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From logarithmic to subdiffusive polynomial fluctuations for internal DLA and related growth models*

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Alexandre Gaudillièrè²

Dedicated to Joel Lebowitz, for his 80th birthday.

Abstract

We consider a family of cluster growth models on \mathbb{Z}^d , which generalize internal diffusion limited aggregation (internal DLA). The latter model is simplest to describe: random walks start at the origin, one at a time, and stop moving when reaching a site not occupied by previous walks. In this case, it is known that the asymptotic shape of the cluster is a sphere, and when dimension is 2 or more, the fluctuations of its radius are at most of order of the radius to the power $1/3$.

For our cluster growth models, we obtain upper and lower bounds for the inner and outer deviation probabilities from the asymptotic shape. Since our growth models are all coupled to internal DLA, we obtain that fluctuations for internal DLA are at most a power of $\log(n)$ in dimension $d \geq 2$.

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1 Introduction

We introduce a family of cluster growth models for which a control of the fluctuations of the cluster shape is easily obtained. These growth models are built so that the asymptotic shape is spherical, but still they exhibit a large diversity of fluctuations parametrized by a certain width ranging from a large constant to a power $1/3$ of the radius of the asymptotic shape. Moreover, all these clusters are coupled to internal DLA, and, as a consequence, we obtain logarithmic bounds on the fluctuations for internal DLA.

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The internal DLA cluster of volume N , say $A(N)$, is obtained inductively as follows. Initially, we assume that the explored region is empty, that is $A(0) = \emptyset$. Then, consider N independent discrete-time random walks S_1, \dots, S_N starting from 0. Assume $A(k-1)$ is obtained, and define

$$\tau_k = \inf \{t \geq 0 : S_k(t) \notin A(k-1)\}, \quad \text{and} \quad A(k) = A(k-1) \cup \{S_k(\tau_k)\}. \quad (1.1)$$

In such a particle system, we call explorers the particles. We say that the k -th explorer is *settled* on $S_k(\tau_k)$ after time τ_k , and is *unsettled* before time τ_k . The cluster $A(N)$ is the positions of the N settled explorers.

The mathematical model of internal DLA was introduced first in the chemical physics literature by Meakin and Deutch [10]. There are many industrial processes that look like internal DLA (see the nice review paper [4]). The most important seems to be electropolishing, defined as *the improvement of surface finish of a metal effected by making it anodic in an appropriate solution*. There are actually two distinct industrial processes (i) *anodic levelling or smoothing* which corresponds to the elimination of surface roughness of height larger than 1 micron, and (ii) *anodic brightening* which refers to elimination of surface defects which are protruding by less than 1 micron. The latter phenomenon requires an understanding of atom removal from a crystal lattice. It was noted in [10], at a qualitative level, that the model produces smooth clusters, and the authors wrote “it is also of some fundamental significance to know just how smooth a surface formed by diffusion limited processes may be”.

Diaconis and Fulton [1] introduced internal DLA in mathematics. Their model is more general than ours: explorers can start on distinct sites, and the explored region at time 0 is not necessarily empty. They were interested in defining a random growth process by iterating simple operation. They introduced many variations, and treat, among other things, the special one dimensional case.

In dimensions two and more, Lawler, Bramson and Griffeath [7] prove that in order to cover, without holes, a sphere of radius n , we need about the number of sites of \mathbb{Z}^d contained in this sphere. In other words, the asymptotic shape of the cluster is a sphere. Then, Lawler in [6] shows subdiffusive fluctuations. The latter result is formulated in terms of inner and outer errors, which we now introduce with some notation. We denote with $\|\cdot\|$ the euclidean norm on \mathbb{R}^d . For any x in \mathbb{R}^d and r in \mathbb{R} , set

$$B(x, r) = \{y \in \mathbb{R}^d : \|y - x\| < r\} \quad \text{and} \quad \mathbb{B}(x, r) = B(x, r) \cap \mathbb{Z}^d. \quad (1.2)$$

For $\Lambda \subset \mathbb{Z}^d$, $|\Lambda|$ denotes the number of sites in Λ . The inner error $\delta_I(n)$ is such that

$$n - \delta_I(n) = \sup \{r \geq 0 : \mathbb{B}(0, r) \subset A(|\mathbb{B}(0, n)|)\}. \quad (1.3)$$

Also, the outer error $\delta_O(n)$ is such that

$$n + \delta_O(n) = \inf \{r \geq 0 : A(|\mathbb{B}(0, n)|) \subset \mathbb{B}(0, r)\}. \quad (1.4)$$

The main result of [6] reads as follows.

Theorem 1.1 [Lawler] Assume $d \geq 2$. Then,

$$P(\exists n(\omega) : \forall n \geq n(\omega) \delta_I(n) \leq n^{1/3} \log(n)^2) = 1, \quad (1.5)$$

and

$$P(\exists n(\omega) : \forall n \geq n(\omega) \delta_O(n) \leq n^{1/3} \log(n)^4) = 1. \quad (1.6)$$

Since Lawler's paper, published 15 years ago, no improvement of these estimates was achieved, but it is believed that fluctuations are on a much smaller scale than $n^{1/3}$. Besides, (1.5) are almost sure upper bounds on errors, and no lower bound on the inner or outer error has been established. Computer simulations [11, 2] suggest indeed that fluctuations are logarithmic. In addition, Levine and Peres studied a deterministic analogue of internal DLA, the rotor-router model, introduced by J. Propp [3]. They bound, in [9], the inner error $\delta_I(n)$ by $\log(n)$, and the outer error $\delta_O(n)$ by $n^{1-1/d}$.

We now generalize internal DLA by enabling explorers to settle only at some special times. Thus, for each explorer, say for the i -th, we consider a collection of stopping times (with respect to the i -th trajectory), called $\{\sigma_{i,k}, k \in \mathbb{N}\}$ and

$$\tau_k^* = \inf \{\sigma_{i,k} : S_k(\sigma_{i,k}) \notin A^*(k-1)\}, \quad \text{and} \quad A^*(k) = A^*(k-1) \cup \{S_k(\tau_k^*)\}. \quad (1.7)$$

The internal DLA is recovered as we choose $\sigma_{i,k} = k$ for all $i = 1, \dots, N$ and $k \in \mathbb{N}$. We call $\{\sigma_{i,k}, k \in \mathbb{N}\}$ the *flashing times* associated to the i -th explorer, and $\{S_i(\sigma_{i,k}), k \in \mathbb{N}\}$ its *flashing positions*.

In this paper, we consider stopping times of a special form, linked with the spherical nature of the internal DLA cluster. An illustration with one *flashing explorer's* trajectory is made in Figure 1.

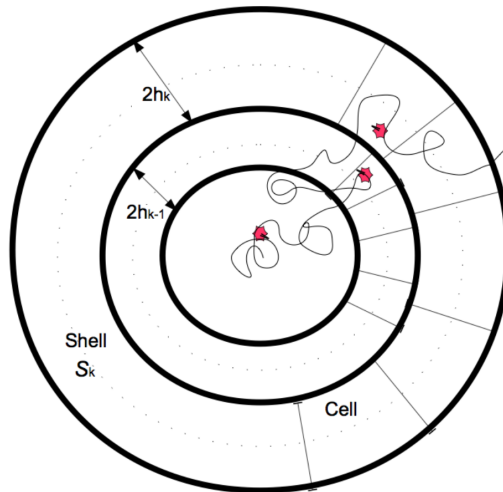


Figure 1: Cell decomposition, and flashing positions as stars.

The precise definition of the *flashing times* requires additional notation, which we postpone to Section 3. We describe here key features of flashing processes. We first choose a

sequence of widths, say $\mathcal{H} = \{h_n, n \in \mathbb{N}\}$, and then partition \mathbb{Z}^d into concentric shells $\{\mathcal{S}_n, n \in \mathbb{N}\}$, whose respective widths are $\{2h_n, n \in \mathbb{N}\}$. Each shell is in turn partitioned into cells, which are brick-like domain, of side length equal to the width of the shell. The flashing times are chosen such that (i) an explorer *flashes* at most once in each shell, (ii) the flashing position, in a shell, is essentially uniform over the cell an explorer first hits upon entering the shell, and (iii) when an explorer leaves a shell, it cannot afterward *flash* in it.

For a given sequence \mathcal{H} , we call the process just described the \mathcal{H} -flashing process. Note that feature (ii) is the seed of a deep difference with internal DLA. *The mechanism of covering a cell, for the flashing process, is very much the same as completing an album in the classical coupon-collector process.* Thus, we need of the order of $V \log(V)$ explorers to cover a cell of volume V . For internal DLA, with explorers started at the origin, we need only of order V explorers to cover a sphere of volume V as shown in [7], and we believe that we need a number of explorers of order $|\mathcal{C}|$ to cover a cell \mathcal{C} . In addition, feature (ii) allows to *localize* the covering mechanism, in the sense that a particle entering a shell cannot flash outside the cell it entered by into that shell. Finally, feature (iii) is essential for having a useful coupling between flashing and internal DLA processes.

Theorem 1.2 *Assume that N is an integer, and \mathcal{H} is a sequence of positive integers. There is a coupling between the two processes such that, for $k \geq 1$,*

- if $A^*(N) \subset \cup_{j < k} \mathcal{S}_j$, then $A(N) \subset \cup_{j < k} \mathcal{S}_j$,
- if $\cup_{j < k} \mathcal{S}_j \subset A^*(N)$, then $\cup_{j < k} \mathcal{S}_j \subset A(N)$.

An \mathcal{H} -flashing process, with $h_j \geq h_0$ for $j \geq 0$, and h_0 a large constant, produces a cluster $A^*(N)$, for which we bound easily the inner error, $\delta_I^*(n)$. Then, to bound the outer error, $\delta_O^*(n)$, we follow the approach of [6], though with a slightly simpler proof.

Theorem 1.3 *Assume that for $j \geq 1$, $h_j \leq h_{j+1} \leq (1 + \frac{1}{2^j})h_j$, with a large h_0 . For a positive constant A_d^* , we have*

$$P(\exists n(\omega) : \forall n \geq n(\omega) \delta_I^*(n) \leq A_d^* h(n) \log(n)) = 1, \quad (1.8)$$

and,

$$P(\exists n(\omega) : \forall n \geq n(\omega) \delta_O^*(n) \leq A_d^* h(n) \log^2(n)) = 1, \quad (1.9)$$

where $h(n) = \max\{h_k \in \mathbb{R} : r_k \leq n\}$.

Finally, we establish lower bound on the inner and outer error.

Theorem 1.4 *Assume that h_0 is large enough. Then, there is a constant a_d^* such that*

$$P(\exists n(\omega) : \forall n \geq n(\omega) \delta_I^*(n) \geq a_d^* h(n) \log(h(n))) = 1, \quad (1.10)$$

and,

$$P(\exists n(\omega) : \forall n \geq n(\omega) \delta_O^*(n) \geq a_d^* h(n) \log(n)) = 1. \quad (1.11)$$

Theorem 1.2 and Theorem 1.3, with the choice $h_j = h_0$ for all $j > 0$, imply the following Proposition 1.5 for internal DLA.

