_i))$;
2. $P \in L^2((0, T); H^1(\Omega_i))$ for $i \in \{1, 2\}$, with for a.e. $t \in (0, T)$, $\int_{\Omega} P(x, t) dx = 0$;
3. there exists a measurable function π mapping $\Gamma \times (0, T)$ to $\overline{\mathbb{R}}$ such that, for almost all $(x, t) \in \Gamma \times (0, T)$,

$$\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \text{ and } P_1 - \lambda_{w,1}(\pi) = P_2 - \lambda_{w,2}(\pi);$$

4. for all $\psi \in L^2((0, T); H^1(\Omega))$, one has

$$\begin{aligned} & \iint_{Q_T} \phi s \partial_t \psi dx dt + \int_{\Omega} \phi s_0 \psi(\cdot, 0) dx \\ & + \sum_{i \in \{1,2\}} \iint_{Q_{i,T}} \mathbf{K}_i \left(\frac{kr_{o,i}(s)}{\mu_o} (\nabla P - \rho_o \mathbf{g}) + \nabla \varphi_i(s) \right) \cdot \nabla \psi dx dt = 0; \end{aligned} \quad (22)$$

5. for all $\psi \in L^2((0, T); H^1(\Omega))$, one has

$$\sum_{i \in \{1,2\}} \iint_{Q_{i,T}} \left(\mathbf{K}_i (M_i(s) \nabla P - \zeta_i(s) \mathbf{g}) \right) \cdot \nabla \psi dx dt = 0. \quad (23)$$

The paper is devoted to the proof of the following theorem.

Theorem 1 (main result) Under Assumptions 1 and 2, there exists a weak solution to the problem (12)-(14), (17)-(21) in the sense of Definition 1.1.

It is well known that for suitable initial and boundary conditions, the flow governed by the equations (13)–(14) admits a solution (see e.g. [ALV84], [AD85], [CJ86], [AKM90], [Arb92] or [Che01]) in the case where the physical characteristics of the domain do not depend on space, or at least sufficiently smoothly. In the case considered here, the difficulty will come from the fact that the physical properties of the medium Ω —particularly the capillary pressure curve— are discontinuous with respect to space at the interface Γ . The effects of space depending capillarities have been widely studied during the last years. Analytical results have been provided by [ABE96], [vDMdN95], [BDPvD03], [Can08], [CGP09]. Effective models have been provided in [BH95a], [vDMP02], [vDEHP07] and [Sch08] using homogenization techniques. Some numerical schemes have been introduced [EEN98], [EMS09] and studied [EEM06], [Can09], [BCH]. It has been pointed out in [Can10a], [Can10b] and [Can10c] that the orientation of the capillary forces at the interface has a strong influence on the qualitative behavior of the saturation profile.

1.3 Organization of the paper

In Section 2, we introduce a simplified problem, where the pressure of both phase is (strongly) continuous at the interface. This can be done under a compatibility condition of the capillary forces, that is

$$\pi_1(0) = \pi_2(0) \in \mathbb{R}, \quad \pi_1(1) = \pi_2(1) \in \mathbb{R}. \quad (24)$$

If the functions π_i satisfy the above condition, the capillarity curves are said to be matching. In that case, the existence of a weak solution is proven using a spatial regularization of the function $x \mapsto \pi(\cdot, x)$, i.e. by introducing a thin transition layer between the two rocks.

Section 3 is devoted to the end of the proof of Theorem 1. We will show that the problem with non-matching capillarity curves, i.e. when the condition (24) is not satisfied, can be approximated by problems with matching capillarity curves studied in Section 2. Compactness properties on the family of approximate solutions will allow us to exhibit a weak solution in the sense of Definition 1.1 as a limit value.

2 The problem with matching capillarity curves

In this section, we assume that the capillary pressure functions π_i belong to $C^1([0, 1]; \mathbb{R})$, and fulfill the relation (24), so that the relation (19) turns to

$$\pi_1(s_1) = \pi_2(s_2) \text{ and } P_1 - \lambda_{w,1}(\pi_1(s_1)) = P_2 - \lambda_{w,2}(\pi_2(s_2)). \quad (25)$$

So the pressure of each phase is continuous at the interface Γ , i.e.

$$p_{o,1} = p_{o,2}, \quad p_{w,1} = p_{w,2}. \quad (26)$$

Theorem 2 *Under assumption (24), there exists a weak solution (s, P) to the problem (12)–(14), (17), (18), (20), (21), (25) in the sense of Definition 1.1.*

Remark 2.1 *The result stated in Theorem 2 is very close to the main result of the paper [BH95b]. However, it seems that there is a technical mistake in the proof suggested in [BH95b] and detailed in [Hid93]. For this reason, we choose to give another proof of this theorem. But we stress the fact that the main result proposed in [BH95b] is true and that numerous ideas presented here have already been proposed in [BH95b], [Hid93]. In particular, the homogenization result published in [BH95a] relies on correct preliminaries.*

2.1 The regularized problems

Let $\varepsilon > 0$. In order to obtain regular phase pressures, we regularize the problem in the following form: find $(s^\varepsilon, p_o^\varepsilon, p_w^\varepsilon)$ such that

$$\left\{ \begin{array}{ll} \phi_i \partial_t s^\varepsilon - \operatorname{div} \left(\mathbf{K}_i \frac{kr_{o,i}(s^\varepsilon)}{\mu_o} (\nabla p_o^\varepsilon - \rho_o \mathbf{g}) \right) = \varepsilon \Delta \pi_i(s^\varepsilon) & \text{in } Q_{i,T}, \\ -\phi_i \partial_t s^\varepsilon - \operatorname{div} \left(\mathbf{K}_i \frac{kr_{w,i}(s^\varepsilon)}{\mu_w} (\nabla p_w^\varepsilon - \rho_w \mathbf{g}) \right) = -\varepsilon \Delta \pi_i(s^\varepsilon) & \text{in } Q_{i,T}, \\ p_o^\varepsilon - p_w^\varepsilon = \pi_i(s^\varepsilon) & \text{in } Q_{i,T}, \\ p_{o,1}^\varepsilon = p_{o,2}^\varepsilon, \quad p_{w,1}^\varepsilon = p_{w,2}^\varepsilon & \text{on } \Gamma \times (0, T), \\ \sum_{i \in \{1,2\}} \left(\mathbf{K}_i \frac{kr_{o,i}(s^\varepsilon)}{\mu_o} (\nabla p_o^\varepsilon - \rho_o \mathbf{g}) + \varepsilon \nabla \pi_i(s^\varepsilon) \right) \cdot \mathbf{n} = 0 & \text{on } \Gamma \times (0, T), \\ \sum_{i \in \{1,2\}} \left(\mathbf{K}_i \frac{kr_{w,i}(s^\varepsilon)}{\mu_w} (\nabla p_w^\varepsilon - \rho_w \mathbf{g}) - \varepsilon \nabla \pi_i(s^\varepsilon) \right) \cdot \mathbf{n} = 0 & \text{on } \Gamma \times (0, T), \\ \left(\mathbf{K}_i \frac{kr_{o,i}(s^\varepsilon)}{\mu_o} (\nabla p_o^\varepsilon - \rho_o \mathbf{g}) + \varepsilon \nabla \pi_i(s^\varepsilon) \right) \cdot \mathbf{n} = 0 & \text{on } (\partial\Omega \cap \partial\Omega_i) \times (0, T), \\ \left(\mathbf{K}_i \frac{kr_{w,i}(s^\varepsilon)}{\mu_w} (\nabla p_w^\varepsilon - \rho_w \mathbf{g}) - \varepsilon \nabla \pi_i(s^\varepsilon) \right) \cdot \mathbf{n} = 0 & \text{on } (\partial\Omega \cap \partial\Omega_i) \times (0, T), \\ s^\varepsilon(\cdot, 0) = s_0 & \text{in } \Omega. \end{array} \right. \quad (27)$$

In order to use an existing result (Theorem 1 in [Arb92] or Theorem 2.1 in [Che01]), we introduce a smooth regularization of Ω , consisting in introducing a thin transition layer to replace Γ . Let $\delta > 0$, we define the Lipschitz continuous function H^δ on Ω by

$$H^\delta(x) = \frac{1}{2} \left(1 - \min \left(\frac{d(x, \Omega_1)}{\delta}, 1 \right) + \min \left(\frac{d(x, \Omega_2)}{\delta}, 1 \right) \right)$$

so that $H^\delta(x) = 1$ if $d(x, \Omega_2) \geq \delta$ and $H^\delta(x) = 0$ if $d(x, \Omega_1) \geq \delta$. Let $f \in \{\mathbf{K}, \phi, kr_\alpha, \pi\}$ piecewise constant on Ω with respect to space, we define the function

$$f^\delta : (s, x) \mapsto f_1(s)H^\delta(x) + f_2(s)(1 - H^\delta(x))$$

which has been built in order to be Lipschitz continuous with respect to the space variable x . For $g \in \{M, \varphi, \zeta, \lambda_w\}$, we denote by g^δ the function obtained by using $kr_\alpha^\delta, \pi^\delta$ instead of kr_α, π in the definition of g .

