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A posteriori error estimates for a finite element approximation of transmission problems with sign changing coefficients

Serge Nicaise and Juliette Venel *

September 16, 2010

Abstract

We perform the a posteriori error analysis of residual type of a transmission problem with sign changing coefficients. According to [6] if the contrast is large enough, the continuous problem can be transformed into a coercive one. We further show that a similar property holds for the discrete problem for any regular meshes, extending the framework from [6]. The reliability and efficiency of the proposed estimator is confirmed by some numerical tests.

Key Words A posteriori estimator, non positive definite diffusion problems.

AMS (MOS) subject classification 65N30; 65N15, 65N50,

1 Introduction

Recent years have witnessed a growing interest in the study of diffusion problems with a sign changing coefficient. These problems appear in several areas of physics, for example in electromagnetism [12, 15, 16, 18, 19]. Thus some mathematical investigations have been performed and concern existence results [7, 19] and numerical approximations by the finite element methods [19, 4, 5, 6], with some a priori error analyses. But for such problems the regularity of the solution may be poor and/or unknown and consequently an a posteriori error analysis would be more appropriate. This analysis is the aim of the present paper.

For continuous Galerkin finite element methods, there now exists a large amount of literature on a posteriori error estimations for (positive definite) problems in mechanics or electromagnetism. Usually locally defined a posteriori error estimators are designed. We refer the reader to the monographs [2, 3, 17, 21] for a good overview on this topic.

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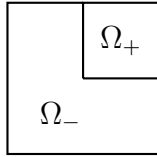


Figure 1: The domain Ω

In contrast to the recent paper [6] we will not use quasi-uniform meshes that are not realistic for an a posteriori error analysis. That is why we improve their finite element analysis in order to allow only regular meshes in Ciarlet's sense [8].

The paper is structured as follows: We recall in Section 2 the "diffusion" problem and the technique from [6] that allows to establish its well-posedness for sufficiently large contrast. In Section 3, we prove that the discrete approximation is well-posed by introducing an ad-hoc discrete lifting operator. The a posteriori error analysis is performed in Section 4, where upper and lower bounds are obtained. Finally in Section 5 some numerical tests are presented that confirm the reliability and efficiency of our estimator.

Let us finish this introduction with some notations used in the remainder of the paper: On D , the $L^2(D)$ -norm will be denoted by $\|\cdot\|_D$. The usual norm and semi-norm of $H^s(D)$ ($s \geq 0$) are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$, respectively. In the case $D = \Omega$, the index Ω will be omitted. Finally, the notations $a \lesssim b$ and $a \sim b$ mean the existence of positive constants C_1 and C_2 , which are independent of the mesh size and of the considered quantities a and b such that $a \leq C_2 b$ and $C_1 b \leq a \leq C_2 b$, respectively. In other words, the constants may depend on the aspect ratio of the mesh and the diffusion coefficient (see below).

2 The boundary value problem

Let Ω be a bounded open domain of \mathbb{R}^2 with boundary Γ . We suppose that Ω is split up into two sub-domains Ω_+ and Ω_- with a Lipschitz boundary that we suppose to be polygonal in such a way that

$$\bar{\Omega} = \bar{\Omega}_+ \cup \bar{\Omega}_-, \quad \Omega_+ \cap \Omega_- = \emptyset,$$

see Figure 1 for an example.

We now assume that the diffusion coefficient a belongs to $L^\infty(\Omega)$ and is positive (resp. negative) on Ω_+ (resp. Ω_-). Namely there exists $\epsilon_0 > 0$ such that

$$a(x) \geq \epsilon_0, \quad \text{for a. e. } x \in \Omega_+, \quad (1)$$

$$a(x) \leq -\epsilon_0, \quad \text{for a. e. } x \in \Omega_-. \quad (2)$$

In this situation we consider the following second order boundary value problem with Dirichlet boundary conditions:

$$\begin{cases} -\operatorname{div}(a \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \quad (3)$$