Proposition 1.5 *Assume $d \geq 2$. There is a positive constant A_d such that,*

$$P(\exists n(\omega) : \forall n \geq n(\omega) \delta_I(n) \leq A_d \log(n)) = 1, \quad (1.12)$$

and,

$$P(\exists n(\omega) : \forall n \geq n(\omega) \delta_O(n) \leq A_d \log^2(n)) = 1, \quad (1.13)$$

We present now a couple of non-rigorous but heuristic remarks.

On previous bounds for internal DLA. We first recall the approach of Lawler, Bramson and Griffeath in [7]. Send $N = |\mathbb{B}(0, n)|$ explorers from the origin. Let $A_n(N)$ be the cluster formed by explorers settled before they escape $\mathbb{B}(0, n)$, and for $z \in \mathbb{B}(0, n)$ let $W_n(z)$ be the number of explorers who visit z before escaping $\mathbb{B}(0, n)$. The approach of [7] is based on the following observations. (i) If explorers would not settle, they would just be independent random walks; (ii) exactly one explorer occupies each site of the cluster. When we launch one explorer from each site of $\Lambda \subset \mathbb{Z}^d$, we denote by $M_n(\Lambda, z)$ the number of crossings of site z achieved by all these explorers. Also, define $M_n(N \mathbb{1}_0, z)$ as $W_n(z) + M_n(A_n(N), z)$. Then, observations (i) and (ii) imply that $M_n(N \mathbb{1}_0, z)$ is equal in law to the number of walks crossing z out of N independent walks started on 0. Note that both $M_n(A_n(N), z)$ and $M_n(N \mathbb{1}_0, z)$ are sums of independent Bernoulli variables, but since $A_n(N)$ is random, the expectation of the former is not trivial. Note now that $A_n(N) \subset \mathbb{B}(0, n)$ so that

$$W_n(z) \geq M_n(N \mathbb{1}_0, z) - M_n(\mathbb{B}(0, n), z). \quad (1.14)$$

From (1.14), [7] use the following rough inequality

$$P(W_n(z) = 0) \leq P(M_n(N \mathbb{1}_0, z) < a) + P(M_n(\mathbb{B}(0, n), z) \geq a). \quad (1.15)$$

The probabilities on the right hand side of (1.15) are small if $E[M_n(\mathbb{B}(0, n), z)] \ll a \ll E[M_n(N \mathbb{1}_0, z)]$. It remains to estimate from below for $\mu = E[M_n(N \mathbb{1}_0, z) - M_n(\mathbb{B}(0, n), z)]$. Indeed, choosing a equal to the arithmetic mean of $E[M_n(N \mathbb{1}_0, z)]$ and $E[M_n(\mathbb{B}(0, n), z)]$ and writing \bar{X} for $X - E[X]$, we obtain from (1.15),

$$P(W_n(z) = 0) \leq P\left(\bar{M}_n(N \mathbb{1}_0, z) < -\frac{\mu}{2}\right) + P\left(\bar{M}_n(\mathbb{B}(0, n), z) \geq \frac{\mu}{2}\right). \quad (1.16)$$

One first need sharp estimates for μ which are obtained in [7] by sharp potential analysis estimates. These sharp estimates requires the special feature that all particles start at the origin. Then, one has a gaussian-type bound for a sum of independent Bernoulli, and since the variance of $M_n(N \mathbb{1}_0, z)$ is bounded by its mean, one obtains for some positive constant c ,

$$P(W_n(z) = 0) \leq 2 \exp\left(-c \frac{(E[M_n(N \mathbb{1}_0, z) - M_n(\mathbb{B}(0, n), z)])^2}{E[M_n(N \mathbb{1}_0, z)]}\right). \quad (1.17)$$

Now, Lawler in [6] shows that

$$E[M_n(N \mathbb{1}_0, z) - M_n(\mathbb{B}(0, n), z)] \sim (n - \|z\|)^2, \quad \text{and} \quad E[M_n(N \mathbb{1}_0, z)] \sim n(n - \|z\|).$$

This implies that $P(W_n(z) = 0)$ is summable if $(n - \|z\|) \geq n^{1/3}$.

On logarithmic fluctuations. Our first observation is that we can improve on the rough bound (1.15). Let us describe this observation in the more general context of flashing processes. These processes are characterized by \mathcal{H} , a sequence of widths for consecutive concentric shells, which in turns are divided into cells. Now, we observe that a site has good chances to lie inside the cluster if some cell, say \mathcal{C} about this site, is crossed by *many* explorers. The notation $W(\mathcal{C})$ refers to the number of explorers visiting \mathcal{C} . We drop the index n appearing in $W_n(z)$ since there are no more constraint on not escaping the ball $\mathbb{B}(0, n)$. Now, the shape result of [7] suggests that for some positive constant α_d , (independent of the volume of \mathcal{C})

$$W(\mathcal{C}) \geq \alpha_d |\mathcal{C}| \implies \mathcal{C} \subset A(N) \text{ with a large probability.} \quad (1.18)$$

Note that in [7], all the explorers start from the origin, whereas here, we only know that they *cross* \mathcal{C} . For internal DLA, estimating the probability that \mathcal{C} is not covered, when \mathcal{C} is large and $W(\mathcal{C}) \geq \alpha_d |\mathcal{C}|$ raises a difficulty which vanishes when considering flashing processes.

We now make our heuristic discussion more precise. For a scale h and an integer $K > 1$, to be determined, assume that $\mathbb{B}(0, n - Kh)$ is covered by settled explorers. Partition the shell $\mathcal{S} = \mathbb{B}(0, n - (K - 1)h) \setminus \mathbb{B}(0, n - Kh)$ into about $(n/h)^{d-1}$ cells, each of volume h^d . We think of cells as brick-like domain, of side length equal to the width of the shell. It is also convenient to stop the explorers as they reach the boundary of $\mathbb{B}(0, n - Kh)$. Thus, with such a stopped process, explorers are either settled inside the $\mathbb{B}(0, n - Kh)$ or unsettled but stopped on its boundary, that we denote by $\partial\mathbb{B}(0, n - Kh)$. What we have called earlier *the number of explorers crossing \mathcal{C}* is taken here to be the unsettled explorers stopped on $\mathcal{C} \cap \partial\mathbb{B}(0, n - Kh)$.

Assuming (1.18) holds, it remains to show that the probability of the event $\{\exists \mathcal{C} \in \mathcal{S} : W(\mathcal{C}) < \alpha_d |\mathcal{C}|\}$ is small. We improve on (1.15) by using the independence between $W(\mathcal{C})$ and $M(\mathbb{B}(0, n - Kh), \mathcal{C})$, and that (1.14) should rather be thought as

$$W(\mathcal{C}) + M(\mathbb{B}(0, n - Kh), \mathcal{C}) \geq M(N \mathbb{1}_0, \mathcal{C}). \quad (1.19)$$

Readily, (1.19) yields for any $t, \xi > 0$

$$\begin{aligned} P(W(\mathcal{C}) \leq \xi) &\leq e^{t\xi} \times \frac{E[\exp(-t(W(\mathcal{C}) + M(\mathbb{B}(0, n - Kh), \mathcal{C})))]}{E[\exp(-tM(\mathbb{B}(0, n - Kh), \mathcal{C}))]} \\ &\leq e^{-t(\mu - \xi)} \frac{E[\exp(-t\bar{M}(N \mathbb{1}_0, \mathcal{C}))]}{E[\exp(-t\bar{M}(\mathbb{B}(0, n - Kh), \mathcal{C}))]}. \end{aligned} \quad (1.20)$$

Using that $M(N \mathbb{1}_0, z)$ and $M(\mathbb{B}(0, n - Kh), z)$ are sums of independent Bernoulli, we show that (1.20) implies a gaussian type lower tail

$$P(W(\mathcal{C}) < \alpha_d |\mathcal{C}|) \leq \exp\left(-\frac{(\mu - \alpha_d |\mathcal{C}|)^2}{c\nu}\right), \quad (1.21)$$

for a positive constant c , and where ν

$$\nu = \text{var}(M(N \mathbb{1}_0, \mathcal{C})) - \text{var}(M(\mathbb{B}(0, n - Kh), \mathcal{C})). \quad (1.22)$$

We show that both μ and ν are of order $K|\mathcal{C}|$. Then, $P(W(\mathcal{C}) < \alpha_d|\mathcal{C}|)$ is summable as soon as $K|\mathcal{C}| \geq A \log(n)$. This latter inequality holds when, for instance, $K = \log(n)$ and $h^d = A$, or better when $K = A$ and $h^d = \log(n)$.

Let us explain the estimation of μ and ν . Both $M(\mathbb{B}(0, n - Kh), \mathcal{C})$ and $M(N \mathbb{1}_0, \mathcal{C})$ are sums of independent Bernoulli random variables, and if τ is the first time a random walk exits $\mathbb{B}(0, n - Kh)$, then (with $N = |\mathbb{B}(0, n)|$),

$$\text{var}(M(N \mathbb{1}_0, \mathcal{C})) = NP_0(S(\tau) \in \mathcal{C})(1 - P_0(S(\tau) \in \mathcal{C})), \quad (1.23)$$

and,

$$\text{var}(M(\mathbb{B}(0, n - Kh), \mathcal{C})) = \sum_{y \in \mathbb{B}(0, n - Kh)} P_y(S(\tau) \in \mathcal{C})(1 - P_y(S(\tau) \in \mathcal{C})). \quad (1.24)$$

From (1.22) (1.23) and (1.24), we have

$$\nu \leq E[M(N \mathbb{1}_0, \mathcal{C}) - M(\mathbb{B}(0, n - Kh), \mathcal{C})] + \sum_{y \in \mathbb{B}(0, n - Kh)} P_y(S(\tau) \in \mathcal{C})^2. \quad (1.25)$$

We then show that for some constants κ_1 and c

$$E[M(N \mathbb{1}_0, \mathcal{C}) - M(\mathbb{B}(0, n - Kh), \mathcal{C})] \leq \kappa_1 K|\mathcal{C}|, \quad \text{and} \quad \sum_{y \in \mathbb{B}(0, n - Kh)} P_y(S(\tau) \in \mathcal{C})^2 \leq c|\mathcal{C}|. \quad (1.26)$$

On lower bound for internal error It is useful to organize the flow of explorers in the flashing process into *exploration waves*, in the way of Section 3 (the upper bound) of [6]. That is, in the k -th exploration wave, the explorers either stop as they reach the *bulk* of \mathcal{S}_k , or settle before reaching \mathcal{S}_k . Consider now the exploration wave associated with the last shell contained in $\mathbb{B}(0, n - h(n) \log(h(n)))$, say \mathcal{S}_{k^*} . Assume that at this time, the cluster fills $\mathbb{B}(0, n - h(n) \log(h(n))) \setminus \mathcal{S}_{k^*}$. From our definition, \mathcal{S}_{k^*} has width of order h_{k^*} and receives a number of stopped explorers of order its volume times $\log(h(n))$. There is necessarily more than half of the cells in \mathcal{S}_{k^*} which receives much less than their own volume times $\log(h(n))$, and for a coupon-collector process, it is very unlikely that the explorers stopped in this very cell can cover the bulk of this cell before escaping \mathcal{S}_{k^*} . By feature (ii) the covering of far away cells is independent, and the estimates are therefore simple. Note that by feature (iii) if a hole is left in \mathcal{S}_{k^*} after the explorers leave \mathcal{S}_{k^*} , this hole remains uncovered forever.