We define the fully regularized problem by: find $(s^{\varepsilon, \delta}, p_o^{\varepsilon, \delta}, p_w^{\varepsilon, \delta})$ such that

$$\left\{ \begin{array}{ll} \phi^\delta \partial_t s^{\varepsilon, \delta} - \operatorname{div} \left(\mathbf{K}^\delta \frac{kr_o^\delta(s^{\varepsilon, \delta})}{\mu_o} (\nabla p_o^{\varepsilon, \delta} - \rho_o \mathbf{g}) \right) = \varepsilon \Delta \pi^\delta(s^{\varepsilon, \delta}) & \text{in } Q_T, \\ -\phi^\delta \partial_t s^{\varepsilon, \delta} - \operatorname{div} \left(\mathbf{K}^\delta \frac{kr_w^\delta(s^{\varepsilon, \delta})}{\mu_w} (\nabla p_w^{\varepsilon, \delta} - \rho_w \mathbf{g}) \right) = -\varepsilon \Delta \pi^\delta(s^{\varepsilon, \delta}) & \text{in } Q_T, \\ p_o^{\varepsilon, \delta} - p_w^{\varepsilon, \delta} = \pi^\delta(s^{\varepsilon, \delta}) & \text{in } Q_T, \\ \left(\mathbf{K}^\delta \frac{kr_o^\delta(s^{\varepsilon, \delta})}{\mu_o} (\nabla p_o^{\varepsilon, \delta} - \rho_o \mathbf{g}) + \varepsilon \nabla \pi^\delta(s^{\varepsilon, \delta}) \right) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ \left(\mathbf{K}^\delta \frac{kr_w^\delta(s^{\varepsilon, \delta})}{\mu_w} (\nabla p_w^{\varepsilon, \delta} - \rho_w \mathbf{g}) - \varepsilon \nabla \pi^\delta(s^{\varepsilon, \delta}) \right) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ s^{\varepsilon, \delta}(\cdot, 0) = s_0 & \text{in } \Omega. \end{array} \right. \quad (28)$$

Proposition 2.1 *There exist $s^{\varepsilon, \delta} \in L^\infty(Q_T, [0, 1])$ and $p_o^{\varepsilon, \delta}, p_w^{\varepsilon, \delta} \in L^2((0, T); H^1(\Omega))$ solution to the system (28). Furthermore, the following energy estimate holds: there exists C depending only on $\phi, \mathbf{K}, \rho_\alpha, \mu_\alpha, \Omega, T, \mathbf{g}$ (but neither on ε nor on δ) such that*

$$\frac{m_{\mathbf{K}}}{2} \sum_{\alpha \in \{o, w\}} \iint_{Q_T} \frac{kr_{\alpha,i}(s^{\varepsilon, \delta})}{\mu_\alpha} (\nabla p_\alpha^{\varepsilon, \delta})^2 dxdt + \varepsilon \iint_{Q_T} (\nabla \pi(s^{\varepsilon, \delta}))^2 dxdt \leq C \left(1 + \sup_{x \in \Omega} \|\pi(\cdot, x)\|_{L^1(0,1)} \right). \quad (29)$$

Denoting by

$$P^{\varepsilon, \delta} = p_w^{\varepsilon, \delta} + \lambda_w^\delta(\pi^\delta(s^{\varepsilon, \delta})), \quad (30)$$

one can furthermore require that

$$\int_{\Omega} P^{\varepsilon, \delta}(x, t) dx = 0 \quad \text{for a.e. } t \in [0, T]. \quad (31)$$

Proof: The existence proofs carried out in [Arb92] and [Che01], dealing with the case $\varepsilon = 0$, can be mimicked for $\varepsilon > 0$. This particularly yields the existence of $s^{\varepsilon, \delta} \in L^\infty(Q_T, [0, 1])$ and $p_o^{\varepsilon, \delta}, p_w^{\varepsilon, \delta} \in L^2((0, T); H^1(\Omega))$ satisfying the system (28).

Choosing $p_o^{\varepsilon, \delta}$ as test function in the first equation, $p_w^{\varepsilon, \delta}$ in the second one and summing yields:

$$\begin{aligned} & \left\langle \phi^\delta \partial_t s^{\delta, \varepsilon}, \pi^\delta(s^{\delta, \varepsilon}) \right\rangle + \sum_{\alpha \in \{o, w\}} \iint_{Q_T} \left(\frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} \mathbf{K}^\delta \nabla p_\alpha^{\delta, \varepsilon} \cdot \nabla p_\alpha^{\delta, \varepsilon} \right) dx dt \\ & + \varepsilon \iint_{Q_T} |\nabla \pi^\delta(s^{\delta, \varepsilon})|^2 dx dt - \sum_{\alpha \in \{o, w\}} \iint_{Q_T} \frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} \mathbf{K}^\delta \nabla p_\alpha^{\delta, \varepsilon} \cdot \rho_\alpha \mathbf{g} dx dt = 0. \end{aligned} \quad (32)$$

Denoting by $\Pi^\delta(s, x) = \int_0^s \pi^\delta(a, x) da$, it is classical (see e.g. Lemma 4 in [Car99]) that

$$\begin{aligned} \left\langle \phi^\delta \partial_t s^{\delta, \varepsilon}, \pi^\delta(s^{\delta, \varepsilon}) \right\rangle &= \int_{\Omega} \phi^\delta(x) \Pi^\delta(s^{\delta, \varepsilon})(x, T) dx - \int_{\Omega} \phi^\delta(x) \Pi^\delta(s_0)(x) dx \\ &\geq -2 \int_{\Omega} \phi^\delta(x) \int_0^1 |\pi^\delta(a, x)| da dx \\ &\geq -2|\Omega| \left(\max_i \phi_i \right) \left(\max_i \|\pi_i\|_{L^1(0,1)} \right). \end{aligned} \quad (33)$$

Since each \mathbf{K}_i is a symmetric positive definite matrix, $\mathbf{K}^\delta(x)$ also for all $x \in \Omega$. We denote by $m_{\mathbf{K}^\delta}(x)$ (resp. $M_{\mathbf{K}^\delta}(x)$) its smaller (resp. larger) eigenvalue. Then it is easy to check that for all $x \in \Omega$, one has

$$m_{\mathbf{K}} = \min_{i \in \{1,2\}} m_{\mathbf{K}_i} \leq m_{\mathbf{K}^\delta}(x), \quad M_{\mathbf{K}} = \max_{i \in \{1,2\}} M_{\mathbf{K}_i} \geq M_{\mathbf{K}^\delta}(x).$$

This provides that for $\alpha \in \{o, w\}$, one has

$$\iint_{Q_T} \left(\frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} \mathbf{K}^\delta \nabla p_\alpha^{\delta, \varepsilon} \cdot \nabla p_\alpha^{\delta, \varepsilon} \right) dx dt \geq m_{\mathbf{K}} \iint_{Q_T} \frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} |\nabla p_\alpha^{\delta, \varepsilon}|^2 dx dt. \quad (34)$$

From Cauchy-Schwarz inequality, one has

$$\begin{aligned} & \iint_{Q_T} \frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} \mathbf{K}^\delta \nabla p_\alpha^{\delta, \varepsilon} \cdot \rho_\alpha \mathbf{g} dx dt \\ & \leq \left(\iint_{Q_T} \frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} \mathbf{K}^\delta \nabla p_\alpha^{\delta, \varepsilon} \cdot \nabla p_\alpha^{\delta, \varepsilon} dx dt \right)^{\frac{1}{2}} \rho_\alpha \left(\iint_{Q_T} \frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} \mathbf{K}^\delta \mathbf{g} \cdot \mathbf{g} dx dt \right)^{\frac{1}{2}} \\ & \leq M_{\mathbf{K}} \frac{\rho_\alpha}{\sqrt{\mu_\alpha}} \|\mathbf{g}\|_{Q_T}^{\frac{1}{2}} \left(\iint_{Q_T} \frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} |\nabla p_\alpha^{\delta, \varepsilon}|^2 dx dt \right)^{\frac{1}{2}}. \end{aligned}$$

Using that for $a, b \in \mathbb{R}$, one has

$$ab \leq m_{\mathbf{K}} \frac{a^2}{2} + \frac{b^2}{2m_{\mathbf{K}}},$$

we obtain the existence of C depending only on \mathbf{K} , ρ_α , μ_α , Ω , T , \mathbf{g} such that

$$\iint_{Q_T} \frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} \mathbf{K}^\delta \nabla p_\alpha^{\delta, \varepsilon} \cdot \rho_\alpha \mathbf{g} dx dt \leq \frac{m_{\mathbf{K}}}{2} \iint_{Q_T} \frac{kr_\alpha^\delta(s^{\delta, \varepsilon})}{\mu_\alpha} |\nabla p_\alpha^{\delta, \varepsilon}|^2 dx dt + C. \quad (35)$$

The inequality (29) is a consequence of (32), (33), (34) and (35).

Since the function $p_w^{\varepsilon, \delta}$ (and thus $p_o^{\varepsilon, \delta}$) is defined up to a function depending on time, one can choose this function so that (31) holds. \square

Lemma 2.2 *There exists C^ε depending only on π , ϕ , \mathbf{K} , ρ_α , μ_α , Ω , T , \mathbf{g} , α_M and ε (but not on δ) such that*

$$\iint_{Q_T} \left(\nabla p_\beta^{\varepsilon, \delta} \right)^2 dx dt \leq C^\varepsilon, \quad \text{for } \beta \in \{o, w\}.$$

Proof: We will prove this estimate only for the oil pressure, since obtaining it for the water pressure is similar.

$$\begin{aligned} \iint_{Q_T} (\nabla p_o^{\varepsilon,\delta})^2 dxdt &\leq \frac{1}{\alpha_M} \iint_{Q_T} \left(\frac{kr_o^\delta(s^{\varepsilon,\delta})}{\mu_o} + \frac{kr_w^\delta(s^{\varepsilon,\delta})}{\mu_w} \right) (\nabla p_o^{\varepsilon,\delta})^2 dxdt \\ &\leq \frac{1}{\alpha_M} \iint_{Q_T} \left[\frac{kr_o^\delta(s^{\varepsilon,\delta})}{\mu_o} (\nabla p_o^{\varepsilon,\delta})^2 + \frac{kr_w^\delta(s^{\varepsilon,\delta})}{\mu_w} (\nabla p_w^{\varepsilon,\delta} + \nabla \pi^\delta(s^{\varepsilon,\delta}))^2 \right] dxdt. \end{aligned}$$

