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Safety Stock Optimization in Two-Echelon Assembly Systems: Normal Approximation Models

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Abstract: This paper tackles the problem of optimizing safety stocks in a two-echelon assembly system. It presents and discusses several approximation models for the assembly lead-time under the assumption of normality of the assembly demand and normality of components' nominal lead times. These approximation models are subsequently used to optimize safety stocks throughout a two-echelon assembly system. They are then tested on a particular two-echelon *N*-identical component assembly system. The obtained results are compared with the results of a discrete event simulation. Finally, it is shown that lead-times and safety stock results already obtained for a two-echelon distribution system can also be derived without difficulty from those of two-echelon assembly systems.

Keywords: Safety stocks, two-echelon assembly, decision support systems.

1. Introduction

This paper deals with safety stock optimization in a 2-echelon assembly system. As illustrated in figure 1 below, a typical system that is analyzed consists of *N* independent component-supply processes and a manufacturing process for an assembly made out of these *N* components. In the general case of different component suppliers, lead-times and lot sizes of the components vary from one supplier to another. Companies tend, thus, to build significant safety stocks for these components in order to decouple the assembly operation from the components' supply processes. Also, since the (external) independent customer demand is for assemblies only, companies tend also to build a significant safety stock of assemblies to hedge against uncertainty the external demand

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and variability of the assembly lead-time. The issue now is how to reduce these safety stocks while keeping the same performance level of the system.

Figure 1: Example of assembly system with *N* components

If safety stocks of the components are reduced, this will increase the number of component back-orders as well as the time to resolve them. Therefore, a first question we address is how to characterize the behaviour of each individual component. To answer this question we will build upon the results already developed in Desmet *et al*. (2008). Furthermore, since the assembly process can start only when all components are available, that is after the largest back-order service time, a second question we address is related to the issue of combining stockout effects of these components. From this analysis the "incoming service time" to the assembly operation will be derived. Finally, the problem of optimizing safety stocks in an assembly system is addressed. Clearly, reducing safety stocks of the components increases the "incoming service time" to the assembly operation. Yet, to maintain a fixed service level for the customer safety stock of the assembly should be increased. The key question is thus, which levels of safety stocks of components and assembly will lead to an optimal safety stock situation for the system.

In what follows we consider an assembly system where an assembly is manufactured out of N components with a non-identical supply. Without loss of generality, we assume one unit of each component is required for producing one unit of the assembly. Demand for the assembly is assumed to be normally distributed with an average μ_{d_a} and a variance $\sigma^2_{d_a}$. The assembly stock is controlled by means of a continuous review policy (Q_a, R_a) (where Q_a is the order quantity and R_a is the reorder point). The nominal lead time for the assembly operation, denoted by L_a^0, is assumed to be normally distributed with average $\mu_{L_a^0}$ and variance $\sigma^2_{L_a^0}$. The nominal lead time assumes all components are available in the requested quantities, i.e. multiples of Q_a. In case the requested quantity of one or more components is not available, this quantity is back-ordered for the respective components for the full amount until sufficient components become available. The actual lead time for the assembly operation, which we denote by L_a, includes a waiting time for all components to become available. In what follows we call this resulting waiting time the incoming service time to the assembly operation and is denoted by S_a^I.

The dependent demand for component *i*, denoted by d_i, can be characterized by a binomial distribution. The probability p_i for component demand to occur is given by

$p_i = \frac{\mu_{d_a}}{Q_a}$. The number of trials n_i equals the number of assemblies so $n_i = 1$ in this case. As a result the number of orders o_i for a component i is binomially distributed with average μ_{o_i} and variance $\sigma_{o_i}^2$ given by $\mu_{o_i} = \frac{\mu_{d_a}}{Q_a}$ and $\sigma_{o_i} = \sqrt{\frac{\mu_{d_a}}{Q_a}\left(1 - \frac{\mu_{d_a}}{Q_a}\right)}$. The binomial demand distribution for component i follows by multiplying with the demand size Q_a and leads to $\mu_{d_i} = \mu_{d_a}$ and $\sigma_{d_i} = Q_a\sqrt{\frac{\mu_{d_a}}{Q_a}\left(1 - \frac{\mu_{d_a}}{Q_a}\right)}$. We use the resulting average and variance for a normal approximation of the demand for the component i. The reorder policy for each component i is also assumed to be based on a (Q_i, R_i) policy. In addition, replenishment sources of the components are assumed to have infinite capacities. This implies that the actual replenishment lead times equal the nominal lead times, assumed to be normally distributed with average μ_{L_i} and variance $\sigma_{L_i}^2$. Finally, it is assumed that the system carries safety stocks, both for the components (i) and the assembly (a), that are proportional to target service levels f_i and f_a respectively. In this case volume fill rates are used to measure service levels. Safety stocks take into account uncertainty on demands and uncertainty on the actual replenishment lead times. The following single-echelon safety stock model is used (see Axsäter (2000)):

$$SS_{i,a} = k\left(f_{i,a}, Q_{i,a}, \mu_{d_{i,a}}, \sigma_{d_{i,a}}, \mu_{L_{i,a}}, \sigma_{L_{i,a}}\right) \cdot \sqrt{\mu_{L_{i,a}}\sigma_{d_{i,a}}^2 + \mu_{d_{i,a}}^2\sigma_{L_{i,a}}^2}$$