Outline of the paper The rest of the paper is organized as follows. Section 2 introduces the main notation, and recall well known useful facts. In Section 3, we build the flashing process, give an alternative construction, and prove Theorem 1.2. In Section 4, we obtain a sharp estimate on the expected number of explorers crossing a given cell, and prove feature (ii) of the flashing times. Both proofs are based on classical potential theory estimates. In Section 5, we prove Theorems 1.3 and 1.4. Finally, in the Appendix, Sébastien Blachère gives a sharp estimate on the expected time spent in an annulus by a random walk.

2 Notation and useful tools

2.1 Notation

We say that $z, z' \in \mathbb{Z}^d$ are nearest neighbors when $\|z - z'\| = 1$, and we write $z \sim z'$. For any subset $\Lambda \subset \mathbb{Z}^d$, we define

$$\partial\Lambda = \{z \in \mathbb{Z}^d \setminus \Lambda : \exists z' \in \Lambda, z' \sim z\}. \quad (2.1)$$

For any $r \leq R$ we define the annulus

$$A(r, R) = B(0, R) \setminus B(0, r) \quad \text{and} \quad \mathbb{A}(r, R) = A(r, R) \cap \mathbb{Z}^d \quad (2.2)$$

A trajectory γ is a discrete nearest-neighbor path on \mathbb{Z}^d . That is $\gamma : \mathbb{N} \rightarrow \mathbb{Z}^d$ with $\gamma(t) \sim \gamma(t+1)$ for all t . The law of the simple random walk started in z , is denoted with \mathbb{P}_z . For a subset Λ in \mathbb{Z}^d , and a trajectory γ , we define the hitting time of Λ as

$$H(\Lambda; \gamma) = \min\{t \geq 0 : \gamma(t) \in \Lambda\}.$$

We often omit γ in the notation when no confusion is possible. We use the shorthand notation

$$B_n = B(0, n), \quad \mathbb{B}_n = \mathbb{B}(0, n), \quad H_R = H(B_R^c), \quad \text{and} \quad H_z = H(\{z\}).$$

For any a, b in \mathbb{R} we write $a \wedge b = \min\{a, b\}$, and $a \vee b = \max\{a, b\}$. Let Γ be a finite collection of trajectories on \mathbb{Z}^d . For $R > 0$, z in \mathbb{Z}^d and Λ a subset of \mathbb{Z}^d , we call $M(\Gamma, R, z)$ (resp. $M(\Gamma, R, \Lambda)$) the number of trajectories which exit $\mathbb{B}(0, R)$ on z (resp. in Λ):

$$M(\Gamma, R, z) = \sum_{\gamma \in \Gamma} \mathbf{1}_{\{\gamma(H_R) = z\}}, \quad \text{and} \quad M(\Gamma, R, \Lambda) = \sum_{z \in \Lambda} M(\Gamma, R, z). \quad (2.3)$$

When we deal with a collection of independent random trajectories, we rather specify its initial configuration $\eta \in \mathbb{N}^{\mathbb{Z}^d}$, so that $M(\eta, R, z)$ is the number of random walks starting from η and hitting $\mathbb{B}(0, R)^c$ on z . Two types of initial configurations are important here: (i) the configuration $n\mathbf{1}_{z^*}$ formed by n walkers starting on a given site z^* , (ii) for $\Lambda \subset \mathbb{Z}^d$, the configuration $\mathbf{1}_\Lambda$ that we simply identify with Λ . For any configuration $\eta \in \mathbb{N}^{\mathbb{Z}^d}$ we write

$$|\eta| = \sum_{z \in \mathbb{Z}^d} \eta(z). \quad (2.4)$$

For any $\Lambda \subset \mathbb{Z}^d$, we define Green's function restricted to Λ , G_Λ , as follows. For $x, y \in \Lambda$

$$G_\Lambda(x, y) = \mathbb{E}_x \left[\sum_{0 \leq n < H(\Lambda^c)} \mathbf{1}_{\{S(n) = y\}} \right]. \quad (2.5)$$

In dimension 3 or more, Green's function on the whole space is well defined and denoted G . That is, for any $x, y \in \mathbb{Z}^d$

$$G(x, y) = \mathbb{E}_x \left[\sum_{n \geq 0} \mathbf{1}_{\{S(n) = y\}} \right]. \quad (2.6)$$

In dimension 2, the potential kernel plays the role of Green's function

$$a(x, y) = \mathbb{E}_x \left[\sum_{l=0}^{\infty} \mathbf{1}_{\{S(l) = x\}} - \mathbf{1}_{\{S(l) = y\}} \right]. \quad (2.7)$$

2.2 Some useful tools

We recall here some well known facts. Some of them are proved for the reader's convenience. This section can be skipped at a first reading.

In [7], the authors emphasized the fact that the spherical limiting shape of internal DLA was intimately linked to strong isotropy properties of Green's function. This isotropy is expressed by the following asymptotics (Theorem 4.3.1 of [8]). In $d \geq 3$, there is a constant K_g , such that for any $z \neq 0$,

$$\left| G(0, z) - \frac{C_d}{\|z\|^{d-2}} \right| \leq \frac{K_g}{\|z\|^d} \quad \text{with} \quad C_d = \frac{2}{v_d(d-2)}, \quad (2.8)$$

where v_d stands for the volume of the euclidean unit ball in \mathbb{R}^d . The first order expansion (2.8) is proved in [8] for general symmetric walks with finite $d+3$ moments and vanishing third moment. All the estimates we use are eventually based on (2.8) and we emphasize the fact that the estimate is uniform in $\|z\|$. There is a similar expansion for the potential kernel. Theorem 4.4.4. of [8] establishes that for $z \neq 0$, (with γ the Euler constant)

$$\left| a(0, z) - \frac{2}{\pi} \log(\|z\|) - \frac{2\gamma + \log(8)}{\pi} \right| \leq \frac{K_g}{\|z\|^2}. \quad (2.9)$$

The following lemma is also useful.

Lemma 2.1 *Each z^* in $\mathbb{Z}^d \setminus \{0\}$ has a nearest-neighbor z (i.e. $z^* \sim z$) such that*

$$\|z\| \leq \|z^*\| - \frac{1}{2\sqrt{d}}. \quad (2.10)$$

Proof. Without loss of generality we can assume that all the coordinates of z^* are non-negative. Let us denote by b the maximum of these coordinates and note that

$$\|z^*\|^2 \leq db^2, \quad \text{and} \quad b \geq 1. \quad (2.11)$$

Denote by z the nearest-neighbor obtained from z^* by decreasing by one unit a maximum coordinate. Using (2.11)

$$\|z^*\|^2 - \|z\|^2 = b^2 - (b-1)^2 = 2b-1 \geq b \geq \frac{\|z^*\|}{\sqrt{d}}. \quad (2.12)$$

Note that (2.10) follows from $2\|z^*\|(\|z^*\| - \|z\|) \geq \|z^*\|^2 - \|z\|^2$, and (2.12). ■

We state now a handy estimate dealing with sums of independent Bernoulli variables.

Lemma 2.2 *Let $\{X_n, Y_n, n \in \mathbb{N}\}$ be independent 0-1 Bernoulli variables. For integer n, m let $S = X_1 + \dots + X_n$ and $S' = Y_1 + \dots + Y_m$. Define for $t \in \mathbb{R}$*

$$f(t) = e^t - 1 - t, \quad \text{and} \quad g(t) = (e^t - 1)^2.$$

If $0 \leq t \leq \log(2)$, then

$$\frac{E[\exp(t(S - E[S]))]}{E[\exp(t(S' - E[S']))]} \leq \exp\left(f(t)E[S - S'] + g(t)\sum_{i=1}^m E[Y_i]^2\right). \quad (2.13)$$

Assume now that for $\kappa > 1$, $\sup_n E[Y_n] \leq \frac{\kappa-1}{\kappa}$. If $t \leq 0$, then

$$\frac{E[\exp(t(S - E[S]))]}{E[\exp(t(S' - E[S']))]} \leq \exp\left(f(t)E[S - S'] + \frac{\kappa}{2}g(t)\sum_{i=1}^m E[Y_i]^2\right). \quad (2.14)$$

Proof. Let X be a Bernoulli variable, and $p = E[X]$. Using (o) $e^x \geq 1 + x$ for any $x \in \mathbb{R}$, we have

$$\begin{aligned} E[\exp(t(X - E[X]))] &= pe^{t(1-p)} + (1-p)e^{-tp} \\ &= e^{-pt}(1 + p(e^t - 1)) \\ &\leq \exp(f(t)E[X]). \end{aligned} \quad (2.15)$$

For a lower bound, we distinguish two cases.

First, assume $t \geq 0$. We claim that $\exp(x - x^2) \leq 1 + x$ for $0 \leq x \leq 1$. Indeed, we use three obvious inequalities: (o) already given, (i) for $x \leq 1$, $1 + x + x^2 \geq e^x$, and (ii) $(1 + x^2)(1 + x) \geq 1 + x + x^2$. Thus,

$$e^{x^2}(1 + x) \geq (1 + x^2)(1 + x) \geq 1 + x + x^2 \geq e^x.$$

This yields the claim. Now, set $x = p(e^t - 1)$, so that $x \leq 1$ when $e^t \leq 2$ and the equality (2.15) yields

$$E[\exp(t(X - E[X]))] \geq \exp(-tp + p(e^t - 1) - p^2(e^t - 1)^2) = e^{f(t)p - g(t)p^2}. \quad (2.16)$$

Assume now that $t \leq 0$, and for $\kappa > 1$, $p < \frac{\kappa-1}{\kappa}$. We claim that for $0 \leq x \leq \frac{\kappa-1}{\kappa}$

$$\exp\left(-x - \frac{\kappa}{2}x^2\right) \leq 1 - x. \quad (2.17)$$

Indeed, we have an additional inequality (iii) $1 - x + \frac{x^2}{2} \geq \exp(-x)$ when $x \geq 0$. Note also that

$$\left(1 + \frac{\kappa}{2}x^2\right)(1 - x) \geq 1 - x + \frac{x^2}{2} \iff x \leq \frac{\kappa - 1}{\kappa}.$$

Thus

$$e^{\frac{\kappa}{2}x^2}(1 - x) \geq \left(1 + \frac{\kappa}{2}x^2\right)(1 - x) \geq 1 - x + \frac{x^2}{2} \geq e^{-x}. \quad (2.18)$$

Now, set $x = -p(e^t - 1) \geq 0$, so that $x \leq \frac{\kappa-1}{\kappa}$. We obtain

$$E[\exp(t(X - E[X]))] \geq \exp\left(-tp + p(e^t - 1) - \frac{\kappa}{2}p^2(e^t - 1)^2\right) = e^{f(t)p - \frac{\kappa}{2}g(t)p^2} \quad (2.19)$$

The statements (2.13), and (2.14) follow at once. ■

3 The flashing process

In this section, we construct the flashing process. We then present a useful alternative construction of the same process. Finally, we prove Theorem 1.2 that couples the two processes.