Since $(a+b)^2 \leq 2(a^2+b^2)$, and since $0 \leq kr_w^\delta(s) \leq 1$, one obtains

$$\begin{aligned} \iint_{Q_T} (\nabla p_o^{\varepsilon,\delta})^2 dxdt &\leq \frac{1}{\alpha_M} \iint_{Q_T} \left[\frac{kr_o^\delta(s^{\varepsilon,\delta})}{\mu_o} (\nabla p_o^{\varepsilon,\delta})^2 + 2 \frac{kr_w^\delta(s^{\varepsilon,\delta})}{\mu_w} (\nabla p_w^{\varepsilon,\delta})^2 + \frac{1}{\mu_w} (\nabla \pi^\delta(s^{\varepsilon,\delta}))^2 \right] dxdt. \end{aligned}$$

We conclude by using the energy estimate (29). \square

Let $h > 0$, then we define $\Omega_i^h = \{x \in \Omega_i \text{ s.t. } \text{dist}(x, \Omega_j) > h \text{ for } j \neq i\}$ and $Q_{i,T}^h = \Omega_i^h \times (0, T)$. On the set Ω_i^δ , the functions f^δ is equal to f for all $f \in \{kr_\alpha, \pi, \dots\}$. This particularly yields that in $Q_{i,T}^\delta$, the two first equations of the system (28) can be rewritten under the form

$$\phi_i \partial_t s^{\varepsilon,\delta} - \text{div} \left(\mathbf{K}_i \left(\frac{kr_{o,i}(s^{\varepsilon,\delta})}{\mu_o} (\nabla P^{\varepsilon,\delta} - \rho_o \mathbf{g}) + \nabla \varphi_i(s^{\varepsilon,\delta}) \right) \right) = \varepsilon \Delta \pi_i(s^{\varepsilon,\delta}), \quad (36)$$

$$- \text{div} \left(\mathbf{K}_i (M_i(s^{\varepsilon,\delta}) \nabla P^{\varepsilon,\delta} - \zeta_i(s^{\varepsilon,\delta}) \mathbf{g}) \right) = 0. \quad (37)$$

Lemma 2.3 *There exists C depending only on $\phi, \mathbf{K}, \rho_\alpha, \mu_\alpha, \Omega, T, \mathbf{g}, \alpha_M$ (but neither on ε nor on δ) such that for all $\varepsilon, \delta > 0$,*

$$\iint_{Q_{i,T}^\delta} (\nabla P^{\varepsilon,\delta})^2 dxdt \leq C \left(1 + \max_{i \in \{1,2\}} \|\pi_i\|_{L^1(0,1)} \right).$$

Proof: One has

$$\begin{aligned} \iint_{Q_{i,T}^\delta} (\nabla P^{\varepsilon,\delta})^2 dxdt &\leq \frac{1}{\alpha_M^2} \iint_{Q_{i,T}^\delta} (M_i(s^{\varepsilon,\delta}) \nabla P^{\varepsilon,\delta})^2 dxdt \\ &\leq \frac{1}{\alpha_M^2} \iint_{Q_{i,T}^\delta} \left(\frac{kr_{o,i}(s^{\varepsilon,\delta})}{\mu_o} \nabla p_o^{\varepsilon,\delta} + \frac{kr_{w,i}(s^{\varepsilon,\delta})}{\mu_w} \nabla p_w^{\varepsilon,\delta} \right)^2 dxdt \\ &\leq \frac{2}{\min(\mu_o, \mu_w) \alpha_M^2} \iint_{Q_{i,T}^\delta} \left(\frac{kr_{o,i}(s^{\varepsilon,\delta})}{\mu_o} (\nabla p_o^{\varepsilon,\delta})^2 + \frac{kr_{w,i}(s^{\varepsilon,\delta})}{\mu_w} (\nabla p_w^{\varepsilon,\delta})^2 \right) dxdt. \end{aligned}$$

We conclude by using Proposition 2.1. \square

Lemma 2.4 *There exists C depending only on $\phi, \mathbf{K}, \rho_\alpha, \mu_\alpha, \Omega, T, \mathbf{g}, \alpha_M$ (but neither on ε nor on δ) such that for all $\varepsilon, \delta > 0$, one has:*

$$\iint_{Q_{i,T}^\delta} (\nabla \varphi_i(s^{\varepsilon,\delta}))^2 dxdt \leq C \left(1 + \max_{i \in \{1,2\}} \|\pi_i\|_{L^1(0,1)} \right)$$

Proof: This estimate is only a consequence of the fact that in $Q_{i,T}^\delta$,

$$\nabla \varphi_i(s^{\varepsilon,\delta}) = \frac{kr_{o,i}(s^{\varepsilon,\delta})}{\mu_o} \left((\nabla p_o^{\varepsilon,\delta} - \rho_o \mathbf{g}) - \nabla P^{\varepsilon,\delta} \right).$$

We conclude by using Proposition 2.1 and Lemma 2.3. \square

Lemma 2.5 *Let $\tau \in (0, T)$ and let $h > 0$, then there exists C^h depending on $\phi, \mathbf{K}, \rho_\alpha, \mu_\alpha, \Omega, T, \mathbf{g}, L_{\varphi_i}, \alpha_M$ and h such that for all $\varepsilon > 0$ and for all $\delta \in (0, h)$,*

$$\iint_{Q_{i,T-\tau}^{2h}} \left(\varphi_i(s^{\varepsilon,\delta})(\cdot, \cdot + \tau) - \varphi_i(s^{\varepsilon,\delta}) \right)^2 dxdt \leq \tau C^h \left(1 + \max_{i \in \{1,2\}} \|\pi_i\|_{L^1(0,1)} \right). \quad (38)$$

Proof: Let ξ^h be a nonnegative smooth function equal to 1 in Ω_i^{2h} and equal to 0 in $(\Omega_i^h)^c$. Since φ_i is Lipschitz continuous, one has

$$\begin{aligned}
& \iint_{Q_{i,T-\tau}^{2h}} \left(\varphi_i(s^{\varepsilon,\delta})(x, t + \tau) - \varphi_i(s^{\varepsilon,\delta})(x, t) \right)^2 dx dt \\
& \leq \iint_{Q_{i,T-\tau}^h} \xi^h(x) \left(\varphi_i(s^{\varepsilon,\delta})(x, t + \tau) - \varphi_i(s^{\varepsilon,\delta})(x, t) \right)^2 dx dt \\
& \leq L_{\varphi_i} \iint_{Q_{i,T-\tau}^h} \xi^h(x) \left(\varphi_i(s^{\varepsilon,\delta})(x, t + \tau) - \varphi_i(s^{\varepsilon,\delta})(x, t) \right) (s^{\varepsilon,\delta}(x, t + \tau) - s^{\varepsilon,\delta}(x, t)) dx dt \\
& \leq -L_{\varphi_i} \iint_{Q_{i,T-\tau}^h} \left[\int_0^\tau \left(\mathbf{K}_i \frac{kr_{o,i}(s^{\varepsilon,\delta})(x, t + \theta)}{\mu_o} \nabla p_o^{\varepsilon,\delta}(x, t + \theta) + \varepsilon \nabla \pi_i(s^{\varepsilon,\delta})(x, t + \theta) \right) d\theta \right] dx dt \\
& \leq L_{\varphi_i} 4\tau \|\nabla(\xi^h \varphi_i(s^{\varepsilon,\delta}))\|_{L^2(Q_{i,T}^h)} \left(\iint_{Q_{i,T}^h} \left(\left(\mathbf{K}_i \frac{kr_{o,i}(s^{\varepsilon,\delta})}{\mu_o} \nabla p_o^{\varepsilon,\delta} \right)^2 + \varepsilon \left(\nabla \pi_i(s^{\varepsilon,\delta}) \right)^2 \right) dx dt \right)^{\frac{1}{2}}.
\end{aligned}$$

There exists C^h depending only on Ω and h such that

$$\|\nabla(\xi^h \varphi_i(s^{\varepsilon,\delta}))\|_{L^2(Q_{i,T}^h)} \leq C^h \|\varphi_i\|_\infty + \sqrt{2} \|\nabla \varphi_i(s^{\varepsilon,\delta})\|_{L^2(Q_{i,T}^h)}.$$

One concludes by using Proposition 2.1, Lemma 2.4 and the fact that $\|\varphi_i\|_\infty \leq L_{\varphi_i}$. \square

We have now all the necessary estimates to consider the limit $\delta \rightarrow 0$ of our problem.

Proposition 2.6 *There exists $s^\varepsilon \in L^\infty(Q_T; [0, 1])$, $p_o^\varepsilon, p_w^\varepsilon \in L^2((0, T); H^1(\Omega))$ solution to the system (27). Moreover, there exists C depending only on $\phi, \mathbf{K}, \rho_\alpha, \mu_\alpha, \Omega, T, \mathbf{g}, \alpha_M$ such that*

$$\frac{m_{\mathbf{K}}}{2} \sum_{\alpha \in \{o, w\}} \iint_{Q_T} \frac{kr_{\alpha,i}(s^\varepsilon)}{\mu_\alpha} (\nabla p_\alpha^\varepsilon)^2 dx dt + \varepsilon \iint_{Q_T} (\nabla \pi(s^\varepsilon))^2 dx dt \leq C \left(1 + \sup_{x \in \Omega} \|\pi(\cdot, x)\|_{L^1(0,1)} \right). \quad (39)$$