for the components (i) and the assembly (a), and where the function k is obtained from:

$$f_{i,a} = 1 - \frac{\sqrt{\mu_{L_{i,a}}\sigma_{d_{i,a}}^2 + \mu_{d_{i,a}}^2\sigma_{L_{i,a}}^2}}{Q_{i,a}} E(k)$$

where $E(k) = \varphi_s(k) - k\left[1 - \Phi_s(k)\right]$, $\varphi_s(k)$ and $\Phi_s(k)$ are normal distribution probability density and cumulative probability functions respectively. Using these pieces of information, the system's safety stock can be expressed as:

$$SS_{tot} = \sum_{i=1}^{N} k_i\left(f_i, Q_i, \mu_{d_i}, \sigma_{d_i}, \mu_{L_i}, \sigma_{L_i}\right)\sqrt{\mu_{L_i}\sigma_{d_i}^2 + \mu_{d_i}^2\sigma_{L_i}^2} + k_a\left(f_a, Q_a, \mu_{d_a}, \sigma_{d_a}, \mu_{L_a}, \sigma_{L_a}\right)\sqrt{\mu_{L_a}\sigma_{d_a}^2 + \mu_{d_a}^2\sigma_{L_a}^2} \quad (1)$$

The main goal of our analysis is to model the effect of a reduction in component fill rates f_i on the system's safety stock for a fixed assembly fill rate f_a. As one can observe from equation (1), decreasing the component fill rates leads to reductions in the component safety stocks. However this may increase the average and variance of the incoming service time to the assembly operation. This increases thus the average and

variance of the actual assembly lead time which in turn results in an increase in assembly safety stock, if one wants to maintain a target fill rate for the assembly. The question is now which level of the component fill-rates minimizes the system's safety stock while maintaining the targeted assembly service level.

In the next paragraphs we present a brief literature review and discuss the issues we attempt to address. We then introduce an approximation for the incoming service time to the assembly operation and test our results with a discrete event simulation. This allows us to assess the effect of reducing the component fill rate on the system's safety stock. We conclude this analysis by showing that safety stock optimization results for distribution systems follow as special cases of the results from the assembly system.

2. A Brief Literature Review

In this section, we provide a concise review of some papers addressing safety stock allocation and optimization in multi-stage systems. For recent surveys on multi-echelon production and inventory systems we refer to van Houtum (2006), de Kok and Fransoo (2003), and Axsäter (2003). The specific case of repairables is addressed in Sherbrooke (2004) and in Muckstadt (2005). As van Houtum (2006) reported, optimal policies are only known for simple networks and under specific assumptions, e.g. for an assembly system assuming constant instead of stochastic lead times and base stock policies instead of using a fixed lot size (R,Q) policy. For more realistic supply chains, a large part of the literature deals with heuristics. Because of the complexity in the interactions, simulation is usually used to evaluate the performance of these heuristics and to generate insights on the behaviour of multi-echelon production-distribution inventory networks. Some comprehensive simulation studies were carried by Merkuryev et al. (2009, 2008, 2007), Olhager and Persson (2006), Köchel and Nielander (2005), Swaminathan et al. (1998).

From a much broader perspective, the literature on multi-echelon systems can be classified in three large sets. A first set of papers focuses on optimizing lot sizes. In contrast to the literature addressing safety stock, papers in this set assume constant or deterministic demand. Related influential papers are, among others, Muckstadt and Roundy (1993), Roundy (1985, 1986, 1989); and some pioneering papers are Crowston and Wagner (1973b) and Crowston et al. (1973a). Additional relevant papers include Mohamed et al. (2004), Moreira and Bispo (1999), Modarres and Taimury (1997) which extend the results of Muckstadt and Roundy (1987). A second set of papers focuses on optimizing control policies. The pioneering work in this area is that by Clark and Scarf (1960) for serial systems. For assembly systems, de Kok and Visschers (1999) introduced the so called synchronized base stock policies with service constraints. Recently, amongst

others, Siajadi *et. al.* (2005) analyzed a joint replenishment problem in an assembly system and Van den Broecke *et. al.* (2003) discussed cyclic schedules in a multi-stage multi-product system. A third set of papers, to which this paper contributes, focuses on safety stocks optimization in supply chains. These papers address the key research questions for assembly systems introduced in the previous paragraph: characterizing stockouts for components and combining potential stockouts of the N components going in the assembly. Key papers are Schwarz et al. (1985), Graves and Willems (2000) and Ettl et al. (2000).

Schwarz et al. (1985) build on Deuermeyer and Schwarz (1981). They characterize the expected value of the service time for back-orders in the central warehouse of a distribution system. For that they assume Poisson distributed demand to the warehouse, a constant lead time and a (Q,R) policy for reordering. They derive an expression for the average number of back-orders and then use the Theorem of Little to derive the average service time for back-orders. Their work has not been extended to assembly systems. In this paper we assume normally distributed demand and lead times. We also consider a normal approximation for service times, instead of only the average.