3.1 Construction of the process

Partitioning the lattice. We are given a sequence $\mathcal{H} = \{h_n, n \in \mathbb{N}\}$. We partition the lattice into shells ($\mathcal{S}_j : j \geq 0$). For an illustration, see Figure 1. For a given parameter $h_0 > 0$ the first shell \mathcal{S}_0 is the ball $\mathbb{B}(0, h_0)$. The next shells are the annuli

$$\mathcal{S}_j = \mathbb{A}(r_j - h_j, r_j + h_j), \quad j \geq 1, \quad (3.1)$$

Where $\{r_j, j \geq 1\}$ is defined inductively by $r_1 = h_0 + h_1$, and for $j \geq 1$

$$r_{j+1} - h_{j+1} = r_j + h_j. \quad (3.2)$$

In Section 5, we need that (o) $\{h_n\}$ is increasing, (i) $j \mapsto h_j/r_j$ is decreasing, and (ii) $h_j = O(r_j^{1/3})$. These properties are a straightforward consequence of our hypothesis $h_j \leq h_{j+1} \leq (1 + \frac{1}{2j})h_j$. Actually we will only need these properties and our hypothesis is no more than a sufficient condition.

We also define

$$\Sigma_0 = \{0\} \quad \text{and} \quad \Sigma_j = \partial\mathbb{B}(0, r_j), \quad j \geq 1. \quad (3.3)$$

Flashing times. Consider $\{X_j, Y_j, j \geq 0\}$ a sequence of independent Bernoulli variables such that

$$P(X_j = 1) = 1 - P(X_j = 0) = \frac{1}{h_j^d}, \quad (3.4)$$

$$P(Y_j = 1) = 1 - P(Y_j = 0) = \begin{cases} 1 & \text{if } j = 0, \\ \frac{1}{2} & \text{if } j \geq 1, \end{cases} \quad (3.5)$$

Consider also a sequence of continuous independent variables $\{R_j, j \geq 0\}$ each of which has density $g_j : [0, h_j] \rightarrow \mathbb{R}^+$ with

$$g_j(h) = \frac{dh^{d-1}}{h_j^d}. \quad (3.6)$$

For $j \geq 0$, and z_j in Σ_j , let S be a random walk starting in z_j , and define a stopping time σ as follows. If $R_j = h$ for some $h \leq h_j$ then

$$\sigma = \begin{cases} 0 & \text{if } X_j = 1, \\ H(\mathbb{B}(z, h \wedge (r_j + h_j - \|z_j\|))^c) & \text{if } X_j = 0 \text{ and } Y_j = 1, \\ H(\mathbb{A}(r_j - h, r_j + h)^c) & \text{if } X_j = 0 \text{ and } Y_j = 0. \end{cases} \quad (3.7)$$

We set $H_j = H(\Sigma_j)$, and we define the stopping times ($\sigma_j : j \geq 0$) as

$$\sigma_j = H_j + \sigma(S \circ \theta_{H_j}), \quad (3.8)$$

where θ stands for the usual time-shift operator. For $j \geq 0$ we note that, by construction, $S(t) \in \mathcal{S}_j$ for all t such that $H_j \leq t < \sigma$ and we say that σ_j is a *flashing time* when $S(\sigma_j)$ is contained in the intersection between \mathcal{S}_j and the cone with base $B(S(H_j), h_j/2)$. We call such an intersection *a cell centered at $S(H_j)$* , that we denote $\mathcal{C}(S(H_j))$. In other words, for any $z \in \Sigma_j$

$$\mathcal{C}(z) = \mathcal{S}_j \cap \{x \in \mathbb{R}^d : \exists \lambda \geq 0, \exists y \in B(z, h_j/2), x = \lambda y\}. \quad (3.9)$$

The uniform hitting property The main property of the hitting time σ constructed above is the following proposition, which yields property $\mathcal{P}b$ of the flashing process to be defined soon.

Proposition 3.1 *There are two positive constants $\alpha_1 < \alpha_2$, such that, for $j \geq 0$, $z_j \in \Sigma_j$, and $z^* \in \mathcal{C}(z_j)$.*

$$\frac{\alpha_1}{h_j^d} \leq \mathbb{P}_{z_j}(S(\sigma) = z^*) \leq \frac{\alpha_2}{h_j^d}. \quad (3.10)$$

The proof of Proposition 3.1 is given in Section 4.

The flashing process. Consider a family of N independent random walks $(S_i : 1 \leq i \leq N)$ with their hitting times, and flashing times $(H_{i,j}, \sigma_{i,j} : j \geq 0)$. Let also $z_{i,j} = S_i(H_{i,j})$ be the first hitting position of S_i on Σ_i .

We define the cluster inductively. Set $A^*(0) = \emptyset$. For $i \geq 1$, we define τ_i^* as the first flashing time associated with S_i when the explorer stands outside $A^*(i-1)$. In other words,

$$\tau_i^* = \min \{\sigma_{i,j} : j \geq 0, S_i(\sigma_{i,j}) \in \mathcal{C}(z_{i,j}) \cap A^*(i-1)^c\}, \quad (3.11)$$

and

$$A^*(i) = A^*(i-1) \cup \{S_i(\tau_i^*)\}. \quad (3.12)$$

3.2 Exploration Waves

Rather than building $A^*(N)$ following the whole journey of one explorer after another, we can build $A^*(N)$ as an increasing union of clusters formed by stopping explorers on successive shells. Similar wave constructions are introduced in [7] and [6], with an equality in law between alternative constructions. However, the features of the flashing process are such that in our case the two constructions are strictly equivalent. We use this alternative construction in the proof of Theorem 1.2.

We denote by $\xi_k \in (\mathbb{Z}^d)^N$ the explorers positions after the k -th wave. We denote by $\mathcal{A}_k^*(N)$ and the set of sites where settled explorers are after the k -th wave. Our construction will be such that

$$\xi_k(i) \notin \Sigma_k \Leftrightarrow \xi_k(i) \in \cup_{j < k} \mathcal{S}_j \Leftrightarrow \xi_k(i) \in \mathcal{A}_k^*(N). \quad (3.13)$$

For $k = 0$ we set $\xi_0(i) = 0$, and $\mathcal{A}_0^*(i) = \emptyset$, for $1 \leq i \leq N$. Then, for all $k \geq 0$, we set $\mathcal{A}_{k+1}^*(0) = \mathcal{A}_k^*(N)$. For i in $\{1, \dots, N\}$, we set the following.

- If $\xi_k(i) \notin \Sigma_k$, then

$$\xi_{k+1}(i) = \xi_k(i) \in \cup_{j < k} \mathcal{S}_j, \quad \text{and} \quad \mathcal{A}_{k+1}^*(i) = \mathcal{A}_{k+1}^*(i-1).$$

- If $\xi_k(i) \in \Sigma_k$ and $S_i(\sigma_{i,k}) \in \mathcal{C}(z_{i,k}) \cap \mathcal{A}_k^*(i-1)^c$, then

$$\xi_{k+1}(i) = S_i(\sigma_{i,k}) \in \mathcal{S}_k, \quad \text{and} \quad \mathcal{A}_{k+1}^*(i) = \mathcal{A}_{k+1}^*(i-1) \cup \{S_i(\sigma_{i,k})\}.$$

- If $\xi_k(i) \in \Sigma_k$ and $S_i(\sigma_{i,k}) \notin \mathcal{C}(z_{i,k}) \cap \mathcal{A}_k^*(i-1)^c$, then

$$\xi_{k+1}(i) = S_i(H_{i,k+1}) \in \Sigma_{k+1}, \quad \text{and} \quad \mathcal{A}_{k+1}^*(i) = \mathcal{A}_{k+1}^*(i-1).$$

In words, for each $k \geq 1$, during the k -th wave of exploration, the unsettled explorers move one after the other in the order of their labels until either settling in \mathcal{S}_{k-1} , or reaching Σ_k where they stop. We then define $\mathcal{A}^*(N)$ by

$$\mathcal{A}^*(N) = \bigcup_{k \geq 1} \mathcal{A}_k^*(N). \quad (3.14)$$

We explain now why this construction yields the same cluster as our previous definition. An explorer cannot settle inside a shell it has left, and thus cannot settle in any shell \mathcal{S}_j with $j < k$ if it reaches Σ_k . Now, since each wave of exploration is organized according to the label ordering, the fact that an explorer has to wait for the following explorers before proceeding its journey beyond Σ_k does not interfere with the site where it eventually settles. We define $\mathcal{G}_k = \sigma(\xi_0, \dots, \xi_k)$, for $k \in \mathbb{N}$.

3.3 Coupling internal DLA and flashing processes

Proposition 3.2 *For all $N \geq 0$ there is a coupling and a one to one map ψ_N between $A(N)$ and $A^*(N)$ such that for all $k \geq 1$*

$$\psi_N(A(N) \cap \mathbb{B}_{r_k+h_k}^c) \subset A^*(N) \cap \mathbb{B}_{r_k+h_k}^c. \quad (3.15)$$

This proposition has two important consequences.

- If $A^*(N) \subset \mathbb{B}_{r_k+h_k}$, then $A(N) \subset \mathbb{B}_{r_k+h_k}$. Indeed, any site in $A(N)$ outside $\mathbb{B}_{r_k+h_k}$ produces, through ψ_N , a site in $A^*(N)$ outside $\mathbb{B}_{r_k+h_k}$.
- If $\mathbb{B}_{r_k+h_k} \subset A^*(N)$, then $\mathbb{B}_{r_k+h_k} \subset A(N)$. Indeed, those sites in $A(N)$ that are mapped through ψ_N on $A^*(N) \cap \mathbb{B}_{r_k+h_k}^c = \mathbb{B}_{r_k+h_k}^c$ are necessarily contained in $\mathbb{B}_{r_k+h_k}$. Since their number is $|\mathbb{B}_{r_k+h_k}^c|$ and ψ_N is one to one, they completely cover $\mathbb{B}_{r_k+h_k}^c$.

Proof. Before giving a precise definition of our coupling, we describe its main features in words. We launch N independent random walks, and build inductively the associated clusters $A(1), A(2), \dots, A(N)$. In doing so, we use the increments of these random walks to define, step by step, N flashing trajectories S_1^*, \dots, S_N^* up to some times $\bar{t}_1, \dots, \bar{t}_N$.