Furthermore, for $i \in \{1, 2\}$, one has $P^\varepsilon, \varphi_i(s^\varepsilon) \in L^2((0, T); H^1(\Omega_i))$ with

$$P_1^\varepsilon - \lambda_{w,1}(\pi_1(s_1^\varepsilon)) = P_2^\varepsilon - \lambda_{w,2}(\pi_2(s_2^\varepsilon)), \quad (40)$$

$$\int_\Omega P^\varepsilon(x, t) dx = 0 \quad \text{for a.e. } t \in [0, T], \quad (41)$$

and $(s^\varepsilon, P^\varepsilon)$ that satisfies the following system: $\forall \psi \in \mathcal{D}(\overline{\Omega}_i \times [0, T])$,

$$\begin{aligned}
& \iint_{Q_T} \phi s^\varepsilon \partial_t \psi dx dt + \int_{\Omega_i} \phi s_0 \psi(\cdot, 0) dx \\
& - \sum_{i \in \{1,2\}} \iint_{Q_{i,T}} \mathbf{K}_i \left(\frac{kr_{o,i}(s^\varepsilon)}{\mu_o} (\nabla P^\varepsilon - \rho_o \mathbf{g}) + \nabla \varphi_i(s^\varepsilon) + \varepsilon \nabla \pi_i(s^\varepsilon) \right) \cdot \nabla \psi dx dt = 0; \quad (42)
\end{aligned}$$

$$\sum_{i \in \{1,2\}} \iint_{Q_{i,T}} \mathbf{K}_i (M_i(s^\varepsilon) \nabla P^\varepsilon - \zeta_i(s^\varepsilon) \mathbf{g}) \cdot \nabla \psi dx dt = 0. \quad (43)$$

The following energy estimate holds:

$$\iint_{Q_{i,T}} (\nabla P^\varepsilon)^2 dx dt + \iint_{Q_{i,T}} (\nabla \varphi_i(s^\varepsilon))^2 dx dt \leq C \left(1 + \max_{i \in \{1,2\}} \|\pi_i\|_{L^1(0,1)} \right). \quad (44)$$

Let $h > 0$ and $\tau \in (0, T)$, then there exists C^h depending only on $\phi, \mathbf{K}, \rho_\alpha, \mu_\alpha, \Omega, T, \mathbf{g}, L_{\varphi_i}, \alpha_M$ and h such that

$$\iint_{Q_{i,T-\tau}^h} (\varphi_i(s^\varepsilon)(\cdot, \cdot + \tau) - \varphi_i(s^\varepsilon))^2 dx dt \leq \tau C^h \left(1 + \max_{i \in \{1,2\}} \|\pi_i\|_{L^1(0,1)} \right). \quad (45)$$

Proof: Let ε be a fixed strictly positive parameter. First of all, since for all $\delta > 0$, $0 \leq s^{\varepsilon,\delta} \leq 1$ a.e. in Q_T , there exists $s^\varepsilon \in L^\infty(Q_T; [0, 1])$ such that, up to a subsequence,

$$s^{\varepsilon,\delta} \rightarrow s^\varepsilon \quad \text{in the } L^\infty(Q_T) \text{ weak-} \star \text{ sense, and } 0 \leq s^\varepsilon \leq 1 \text{ a.e. in } Q_T. \quad (46)$$

Let $h > 0$. It follows from Lemma 2.4 that for all $\delta \in (0, h)$,

$$\iint_{Q_{i,T}^h} (\nabla \varphi_i(s^{\varepsilon,\delta}))^2 dx dt \leq C.$$

Then in particular, for all $\xi > 0$, one has the following estimate on the space-translates of $\varphi_i(s^{\varepsilon,\delta})$:

$$\iint_{Q_{i,T}^{h+\xi}} \left(\varphi_i(s^{\varepsilon,\delta})(\cdot + \xi, \cdot) - \varphi_i(s^{\varepsilon,\delta}) \right)^2 dxdt \leq C\xi^2 \quad (47)$$

where C does not depend on ε, δ, ξ (see e.g. [Bré83]). Using moreover Lemma 2.5 allows to use the Kolmogorov compactness criterion (see e.g. [Bré83]) that provides that $(\varphi_i(s^{\varepsilon,\delta}))_{\delta \in (0,h)}$ is relatively compact in $L^2(Q_{i,T}^h)$. Hence, up to a subsequence, there exists a function $f \in L^2(Q_{i,T}^h)$ such that $\varphi_i(s^{\varepsilon,\delta})$ converges almost everywhere in $Q_{i,T}^h$ towards f . Since φ_i^{-1} is continuous, one obtains that $s^{\varepsilon,\delta}$ converges almost everywhere in $Q_{i,T}^h$ towards $\varphi_i^{-1}(f) = s^\varepsilon$. Since this convergence results holds for all $h > 0$, one obtains that, up to a subsequence,

$$s^{\varepsilon,\delta} \rightarrow s^\varepsilon \quad \text{a.e. in } Q_T. \quad (48)$$

Because of the definition (30) of the global pressure $P^{\varepsilon,\delta}$ and thanks to (31), one has, for almost every $t \in [0, T]$ that

$$\int_{\Omega} p_w^{\varepsilon,\delta}(x, t) dx = \int_{\Omega} \lambda_w^\delta(\pi^\delta(s^{\varepsilon,\delta})) dx.$$

Hence, since we have supposed in this section that $\pi_i \in C^1([0, 1]; \mathbb{R})$ and since $0 \leq (\lambda_w^\delta)' \leq 1$, we obtain that

$$\left| \int_{\Omega} p_w^{\varepsilon,\delta}(x, t) dx \right| \leq \|\pi_1\|_\infty |\Omega|. \quad (49)$$

Similarly, using the fact that $P^{\varepsilon,\delta} = p_o^{\varepsilon,\delta} + \lambda_o^\delta(\pi^\delta(s^{\varepsilon,\delta}))$ provides that for almost every $t \in [0, T]$, one has

$$\left| \int_{\Omega} p_o^{\varepsilon,\delta}(x, t) dx \right| \leq \|\pi_1\|_\infty |\Omega|. \quad (50)$$

Thanks to Lemma 2.2 and Poincaré-Wirtinger inequality, one can claim the existence of C^ε which is not depending δ such that

$$\left\| p_\beta^{\varepsilon,\delta} - \int_{\Omega} p_\beta^{\varepsilon,\delta}(x, \cdot) dx \right\|_{L^2((0,T); H^1(\Omega))} \leq C^\varepsilon, \quad \text{for } \beta \in \{o, w\}.$$

This yields, using (49)-(50), that $(p_\beta^{\varepsilon,\delta})_\delta$ is uniformly bounded in $L^2((0, T); H^1(\Omega))$. Thus there exists p_β^ε belonging to $L^2((0, T); H^1(\Omega))$ such that, up to a subsequence,

$$p_\beta^{\varepsilon,\delta} \rightarrow p_\beta^\varepsilon \quad \text{weakly in } L^2((0, T); H^1(\Omega)). \quad (51)$$

In particular, $\pi^\delta(s^{\varepsilon,\delta}) = p_o^{\varepsilon,\delta} - p_w^{\varepsilon,\delta}$ also converges weakly in $L^2((0, T); H^1(\Omega))$ and strongly in $L^2(Q_T)$ towards $\pi(s^\varepsilon)$ thanks to (48). In order to check that P^ε satisfies the equation (41), it suffices to verify that $P^{\varepsilon,\delta}$ tends weakly to P^ε in $L^2(Q_T)$. This convergence can be directly established using the definition (30) and (48)-(51).

The first and the fourth equation of the system (28) can be rewritten: $\forall \psi \in \mathcal{D}(\overline{\Omega} \times [0, T])$,

$$\begin{aligned} & \iint_{Q_T} \phi^\delta s^{\varepsilon,\delta} \partial_t \psi dxdt + \int_{\Omega} \phi^\delta s_0 \psi(\cdot, 0) dx \\ & - \iint_{Q_T} \mathbf{K}^\delta \frac{kr_o^\delta(s^{\varepsilon,\delta})}{\mu_o} (\nabla p_o^{\varepsilon,\delta} - \rho_o \mathbf{g}) \nabla \psi dxdt = \varepsilon \iint_{Q_T} \nabla \pi^\delta(s^{\varepsilon,\delta}) \nabla \psi dxdt \end{aligned} \quad (52)$$

while the second and the fifth turn to: $\forall \psi \in \mathcal{D}(\overline{\Omega} \times [0, T])$,

$$\begin{aligned} & \iint_{Q_T} \phi^\delta s^{\varepsilon,\delta} \partial_t \psi dxdt + \int_{\Omega} \phi^\delta s_0 \psi(\cdot, 0) dx \\ & + \iint_{Q_T} \mathbf{K}^\delta \frac{kr_w^\delta(s^{\varepsilon,\delta})}{\mu_w} (\nabla p_w^{\varepsilon,\delta} - \rho_w \mathbf{g}) \nabla \psi dxdt = \varepsilon \iint_{Q_T} \nabla \pi^\delta(s^{\varepsilon,\delta}) \nabla \psi dxdt. \end{aligned} \quad (53)$$

Since ϕ^δ and \mathbf{K}^δ converge almost everywhere respectively towards ϕ and \mathbf{K} , and since, thanks to (48), $kr_\beta^\delta(s^{\varepsilon,\delta})$ tends almost everywhere —thus strongly in $L^p(Q_T)$ for all $p \in [1, \infty)$ — towards $kr_\beta(s^\varepsilon)$, one can pass to the limit in (52)-(53) using (48) and (51), obtaining

$$\begin{aligned} & \iint_{Q_T} \phi s^\varepsilon \partial_t \psi dxdt + \int_{\Omega} \phi s_0 \psi(\cdot, 0) dx \\ & - \iint_{Q_T} \mathbf{K} \frac{kr_o(s^\varepsilon)}{\mu_o} (\nabla p_o^\varepsilon - \rho_o \mathbf{g}) \nabla \psi dxdt = \varepsilon \iint_{Q_T} \nabla \pi(s^\varepsilon) \nabla \psi dxdt \end{aligned} \quad (54)$$

and

$$\begin{aligned} & \iint_{Q_T} \phi s^\varepsilon \partial_t \psi dx dt + \int_{\Omega} \phi s_0 \psi(\cdot, 0) dx \\ & + \iint_{Q_T} \mathbf{K} \frac{kr_w(s^\varepsilon)}{\mu_w} (\nabla p_w^\varepsilon - \rho_w \mathbf{g}) \nabla \psi dx dt = \varepsilon \iint_{Q_T} \nabla \pi(s^\varepsilon) \nabla \psi dx dt. \end{aligned} \quad (55)$$

Thanks to (48) and (51), one can also pass in the limit in the third equation of the system (28), leading to

$$p_o^\varepsilon - p_w^\varepsilon = \pi(s^\varepsilon),$$

then $s^\varepsilon, p_o^\varepsilon$ and p_w^ε are solutions to (27).