The variational formulation of (3) involves the bilinear form

$$B(u, v) = \int_{\Omega} a \nabla u \cdot \nabla v$$

and the Hilbert space

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma\}.$$

Due to the lack of coercivity of B on $H_0^1(\Omega)$ (see [7, 5, 6]), this problem does not fit into a standard framework. In [5, 6], the proposed approach is to use a bijective and continuous linear mapping \mathbb{T} from $H_0^1(\Omega)$ into itself that allows to come back to the coercive framework. Namely these authors assume that $B(u, \mathbb{T}v)$ is coercive in the sense that there exists $\alpha > 0$ such that

$$B(u, \mathbb{T}u) \geq \alpha \|u\|_{1,\Omega}^2 \quad \forall u \in H_0^1(\Omega). \quad (4)$$

Hence given $f \in L^2(\Omega)$, by the Lax-Milgram theorem the problem

$$B(u, \mathbb{T}v) = \int_{\Omega} f \mathbb{T}v \quad \forall v \in H_0^1(\Omega), \quad (5)$$

has a unique solution $u \in H_0^1(\Omega)$. Since \mathbb{T} is an isomorphism, the original problem

$$B(u, v) = \int_{\Omega} f v \quad \forall v \in H_0^1(\Omega), \quad (6)$$

has also a unique solution $u \in H_0^1(\Omega)$.

In [6], the mapping \mathbb{T} is built by using a trace lifting operator \mathcal{R} from $H_{00}^{1/2}(\Sigma)$ into $H^1(\Omega_-)$, where $\Sigma = \partial\Omega_- \cap \partial\Omega_+$ is the interface between Ω_- and Ω_+ ,

$$H_{\pm}^1(\Omega_{\pm}) = \{u \in H^1(\Omega_{\pm}) : u = 0 \text{ on } \partial\Omega_{\pm} \setminus \Sigma\},$$

and

$$H_{00}^{1/2}(\Sigma) = \{u|_{\Sigma} : u \in H^1(\Omega_-)\} = \{u|_{\Sigma} : u \in H^1(\Omega_+)\}$$

is the space of the restrictions to Σ of functions in $H^1(\Omega_-)$ (or in $H^1(\Omega_+)$). This last space may be equipped with the norms

$$\|p\|_{1/2,\pm} = \inf_{\substack{u \in H_{\pm}^1(\Omega_{\pm}) \\ p = u|_{\Sigma}}} |u|_{1,\Omega_{\pm}}.$$

With the help of such a lifting, a possible mapping \mathbb{T} is given by (see [6])

$$\mathbb{T}v = \begin{cases} v_+ & \text{in } \Omega_+, \\ -v_- + 2\mathcal{R}(v_+|_{\Sigma}) & \text{in } \Omega_-, \end{cases}$$

where v_{\pm} denotes the restriction of v to Ω_{\pm} . With this choice, it is shown in Proposition 3.1 of [6] that (4) holds if

$$K_{\mathcal{R}} = \sup_{\substack{v \in H_{\pm}^1(\Omega_{\pm}) \\ v \neq 0}} \frac{|B_-(\mathcal{R}(v|_{\Sigma}), \mathcal{R}(v|_{\Sigma}))|}{B_+(v, v)} < 1, \quad (7)$$

where $B_{\pm}(u, v) = \int_{\Omega_{\pm}} a \nabla u \cdot \nabla v$.

For concrete applications, one can make the following particular choice for \mathcal{R} , that we denote by \mathcal{R}_p : for any $\varphi \in H_{00}^{1/2}(\Sigma)$ we define $\mathcal{R}_p(\varphi) = w$ as the unique solution $w \in H^1(\Omega_-)$ of

$$\Delta w = 0 \text{ in } \Omega_-, \quad w = \varphi \text{ on } \Sigma.$$

With this choice, one obtains that $K_{\mathcal{R}_p} < 1$ if the contrast

$$\frac{\min_{\Omega_-} |a|}{\max_{\Omega_+} a}$$

is large enough, we refer to Section 3 of [6] for more details.