Let us describe informally step $i + 1$ of the induction. Assume that S_1^*, \dots, S_i^* are defined up to some times $t_1 \leq \bar{t}_1, \dots, t_i \leq \bar{t}_i$, and that each site of $A(i)$ is covered by exactly one $S_k^*(t_k)$ with $1 \leq k \leq i$. We can think of $S_1^*(t_1), \dots, S_i^*(t_i)$ as the positions of stopped flashing explorers, some of them stopped at a flashing time – say on *blue* sites – some of them not – say on *red* sites. Then, we add the $i + 1$ -th explorer and flashing explorer. We set $S_{i+1}^*(0) = S_{i+1}(0) = 0$. We add new increments to S_{i+1} as well as to the trajectory of one flashing explorer, say with label j in $\{1; \dots; i+1\}$, in such a way that the current positions of the walker $i + 1$ and that of the flashing explorer j coincide. The label j is defined inductively as follows. Initially, $j = i + 1$. Assume that the walker $i + 1$ steps on a red or blue site inside $A(i)$. This site is occupied by exactly two stopped flashing explorers, j and j' (and all other red and blue sites of $A(i)$ are occupied by exactly one flashing explorer). Since flashing explorers can settle at their flashing times, it makes sense, when j is flashing, to add the next increment to the trajectory of flashing explorer j' rather than j . We do so in two cases. First, when this happens on a red site. In this case, we turn blue that site since j is stopped at a flashing time. Second, when this happens on a blue site, say z , and $j' > j$. Note that in this case, both explorers flash on z but explorer j reaches z before explorer j' when launched in their label order. Our choice is such that the eventual cluster $A^*(N)$ has the correct law. In all other cases, we keep adding the increments of S_{i+1} to the same flashing trajectory. It is important to note that the value of the increment does not depend on the index of the trajectory we choose to extend. Walker $i + 1$ eventually steps outside $A(i)$. When the last walker steps outside $A(N - 1)$ we have

$$A(N) = \{S_1^*(\bar{t}_1); \dots; S_N^*(\bar{t}_N)\} \quad \text{with} \quad |A(N)| = N. \quad (3.16)$$

To define $A^*(N)$ we launch again, in their labels order, our flashing explorers from their current positions (possibly some or none of them since some or all of them can already have reached their settling position). We then get

$$A^*(N) = \{S_1^*(\tau_1^*); \dots; S_N^*(\tau_N^*)\} \quad \text{with} \quad |A^*(N)| = N \quad \text{and} \quad \tau_k^* \geq \bar{t}_k, \quad \text{for all } k. \quad (3.17)$$

Then, since a flashing explorer that visited some site beyond a given shell cannot settle in that shell, the one to one map

$$\psi_N : S_k^*(\bar{t}_k) \in A(N) \mapsto S_k^*(\tau_k^*) \in A^*(N) \quad (3.18)$$

satisfies, for all k and l ,

$$S_k^*(\bar{t}_k) \notin \bigcup_{m < l} \mathcal{S}_m \quad \Rightarrow \quad S_k^*(\tau_k^*) = \psi_N(S_k^*(\bar{t}_k)) \notin \bigcup_{m < l} \mathcal{S}_m. \quad (3.19)$$

We now give a precise definition of our coupling. To avoid heavy notation we write the coupling algorithm as a pseudo-code with a couple of remarks to check that everything is well defined. First of all, we draw N independent sequences of independent Bernoulli and continuous random variables $((X_{k,l}, Y_{k,l}, R_{k,l} : l \geq 0) : 1 \leq k \leq N)$ as in Section 3. In addition, we call $(U_k : k \geq 1)$ the sequence of the increments of a generic independent simple random walk on \mathbb{Z}^d . From these two sources of randomness we extract our explorer and flashing explorer trajectories with their associated clusters. The flashing times will be adapted to the flashing explorer trajectories as in Section 3.

Our pseudo-code is made of two loops of size N that make precise the previous description. With the first loop we build our N random walk trajectories $((S_i(t) : 0 \leq t \leq \tau_i) : 1 \leq i \leq N)$ with their associated clusters $A(1), \dots, A(N)$. Step by step, within this first loop, we also define pieces of the flashing explorers trajectories S_1^*, \dots, S_N^* . With the second loop we complete the trajectories of the flashing explorers to build the associated cluster $A^*(N)$. During the algorithm, $t_k \in \mathbb{N}$ stands for the time up to which the trajectory of flashing explorer k has been defined ($k \in \{1; \dots; N\}$). We use the same t for the time governing the evolution of each simple random walk S_i . The index j is updated before adding each random walk increment to the partial sum of S_i and S_j^* . The updating procedure uses the index j' described above and we denote by $\Delta = U$ the increment. Each encountered U stands for the first unused random variable in the sequence $(U_k : k \geq 1)$.

The main advantage of the pseudo-code formalism is that it allows, through the assignment operator ' \leftarrow ', expressions of the kind $j \leftarrow \max(j, j')$ or $t_j \leftarrow t_j + 1$ rather than $j(\theta+1) = \max(j(\theta), j'(\theta))$ and $t_{j(\theta+1)}(\theta+1) = t_{j(\theta)}(\theta) + 1$ with θ a discrete parameter ordering the sequence of our elementary moves. It makes also implicit identities like $t_k(\theta+1) = t_k(\theta)$ for any quantity t_k that does not need to be updated. Our following pseudo-code can be rewritten in a classical inductive way with θ running through $\{(i, t) \in \{1; \dots; N\} \times \mathbb{N} : t \leq \tau_i\}$ according to lexicographic order. Marks ^(a) and ^(b) refer to remarks (a) and (b) below.

$$\begin{aligned}
& A(0) \leftarrow \emptyset; \\
& \text{For } i = 1 \text{ to } N \\
& \quad \left\{ \begin{array}{l}
j \leftarrow i; \\
t \leftarrow 0; \quad t_j \leftarrow 0; \\
S_i(t) \leftarrow 0; \quad S_j^*(t_j) \leftarrow 0; \quad \text{(Note that } S_i(t) = S_j^*(t_j)) \\
\text{While } S_i(t) \in A(i-1) \\
\quad \left\{ \begin{array}{l}
\text{If } t_j \text{ is a flashing time for explorer } j^{(a)}, \text{ then} \\
\quad \left\{ \begin{array}{l}
j' \leftarrow \text{unique index}^{(b)} k \in \{1; \dots; i\} \setminus \{j\} \\
\text{such that } S_k^*(t_k) = S_j^*(t_j) = S_i(t); \\
\text{If } t_{j'} \text{ is not a flashing time for explorer } j'^{(a)}, \text{ then } j \leftarrow j'; \\
\text{Otherwise } j \leftarrow \max(j, j'); \\
\Delta \leftarrow U; \quad S_i(t+1) \leftarrow S_i(t) + \Delta; \quad S_j^*(t_j+1) \leftarrow S_j^*(t_j) + \Delta; \\
\text{(so that } S_i(t+1) = S_j^*(t_j+1)) \\
t \leftarrow t+1; \quad t_j \leftarrow t_j+1; \\
A(i) \leftarrow A(i-1) \cup \{S_i(t)\}; \\
i \leftarrow i+1;
\end{array} \right. \\
\end{array} \right. \\
& A^*(0) \leftarrow \emptyset; \\
& \text{For } k = 1 \text{ to } N \\
& \quad \left\{ \begin{array}{l}
\text{While } t_k \text{ is not a flashing time for explorer } k^{(a)} \text{ or } S_k^*(t_k) \in A_{k-1}^* \\
\quad \left\{ \begin{array}{l}
S_k^*(t_k+1) \leftarrow S_k^*(t_k) + U; \\
t_k \leftarrow t_k+1; \\
A^*(k) \leftarrow A^*(k-1) \cup \{S_k^*(t_k)\}; \\
k \leftarrow k+1;
\end{array} \right. \\
\end{array} \right.
\end{aligned}$$

Remarks:

- (a) Recall that for $l = j, j'$ or k , S_l^* is defined up to time t_l as well as its associated flashing times.
- (b) One checks by induction on i that just after the instruction “ $A(i) \leftarrow A(i-1) \cup \{S_i(t)\}$ ” we have

$$A(i) = \{S_1^*(t_1); \dots; S_i^*(t_i)\} \quad \text{and} \quad |A(i)| = i. \quad (3.20)$$

To do so, one checks by induction on $t < \tau_i$, that

$$A(i-1) = \{S_1^*(t_1); \dots; S_{j-1}^*(t_{j-1}); S_{j+1}^*(t_{j+1}); \dots; S_i^*(t_i)\} \quad \text{and} \quad |A(i-1)| = i-1. \quad (3.21)$$

Since we always have $S_j^*(t_j) = S_i(t)$ this proves by induction that j' is well defined.

The key observation is that for each increment U , the index of the explorer that follows this increment depends on the whole previous construction, but the value of U does not depend on it. As a consequence, we build independent random walks S_1, \dots, S_N coupled with independent flashing random walks S_1^*, \dots, S_N^* . Then, one simply checks by induction on i and k that

$$A(i) = \{S_1(\tau_1); \dots; S_i(\tau_i)\} \quad \text{and} \quad A^*(k) = \{S_1^*(\tau_1^*); \dots; S_k^*(\tau_k^*)\} \quad (3.22)$$

for all $1 \leq i, k \leq N$.

Finally, define $(\bar{t}_1, \dots, \bar{t}_N)$ and $(\tau_1^*, \dots, \tau_N^*)$ the values of (t_1, \dots, t_N) at the end of the first and last cycle respectively. Since t_1, \dots, t_N can only increase during our loops, we have $\tau_k^* \geq \bar{t}_k$ for all k . Then (3.16) and (3.17) follow from (3.20) and (3.22) and we get the desired property (3.19) for the one to one map defined in (3.18). \blacksquare

4 Potential theory estimates

We gather in this section two new results dealing with hitting probability of sets. The first one relies on a discrete mean value theorem for Green's function. This latter theorem relies on Green's function estimates in [6], and Proposition A.1 given in the Appendix. The second result is Proposition 3.1, which we prove in Section 4.2. The set we wish to hit is not a sphere, and the proof is inspired by Lemma 5 of [7], which only gives an upper bound. We have also included an improvement of Lemma 5(b) of [7] which we need.

4.1 A discrete mean value theorem

Our main result in this section is the following.

Theorem 4.1 *Let $\{\Delta_n, n \in \mathbb{N}\}$ be a positive sequence, with $K_0 \leq \Delta_n \leq Kn^{1/3}$ for some constants K_0, K , and set $r_n = n - \Delta_n$. There is a constant K_a , such that for any $\Lambda \subset \partial\mathbb{B}_n$*

$$\left| E[M(|\mathbb{B}_{r_n}| \mathbf{1}_0, n, \Lambda)] - E[M(\mathbb{B}_{r_n}, n, \Lambda)] \right| \leq K_a |\Lambda|. \quad (4.1)$$

Note that (4.1) follows if we have for any $z^* \in \partial\mathbb{B}_n$

$$|\mathbb{B}_{r_n}| \times \mathbb{P}_0(S(H_n) = z^*) - \sum_{y \in \mathbb{B}_{r_n}} \mathbb{P}_y(S(H_n) = z^*) \leq K_a. \quad (4.2)$$

We now recall a classical decomposition (Lemma 6.3.6 of [8]). For a finite subset Λ , $y \in \Lambda$, and $z^* \in \partial\Lambda$

$$\mathbb{P}_y(S(H(\partial\Lambda)) = z^*) = \frac{1}{2d} \sum_{z \in \Lambda, z \sim z^*} G_\Lambda(y, z). \quad (4.3)$$

By (4.2) and (4.3) with $\Lambda = \mathbb{B}_n$, Theorem 4.1 reduces to proving a discrete mean value theorem which we formulate next. We keep the same notation as Theorem 4.1.