Graves and Willems (2000) discuss safety stock optimization in spanning tree assembly systems. Their model assumes a bounded demand, a base stock policy with common review cycle for reordering, and constant nominal lead times. The bounded demand assumption makes sure that demand will not exceed a given demand upper bound. In extending a result of Simpson (1958), these assumptions allow deriving an extremum characteristic for the outgoing service time of a component, defined as the time between receipt of an order for a component and its actual availability. The optimal outgoing service time equals either the nominal replenishment lead time for the component or zero, corresponding with a safety stock respectively equal to zero or equal to the demand bound. Thus, under the above assumptions, the outgoing service times are constant. As a result, for Graves and Willems (2000), the incoming service time to the assembly operation simply follows as the maximum of the outgoing service times of the components. This bounded demand assumption, as report in Graves and Willems (2003), is delicate. It explicitly ignores a part of the demand variability that companies actually face and must handle in practice. Also, base stock policies with common review cycle ignore the batching requirement that may occur in practice. In this paper the bounded demand assumption is relaxed, (R, Q) policies are allowed, and lead times are stochastic. This of course complicates the characterisation of the incoming service time as it becomes a maximum of distributions instead of the maximum of constant variables.

Ettl *et. al.* (2000), building upon Lee and Billington (1993), also assume base stock policies, with the same constraint for modelling real life supply chains. They do not

assume bounded demand and they allow for stochastic lead times but with known distribution. They approximate the outgoing service time distribution for a component as a proportion of the nominal lead time distribution. For the proportion they derive an upper bound, on the average outgoing service time, making some assumptions such as a lot size of 1 and M/M/∞ queue. For the incoming service time to the assembly operation, Ettl *et. al.* assume there are no concurrent stockouts of components; that is if an assembly order must wait for material it because of the shortage of exactly one component. This assumption reduces the computation of the incoming service time distribution to the weighted sum of the nominal lead time distribution and the convolution of the nominal and the outgoing service time distribution for each component. This same assumption implies minimal service levels and safety stock levels for the components, which limits the optimization potential, as illustrated in Graves and Willems (2003). In this paper, we allow for concurrent stockouts resulting in 2^N possible combinations. We will derive approximations that allow for treating assembly systems with a large number of inputs, while allowing concurrent stockouts and avoiding minimum service level constraints

3. Approximation Models for Safety Stock Optimization

In the following two subsections, we first develop an approximation model for the service time of back-orders for a subset A of components. We then incorporate this in an approximation of the actual assembly lead time using a normal distribution.

3.1 Normal Approximation for Back-Order Service Time of a Subset of Components

Consider an assembly system (such as on the figure 1), and consider a subset A of components experiencing a concurrent stockout. We denote the cardinality of A by m, with $0 \le m \le N$. We assume that the assembly operation can start only when all of the m inputs are available, or likewise, after the maximum of the backorder service times of the m components of A. We call the resulting service time the "backorder service time of the subset A" and we denote it by S_A^B. Desmet *et. al.* (2008) characterized the backorder service time of a component i using an exponential distribution with expected value $\mu_{S_i^B}$ following through numerical integration from the following expression:

$$\mu_{S_i^B} = \frac{\int_{-\infty}^{+\infty} \frac{\sigma_{L_i}.E\left(\frac{t-\mu_{L_i}}{\sigma_{L_i}}\right)}{1-\Phi\left(t;\mu_{L_i};\sigma_{L_i}^2\right)}.\left[1-\Phi\left(R_i;t.\mu_{d_i};t.\sigma_{d_i}^2\right)\right]\left[1-\Phi\left(t;\mu_{L_i};\sigma_{L_i}^2\right)\right].dt}{\int_{-\infty}^{+\infty}\left[1-\Phi\left(R_i;t.\mu_{d_i};t.\sigma_{d_i}^2\right)\right]\left[1-\Phi\left(t;\mu_{L_i};\sigma_{L_i}^2\right)\right].dt}$$

The backorder service time of the subset A then follows as the maximum distribution of m exponentially distributed variables. In the appendix we show that the *p.d.f.* and *c.d.f.* of the maximum distribution of m independent exponentially distributed variables are given by:

$$\begin{cases} f(y) = \sum_i e\left(y,\lambda_{x_i}\right)\prod_{j\neq i} E\left(y,\lambda_{x_j}\right) \\ F(y) = \prod_i E\left(y,\lambda_{x_i}\right) \end{cases}$$

where e and E are the exponential probability and cumulative probability distribution functions respectively. This can be applied to the backorder service time of the subset A with $\lambda_{x_i} = 1/\mu_{S_i^B}$.

We can conservatively approximate this exact distribution by the m-identical distribution with $\lambda_x = \frac{1}{\max\left\{\mu_{S_i^B} \middle| i \in A\right\}}$, for which the above expressions simplify to

$$\begin{cases} f(y) = me\left(y, \frac{1}{\max\left\{\mu_{S_i^B} \middle| i \in A\right\}}\right) E\left(y, \frac{1}{\max\left\{\mu_{S_i^B} \middle| i \in A\right\}}\right)^{m-1} \\ F(y) = E\left(y, \frac{1}{\max\left\{\mu_{S_i^B} \middle| i \in A\right\}}\right)^m \end{cases}.$$