Remark 2.1 *Note that in [5, 6] the authors consider sub-domains Ω_+ and Ω_- with a pseudo-Lipschitz boundary. However the previous arguments from [6] (shortly summarized above) are not valid in this case since the space $H^1(\Omega_+)$ equipped with the norm $|\cdot|_{1, \Omega_+}$ is not complete.*

3 The discrete approximated problem

Here we consider the following standard Galerkin approximation of our continuous problem. We consider a triangulation \mathcal{T} of Ω , that is a "partition" of Ω made of triangles T (closed subsets of $\bar{\Omega}$) whose edges are denoted by e . We assume that this triangulation is regular, i.e., for any element T , the ratio h_T/ρ_T is bounded by a constant $\sigma > 0$ independent of T and of the mesh size $h = \max_{T \in \mathcal{T}} h_T$, where h_T is the diameter of T and ρ_T the diameter of its largest inscribed ball. We further assume that \mathcal{T} is conforming with the partition of Ω , i.e., each triangle is assumed to be either included into $\bar{\Omega}_+$ or into $\bar{\Omega}_-$. With each edge e of the triangulation, we denote by h_e its length and n_e a unit normal vector (whose orientation can be arbitrary chosen) and the so-called patch $\omega_e = \cup_{e \subset T} T$, the union of triangles having e as edge. We similarly associate with each vertex x , a patch $\omega_x = \cup_{x \in T} T$. For a triangle T , n_T stands for the outer unit normal vector of T . \mathcal{E} (resp. \mathcal{N}) represents the set of edges (resp. vertices) of the triangulation. In the sequel, we need to distinguish between edges (or vertices) included into Ω or into Γ , in other words, we set

$$\begin{aligned} \mathcal{E}_{int} &= \{e \in \mathcal{E} : e \subset \Omega\}, \\ \mathcal{E}_{\Gamma} &= \{e \in \mathcal{E} : e \subset \Gamma\}, \\ \mathcal{N}_{int} &= \{x \in \mathcal{N} : x \in \Omega\}. \end{aligned}$$

Problem (6) is approximated by the continuous finite element space:

$$V_h = \{v_h \in H_0^1(\Omega) : v_h|_T \in \mathbb{P}_{\ell}(T), \forall T \in \mathcal{T}\}, \quad (8)$$

where ℓ is a fixed positive integer and the space $\mathbb{P}_\ell(T)$ consists of polynomials of degree at most ℓ .

The Galerkin approximation of problem (6) reads now: Find $u_h \in V_h$, such that

$$B(u_h, v_h) = \int_{\Omega} f v_h \quad \forall v_h \in V_h. \quad (9)$$

Since there is no reason that the bilinear form would be coercive on V_h , as in [6] we need to use a discrete mapping \mathbb{T}_h from V_h into itself defined by (see [6])

$$\mathbb{T}_h v_h = \begin{cases} v_{h+} & \text{in } \Omega_+, \\ -v_{h-} + 2\mathcal{R}_h(v_{h+|\Sigma}) & \text{in } \Omega_-, \end{cases}$$

where \mathcal{R}_h is a discrete version of the operator \mathcal{R} . Here contrary to [6] and in order to avoid the use of quasi-uniform meshes (meaningless in an a posteriori error analysis), we take

$$\mathcal{R}_h = I_h \mathcal{R}, \quad (10)$$

where I_h is a sort of Clément interpolation operator [9] and \mathcal{R} is any trace lifting operator from $H_{00}^{1/2}(\Sigma)$ into $H_-^1(\Omega_-)$ (see the previous section). More precisely for $\varphi_h \in H_h(\Sigma) = \{v_{h|\Sigma} : v_h \in V_h\}$, we set

$$I_h \mathcal{R}(\varphi_h) = \sum_{x \in \mathcal{N}_-} \alpha_x \lambda_x,$$

where $\mathcal{N}_- = \mathcal{N}_{int} \cap \bar{\Omega}_-$, λ_x is the standard hat function (defined by $\lambda_x \in V_h$ and satisfying $\lambda_x(y) = \delta_{xy}$) and $\alpha_x \in \mathbb{R}$ are defined by

$$\alpha_x = \begin{cases} |\omega_x|^{-1} \int_{\omega_x} \mathcal{R}(\varphi_h) & \text{if } x \in \mathcal{N}_{int} \cap \Omega_-, \\ \varphi_h(x) & \text{if } x \in \mathcal{N}_{int} \cap \Sigma, \end{cases}$$

where we recall that ω_x is the patch associated with x , which is simply the support of λ_x . Note that I_h coincides with the Clément interpolation operator I_{Cl} for the nodes in Ω_- and only differs on the nodes on Σ . Indeed let us recall the definition of $I_{Cl} \mathcal{R}(\varphi_h)$ (defined in a Scott-Zhang manner [20] for the points belonging to Σ):

$$I_{Cl} \mathcal{R}(\varphi_h) = \sum_{x \in \mathcal{N}_-} \beta_x \lambda_x$$

with

$$\beta_x = \begin{cases} |\omega_x|^{-1} \int_{\omega_x} \mathcal{R}(\varphi_h) & \text{if } x \in \mathcal{N}_{int} \cap \Omega_-, \\ |e_x|^{-1} \int_{e_x} \mathcal{R}(\varphi_h) d\sigma & \text{if } x \in \mathcal{N}_{int} \cap \Sigma \text{ with } e_x = \omega_x \cap \Sigma. \end{cases}$$