Proposition 4.2 For $z \in \mathbb{B}_n$, and $n - \|z\| \leq 1$,

$$|\mathbb{B}_{r_n}| \times G_n(0, z) - \sum_{y \in \mathbb{B}_{r_n}} G_n(y, z) \leq K_a. \quad (4.4)$$

Remark 4.3 Note that a related (but distinct) property was also at the heart of [7]. Namely, for $\epsilon > 0$, and n large enough, if $z \in \mathbb{B}_n$, and $n - \|z\| \geq \epsilon n$,

$$|\mathbb{B}_n| \times G_n(0, z) \geq \sum_{y \in \mathbb{B}_n} G_n(y, z). \quad (4.5)$$

Proof. In dimension $d \geq 3$, we use that $G_n(0, z) = G(0, z) - \mathbb{E}_z[G(0, S(H_n))]$ (Proposition 1.5.8 of [5]), and (2.8), one obtains by Taylor expansion (2.8) that for a constant K_1 (independent on n)

$$\left| v_d G_n(0, z) - 2 \frac{\alpha(z)}{n^{d-1}} \right| \leq \frac{K_1}{n^d}, \quad \text{where } \alpha(z) = \mathbb{E}_z[\|S(H_n)\| - \|z\|]. \quad (4.6)$$

In dimension 2, we use that $G_n(0, z) = -a(0, z) + E_z[a(0, S(H_n))]$ and (2.9) to obtain $\pi G_n(0, z) = 2\alpha(z)/n + O(1/n^2)$.

Now, $r_n^d = n^d - d\Delta_n n^{d-1} + O(\Delta_n^2 n^{d-2})$, so that using (4.6), and the hypothesis $\Delta_n = O(n^{1/3})$, and $0 \leq n - \|z\| \leq 1$

$$\begin{aligned} \|\mathbb{B}_{r_n}\| G_n(0, z) &= (r_n^d + O(r_n^{d-1})) \left(2 \frac{\alpha(z)}{n^{d-1}} + O\left(\frac{1}{n^d}\right) \right) \\ &= (n^d - d\Delta_n n^{d-1} + O(\Delta_n^2 n^{d-2}) + O(n^{d-1})) \left(2 \frac{\alpha(z)}{n^{d-1}} + O\left(\frac{1}{n^d}\right) \right) \\ &= 2\alpha(z)(n - d\Delta_n) + O(1) \end{aligned} \quad (4.7)$$

A martingale argument (see Lemma 2 of [6]) yields for a constant K_l

$$\left| \sum_{y \in \mathbb{B}_n} G_n(y, z) - 2\alpha(z)n \right| \leq K_l. \quad (4.8)$$

Proposition A.1 of the Appendix reads here as follows. There is K_b such that for $z \in \mathbb{B}_n$ with $n - \|z\| \leq 1$

$$\left| \sum_{y \in \mathcal{A}(r_n, n)} G_n(y, z) - 2\alpha_0(z) d\Delta_n \right| \leq K_b, \quad \text{where } \alpha_0(z) = \mathbb{E}_z [\|S(H_n)\| - \|z\| | H_n < H(B_{r_n})]. \quad (4.9)$$

Now, combining (4.8) and (4.9) we obtain (when $0 \leq n - \|z\| \leq 1$)

$$\left| \sum_{y \in \mathbb{B}_{r_n}} G_n(y, z) - 2n\alpha(z) + 2\alpha_0(z) d\Delta_n \right| \leq K_l + K_b. \quad (4.10)$$

We combine (4.6) and (4.10) we obtain for a constant K_2 ,

$$\left| \left(|\mathbb{B}_{r_n}| G_n(0, z) - \sum_{y \in \mathbb{B}_{r_n}} G_n(y, z) \right) + 2(\alpha_0(z) - \alpha(z)) d\Delta_n \right| \leq K_2. \quad (4.11)$$

We bound $|\alpha_0(z) - \alpha(z)|$ by the following expression

$$\mathbb{P}_z (H(B_{r_n}) < H_n) \times (\alpha_0(z) + \mathbb{E}_z [\|S(H_n)\| - \|z\| | H_n > H(B_{r_n})]). \quad (4.12)$$

Now, it is a classical estimate (see (A.5)) that there is K_0 such that for any $z \in \mathbb{A}(n-1, n)$,

$$\mathbb{P}_z (H(B_{r_n}) < H_n) \leq \frac{K_0}{\Delta_n}. \quad (4.13)$$

Thus,

$$\Delta_n |\alpha_0(z) - \alpha(z)| \leq 2\Delta_n \mathbb{P}_z (H(B_{r_n}) < H_n) \leq 2K_0. \quad (4.14)$$

The desired result follows at once. \blacksquare

4.2 Estimates on the harmonic measure

We recall a rough but useful result about the exit site distribution from a sphere. This is Lemma 1.7.4 of [5].

Lemma 4.4 *There are two positive constants c_1, c_2 such that for any $z \in \partial B(0, n)$, and $n > 0$*

$$\frac{c_1}{n^{d-1}} \leq \mathbb{P}_0(S(H_n) = z) \leq \frac{c_2}{n^{d-1}}. \quad (4.15)$$

When the starting point is not the origin, we first recall Lemma.5(b) of [7]. There is a constant J_d , such that for $z \in \mathbb{B}_r$ and $z^* \in \partial \mathbb{B}_r$, we have

$$P_z(S(H_r) = z^*) \leq \frac{J_d}{(\|z^*\| - \|z\|)^{d-1}}. \quad (4.16)$$

The following improvement of Lemma.5(b) of [7] will be needed later.

Lemma 4.5 *There is a positive constant κ_G such that, for all $r > 0$, if $z \in \mathbb{B}_r$, $z^* \in \partial\mathbb{B}_r$, then*

$$P_z(S(H_r) = z^*) \leq \frac{\kappa_G}{\|z - z^*\|^{d-1}}. \quad (4.17)$$

Proof. We adapt the proof of Lemma.5(b) of [7]. Set $D := \|z - z^*\|$. Let O' be a closest point from $(1 + \frac{D}{4r})z^*$ in $\mathbb{B}(z^*, \frac{D}{4})$. We define $B'_1 := B(O', \frac{D}{4})$, $B'_2 := B(O', \frac{D}{2})$, and we note that

$$\|z - z^*\| \leq \|z - O'\| \quad (4.18)$$

and, for all x in $\partial B'_2$, $\|z - z^*\| \leq \|z - x\| + \|x - z^*\|$ implies that

$$\min_{x \in \partial B'_2} \|z - x\| \geq \frac{D}{3}. \quad (4.19)$$

Now, define

$$\tau := \inf \{t > 0 : S(t) \in \{z\} \cup B_r^c\}, \quad \text{and} \quad \tau' := \inf \{t > 0 : S(t) \in B'_1 \cup \partial B'_2\}.$$

By a last exit decomposition together with the strong Markov property

$$\begin{aligned} P_z(S(H_r) = z^*) &= G_r(z, z) P_{z^*}(S(\tau) = z) \\ &\leq G_r(z, z) P_{z^*}(S(\tau') \in B'_2) \max_{x \in \partial B'_2} P_x(S(\tau) = z) \\ &= P_{z^*}(S(\tau') \in B'_2) \max_{x \in \partial B'_2} G_r(x, z) \\ &\leq P_{z^*}(S(\tau') \in B'_2) \max_{x \in \partial B'_2} G_{r+D}(x, z). \end{aligned} \quad (4.20)$$

A gambler's ruin estimate gives

$$P_{z^*}(S(\tau') \in B'_2) \leq \frac{c}{D} \quad (4.21)$$

for some positive constant c . Then we just have to prove that there is a constant c such that for all x satisfying $\|x - z\| \geq \frac{D}{3}$

$$G_{r+D}(x, z) \leq \frac{c}{D^{d-2}} \quad (4.22)$$

Let us define $V := B(z, \frac{D}{4})$. $G(x, \cdot)$ is harmonic on V and, by Harnack's inequality, we have

$$G_{r+D}(x, z) \leq \frac{c}{D^d} \sum_{y \in V} G_{r+D}(x, y) \leq \frac{c}{D^d} E_x[Y] \quad (4.23)$$

where c is a positive constant and Y is the local time associated with V and H_{r+D} :

$$Y := \sum_{t < H_{r+D}} \mathbf{1}_V(S(t)). \quad (4.24)$$

This gives

$$G_{r+D}(x, z) \leq \frac{c}{D^d} \sup_{y \in V} E_y[Y]. \quad (4.25)$$

It remains to show that $\sup_{y \in V} E_y[Y] \leq JD^2$, for some positive constant J . This is identical to (2.10) of [7], and we omit this last step. \blacksquare

We now prove our main result of this section.

Proof of Proposition 3.1

For $j \geq 0$, consider z_j in Σ_j . We show that for suitable positive constants α_1, α_2 , and for all z^* in $\mathcal{C}(z_j)$

$$\frac{\alpha_1}{h_j^d} \leq \mathbb{P}_{z_j}(S(\sigma_j) = z^*) \leq \frac{\alpha_2}{h_j^d} \quad (4.26)$$

We denote by S a random walk with $S(0) = z_j$. First, $S(\sigma_j) = z_j$ when $X_j = 1$. This happens with probability $1/h_j^d$, and gives the result when $z^* = z_j$. Now, consider $z^* \in \mathcal{C}(z_j) \setminus \{z_j\}$. We recall that the unbiased Bernoulli Y_j decides whether we flash on $\partial B(z_j, R_j)$ or on $\partial \mathbb{A}(r_j - R_j, r_j + R_j)$, where R_j has density g_j given in (3.6).

Step 1: Proof of the upper bound in (4.26). We assume that $S(\sigma_j) = z^*$ with $S(0) = z_j$. The following obvious facts follow from Lemma 2.10.

$$(i) \quad z^* \in \partial \mathbb{B}(z_j, \|z^* - z_j\|), \quad \text{and} \quad z^* \in \partial \mathbb{A}(r_j - \|z^*\| - r_j, r_j + \|z^*\| - r_j).$$

$$(ii) \quad z^* \notin \partial \mathbb{B}(z_j, \|z^* - z_j\| - 1), \quad \text{and} \quad z^* \notin \partial \mathbb{A}(r_j - \|z^*\| - r_j + 1, r_j + \|z^*\| - r_j - 1).$$

This means that if $Y_j = 1$, then $R_j \in]\|z^* - z_j\| - 1, \|z^* - z_j\|]$, whereas if $Y_j = 0$, then $R_j \in]\|z^*\| - r_j - 1, \|z^*\| - r_j]$. Also, there is a constant C such that

$$(i) \quad P(Y_j = 1, R_j \in]\|z^* - z_j\| - 1, \|z^* - z_j\|]) \leq C \frac{\|z^* - z_j\|^{d-1}}{h_j^d}, \quad (4.27)$$

and

$$(ii) \quad P(Y_j = 0, R_j \in]\|z^*\| - r_j - 1, \|z^*\| - r_j]) \leq C \frac{\|z^*\| - r_j^{d-1}}{h_j^d}. \quad (4.28)$$

In the case $z^* \in \partial \mathbb{B}(z_j, \partial R_j)$, the upper bound (4.29) then follows from (i) of (4.27), and (4.15) of Section 2.2. We consider now $Y_j = 0$. To simplify the notation we set for $h > 0$,

$$\mathcal{D}_h = \mathbb{A}(r_j - h, r_j + h), \quad \text{and} \quad \tilde{\mathcal{D}}_h = \mathbb{A}(r_j - \frac{h}{2}, r_j + \frac{h}{2}),$$

and define two stopping times

$$\tau = \inf \{n \geq 0 : S(n) \in \partial \mathcal{D}_h \cup \{z_j\}\}, \quad \text{and} \quad \tau^+ = \inf \{n \geq 1 : S(n) \in \mathcal{D}_h^c \cup \{z_j\}\}.$$

It is enough to prove that for some constant c , and for h such that $z^* \in \partial \mathcal{D}_h$, (and $h \in]\|z^*\| - r_j - 1, \|z^*\| - r_j]$)

$$\mathbb{P}_{z_j}(S(H(\mathcal{D}_h^c)) = z^*) \leq \frac{c}{h^{d-1}}. \quad (4.29)$$

Exiting from \mathcal{D}_h at z^* implies that the walk exit from $\partial \mathbb{B}(0, r_j + h)$ at z^* . Lemma 5(b) of [7] yields then (4.29).