From these expressions we can calculate the average and variance as:

$$\begin{cases} \mu_{S_A^B} = \int_{-\infty}^{+\infty} y\, me\left(y, \frac{1}{\max\left\{\mu_{S_i^B} \middle| i \in A\right\}}\right) E\left(y, \frac{1}{\max\left\{\mu_{S_i^B} \middle| i \in A\right\}}\right)^{m-1} dy \\ \sigma^2_{S_A^B} = \int_{-\infty}^{+\infty} \left(y - \mu_{S_A^B}\right)^2 me\left(y, \frac{1}{\max\left\{\mu_{S_i^B} \middle| i \in A\right\}}\right) E\left(y, \frac{1}{\max\left\{\mu_{S_i^B} \middle| i \in A\right\}}\right)^{m-1} dy \end{cases} \quad (2)$$

which we finally use for a normal approximation of the backorder service time of the subset *A*. This equation (2) can be evaluated numerically. In the actual evaluation of the approximation model, the numerical integration is carried out over the interval $\left[0, \max\left\{\mu_{L_i} \middle| i \in A\right\} + 6\max\left\{\sigma_{L_i} \middle| i \in A\right\}\right]$, using an iterative algorithm of Romberg (Ueberhuber, 1997).

3.2 Normal Approximations for Actual Assembly Lead-Time

The incoming service time to the assembly process, denoted by S_a^I, is found in weighing the stockout effect of all subsets A with the probability that the concurrent stockout of the components in this subset occurs. If f_j is the order line based service level for stock point *j*, the probability that exactly and only the components in subset A experience a stockout is given by $\prod_{j \notin A} f_j \prod_{j \in A} \left(1 - f_j\right)$. As is done in Desmet *et. al.* (2008) we approximate the order line fill rate by the volume fill rate. The probability distribution function of S_a^I follows as then:

$$f(s) = \sum_A \prod_{j \notin A} f_j \prod_{j \in A} \left(1 - f_j\right) \varphi\left(s; \mu_{S_B^A}; \sigma^2_{S_B^A}\right)$$

This resulting probability distribution function is generally not normally distributed. We can again approximate it by a normal distribution with as average the probability weighted averages and as variance the probability weighted variances. That is

$$\begin{cases} \mu_{S_a^I} = \sum_A \prod_{j \notin A} f_j \prod_{j \in A} \left(1 - f_j\right) \mu_{S_A^B} \\ \sigma^2_{S_a^I} = \sum_A \prod_{j \notin A} f_j \prod_{j \in A} \left(1 - f_j\right) \sigma^2_{S_A^B} \end{cases} \quad (3)$$

where $\mu_{S_A^B}$ and $\sigma^2_{S_A^B}$ are obtained from equation (2).

Unfortunately even with these normal approximations, equation (3) is still computationally cumbersome, especially when a large number of components is

involved. The expressions have an exponential combinatorial complexity which we can alleviate with further approximations.

First, we can observe from equation (2) that if $A \subseteq A'$ this leads to $\mu_{S_A^B} \leq \mu_{S_{A'}^B}$, we can thus obtain a first rough approximation, with a scalar combinatorial complexity, to equations (3) as:

$$\begin{cases} \mu_{S_a^I} = (1 - f_1 \ldots f_N)\mu_{S^B_{\{1,\ldots,N\}}} \\ \sigma^2_{S_a^I} = (1 - f_1 \ldots f_N)\sigma^2_{S^B_{\{1,\ldots,N\}}} \end{cases} \tag{4}$$

A second and finer approximation to equation (3) can be obtained, by relabeling the components i so that $\mu_{S_i^B} \leq \mu_{S_j^B}$ for $i \leq j$ and defining $A_i = \{1,\ldots,i\}$, as:

$$\begin{aligned} \mu_{S_a^I} &= \sum_A \prod_{j \notin A} f_j \prod_{j \in A} (1 - f_j)\mu_{S_A^B} \\ &\leq (1 - f_1) f_2 \ldots f_N \mu_{S^B_{A_1}} + f_1 (1 - f_2) f_3 \ldots f_N \mu_{S^B_{A_2}} + \ldots \\ &\qquad + (1 - f_1)(1 - f_2) f_3 \ldots f_N \mu_{S^B_{A_2}} + \ldots \\ &= (1 - f_1) f_2 \ldots f_N \mu_{S^B_{A_1}} + (1 - f_2) f_3 \ldots f_N \mu_{S^B_{A_2}} + (1 - f_3) f_4 \ldots f_N \mu_{S^B_{A_3}} + \ldots + (1 - f_N)\mu_{S^B_{A_N}} \end{aligned}$$

The equations for the variance follow in the same way as above. Notice that this second approximation to equation (3), shown below, has a linear combinatorial complexity:

$$\begin{cases} \mu_{S_a^I} = (1 - f_1) f_2 \ldots f_N \mu_{S^B_{A_1}} + (1 - f_2) f_3 \ldots f_N \mu_{S^B_{A_2}} + (1 - f_3) f_4 \ldots f_N \mu_{S^B_{A_3}} + \ldots + (1 - f_N)\mu_{S^B_{A_N}} \\ \sigma^2_{S_a^I} = (1 - f_1) f_2 \ldots f_N \sigma^2_{S^B_{A_1}} + (1 - f_2) f_3 \ldots f_N \sigma^2_{S^B_{A_2}} + (1 - f_3) f_4 \ldots f_N \sigma^2_{S^B_{A_3}} + \ldots + (1 - f_N)\sigma^2_{S^B_{A_N}} \end{cases} \tag{5}$$