Since in $Q_{i,T}$, the function P^ε has been built so that

$$M_i(s^\varepsilon) \nabla P^\varepsilon = \frac{kr_{o,i}(s^\varepsilon)}{\mu_o} \nabla p_o^\varepsilon + \frac{kr_{w,i}(s^\varepsilon)}{\mu_w} \nabla p_w^\varepsilon,$$

classical calculations (see e.g. [AKM90], [CJ86]) yield that the weak formulation (54)–(55) is equivalent to (42)–(43).

Let $h > 0$, then for all $\delta \in (0, h)$, one has

$$\nabla P^{\varepsilon, \delta} = \nabla p_w^{\varepsilon, \delta} + \frac{kr_{o,i}(s^{\varepsilon, \delta})}{kr_{o,i}(s^{\varepsilon, \delta}) + \frac{\mu_o}{\mu_w} kr_{w,i}(s^{\varepsilon, \delta})} \nabla \pi_i(s^{\varepsilon, \delta}) \quad \text{a.e. in } Q_T^h.$$

Thus it follows from (48)–(51) that $\nabla P^{\varepsilon, \delta}$ converges towards ∇P^ε weakly in $L^2(Q_T^h)$ as δ tends to 0. This ensures that

$$\iint_{Q_{i,T}^h} (\nabla P^\varepsilon)^2 dx \leq \liminf_{\delta \rightarrow 0} \iint_{Q_{i,T}^h} (\nabla P^{\varepsilon, \delta})^2 dx.$$

Thanks to Lemma 2.3, one obtains that for all $h > 0$,

$$\iint_{Q_{i,T}^h} (\nabla P^\varepsilon)^2 dx \leq C \left(1 + \max_{i \in \{1,2\}} \|\pi_i\|_{L^1(0,1)} \right).$$

Letting now h tend to 0 provides the estimate (44). The estimates (39) and (45) are directly provided by letting δ tend to 0 in the estimates (29) and (38).

Since $P^{\varepsilon, \delta}$ converges weakly in $L^2(Q_T)$ towards P^ε , the zero mean condition (31) is conserved as $\delta \rightarrow 0$, giving (41). Since $p_w^\varepsilon \in L^2((0, T); H^1(\Omega))$, then it is continuous on $\Gamma \times (0, T)$ in the sense that its traces from $Q_{1,T}$ and $Q_{2,T}$ coincide. Using the definition (15) of the global pressure provides (40). \square

2.2 Proof of Theorem 2

The goal of this section is to let tend ε to 0 in the system (27). We first give the following technical lemma, that ensures that the global pressure jump at the interface remains uniformly bounded, and that remains valid for non-matching capillary pressure functions.

Lemma 2.7 *Let π_1, π_2 be increasing functions belonging to $C^1((0, 1); \mathbb{R}) \cap L^1((0, 1))$, then the function $p \mapsto Z(p) = \lambda_{w,1}(p) - \lambda_{w,2}(p)$, where the functions $\lambda_{w,i}$ are defined by (15), is uniformly bounded on \mathbb{R} by a constant depending only on $kr_{\alpha,i}$, μ_α and $\|\pi_i\|_{L^1((0,1))}$ ($i \in \{1,2\}$, $\alpha \in \{o,w\}$). Moreover, $Z(p)$ admits a finite limits as $p \rightarrow \pm\infty$.*

Proof: Denote by \mathcal{L}_i the Lipschitz constant of $s \mapsto \frac{kr_{o,i}(s)}{kr_{o,i}(s) + \frac{\mu_o}{\mu_w} kr_{w,i}(s)}$, then, for all $p \leq 0$,

$$\lambda_{w,i}(p) = \int_0^p \frac{kr_{o,i}(\pi_i^{-1}(a))}{kr_{o,i}(\pi_i^{-1}(a)) + \frac{\mu_o}{\mu_w} kr_{w,i}(\pi_i^{-1}(a))} da \geq -\mathcal{L}_i \int_0^p |\pi_i^{-1}(a)| da.$$

Then it follows from (7) that

$$0 \geq \lambda_{w,i}(p) \geq -\mathcal{L}_i \|\pi_i^{-1}\|_{L^1(\mathbb{R}_-)} \geq -\mathcal{L}_i \|\pi_i\|_{L^1((0,1))}, \quad \forall p \leq 0,$$

and thus that

$$|Z(p)| \leq \mathcal{L}_i \|\pi_i\|_{L^1((0,1))}, \quad \forall p \leq 0.$$

In order to deal with the case $p \geq 0$, we remark that, thanks to (16), $Z(p)$ is also equal to $\lambda_{o,1}(p) - \lambda_{o,2}(p)$, where

$$\lambda_{o,i}(p) = \int_0^p \left(\frac{kr_{o,i}(\pi_i^{-1}(a))}{kr_{o,i}(\pi_i^{-1}(a)) + \frac{\mu_o}{\mu_w} kr_{w,i}(\pi_i^{-1}(a))} - 1 \right) da \geq \mathcal{L}_i \int_0^p |\pi_i^{-1}(a) - 1| da.$$

Hence, for all $p \geq 0$,

$$0 \geq \lambda_{o,i}(p) \geq -\mathcal{L}_i \|\pi_i^{-1} - 1\|_{L^1(\mathbb{R}_+)} \geq -\mathcal{L}_i \|\pi_i\|_{L^1((0,1))}.$$

This yields that

$$|Z(p)| \leq \mathcal{L}_i \|\pi_i\|_{L^1((0,1))}, \quad \forall p \geq 0.$$

The fact that $Z(p)$ admits finite limits as $p \rightarrow -\infty$ comes from the fact that so does $\lambda_{w,i}(p)$, while $\lambda_{o,i}(p)$ admits a finite limit as $p \rightarrow +\infty$, ensuring that it is also the case for $Z(p)$. \square

Lemma 2.8 *Denote by $m_i(P^\varepsilon)(t) = \frac{1}{|\Omega_i|} \int_{\Omega_i} P^\varepsilon(x, t) dx$, then there exists C depending only on $kr_{\alpha,j}$, $\|\pi_j\|_{L^1((0,1))}$, ϕ , \mathbf{K} , ρ_α , μ_α , Ω , T , \mathbf{g} ($j \in \{1, 2\}$, $\alpha \in \{o, w\}$) such that*

$$\|m_i(P^\varepsilon)\|_{L^2((0,T))} \leq C.$$

Proof: It follows from (41) that for almost all $t \in (0, T)$, one has

$$|\Omega_1| m_1(P^\varepsilon)(t) + |\Omega_2| m_2(P^\varepsilon)(t) = 0. \quad (56)$$

Thanks to (40), the following relation holds almost everywhere on $\Gamma \times (0, T)$:

$$m_1(P^\varepsilon) - m_2(P^\varepsilon) = (P_2^\varepsilon - m_2(P^\varepsilon)) - (P_1^\varepsilon - m_1(P^\varepsilon)) + \lambda_{w,2}(\pi_2(s_2^\varepsilon)) - \lambda_{w,1}(\pi_1(s_1^\varepsilon)),$$

ensuring, thanks to (56), that

$$(m_1(P^\varepsilon))^2 \leq C \left((P_2^\varepsilon - m_2(P^\varepsilon))^2 + (P_1^\varepsilon - m_1(P^\varepsilon))^2 + (\lambda_{w,2}(\pi_2(s_2^\varepsilon)) - \lambda_{w,1}(\pi_1(s_1^\varepsilon)))^2 \right).$$

Integrating this relation on $\Gamma \times (0, T)$ provides

$$|\Gamma| \int_0^T (m_1(P^\varepsilon)(t))^2 dt \leq A_2^\varepsilon + A_1^\varepsilon + B^\varepsilon, \quad (57)$$

where

$$A_i^\varepsilon = C \int_0^T \int_\Gamma (P_i^\varepsilon - m_i(P^\varepsilon))^2 dx dt$$

and

$$B^\varepsilon = C \int_0^T \int_\Gamma Z(\pi_i(s_i^\varepsilon))^2 dx dt,$$

where the function Z was introduced in Lemma 2.7. Thanks to Lemma 2.7, there exists C depending only on the prescribed data such that

$$B^\varepsilon \leq C. \quad (58)$$

Thanks to the continuity of the trace operator mapping $H^1(\Omega_i)$ to $L^2(\Gamma)$, there exists C depending only on Ω_i such that

$$\|P_i^\varepsilon - m_i(P^\varepsilon)\|_{L^2(\Gamma \times (0,T))} \leq C \|P^\varepsilon - m_i(P^\varepsilon)\|_{L^2((0,T); H^1(\Omega_i))}.$$

Now, from Poincaré-Wirtinger inequality and estimate (44), one has

$$\|P^\varepsilon - m_i(P^\varepsilon)\|_{L^2((0,T); H^1(\Omega_i))} \leq C \|\nabla P^\varepsilon\|_{L^2(Q_{i,T})} \leq C,$$

where C only depends on the prescribed data. As a consequence, there exists C depending only on the prescribed data such that

$$A_i^\varepsilon \leq C. \quad (59)$$

It follows then from (57)–(59) that

$$\int_0^T (m_1(P^\varepsilon)(t))^2 dt \leq C.$$

The derivation of an $L^2((0, T))$ -estimate on $m_2(P^\varepsilon)$ is similar. \square

We now give the following lemma, which is a straightforward consequence of (44), Lemma 2.8 and the Poincaré-Wirtinger inequality.