The definition of I_h aims at ensuring that

$$I_h \mathcal{R}(\varphi_h) = \varphi_h \text{ on } \Sigma.$$

Let us now prove that \mathcal{R}_h is uniformly bounded.

Theorem 3.1 For all $h > 0$ and $\varphi_h \in H_h(\Sigma)$, one has

$$|\mathcal{R}_h(\varphi_h)|_{1,\Omega_-} \lesssim \|\varphi_h\|_{1/2,-}.$$

Proof: For the sake of simplicity we make the proof in the case $\ell = 1$, the general case is treated in the same manner by using modified Clément interpolation operator. Since \mathcal{R} is bounded from $H_{00}^{1/2}(\Sigma)$ into $H_-^1(\Omega_-)$, one has

$$|\mathcal{R}(\varphi_h)|_{1,\Omega_-} \lesssim \|\varphi_h\|_{1/2,-}. \quad (11)$$

Hence it suffices to show that

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1,\Omega_-} \lesssim \|\varphi_h\|_{1/2,-}. \quad (12)$$

For that purpose, we distinguish the triangles T that have no nodes in $\mathcal{N}_{int} \cap \Sigma$ to the other ones:

1. If T has no nodes in $\mathcal{N}_{int} \cap \Sigma$, then $I_h\mathcal{R}(\varphi_h)$ coincides with $I_{Cl}\mathcal{R}(\varphi_h)$ on T and therefore by a standard property of the Clément interpolation operator, we have

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1,T} = |(I - I_{Cl})\mathcal{R}(\varphi_h)|_{1,T} \lesssim \|\mathcal{R}(\varphi_h)\|_{1,\omega_T}, \quad (13)$$

where the patch ω_T is given by $\omega_T = \bigcup_{T' \cap T \neq \emptyset} T'$.

2. If T has at least one node in $\mathcal{N}_{int} \cap \Sigma$, by the triangle inequality we may write

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1,T} \leq |(I - I_{Cl})\mathcal{R}(\varphi_h)|_{1,T} + |(I_{Cl} - I_h)\mathcal{R}(\varphi_h)|_{1,T}.$$

For the first term of this right-hand side we can still use (13) and therefore it remains to estimate the second term. For that one, we notice that

$$(I_{Cl} - I_h)\mathcal{R}(\varphi_h) = \sum_{x \in T \cap \Sigma} (\alpha_x - \beta_x) \lambda_x \text{ on } T.$$

Hence

$$|(I_{Cl} - I_h)\mathcal{R}(\varphi_h)|_{1,T} \lesssim \sum_{x \in T \cap \Sigma} |\alpha_x - \beta_x|.$$

Since $\mathcal{R}(\varphi_h) = \varphi_h$ on Σ and due to the definition of I_{Cl} , it follows that for $x \in T \cap \Sigma$,

$$|\alpha_x - \beta_x| = \left| \varphi_h(x) - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma \right|.$$

Since all norms are equivalent in finite dimensional spaces, we have for all $v_h \in \mathbb{P}_1(e_x)$,

$$|v_h(x)| \lesssim |e_x|^{-1/2} \|v_h\|_{e_x}. \quad (14)$$

Moreover,

$$|e_x|^{-1/2} \left\| \varphi_h - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma \right\|_{e_x} \lesssim |\varphi_h|_{1/2, e_x}, \quad (15)$$

where here $|\cdot|_{1/2, e_x}$ means the standard $H^{1/2}(e_x)$ -seminorm. Thus Inequalities (14) with $v_h = \varphi_h - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma$ and (15) imply that

$$|\alpha_x - \beta_x| \lesssim |\varphi_h|_{1/2, e_x}.$$

All together we have shown that

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1, T} \lesssim \|\mathcal{R}(\varphi_h)\|_{1, \omega_T \cap \bar{\Omega}_-} + |\varphi_h|_{1/2, \omega_T \cap \Sigma}. \quad (16)$$