Finally we approximate the actual replenishment lead time of the assembly operation by a normal distribution with as average and variance the sum of the averages and variances of the incoming service time and the nominal assembly time:

$$\begin{cases} \mu_{L_a} = \mu_{S_a^I} + \mu_{L_a^0} \\ \sigma^2_{L_a} = \sigma^2_{S_a^I} + \sigma^2_{L_a^0} \end{cases} \tag{6}$$

3.3 Normal Approximations for Safety Stock Optimization

As already shown in the first paragraph, the system's safety stock can be expressed as:

$$SS_{tot} = \sum_{i=1}^{N} k_i\left(f_i, Q_i, \mu_{d_i}, \sigma_{d_i}, \mu_{L_i}, \sigma_{L_i}\right)\sqrt{\mu_{L_i}\sigma_{d_i}^2 + \mu_{d_i}^2\sigma_{L_i}^2} + k_a\left(f_a, Q_a, \mu_{d_a}, \sigma_{d_a}, \mu_{L_a}, \sigma_{L_a}\right)\sqrt{\mu_{L_a}\sigma_{d_a}^2 + \mu_{d_a}^2\sigma_{L_a}^2} \quad (1)$$

In this model the parameters Q_i, Q_a, μ_{L_i}, σ_{L_i}, f_a μ_{d_a}, σ_{d_a}, respectively the component and assembly lot-sizes, the average and standard deviation of the components lead-times, the assembly service level, and the average and standard deviation of the assembly demand are known. The parameters μ_{d_i} and σ_{d_i} are given by $\mu_{d_i} = \mu_{d_a}$ and $\sigma_{d_i} = Q_a\sqrt{\frac{\mu_{d_a}}{Q_a}\left(1-\frac{\mu_{d_a}}{Q_a}\right)}$, as explained in the first paragraph. The component service levels f_i are the variables which must be optimized. The parameters μ_{L_a} and σ_{L_a}, which depend of the variables f_i, are approximated with the normal approximation models discussed in the above two subparagraphs.

4. An Illustrative Example: Comparison with Simulation

In this paragraph, results of the model are compared with the results obtained from discrete event simulation applied to a 10-identical components assembly system. We assume that demand for assemblies is normally distributed with an average of 100 units/day and a standard deviation of 30 units/day. The nominal assembly lead time is assumed to be 5 days with a standard deviation of 1 day. The assembly lot size is 500 pieces, which reflects a traditional batch driven production environment. The target fill rate for assemblies is assumed to be 95%.

Supply lead times for components are assumed to be on average 30 days, with a standard deviation of 6 days. This could reflect component sourcing from Asia. Demand for components is assumed to be normal with an average of 100 and a standard deviation of 200, calculated using the binomial expression $\sigma_{d_i} = Q_a\sqrt{\frac{\mu_{d_a}}{Q_a}\left(1-\frac{\mu_{d_a}}{Q_a}\right)}$ derived in paragraph 1. Component supply is done per 1500 pieces. A common situation for companies is to have a high service level on components to assure a high availability for the assembly operation, say somewhere between 95% and 99%. In what follows we analyze the sensitivity of the component service level on the incoming service time, the actual assembly time and the system safety stock. We compare simulation results (sim) with all three approximations with exponential (exp), linear (lin) and scalar (scal) combinatorial complexity calculated using the equations (3), (5) and (4) respectively.

Figure 2 and 3 compare the analytical results of respectively the average and the standard deviation of the incoming service time to the assembly operation with the corresponding simulation results.

Figure 2. Calculation and simulation results for the average of the incoming service time to the assembly operation

Figure 3. Calculation and simulation results for the standard deviation of the incoming service time to the assembly operation

Our approximations prove to be conservative. For the average, the approximation with exponential combinatorial complexity seems to perform significantly better over those with linear and scalar combinatorial complexities. The difference is not significant for the standard deviation. The approximations with linear and scalar combinatorial complexity are close to each other. One may suspect that this difference would be bigger for a non-identical component system.

The resulting upstream service level as measured in the simulation does not follow the analytically modelled steps of 1.5%. Rather the simulation results seem to cluster around discrete levels of service around 77%, 92%, 94% and 99%. This is due to the discrete character of the demand for components. Using the binomial distribution for demand, derived in the first paragraph, may probably improve the estimate of the service levels but this complicates safety stock calculations.

Figure 4 and 5 compare the analytical results of the average and the standard deviation of the actual assembly lead time with the corresponding simulation results. This time we only show the approximation with exponential combinatorial complexity.

Figure 4 Calculation and simulation results for the average of the actual assembly lead time

Figure 5. Calculation and simulation results for the standard deviation of the actual assembly lead time

Conclusions are comparable to those of the incoming service time. For expected service levels of $(84{,}5\%;85\%;86{,}5\%)$ the service measured in the simulation is $(79\%;78\%;78\%)$, the approximation model shows an average actual assembly lead time of $(12{,}38;11{,}79;11{,}17)$ versus simulation results of $(9{,}14;9{,}00;9{,}34)$. For zero component safety stocks (expected service level of 84,5%) this implies we overestimate the average of the actual lead times with around 30%, for the standard deviation that runs up to 40%.

The following graph (figure 6) shows the evolution of the system's safety stock for changing component service levels shown on the x-axis, and a fixed service level of 95% on assemblies. Safety stocks are calculated using equation (1) for the analytical results of the approximation with exponential combinatorial complexity and the corresponding simulation results.