Step 2: Proof of the lower bound in (4.26). Note the following two facts.

(iii) By Lemma 2.10, z^* has a nearest neighbor, say z ,

$$\|z - z_j\| \leq \|z^* - z_j\| - \frac{1}{4\sqrt{d}}.$$

This means that if $h \in [\|z^* - z_j\| - 1/(4\sqrt{d}), \|z^* - z_j\|]$, then $z^* \in \partial\mathbb{B}(z_j, h)$.

(iv) By Lemma 2.10, z^* has a nearest neighbor, say z ,

$$|\|z\| - r_j| \leq |\|z^*\| - r_j| - \frac{1}{4\sqrt{d}}.$$

This means that if $h \in [|\|z^*\| - r_j| - 1/(4\sqrt{d}), |\|z^*\| - r_j|]$, then $z^* \in \partial\mathcal{D}_h$.

We deal separately with the cases $|\|z^*\| - r_j| < h_j/2$ and $|\|z^*\| - r_j| \geq h_j/2$.

Consider first the case $|\|z^*\| - r_j| < h_j/2$.

$$\begin{aligned} P_{z_j}(S(\sigma_j) = z^*) &\geq P(Y_j = 1, R_j \in [\|z^* - z_j\| - 1/(4\sqrt{d}), \|z^* - z_j\|]) \\ &\quad \times \inf_{h \in [\|z^* - z_j\| - 1/(4\sqrt{d}), \|z^* - z_j\|]} \mathbb{P}_{z_j}(S(H(\partial\mathbb{B}(z_j, h))) = z^*) \\ &\geq \frac{ch_j^{d-1}}{h_j^d} \inf_{h \in [\|z^* - z_j\| - 1/(4\sqrt{d}), \|z^* - z_j\|]} \mathbb{P}_{z_j}(S(H(\partial\mathbb{B}(z_j, h))) = z^*) \\ &\geq \frac{c}{h_j^d}. \end{aligned} \tag{4.30}$$

Thus, for some constant α_1 (that depends on d), we have (4.26).

Consider now the case $|\|z^*\| - r_j| \geq h_j/2$. It is enough to prove, for h such that $z^* \in \partial\mathcal{D}_h$, and for some constant c (that depends on d)

$$P_{z_j}(S(H(\mathcal{D}_h^c)) = z^*) \geq \frac{c}{h_j^{d-1}}. \tag{4.31}$$

Let y^* be the closest site of $\partial\mathbb{B}(0, r_j)$ to the segment $[0, z^*]$, and x^* be the closest site of $\partial\mathbb{B}(0, r_j + h/2)$ to the segment $[0, z^*]$. We set $\Gamma = \mathbb{B}(x^*, \|z^* - x^*\|) \cap \tilde{\mathcal{D}}_h$. It may be that $\mathbb{B}(x^*, \|z^* - x^*\|) \cap \mathcal{D}_h^c \neq \emptyset$, and if so, one would only have to consider a site at a distance 1 from z^* , say $\tilde{z} \in \mathcal{D}_h$, and such that $\mathbb{B}(x^*, \|\tilde{z} - x^*\|) \cap \mathcal{D}_h^c = \emptyset$, and work with \tilde{z} instead of z^* in the sequel. We assume henceforth that $\mathbb{B}(x^*, \|z^* - x^*\|) \cap \mathcal{D}_h^c = \emptyset$.

We set $\Gamma' = \Gamma \cap \partial(\tilde{\mathcal{D}}_h^c)$, and use a last exit decomposition, and the strong Markov property to get

$$\begin{aligned} \mathbb{P}_{z_j}(S(H(\partial\mathcal{D}_h)) = z^*) &= G_{\mathcal{D}_h}(z_j, z_j) \mathbb{P}_{z^*}(S(\tau^+) = z_j) \\ &\geq G_{\mathcal{D}_h}(z_j, z_j) \mathbb{P}_{z^*}(H(\Gamma) < \tau^+) \min_{x \in \Gamma'} \mathbb{P}_x(S(\tau) = z_j) \\ &\geq \mathbb{P}_{z^*}(H(\Gamma) < \tau^+) \min_{x \in \Gamma'} G_{\mathcal{D}_h}(x, z_j). \end{aligned} \tag{4.32}$$

Since $z^* \in \mathcal{C}(z_j)$, y^* and z_j can be connected by 20 overlapping balls of radius $h_j/10$ in such a way that, applying Harnack's inequality 20 times (see Theorem 6.3.9 in [8]) to the

harmonic functions $y \mapsto G_{\mathcal{D}_h}(x, y)$, we can estimate from below the last factor in (4.32). For any $x \in \Gamma'$

$$G_{\mathcal{D}_h}(x, z_j) \geq c_H^{20} G_{\mathcal{D}_h}(x, y^*).$$

We use again Harnack's inequality once on the harmonic functions $x \mapsto G_{\mathcal{D}_h}(x, y^*)$, to obtain

$$\min_{x \in \Gamma'} G_{\mathcal{D}_h}(x, y^*) \geq c_H G_{\mathcal{D}_h}(x', y^*),$$

where $x' \in \Gamma'$ and $\|x' - y^*\| \in [\frac{h}{4} - 1, \frac{h}{4}]$. The purpose of choosing x' is that $\mathbb{B}(x', h/2) \in \mathcal{D}_h$, $y \in \mathbb{B}(x', h/4)$, and $G_{\mathcal{D}_h}(x', y^*) \geq G_{\mathbb{B}(x', h/2)}(x', y^*)$. When dimension is 2, the classical expansion of $G_{\mathbb{B}(x', h/2)}(x', \cdot)$ (see Proposition 6.3.5 of [8]) gives with a constant K_2

$$G_{\mathbb{B}(x', h/2)}(x', y^*) \geq \frac{2}{\pi} \log \left(\frac{h/2}{\|x' - y^*\|} \right) - \frac{K_2}{\|x' - y^*\|} \geq \frac{2}{\pi} \log 2 - \frac{4K_2}{h}. \quad (4.33)$$

When h is large enough, $G_{\mathbb{B}(x', h/2)}(x', y^*) \geq \log(2)/\pi$.

When dimension $d > 2$, there is a constant K_d such that, when h is large enough

$$\begin{aligned} G_{\mathcal{D}_h}(x', y^*) &= G(x', y^*) - \mathbb{E}_{x'} [G(S(H(\mathcal{D}_h^c)), y^*)] \\ &\geq \frac{C_d}{h^{d-2}} (4^{d-2} - 1) - \frac{K}{h_j^d} \geq \frac{K_d}{h_j^{d-2}}. \end{aligned} \quad (4.34)$$

As a consequence of (4.34), we just need to prove that the first factor in (4.32) is of order $1/h_j$ at least. We realize the event $\{H(\Gamma) < \tau^+\}$ in two moves: we first hit the sphere $\mathbb{B}(x^*, R^*/2)$, and then we exit from the cap $\partial\mathbb{B}(x^*, R^*)$ which lies in $\tilde{\mathcal{D}}_h$.

$$\begin{aligned} \mathbb{P}_{z^*} (H(\Gamma) < \tau^+) &\geq \mathbb{P}_{z^*} (H(\mathbb{B}(x^*, R^*/2)) < H(\mathbb{B}^c(x^*, R^*))) \\ &\quad \inf_{y \in \partial\mathbb{B}(x^*, R^*/2)} \mathbb{P}_y \left(H(\mathbb{B}^c(x^*, R^*)) = H(\mathbb{B}^c(x^*, R^*) \cap \tilde{\mathcal{D}}_h) \right). \end{aligned} \quad (4.35)$$

We invoke again Harnack's inequality to have for $y \in \partial\mathbb{B}(x^*, R^*/2)$

$$\begin{aligned} \mathbb{P}_y \left(H(\mathbb{B}^c(x^*, R^*) \cap \tilde{\mathcal{D}}_h) = H(\mathbb{B}^c(x^*, R^*)) \right) \\ \geq c_H \mathbb{P}_{x^*} \left(H(\mathbb{B}^c(x^*, R^*)) = H(\tilde{\mathcal{D}}_h \cap \mathbb{B}^c(x^*, R^*)) \right). \end{aligned} \quad (4.36)$$

We invoke now (4.15) to obtain for some constant K_3

$$\mathbb{P}_0 \left(H(\mathbb{B}^c(x^*, R^*) \cap \tilde{\mathcal{D}}_h) = H(\mathbb{B}^c(x^*, R^*)) \right) \geq c_1 \frac{|\partial\mathbb{B}(x^*, R^*) \cap \tilde{\mathcal{D}}_h|}{|\partial\mathbb{B}(x^*, R^*)|} \geq K_3. \quad (4.37)$$

We gather (4.35), (4.36) and (4.37) to obtain the desired lower bound.

5 The flashing process fluctuations

In this section we prove Theorems 1.3 and 1.4. To do so we use the construction in terms of exploration waves of Section 3.2.