Taking the sum of the square of (13) and of (16), we obtain that

$$|(I - I_h)\mathcal{R}(\varphi_h)|_{1, \Omega_-}^2 \lesssim \|\mathcal{R}(\varphi_h)\|_{1, \Omega_-}^2 + |\varphi_h|_{1/2, \Sigma}^2.$$

We conclude thanks to (11) and to the fact that

$$|\varphi_h|_{1/2, \Sigma} \lesssim \|\varphi_h\|_{1/2, -}.$$

■

This Theorem and Proposition 4.2 of [6] allow to conclude that (9) has a unique solution provided that (7) holds, in particular if the contrast is large enough.

Note that the advantage of our construction of \mathcal{R}_h is that we no more need the quasi-uniform property of the meshes imposed in [6].

4 The a posteriori error analysis

Error estimators can be constructed in many different ways as, for example, using residual type error estimators which measure locally the jump of the discrete flux [21]. A different method, based on equilibrated fluxes, consists in solving local Neumann boundary value problems [2] or in using Raviart-Thomas interpolant [1, 10, 13, 14]. Here since the coercivity constant is not explicitly known, we chose the simplest approach of residual type.

The residual estimators are denoted by

$$\eta_R^2 = \sum_{T \in \mathcal{T}} \eta_{R, T}^2, \quad \eta_J^2 = \sum_{T \in \mathcal{T}} \eta_{J, T}^2, \quad (17)$$

where the indicators $\eta_{R, T}$ and $\eta_{J, T}$ are defined by

$$\begin{aligned} \eta_{R, T} &= h_T \|f_T + \operatorname{div}(a \nabla u_h)\|_T, \\ \eta_{J, T} &= \sum_{e \in \mathcal{E}_{int}: e \subset T} h_e^{1/2} \|[a \nabla u_h \cdot n_e]\|_e, \end{aligned}$$

when f_T is an approximation of f , for instance

$$f_T = |T|^{-1} \int_T f.$$

Note that $\eta_{R, T}^2$ is meaningful if $a|_T \in W^{1,1}(T)$, for all $T \in \mathcal{T}$.

4.1 Upper bound

Theorem 4.1 *Assume that $a \in L^\infty(\Omega)$ satisfies (1)-(2) and that $a|_T \in W^{1,1}(T)$, for all $T \in \mathcal{T}$. Assume further that (7) holds. Let $u \in H_0^1(\Omega)$ be the unique solution of Problem (6) and let u_h be its Galerkin approximation, i.e. $u_h \in V_h$ a solution of (9). Then one has*

$$\|\nabla(u - u_h)\| \lesssim \eta_R + \eta_J + \text{osc}(f), \quad (18)$$

where

$$\text{osc}(f) = \left(\sum_{T \in \mathcal{T}} h_T^2 \|f - f_T\|^2 \right)^{\frac{1}{2}}.$$

Proof: By the coerciveness assumption (4), we may write

$$\|\nabla(u - u_h)\|^2 \lesssim B(u - u_h, \mathbb{T}(u - u_h)). \quad (19)$$

But we notice that the Galerkin relation

$$B(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

holds. Hence by taking $v_h = I_{Cl}\mathbb{T}(u - u_h)$, (19) may be written

$$\|\nabla(u - u_h)\|^2 \lesssim B(u - u_h, (I - I_{Cl})\mathbb{T}(u - u_h)). \quad (20)$$

Now we apply standard arguments, see for instance [21]. Namely applying element-wise Green's formula and writing for shortness $w = (I - I_{Cl})\mathbb{T}(u - u_h)$, we get

$$\begin{aligned} \|\nabla(u - u_h)\|^2 &\lesssim - \sum_{T \in \mathcal{T}} \int_T \text{div}(a \nabla(u - u_h)) w \\ &\quad + \sum_{e \in \mathcal{E}_{int}} \int_e \llbracket a \nabla(u - u_h) \cdot n \rrbracket w \, d\sigma, \end{aligned}$$

reminding that $w = 0$ on Γ . By Cauchy-Schwarz's inequality we directly obtain

$$\begin{aligned} \|\nabla(u - u_h)\|^2 &\lesssim \sum_{T \in \mathcal{T}} \|f + \text{div}(a \nabla u_h)\|_T \|w\|_T \\ &\quad + \sum_{e \in \mathcal{E}_{int}} \|\llbracket a \nabla u_h \cdot n \rrbracket\|_e \|w\|_e. \end{aligned}$$