Figure 6. System safety stock as a function of the component service level

It is clear from figure (6) that the approximation model leads to a conservative estimate of the required assembly safety stocks. Second, the impact of overestimating the average and standard deviation is significantly reduced in the safety stock sensitivity analysis. This is due to the dominance of the component safety stocks in number of units, which is exogenous to the model. Taking into account the value added in the assembly process will increase the importance of the assembly safety stocks.

From a practical perspective, figure (6) indicates that it is better not to hold safety stock for components. Taking into account the value added in the assembly process will act as a counterforce to holding all safety stock in assemblies. Another factor that will act as a counterforce is a penalty cost for not being able to start the assembly. From the theory of constraints we learn that an hour lost in a bottleneck is an hour lost for the system. In case the assembly is a bottleneck, safety stock may be required to reduce the required flexibility in planning otherwise. However, as in the distribution case discussed in Desmet *et. al.* (2008), we can expect the above result will be much more robust than many practitioners are ready to consider. In the end it is the basic supply chain principle of "risk pooling" that would logically predict it is more favourable to regroup risks in 1 stock point, the assembly, rather than spreading it over 11 stock points, the single assembly plus the 10 components.

5. Distribution System as Special Case of the Assembly System

A distribution system with one warehouse and *N* retailers, can be decomposed into *N* assembly systems where the assembly processes are the distribution processes to the *N* retailers. Each of those "assembly" processes requires a single component as input, namely the finished product in the warehouse. When applying the reasoning of paragraph 4 to this situation our set of components $\{1,...,N\}$ becomes a singleton $\{a\}$. There are thus two possible subsets to consider, namely the empty set $\varnothing$ and the singleton $\{a\}$ itself.

Using $\mu_{S_{\varnothing}^{B}} = 0$ and $\mu_{S_{\{a\}}^{B}} = \mu_{S_a^B}$ from equation (2), we can rework equation (3) for the average and present it as

$$\mu_{S_a^I} = \prod_{j\notin\varnothing} f_j \prod_{j\in\varnothing}(1-f_j)\mu_{S_{\varnothing}^{B}} + \prod_{j\notin\{a\}} f_j \prod_{j\in\{a\}}(1-f_j)\mu_{S_{\{a\}}^{B}} = (1-f_a)\mu_{S_a^B}$$

.

A comparable calculation for the variance and substitution in equation (6) leads to

$$\begin{cases} \mu_{L_a} = \mu_{S_a^B} + \mu_{L_a^0} \\ \sigma^2_{L_a} = \sigma^2_{S_a^B} + \sigma^2_{L_a^0} \end{cases}$$

which is the same result derived in Desmet *et. al.* (2008) for a distribution system.

A comparable way of thinking on equations (5) and (4), leads to the same result for the approximations with linear and scalar combinatorial complexity. This result is important as it allows extending the developed approach for assembly systems to the spanning tree networks and more generic networks considered by Graves and Willems (2000) and Ettl et al. (2000). We plan to address this in future research and publications.

6. Concluding remarks

In this paper, we developed a normal approximation model for the actual assembly lead time in a 2-echelon assembly system with *N* components. In a first step we developed a normal approximation model for the service time resulting from the concurrent stockout of a given subset of components. This service time is obtained as the maximum distribution of the back-order service times of the individual components in the subset. The incoming service time to the assembly operation is obtained as the weighted sum of the service times resulting from all component subsets. The weight of each subset is the probability of concurrent stockout of the components in this subset. The exact

distribution of this service time is then approximated by a normal distribution. The actual assembly lead time is obtained as the nominal lead time augmented with this normally approximated incoming service time. This actual assembly lead time expression is used in the safety stock optimization of assembly systems. These normal approximations are tested on an illustrative example involving 10-identical components and compared with the results of a discrete event simulation. The comparison shows that these normal approximations are rather conservative but provide good estimates for the optimal safety stocks. Finally we showed that the safety stock optimization results obtained for distribution systems by Desmet *et. al.* (2008), follow as particular cases of the results for assembly systems developed in this paper. This fact is important, as it is a preliminary step to extending the approach developed for assembly systems to the spanning tree networks and then to more generic networks as tackled in Graves and Willems (2000) and Ettl *et. al.* (2000). We are currently investigating these important extensions.

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Appendix

Distribution of the maximum of n independent exponentially distributed variables:

Let $x_1, \ldots x_n$ be n independent exponentially distributed variables

$$\begin{cases} x_1 & E(\lambda_1) \\ \ldots & \\ x_n & E(\lambda_n) \end{cases}$$

and define y as the maximum of a sample $(x_1, \ldots, x_n)$.

$$y = \max(x_1, \ldots, x_n)$$

The cumulative distribution function F for y can be derived as

$$\begin{aligned} & F(y_0) \\ &= P(y \le y_0) \\ &= P(\max(x_1, \ldots, x_n) \le y_0) \\ &= P((x_1 \le y_0) \wedge \ldots \wedge (x_n \le y_0)) \\ &= P(x_1 \le y_0) \ldots . P(x_n \le y_0) \\ &= E(y_0, \lambda_{x_1}) \ldots . E(y_0, \lambda_{x_n}) \end{aligned}$$

where E is the cumulative exponential distribution function with rate λ. The probability distribution function f for y follows in taking the first derivative

$$\begin{aligned} & f(y) \\ &= \frac{d}{dy} F(y) \\ &= \sum_{i=1}^{n} \left[e(y, \lambda_i) \prod_{j \ne i} E(y, \lambda_j) \right] \end{aligned}$$

where e is the exponential probability distribution function with rate λ.