Lemma 2.9 *There exists $P \in L^2((0, T); H^1(\Omega_i))$ such that, up to a subsequence, P^ε converges towards P weakly in $L^2((0, T); H^1(\Omega_i))$ as ε tends to 0. Moreover,*

$$\|P\|_{L^2((0,T); H^1(\Omega_i))} \leq C,$$

where C only depends on $kr_{\alpha,j}$, $\|\pi_j\|_{L^1((0,1))}$, ϕ , \mathbf{K} , ρ_α , μ_α , Ω , T , \mathbf{g} ($j \in \{1, 2\}$, $\alpha \in \{o, w\}$).

Lemma 2.10 *There exists $s \in L^\infty(Q_T; [0, 1])$ such that, up to a subsequence,*

$$s^\varepsilon \rightarrow s \text{ a.e. in } Q_T \text{ as } \varepsilon \rightarrow 0,$$

$$\varphi_i(s^\varepsilon) \rightarrow \varphi_i(s) \text{ weakly in } L^2((0, T); H^1(\Omega_i)).$$

Moreover,

$$\|\varphi_i(s)\|_{L^2((0,T); H^1(\Omega_i))} \leq C, \quad (60)$$

where C only depends on $kr_{\alpha,j}$, $\|\pi_j\|_{L^1((0,1))}$, ϕ , \mathbf{K} , ρ_α , μ_α , Ω , T , \mathbf{g} ($j \in \{1, 2\}$, $\alpha \in \{o, w\}$), and

$$\iint_{Q_{i,T-\tau}^h} (\varphi_i(s)(\cdot, \cdot + \tau) - \varphi_i(s))^2 dx dt \leq \tau C^h, \quad (61)$$

where C^h depends on $kr_{\alpha,j}$, $\|\pi_j\|_{L^1((0,1))}$, ϕ , \mathbf{K} , ρ_α , μ_α , Ω , T , \mathbf{g} ($j \in \{1, 2\}$, $\alpha \in \{o, w\}$), L_{φ_i} and h .

Proof: It follows from (44) and (45) that the family $(\varphi_i(s^\varepsilon))_\varepsilon$ is sequentially relatively compact in $L^2(Q_{i,T}^h)$ for all $h > 0$. Then there exists $f_i \in L^2(Q_{i,T}^h)$ such that, up to a subsequence,

$$\varphi_i(s^\varepsilon) \rightarrow f_i \text{ a.e. in } Q_{i,T}^h \text{ as } \varepsilon \rightarrow 0.$$

Since this relation stands for all $h > 0$, one can claim that

$$\varphi_i(s^\varepsilon) \rightarrow f_i \text{ a.e. in } Q_{i,T} \text{ as } \varepsilon \rightarrow 0.$$

Using the fact that φ_i^{-1} is continuous, then, by setting $s = \varphi_i^{-1}(f_i)$ in $Q_{i,T}$, we obtain that

$$s^\varepsilon \rightarrow s \text{ a.e. in } Q_T \text{ as } \varepsilon \rightarrow 0.$$

Since $0 \leq s^\varepsilon \leq 1$ almost everywhere in Q_T , we obtain that $s \in L^\infty(Q_T; [0, 1])$. It follows from (44) that $\varphi_i(s^\varepsilon)$ converges, up to a subsequence, towards $\varphi_i(s)$ weakly in $L^2((0, T); H^1(\Omega_i))$, and that the estimate (60) holds for the limit, while the estimate (61) is obtained by passing to the limit in (45). \square

Lemma 2.11 *The function $(x, t) \mapsto \varepsilon\pi(s^\varepsilon(x, t), x)$ tends to 0 in $L^2((0, T); H^1(\Omega))$ as ε tends to 0.*

Proof: We deduce from the estimate (39) that

$$\|\varepsilon\pi(s^\varepsilon, \cdot)\|_{L^2((0, T); H^1(\Omega))} \leq C\varepsilon^{1/2},$$

ensuring the expected convergence. \square

We now state a proposition that ends the proof of Theorem 2.

Proposition 2.12 *Let s, P be the functions built in Lemmas 2.9 and 2.10, then (s, P) is a solution to the problem (12)-(14), (17), (18), (20), (21), (25) in the sense of Definition 1.1.*

Proof: The convergence properties stated in Lemmas 2.9, 2.10 and 2.11 allow to pass to the limit $\varepsilon \rightarrow 0$ in the weak formulations (42) and (43). Since P^ε converges weakly in $L^2((0, T); H^1(\Omega_i))$ for $i \in \{1, 2\}$, it also converges weakly in $L^2(Q_T)$. As a consequence, we deduce from (41) that for all $\psi \in L^2((0, T))$,

$$0 = \int_0^T \left(\int_\Omega P^\varepsilon(x, t) dx \right) \psi(t) dt \xrightarrow{\varepsilon \rightarrow 0} \int_0^T \left(\int_\Omega P(x, t) dx \right) \psi(t) dt,$$

ensuring that for almost all $t \in (0, T)$,

$$\int_\Omega P(x, t) dx = 0.$$

Since $\varphi_i(s^\varepsilon)$ converges towards $\varphi_i(s)$ weakly in $L^2((0, T); H^1(\Omega_i))$ and strongly in $L^2(Q_{i,T})$, then it also converges strongly in $L^2((0, T); H^s(\Omega_i))$ for $s \in (1/2, 1)$. As a consequence, its trace on $\Gamma \times (0, T)$ converges strongly in $L^2(\Gamma \times (0, T))$. The continuity of φ_i^{-1} ensures the convergence of the traces s_i^ε towards s_i almost everywhere on $\Gamma \times (0, T)$ (up to a subsequence) and in $L^p(\Gamma \times (0, T))$ for all $p \in [1, \infty)$. Hence, we can pass to the limit in the relation

$$\pi_1(s_1^\varepsilon) = \pi_2(s_2^\varepsilon) \text{ a.e. on } \Gamma \times (0, T),$$

that gives

$$\pi_1(s_1) = \pi_2(s_2) \text{ a.e. on } \Gamma \times (0, T).$$

Since P^ε converges towards P weakly in $L^2((0, T); H^1(\Omega_i))$, then P_i^ε converges towards P_i weakly in $L^2(\Gamma \times (0, T))$. We can pass to the limit in the relation

$$P_1^\varepsilon - \lambda_{w,1}(\pi_1(s_1^\varepsilon)) = P_2^\varepsilon - \lambda_{w,2}(\pi_2(s_2^\varepsilon)),$$

that takes sense in $L^2(\Gamma \times (0, T))$, and that provides (25). \square

3 Existence of a solution for non-matching capillary curves

In this section, we aim to prove the existence of a weak solution in the case where the capillary pressure curve do not satisfy the assumption (24). As it has been done in [BLS09, CGP09] in the case where the elliptic equation on the pressure can be removed, as for example in the one-dimensional case, the main idea consists in approximating the capillary pressure graphs $\tilde{\pi}_i$ by regularized capillary pressure functions $\pi_{i,n}$ satisfying the matching conditions (24).

Let $(\pi_{i,n})_{n \geq 1} \subset C^1([0, 1]; \mathbb{R})$ be a sequence of approximate capillary pressures satisfying

$$\pi_{1,n}(0) = \pi_{2,n}(0), \quad \pi_{1,n}(1) = \pi_{2,n}(1)$$

such that

$$\pi_{i,n} = \pi_i \quad \text{on} \quad \left[\frac{1}{n}, 1 - \frac{1}{n} \right], \quad (62)$$

such that there exists C_π not depending on n fulfilling

$$\int_0^1 |\pi_{i,n}(a)| da \leq C_\pi, \quad \forall i \in \{1, 2\}, \forall n \geq 1, \quad (63)$$

and such that

$$\pi_{i,n}(0) \rightarrow \min_j \pi_j(0), \quad \pi_{i,n}(1) \rightarrow \max_j \pi_j(0) \quad \text{as } n \rightarrow \infty, \quad (64)$$

the quantities $\min_j \pi_j(0)$ and $\max_j \pi_j(0)$ belonging to $\overline{\mathbb{R}}$. Note that Dini's theorem implies that

$$\pi_{i,n}^{-1} \rightarrow \pi_i^{-1} \text{ uniformly on } \mathbb{R} \text{ as } n \rightarrow \infty. \quad (65)$$

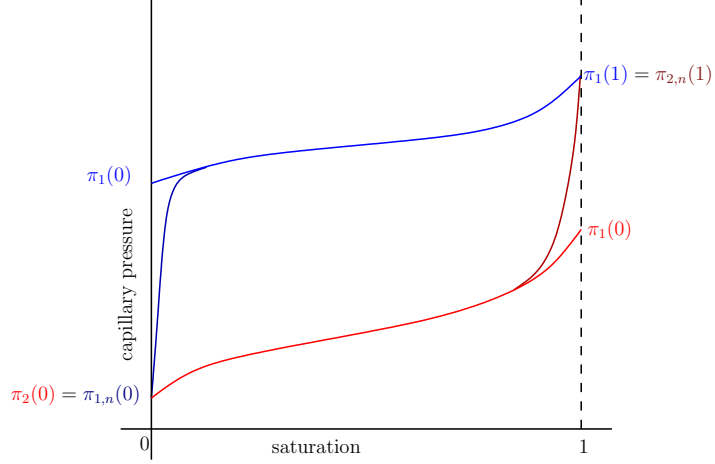


Figure 2: An example of functions $\pi_{i,n}$ fulfilling the matching condition (24) approximating non-matching capillary pressure functions π_i .