By standard interpolation error estimates, we get

$$\begin{aligned} \|\nabla(u - u_h)\|^2 &\lesssim \left(\sum_{T \in \mathcal{T}} h_T^2 \|f + \text{div}(a \nabla u_h)\|_T^2 \right. \\ &\quad \left. + \sum_{e \in \mathcal{E}_{int}} h_e \|\llbracket a \nabla u_h \cdot n \rrbracket\|_e^2 \right)^{1/2} |\mathbb{T}(u - u_h)|_{1,\Omega}. \end{aligned}$$

Since \mathbb{T} is an isomorphism, we conclude that

$$\begin{aligned} \|\nabla(u - u_h)\| &\lesssim \left(\sum_{T \in \mathcal{T}} h_T^2 \|f + \operatorname{div}(a \nabla u_h)\|_T^2 \right. \\ &\quad \left. + \sum_{e \in \mathcal{E}_{int}} h_e \|[a \nabla u_h \cdot n]\|_e^2 \right)^{1/2}. \end{aligned}$$

This leads to the conclusion due to the triangle inequality. \blacksquare

4.2 Lower bound

The lower bound is fully standard since by a careful reading of the proof of Proposition 1.5 of [21], we see that it does not use the positiveness of the diffusion coefficient a . Hence we can state the

Theorem 4.2 *Let the assumptions of Theorems 4.1 be satisfied. Assume furthermore that $a|_T$ is constant for all $T \in \mathcal{T}$. Then for each element $T \in \mathcal{T}$ the following estimate holds*

$$\eta_{R,T} + \eta_{J,T} \lesssim |u - u_h|_{1, \omega_T} + \operatorname{osc}(f, \omega_T),$$

where

$$\operatorname{osc}(f, \omega_T)^2 = \sum_{T' \subset \omega_T} h_{T'}^2 \|f - f'_{T'}\|_{T'}^2.$$

5 Numerical results

5.1 The polynomial solution

In order to illustrate our theoretical predictions, this first numerical test consists in validating our computations on a simple case, using an uniform refinement process. Let Ω be the square $(-1, 1)^2$, $\Omega_+ = (0, 1) \times (-1, 1)$ and $\Omega_- = (-1, 0) \times (-1, 1)$. We assume that $a = 1$ on Ω_+ and $a = \mu < 0$ on Ω_- . In such a situation we can take

$$\mathcal{R}(v_+)(x, y) = v_+(-x, y) \quad \forall (x, y) \in \Omega_-.$$

With this choice we see that

$$K_{\mathcal{R}} = |\mu|,$$

and therefore for $|\mu| < 1$, (4) holds and Problem (6) has a unique solution. We further easily check that the corresponding mapping \mathbb{T} is an isomorphism since $(\mathbb{T})^2 = \mathbb{T}$. Similarly by exchanging the role of Ω_+ and Ω_- , (4) will also hold if $|\mu| > 1$.

Now we take as exact solution

$$\begin{aligned} u(x, y) &= \mu x(x+1)(x-1)(y+1)(y-1) \quad \forall (x, y) \in \Omega_+, \\ u(x, y) &= x(x+1)(x-1)(y+1)(y-1) \quad \forall (x, y) \in \Omega_-, \end{aligned}$$

f being fixed accordingly.

Let us recall that u_h is the finite element solution, and set $e_{L^2}(u_h) = \|u - u_h\|$ and $e_{H^1}(u_h) = \|u - u_h\|_1$ the L^2 and H^1 errors. Moreover let us define $\eta(u_h) = \eta_R + \eta_J$ the estimator and CV_{L^2} (resp. CV_{H^1}) as the experimental convergence rate of the error $e_{L^2}(u_h)$ (resp. $e_{H^1}(u_h)$) with respect to the mesh size defined by $DoF^{-1/2}$, where the number of degrees of freedom is DoF , computed from one line of the table to the following one.