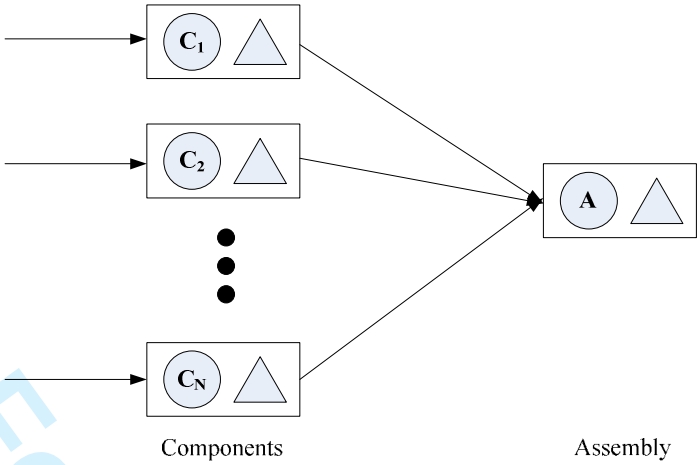

Figure 1: Example of assembly system with *N* components

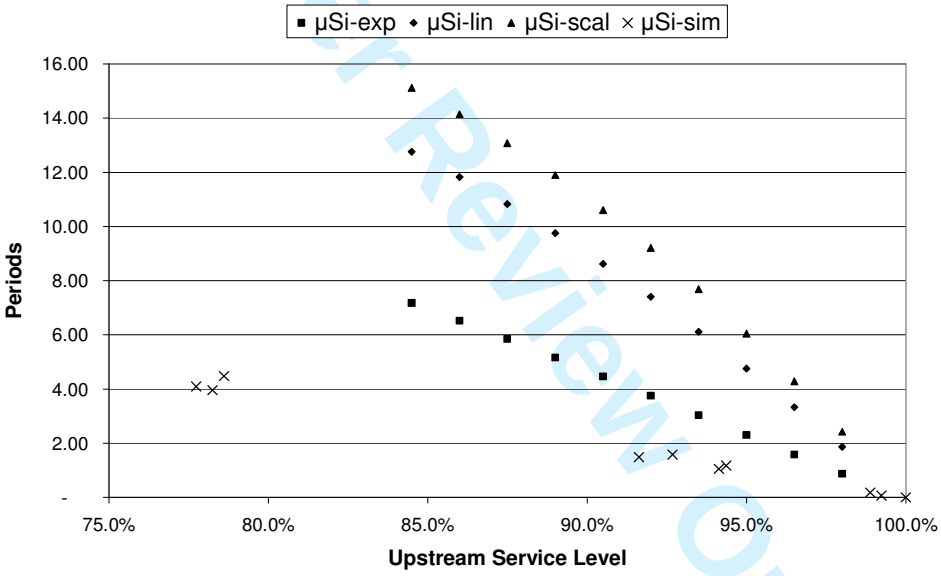

Figure 2. Calculation and simulation results for the average of the incoming service time to the assembly operation

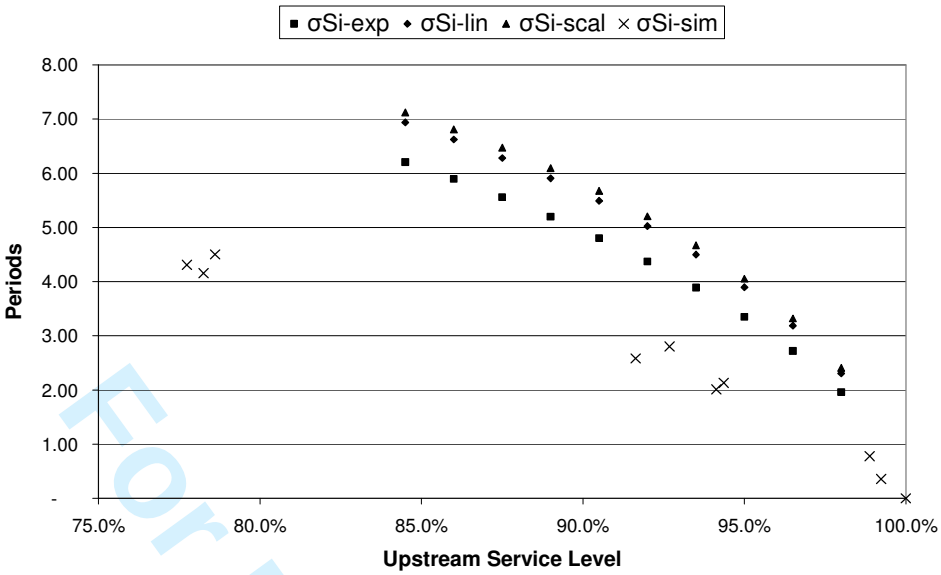

Figure 3. Calculation and simulation results for the standard deviation of the incoming service time to the assembly operation