We denote by

$$\varphi_{i,n}(s) = \int_0^1 \frac{kr_{o,i}(a)kr_{w,i}(a)}{\mu_w kr_{o,i}(a) + \mu_o kr_{w,i}(a)} \pi'_{i,n}(a) da.$$

We assume furthermore that the sequence $(\pi_{i,n})$ is chosen such that

$$\varphi'_{i,n} = \frac{kr_{o,i}kr_{w,i}}{\mu_w kr_{o,i} + \mu_o kr_{w,i}} \pi'_{i,n} \rightarrow \varphi'_i \quad \text{uniformly in } [0, 1] \text{ as } n \rightarrow \infty. \quad (66)$$

This particularly yields that

$$\varphi_{i,n} \rightarrow \varphi_i \text{ uniformly on } [0, 1] \text{ as } n \rightarrow \infty.$$

Moreover, the Lipschitz constant $L_{\varphi_{i,n}}$ of the function $\varphi_{i,n}$ remains uniformly bounded as n tends to ∞ thanks to (66), i.e. there exists a constant $C_\varphi > 0$ such that

$$L_{\varphi_{i,n}} \leq C_\varphi, \quad \forall i \in \{1, 2\}, \forall n \geq 1.$$

One denotes by $\lambda_{w,i,n}$ then function defined by

$$\lambda_{w,i,n}(p) = \int_0^p \frac{kr_{o,i}(\pi_{i,n}^{-1}(a))}{\frac{\mu_o}{\mu_w} kr_{w,i}(\pi_{i,n}^{-1}(a)) + kr_{o,i}(\pi_{i,n}^{-1}(a))} da, \quad \forall p \in \overline{\mathbb{R}},$$

then it follows from (65) that

$$\lambda_{w,i,n} \rightarrow \lambda_{w,i} \text{ uniformly on } \mathbb{R} \text{ as } n \rightarrow \infty. \quad (67)$$

For any fixed n , Theorem 2 ensures the existence of a weak solution (s_n, P_n) to the approximate

problem

$$\left\{ \begin{array}{ll} \phi_i \partial_t s_n - \operatorname{div} \left(\mathbf{K}_i \left(\frac{kr_{o,i}(s_n)}{\mu_o} (\nabla P_n - \rho_o \mathbf{g}) + \nabla \varphi_{i,n}(s_n) \right) \right) = 0 & \text{in } Q_{i,T}, \\ -\operatorname{div} (\mathbf{K}_i (M_i(s_n) \nabla P_n - \zeta_i(s_n) \mathbf{g})) = 0 & \text{in } Q_{i,T}, \\ \pi_{1,n}(s_{1,n}) = \pi_{2,n}(s_{2,n}) & \text{on } \Gamma \times (0, T), \\ P_{1,n} - \lambda_{w,1,n}(\pi_{1,n}(s_{1,n})) = P_{2,n} - \lambda_{w,2,n}(\pi_{2,n}(s_{2,n})) & \text{on } \Gamma \times (0, T), \\ \sum_{i \in \{1,2\}} \mathbf{K}_i \left(\frac{kr_{o,i}(s_n)}{\mu_o} (\nabla P_n - \rho_o \mathbf{g}) + \nabla \varphi_{i,n}(s_n) \right) \cdot \mathbf{n}_i = 0 & \text{on } \Gamma \times (0, T), \\ \sum_{i \in \{1,2\}} \mathbf{K}_i (M_i(s_n) \nabla P_n - \zeta_i(s_n) \mathbf{g}) \cdot \mathbf{n}_i = 0 & \text{on } \Gamma \times (0, T), \\ \mathbf{K}_i \left(\frac{kr_{o,i}(s_n)}{\mu_o} (\nabla P_n - \rho_o \mathbf{g}) + \nabla \varphi_{i,n}(s_n) \right) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ \mathbf{K}_i (M_i(s_n) \nabla P_n - \zeta_i(s_n) \mathbf{g}) \cdot \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, T), \\ s_n(\cdot, 0) = s_0 & \text{in } \Omega, \end{array} \right. \quad (68)$$

where $s_{i,n}$ and $P_{i,n}$ denote the respective traces of $(s_n)|_{\Omega_i}$ and $(P_n)|_{\Omega_i}$ on $\Gamma \times (0, T)$.

3.1 Uniform estimates with respect to n and compactness properties

First of all, since for all $n \geq 1$, one has $s_n \in L^\infty(Q_T; [0, 1])$, there exists $s \in L^\infty(Q_T; [0, 1])$ such that, up to a subsequence,

$$s_n \rightarrow s \text{ in the } L^\infty(Q_T) \text{ weak-} \star \text{ sense.} \quad (69)$$

Thanks to Lemma 2.10 and to the assumption (63) on the sequences $(\pi_{i,n})_n$, there exists C (not depending on n) such that

$$\iint_{Q_{i,T}} (\nabla \varphi_{i,n}(s_n))^2 dx dt \leq C.$$

Moreover, since $(\varphi_{i,n})_n$ converges uniformly towards φ_i on $[0, 1]$, then $(\varphi_{i,n}(s_n))_n$ is uniformly bounded in $L^\infty(Q_T)$. Thus up to a subsequence,

$$\varphi_{i,n}(s_n) \rightarrow \varphi_i(s) \text{ weakly in } L^2((0, T); H^1(\Omega_i)) \text{ as } n \rightarrow \infty. \quad (70)$$

Let $\tau, h > 0$, then it follows from Lemma 2.10 and (63) that there exists C^h (not depending on n) such that

$$\iint_{Q_{i,T-\tau}^h} (\varphi_{i,n}(s_n(\cdot, \cdot + \tau)) - \varphi_{i,n}(s_n))^2 dx dt \leq \tau C^h.$$

Hence the sequence $(\varphi_{i,n}(s_n))_n$ is relatively compact in $L^2(Q_{i,T}^h)$ for all $h > 0$, thus also in $L^2(Q_{i,T})$. Using Minty's trick (see e.g. [CGP09]), we obtain that up to a subsequence,

$$\varphi_{i,n}(s_n) \rightarrow \varphi_i(s) \text{ a.e. in } Q_{i,T}.$$

Since φ_i^{-1} is continuous, one has, up to a new subsequence

$$s_n \rightarrow s \text{ a.e. in } Q_T. \quad (71)$$

Thanks to Lemma 2.9 and (63), we can claim that there exists $P \in L^2((0, T); H^1(\Omega_i))$ such that

$$P_n \rightarrow P \text{ weakly in } L^2((0, T); H^1(\Omega_i)) \text{ as } n \rightarrow \infty. \quad (72)$$

Thus P_n tends to P also weakly in $L^2(Q_T)$, hence, since P_n satisfies

$$\int_{\Omega} P_n(x, t) dx = 0 \text{ for a.e. } t \in (0, T),$$

using again arguments developed in the proof of Proposition 2.12, one obtains for n tending to ∞ that

$$\int_{\Omega} P(x, t) dx = 0 \text{ for a.e. } t \in (0, T).$$

3.2 Recovery of the weak formulations (22) and (23)

Let $\psi \in C_c^\infty(\bar{\Omega} \times [0, T])$, then thanks to (69), one has

$$\iint_{Q_T} \phi s_n \partial_t \psi dx dt \rightarrow \iint_{Q_T} \phi s_n \partial_t \psi dx dt \quad \text{as } n \rightarrow \infty. \quad (73)$$

Thanks to (70), (71) and (72), one has

$$\begin{aligned} \lim_{n \rightarrow \infty} \sum_{i \in \{1,2\}} \iint_{Q_{i,T}} \mathbf{K}_i \left(\frac{kr_{o,i}(s_n)}{\mu_o} (\nabla P_n - \rho_o \mathbf{g}) - \nabla \varphi_{i,n}(s_n) \right) \cdot \nabla \psi dx dt \\ = \sum_{i \in \{1,2\}} \iint_{Q_{i,T}} \mathbf{K}_i \left(\frac{kr_{o,i}(s)}{\mu_o} (\nabla P - \rho_o \mathbf{g}) - \nabla \varphi_i(s) \right) \cdot \nabla \psi dx dt. \end{aligned} \quad (74)$$

The weak formulation (22) is then a direct consequence of (73) and (74).

The same way, in order to recover (23), it suffices to check that thanks to (71) and (72),

$$\mathbf{K}_i (M_i(s_n) \nabla P_n - \zeta_i(s_n) \mathbf{g}) \rightarrow \mathbf{K}_i (M_i(s) \nabla P - \zeta_i(s) \mathbf{g})$$

weakly in $L^2(Q_{i,T})$ as $n \rightarrow \infty$.

3.3 Recovery of the transmission conditions on $\Gamma \times (0, T)$

Since P_n converges weakly towards P in $L^2((0, T); H^1(\Omega_i))$, one has

$$P_{i,n} \rightarrow P_i \quad \text{weakly in } L^2(\Gamma \times (0, T)) \text{ as } n \rightarrow \infty. \quad (75)$$

Since the sequence $(\varphi_{i,n}(s_n))$ converges (up to a subsequence) to $\varphi_i(s)$ weakly in $L^2((0, T); H^1(\Omega_i))$ and strongly in $L^2(Q_{i,T})$ and since Ω_i is supposed to be Lipschitz continuous, then for all $s \in (\frac{1}{2}, 1)$, the sequence $(\varphi_{i,n}(s_n))$ converges strongly in $L^2((0, T); H^s(\Omega_i))$. In particular, the trace $\varphi_{i,n}(s_{i,n})$ converges strongly in $L^2(\Gamma \times (0, T))$ towards $\varphi_i(s_i)$, thus almost everywhere up to a new extraction. Since $\varphi_{i,n}^{-1} \circ \varphi_i$ converges pointwise on $[0, 1]$ towards the identity, then the traces of the saturation $s_{i,n}$ also converge on the interface

$$s_{i,n} \rightarrow s_i \text{ a.e. on } \Gamma \times (0, T). \quad (76)$$

We denote by \mathcal{U} and \mathcal{V} the measurable sets of $\Gamma \times (0, T)$ defined by

$$\mathcal{U} = \{(x, t) \in \Gamma \times (0, T) \mid \{s_1(x, t), s_2(x, t)\} \neq \{0, 1\}\} \quad \text{and} \quad \mathcal{V} = \mathcal{U}^c.$$

It is worth noticing that \mathcal{V} is negligible if $\min_j \pi_j(1) > \max_j \pi_j(0)$.