Computations are performed with $\mu = -3$ using a global mesh refinement process from an initial cartesian grid. First, it can be seen from Table 1 that the convergence rate of the H^1 error norm is equal to one, as theoretically expected (see [6]). Furthermore the convergence rate of the L^2 error norm is 2, which is a consequence of the Aubin-Nitsche trick and regularity results for Problem (3). Finally, the reliability of the estimator is ensured since the ratio in the last column (the so-called effectivity index), converges towards a constant close to 6.5.

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	289	2.37E-02		5.33E-01		6.70
2	1089	5.95E-03	2.08	2.67E-01	1.04	6.59
3	4225	1.49E-03	2.04	1.34E-01	1.02	6.53
4	16641	3.73E-04	2.02	6.68E-02	1.01	6.49
5	32761	1.89E-04	2.01	4.75E-02	1.01	6.48
6	90601	6.79E-05	2.01	2.85E-02	1.00	6.47
7	251001	2.45E-05	2.00	1.71E-02	1.00	6.47

Table 1: The polynomial solution with $\mu = -3$ (uniform refinement).

5.2 A singular solution

Here we analyze an example introduced in [7] and precise some results from [7]. The domain $\Omega = (-1, 1)^2$ is decomposed into two sub-domains $\Omega_+ = (0, 1) \times (0, 1)$, and $\Omega_- = \Omega \setminus \bar{\Omega}_+$, see Figure 1. As before we take $a = 1$ on Ω_+ and $a = \mu < 0$ on Ω_- . According to Section 3 of [7], Problem (6) has a singularity S at $(0, 0)$ if $\mu < -3$ or if $\mu \in (-1/3, 0)$ given in polar coordinates by

$$\begin{aligned}
 S_+(r, \theta) &= r^\lambda (c_1 \sin(\lambda\theta) + c_2 \sin(\lambda(\frac{\pi}{2} - \theta))) && \text{for } 0 < \theta < \frac{\pi}{2}, \\
 S_-(r, \theta) &= r^\lambda (d_1 \sin(\lambda(\theta - \frac{\pi}{2})) + d_2 \sin(\lambda(2\pi - \theta))) && \text{for } \frac{\pi}{2} < \theta < 2\pi,
 \end{aligned}$$

where $\lambda \in (0, 1)$ is given by

$$\lambda = \frac{2}{\pi} \arccos \left(\frac{1 - \mu}{2|1 + \mu|} \right),$$

and the constants c_1, c_2, d_1, d_2 are appropriately defined.

Now we show using the arguments of Section 2 that for $-\frac{1}{3} < \mu < 0$ and $\mu < -3$, the assumption (4) holds. As before we define

$$\mathcal{R}(v_+)(x, y) = \begin{cases} v_+(-x, y) & \forall (x, y) \in (-1, 0) \times (0, 1), \\ v_+(-x, -y) & \forall (x, y) \in (-1, 0) \times (-1, 0), \\ v_+(x, -y) & \forall (x, y) \in (0, 1) \times (-1, 0). \end{cases}$$

This extension defines an element of $H_-^1(\Omega_-)$ such that

$$\mathcal{R}(v_+) = v_+ \quad \text{on } \Sigma.$$

Moreover with this choice we have

$$\sup_{\substack{v \in H_+^1(\Omega_+) \\ v \neq 0}} \frac{|B_-(\mathcal{R}(v), \mathcal{R}(v))|}{B_+(v, v)} = 3|\mu|,$$

and therefore for

$$3|\mu| < 1,$$

we deduce that (4) holds.

To exchange the role of Ω_+ and Ω_- we define the following extension from Ω_- to Ω_+ : for $v_- \in H_-^1(\Omega_-)$, let

$$\mathcal{R}(v_-)(x, y) = v_-(-x, y) + v_-(x, -y) - v_-(-x, -y) \quad \forall (x, y) \in \Omega_+.$$

We readily check that it defines an element of $H_+^1(\Omega_+)$ such that

$$\mathcal{R}(v_-) = v_- \quad \text{on } \Sigma.$$

Moreover with this choice we have (using the estimate $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ valid for all real numbers a, b, c)

$$\sup_{v \in H_-^1(\Omega_-), v \neq 0} \frac{B_+(\mathcal{R}(v), \mathcal{R}(v))}{|B_-(v, v)|} \leq 3/|\mu|,$$

and therefore for

$$3/|\mu| < 1,$$

we deduce that (4) holds.