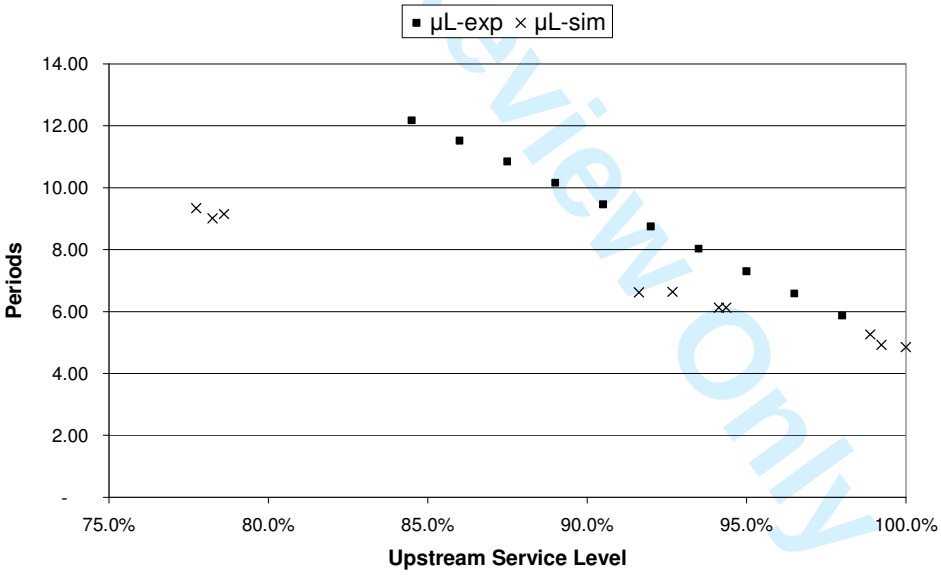

Figure 4 Calculation and simulation results for the average of the actual assembly lead time

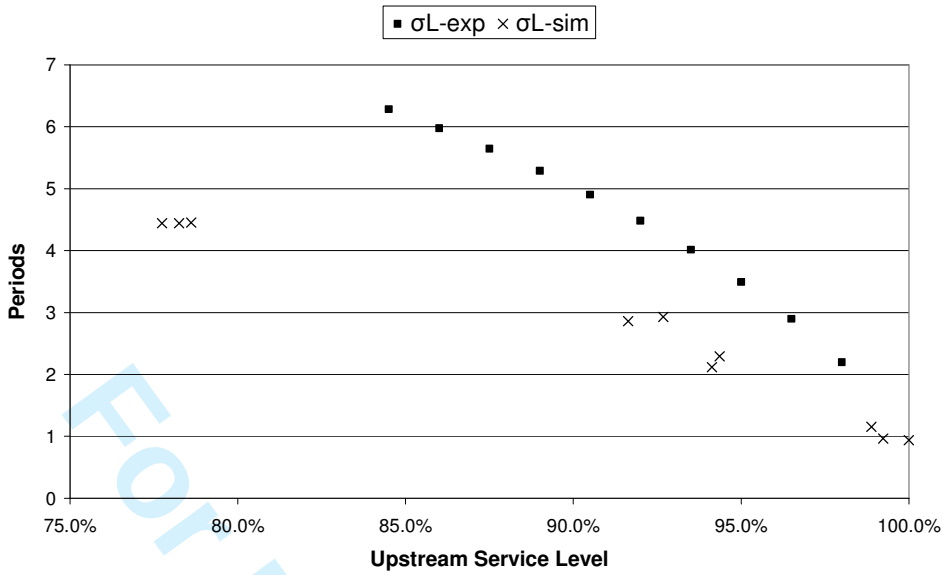

Figure 5. Calculation and simulation results for the standard deviation of the actual assembly lead time

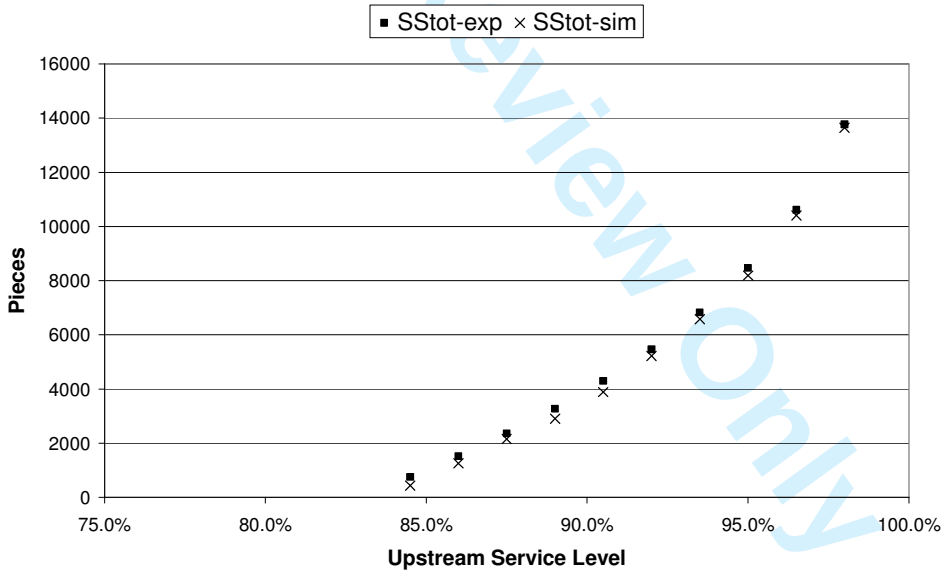

Figure 6. System safety stock as a function of the component service level