In the sequel, we denote by

$$\pi_n := \pi_{1,n}(s_{1,n}) = \pi_{2,n}(s_{2,n}). \quad (77)$$

Lemma 3.1 *There exists a measurable function π mapping \mathcal{U} to $[\min_j \pi_j(0), \max_j \pi_j(1)]$ such that π_n converges almost everywhere to π on \mathcal{U} , and such that*

$$\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \quad \text{and} \quad P_1 - \lambda_{w,1}(\pi) = P_2 - \lambda_{w,2}(\pi).$$

Proof: Let $(x, t) \in \mathcal{U}$ such that $(s_{1,n}(x, t), s_{2,n}(x, t))$ tends to $(s_1(x, t), s_2(x, t))$, then there exists $j \in \{1, 2\}$ such that $\pi_n(x, t) = \pi_{j,n}(s_{j,n}(x, t))$ converges towards $\pi(x, t) = \pi_j(s(x, t)) \in [\pi_j(0), \pi_j(1)]$, ensuring by the way that

$$\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2).$$

One has

$$P_{1,n} - P_{2,n} = \lambda_{w,1,n}(\pi_n) - \lambda_{w,2,n}(\pi_n).$$

The left hand side converges weakly in $L^2(\Gamma \times (0, T))$ towards $P_1 - P_2$, thus also weakly in $L^2(\mathcal{U})$. Thanks to Lemma 2.7, to the almost everywhere convergence on \mathcal{U} of π_n towards π , and to the uniform convergence (67) of $\lambda_{w,i,n}$ towards $\lambda_{w,i}$, the righthand side converges strongly in $L^2(\mathcal{U})$ towards $\lambda_{w,1}(\pi) - \lambda_{w,2}(\pi)$. Then the relation

$$P_1 - P_2 = \lambda_{w,1}(\pi) - \lambda_{w,2}(\pi)$$

holds in $L^2(\mathcal{U})$, thus almost everywhere. \square

As it has been already noticed, the set \mathcal{V} has to be taken in consideration only if

$$\underline{\pi} := \min_i \pi_i(1) < \bar{\pi} := \max_i \pi_i(0). \quad (78)$$

Now, we assume that (78) is fulfilled. For any $\eta > 0$, we denote by T_η the function defined on \mathbb{R} by

$$T_\eta(c) = \min(\bar{\pi} + \eta, \max(\underline{\pi} - \eta, c)) = \begin{cases} \underline{\pi} - \eta & \text{if } c \leq \underline{\pi} - \eta, \\ c & \text{if } c \in [\underline{\pi} - \eta, \bar{\pi} + \eta], \\ \bar{\pi} + \eta & \text{if } c \geq \bar{\pi} + \eta. \end{cases}$$

Lemma 3.2 *There exists $\pi \in L^\infty(\mathcal{V})$ such that, for all $\eta > 0$, up to a subsequence,*

$$T_\eta(\pi_n) \rightarrow \pi \text{ in the } L^\infty(\mathcal{V})\text{-weak-}^* \text{ sense as } n \rightarrow \infty.$$

Proof: First, since the sequence $(T_\eta(\pi_n))_n$ is uniformly bounded on \mathcal{V} , then there exists $\pi^\eta \in L^\infty(\mathcal{V})$ such that, up to a subsequence,

$$T_\eta(\pi_n) \rightarrow \pi^\eta \text{ in the } L^\infty(\mathcal{V})\text{-weak-}^* \text{ sense as } n \rightarrow \infty.$$

It remains to show that π^η does not depend on η . Let $\eta_1, \eta_2 > 0$, then (up to a new subsequence), one has

$$T_{\eta_1}(\pi_n) - T_{\eta_2}(\pi_n) \rightarrow \pi^{\eta_1} - \pi^{\eta_2}.$$

Let $(x, t) \in \mathcal{V}$ such that $s_{1,n}(x, t) \rightarrow s_1(x, t)$ and $s_{2,n}(x, t) \rightarrow s_2(x, t)$, with $\{s_1(x, t), s_2(x, t)\} = \{0, 1\}$, then it follows from (77) that

$$\liminf_n \pi_n(x, t) \geq \min_i \pi_i(1), \quad \limsup_n \pi_n(x, t) \leq \max_i \pi_i(0).$$

As a consequence,

$$T_{\eta_1}(\pi_n(x, t)) - T_{\eta_2}(\pi_n(x, t)) \rightarrow 0 \text{ as } n \rightarrow \infty,$$

ensuring that $\pi^{\eta_1}(x, t) = \pi^{\eta_2}(x, t)$. \square

Lemma 3.3 *Let $\pi \in L^\infty(\mathcal{V})$ be the function defined by Lemma 3.2, then, for a.e. $(x, t) \in \mathcal{V}$, one has*

$$P_1(x, t) - P_2(x, t) = \lambda_{w,1}(\pi(x, t)) - \lambda_{w,2}(\pi(x, t))$$

Proof: First, thanks to (67), we can claim that

$$\lambda_{w,1,n}(\pi_n) - \lambda_{w,2,n}(\pi_n) = \lambda_{w,1}(\pi_n) - \lambda_{w,2}(\pi_n) + \varepsilon(n),$$

with $\lim_{n \rightarrow \infty} \varepsilon(n) = 0$. Thus, since $P_{i,n} \rightarrow P_i$ weakly in $L^2(\mathcal{V})$, it is sufficient to show that

$$\lambda_{w,1}(\pi_n) - \lambda_{w,2}(\pi_n) \rightarrow \lambda_{w,1}(\pi) - \lambda_{w,2}(\pi) \text{ weakly in } L^2(\mathcal{V}) \text{ as } n \rightarrow \infty.$$

Let $\psi \in L^2(\mathcal{V})$, then, denoting by $Z(p) = \lambda_{w,1}(p) - \lambda_{w,2}(p)$, for all $\eta > 0$,

$$\iint_{\mathcal{V}} Z(\pi_n) \psi dx dt = A_n(\eta) + B_n(\eta), \quad (79)$$

where one has

$$\begin{aligned} A_n(\eta) &= \iint_{\mathcal{V}} Z(T_\eta(\pi_n)) \psi dx dt, \\ B_n(\eta) &= \iint_{\mathcal{V}} (Z(\pi_n) - Z(T_\eta(\pi_n))) \psi dx dt. \end{aligned}$$

Fix $\varepsilon > 0$. Since, as stated in Lemma 2.7, $Z(p)$ admits finite limits as $p \rightarrow \pm\infty$, then there exists R such that

$$\eta > R \implies \|Z - Z \circ T_\eta\|_\infty \leq \varepsilon,$$

ensuring that

$$\eta > R \implies |B_n(\eta)| \leq C\varepsilon. \quad (80)$$

We suppose now, without loss of generality, that $\pi_1(1) \leq \pi_2(0)$. Then for almost all $(x, t) \in \mathcal{V}$, $s_2(x, t) = 0$ and $s_1(x, t) = 1$. One has

$$\lambda_{w,2}(T_\eta(\pi_n)) = \int_0^{\pi_2(0)} f_2(a) da + \int_{\pi_2(0)}^{T_\eta(\pi_n)} f_2(a) da,$$

where

$$f_i(p) = \frac{kr_{o,i} \circ \pi_i^{-1}(p)}{\frac{\mu_o}{\mu_w} kr_{w,i} \circ \pi_i^{-1}(p) + kr_{o,i} \circ \pi_i^{-1}(p)}.$$

Note that $f_i(p) = 0$ if $p \leq \pi_i(0)$, and $f_i(p) = 1$ if $p \geq \pi_i(1)$. For almost all $(x, t) \in \mathcal{V}$, one has $\limsup_n \pi_n(x, t) \leq \pi_2(0)$, thus

$$\lim_{n \rightarrow \infty} \lambda_{w,2}(T_\eta(\pi_n(x, t))) = \int_0^{\pi_2(0)} f_2(a) da. \quad (81)$$

Similarly, the relation

$$\lambda_{w,1}(T_\eta(\pi_n)) - T_\eta(\pi_n) = \int_0^{\pi_1(1)} (f_1(a) - 1) da + \int_{\pi_1(1)}^{T_\eta(\pi_n)} (f_1(a) - 1) da$$

yields that, for almost all $(x, t) \in \mathcal{V}$,

$$\lim_{n \rightarrow \infty} \lambda_{w,1}(T_\eta(\pi_n(x, t))) - T_\eta(\pi_n)(x, t) = \int_0^{\pi_1(1)} (f_1(a) - 1) da. \quad (82)$$

As a consequence of (81), (82) and Lemma 3.2, we obtain that, for all $\eta > 0$,

$$\lim_{n \rightarrow \infty} A_n(\eta) = C \iint_{\mathcal{V}} \psi dx dt + \iint_{\mathcal{V}} \pi \psi dx dt = \iint_{\mathcal{V}} Z(\pi) \psi dx dt.$$

Letting now η tend to $+\infty$ provides, thanks to (79) and (80), that

$$\lim_{n \rightarrow \infty} \iint_{\mathcal{V}} Z(\pi_n) \psi dx dt = \iint_{\mathcal{V}} Z(\pi) \psi dx dt.$$

□

In order to conclude the proof of Theorem 1, we gather the results of Lemmas 3.1, 3.2 and 3.3 in the following proposition.

Proposition 3.4 *There exists a measurable function π mapping $\Gamma \times (0, T)$ to $\overline{\mathbb{R}}$ such that, almost everywhere on $\Gamma \times (0, T)$, one has*

$$\pi \in \tilde{\pi}_1(s_1) \cap \tilde{\pi}_2(s_2) \quad \text{and} \quad P_1 - \lambda_{w,1}(\pi) = P_2 - \lambda_{w,2}(\pi).$$

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