For this second test, we take as exact solution the singular function $u(x, y) = S(x, y)$ for $\mu = -5$ and $\mu = -100$, non-homogeneous Dirichlet boundary conditions on Γ are fixed accordingly. First, with uniform meshes, we obtain the expected convergence rate of order λ (resp. 2λ) for the H^1 (resp. L^2) error norm, see Tables 2 and 3. There, for sufficiently fine meshes, we may notice that the effectivity index varies between 1 and 0.6 for $\mu = -5$ or between 9 and 6 for $\mu = -100$. From these results we can say that the effectivity index

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	289	1.60E-02		2.84E-01		2.57
2	1089	8.66E-03	0.93	2.10E-01	0.45	1.94
3	4225	4.63E-03	0.92	1.55E-01	0.45	1.46
4	16641	2.47E-03	0.92	1.13E-01	0.45	1.09
5	32761	1.80E-03	0.92	9.69E-02	0.46	0.95
6	90601	1.13E-03	0.92	7.68E-02	0.46	0.76
7	251001	7.08E-04	0.92	6.08E-02	0.46	0.61

Table 2: The singular solution, $\mu = -5$, $\lambda \approx 0.46$ (uniform refinement).

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	289	6.12E03		1.54E-01		18.77
2	1089	2.59E-03	1.29	9.91E-02	0.66	15.04
3	4225	1.08E-03	1.29	6.35E-02	0.66	12.06
4	16641	4.46E-04	1.29	4.04E-02	0.66	9.66
5	32761	2.88E-04	1.29	3.24E-02	0.66	8.65
6	90601	1.49E-04	1.30	2.32E-02	0.66	7.33
7	251001	7.66E-05	1.30	1.66E-02	0.66	6.21

Table 3: The singular solution, $\mu = -100$, $\lambda \approx 0.66$ (uniform refinement).

depends on μ , this is confirmed by the numerical results obtained by an adaptive algorithm (see below).

Secondly, an adaptive mesh refinement strategy is used based on the estimator $\eta_T = \eta_{R,T} + \eta_{J,T}$, the marking procedure

$$\eta_T > 0.5 \max_{T'} \eta_{T'}$$

and a standard refinement procedure with a limitation on the minimal angle.

For $\mu = -5$ (resp. $\mu = -100$), Table 4 (resp. 5) displays the same quantitative results as before. There we see that the effectivity index is around 3 (resp. 34), which is quite satisfactory and comparable with results from [11, 14]. As before and in these references we notice that it deteriorates as the contrast becomes larger. On these tables we also remark a convergence order of 0.76 (resp. 1) in the H^1 -norm and mainly the double in the L^2 -norm. This yields better orders of convergence as for uniform meshes as expected, the case $\mu = -5$ giving less accurate results due to the high singular behavior of the solution (a similar phenomenon occurs in [11] for instance).

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	81	2.92E-02		3.79E-01		3.39
5	432	3.49E-03	2.54	1.40E-01	1.19	4.18
7	1672	1.25E-03	1.52	8.04E-02	0.82	4.07
10	5136	4.26E-04	1.92	4.90E-02	0.88	3.63
13	20588	1.64E-04	1.37	3.14E-02	0.64	3.32
18	80793	5.50E-05	1.60	1.80E-02	0.81	3.23
24	272923	2.39E-05	1.37	1.17E-02	0.71	2.5

Table 4: The singular solution, $\mu = -5$, $\lambda \approx 0.46$ (local refinement).

k	DoF	$e_{L^2}(u_h)$	CV_{L^2}	$e_{H^1}(u_h)$	CV_{H^1}	$\frac{\eta(u_h)}{e_{H^1}(u_h)}$
1	81	1.41E-02		2.35E-01		23.59
4	363	1.93E-03	2.65	8.77E-02	1.31	34.86
7	1566	4.94E-04	1.86	4.31E-02	0.97	33.10
11	5981	1.23E-04	2.07	2.15E-02	1.04	33.17
16	25452	2.98E-05	1.96	1.05E-02	0.99	34.65
24	106827	7.36E-06	1.95	5.23E-03	0.97	33.89

Table 5: The singular solution, $\mu = -100$, $\lambda \approx 0.66$ (local refinement).

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