5.1 Tiles

We recall that we have defined a *cell* of \mathcal{S}_j in (3.9), as the intersection of a cone with \mathcal{S}_j . We need also a smaller shape. We define, for any z_j in Σ_j , and ϵ_0 small to be defined below in (5.5),

$$\tilde{\mathcal{C}}(z_j) = \mathcal{S}_j \cap \{x \in \mathbb{R}^d : \exists \lambda \geq 0, \exists y \in B(z_j, \epsilon_0 h_j), x = \lambda y\}. \quad (5.1)$$

As in Lemma 12 in [6], concerning locally finite coverings, we claim that, for h_0 large enough, there exist a positive constants c_1 , and, for each $j \geq 0$, a subset $\tilde{\Sigma}_j$ of Σ_j such that for a constant K_F

$$\forall y \in \mathcal{S}_j \quad \left| \left\{ z \in \tilde{\Sigma}_j : y \in \tilde{\mathcal{C}}(z) \right\} \right| \leq K_F \quad \text{and} \quad \mathcal{S}_j = \bigcup_{z_j \in \tilde{\Sigma}_j} \tilde{\mathcal{C}}(z_j). \quad (5.2)$$

For any $z_j \in \Sigma_j$, we call *tile centered at z_j* , the intersections of $\tilde{\mathcal{C}}(z_j)$ with Σ_j . We denote by $\mathcal{T}(z_j)$ a *tile centered at z_j* , and by \mathcal{T}_j the set of tiles associated with the shell \mathcal{S}_j :

$$\mathcal{T}_j = \left\{ \mathcal{T}(z_j) : z_j \in \tilde{\Sigma}_j \right\}. \quad (5.3)$$

Let us explain the reasons for $\epsilon_0 h_j$ in the definition of a tile. It is linked with two fundamental features of the flashing construction. First, for any $z \in \mathcal{S}_j$, there is $\tilde{z}_j \in \tilde{\Sigma}_j$ such that

$$z \in \bigcap \{ \mathcal{C}(y) : y \in \mathcal{T}(\tilde{z}_j) \}. \quad (5.4)$$

Indeed, let $z_j \in \Sigma_j$ be a site realizing the minimum of $\{\|z - y\| : y \in \Sigma_j\}$. There is $\lambda > 0$ and $u \in B(z_j, 1)$, such that $z = \lambda u$. Now, there is $\tilde{z}_j \in \tilde{\Sigma}_j$ such that $\|\tilde{z}_j - z_j\| < \epsilon_0 h_j$, and for any $y \in \mathcal{T}(\tilde{z}_j)$, we have $\|y - z_j\| < 2\epsilon_0 h_j$. Thus, for ϵ_0 small enough so that $1 + 2\epsilon_0 h_j \leq h_j/2$

$$\forall y \in \mathcal{T}(\tilde{z}_j) \quad |u - y| \leq |u - z_j| + |z_j - y| \leq 1 + 2\epsilon_0 h_j \leq \frac{h_j}{2} \quad \text{which implies (5.4).}$$

Secondly, the size of a tile should be such that for some $\kappa > 1$, for any $k \geq 1$, and any tile $\mathcal{T} \in \mathcal{T}_k$

$$\sup_{z \in \mathbb{B}(0, r_k - h_k)} P_z(S(H(\Sigma_k)) \in \mathcal{T}) \leq \frac{\kappa - 1}{\kappa}. \quad (5.5)$$

The inequality (5.5) follows from Lemma 5(b) of [6] which for a constant J_d yields

$$\sup_{z \in \mathbb{B}(0, r_k - h_k)} P_z(S(H(\Sigma_k)) \in \mathcal{T}) \leq J_d \frac{|\mathcal{T}|}{h_k^{d-1}}. \quad (5.6)$$

The choice of ϵ_0 is such that $J_d |\mathcal{T}| \leq \frac{\kappa - 1}{\kappa} h_k^{d-1}$.

5.2 The inner ball

For $n \geq 0$, we take $N = |\mathbb{B}_n|$, we recall that $\mathcal{A}_k^*(N)$ are increasing clusters, and $\mathcal{A}^*(N) = \bigcup_{k \geq 1} \mathcal{A}_k^*(N)$. We consider

$$T^* = \min \{k \geq 1 : \cup_{j < k} \mathcal{S}_j \not\subset \mathcal{A}_k^*(N)\}. \quad (5.7)$$

Note that $\mathcal{A}_k^*(N) \subset \cup_{j < k} \mathcal{S}_j$, so that T^* is the first time, where time is intended as counting the number of exploration waves, the k^{th} wave does not cover all its allowed space. For the flashing process if $\cup_{j < k} \mathcal{S}_j \not\subset \mathcal{A}_k^*(N)$, then for any $k' > k$, we have $\cup_{j < k} \mathcal{S}_j \not\subset \mathcal{A}_{k'}^*(N)$, so that T^* is also the shell label where the first hole of $\mathcal{A}^*(N)$ appear. We have, for l with $r_l < n$,

$$P(T^* \leq l) = P(\mathbb{B}(0, r_l + h_l) \not\subset \mathcal{A}^*(N)) \leq \sum_{k \leq l} P(T^* = k + 1). \quad (5.8)$$

In this section, we estimate from above the probability $P(T^* = k + 1)$ assuming $r_k < n$.

For $k \geq 1$ and $\Lambda \subset \Sigma_k$, we call $W_k(\Lambda)$ the number of unsettled explorers that stand in Λ after the k -th wave, that is

$$W_k(\Lambda) = \sum_{i=1}^N \mathbf{1}_{\Lambda}(\xi_k(i)). \quad (5.9)$$

We now look at the *crossings* of tiles of \mathcal{T}_k . On one hand, we will use that if $W_k(\mathcal{T})$ is *large*, then it is unlikely that a hole appears in the cell containing \mathcal{T} during the $k + 1^{\text{th}}$ -wave. We use for this purpose the fact that covering for the flashing process is similar to filling an album for a coupon-collector model. On the other hand, if r_k is *small* it is unlikely that $W_k(\mathcal{T})$ is *small*. We now make precise what we intend by *small* and *large*. For any positive constant ξ , we write

$$\begin{aligned} P(T^* = k + 1) &= P(T^* = k + 1, \forall \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) \geq \xi) \\ &\quad + P(T^* = k + 1, \exists \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) < \xi) \\ &\leq P(T^* = k + 1 | \forall \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) \geq \xi) + P(\exists \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) < \xi). \end{aligned} \quad (5.10)$$

A coupon-collector estimate. The first term in the right hand side of (5.10) is bounded using a simple coupon-collector argument. Indeed, the event $\{T^* = k + 1\}$ implies that there is one uncovered site in \mathcal{S}_k , say z , when explorers stopped in Σ_k are released. By (5.4), there is $z_k \in \tilde{\Sigma}_k$, such that z is a possible settling position of all explorers stopped in $\mathcal{T}(z_k)$. Now, knowing that $\{W_k(\mathcal{T}(z_k)) \geq \xi\}$, Proposition 3.1 tells us that the probability of not covering this site is less than $(1 - \alpha_2/h_k^d)$ to the power ξ . In other words,

$$P(T^* = k + 1 | \forall \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) \geq \xi) \leq |\mathcal{S}_k| \left(1 - \frac{\alpha_2}{h_k^d}\right)^\xi \leq |\mathcal{S}_k| \exp\left(-\alpha_2 \frac{\xi}{h_k^d}\right). \quad (5.11)$$

Henceforth, we set

$$\xi = Ah^d \log(n) \quad \text{with} \quad h = \sup\{h_k : r_k \leq n\} \quad (5.12)$$

and A large enough so that

$$\sum_{k: r_k < n} P(T^* = k + 1 | \forall \mathcal{T} \in \mathcal{T}_k, W_k(\mathcal{T}) \geq \xi) \leq |\mathbb{B}_n| \exp(-\alpha_2 A \log n) \leq \frac{1}{n^2}. \quad (5.13)$$

Estimating $\{W_k(\mathcal{T}) < \xi\}$. For any $\mathcal{T} \in \mathcal{T}_k$, we consider the counting variable $L_k(\mathcal{T}) = M(\mathbb{B}(0, r_k - h_k), r_k, \mathcal{T})$, and define

$$M_k(\mathcal{T}) = W_k(\mathcal{T}) + M(A_k^*, r_k, \mathcal{T}), \quad \text{so that} \quad M_k(\mathcal{T}) \stackrel{\text{law}}{=} M(N\mathbf{1}_{\{0\}}, r_k, \mathcal{T}). \quad (5.14)$$

The idea of defining M_k and L_k (for the internal DLA process), and bounding W_k by $M_k - L_k$, is introduced in [7]. Our main observation is that $L_k(\mathcal{T})$ is independent of $W_k(\mathcal{T})$, and

$$W_k(\mathcal{T}) + L_k(\mathcal{T}) \geq M_k(\mathcal{T}). \quad (5.15)$$

As a consequence, for any positive constants t and ξ (and with the notation $\bar{X} = X - E[X]$),

$$\begin{aligned} P(W_k(\mathcal{T}) < \xi) &\leq e^{t\xi} \times E[\exp(-tW_k(\mathcal{T}))] \\ &\leq e^{t\xi} \frac{E[\exp(-t(W_k(\mathcal{T}) + L_k(\mathcal{T})))]}{E[\exp(-tL_k(\mathcal{T}))]} \\ &\leq \exp(-t(E[M_k(\mathcal{T}) - L_k(\mathcal{T})] - \xi)) \times \frac{E[\exp(-t(\bar{M}_k(\mathcal{T})))]}{E[\exp(-t\bar{L}_k(\mathcal{T}))]}. \end{aligned} \quad (5.16)$$

Using Lemma 2.2(b) with condition (5.5), we obtain

$$\begin{aligned} \log P(W_k(\mathcal{T}) < \xi) &\leq -t(E[M_k(\mathcal{T}) - L_k(\mathcal{T})] - \xi) + f(-t)E[M_k(\mathcal{T}) - L_k(\mathcal{T})] \\ &\quad + \frac{\kappa}{2}g(-t) \sum_{y \in \mathbb{B}(0, r_k - h_k)} P_y^2(S(H(\Sigma_k)) \in \mathcal{T}). \end{aligned} \quad (5.17)$$

We now proceed in two steps. We show in Step 1 that for some constant κ'

$$E[M_k(\mathcal{T}) - L_k(\mathcal{T})] \geq \kappa'(n^d - (r_k - h_k)^d) \frac{h_k^{d-1}}{r_k^{d-1}}. \quad (5.18)$$

As a consequence there is a constant $\kappa_1 > 0$ such that, for all $\alpha > 0$ and with $k_\alpha := \sup\{j \in \mathbb{N} : r_j < n - \alpha \log n\}$,

$$\inf_{k \leq k_\alpha} E[M_k(\mathcal{T}) - L_k(\mathcal{T})] \geq \kappa_1 \alpha h^d \log n$$

With ξ as in (5.12), $\alpha = 2A/\kappa_1$ and $k^* = k_\alpha$, i.e.,

$$k^* := \sup \left\{ j \in \mathbb{N} : r_j \leq n - \frac{2A}{\kappa_1} h \log(n) \right\},$$

we get, for all $k \leq k^*$,

$$E[M_k(\mathcal{T}) - L_k(\mathcal{T})] \geq 2\xi. \quad (5.19)$$

We show in Step 2, that for a constant C

$$\sum_{y \in \mathbb{B}(0, r_k - h_k)} P_y^2(S(H(\Sigma_k)) \in \mathcal{T}) \leq C E[M_k(\mathcal{T}) - L_k(\mathcal{T})]. \quad (5.20)$$

Suppose for a moment that Step 1 and Step 2 hold. Since $\max(f(-t), g(-t)) \leq ct^2$ when $t \leq 1$, there is $c' > 0$ such that for $k \leq k^*$ and an appropriate t ,

$$\begin{aligned} \log P(W_k(\mathcal{T}) < Ah^d \log(n)) &\leq \left(-\frac{t}{2} + c\left(1 + \frac{C\kappa}{2}\right)t^2\right) E[M_k(\mathcal{T}) - L_k(\mathcal{T})] \\ &\leq -c'E[M_k(\mathcal{T}) - L_k(\mathcal{T})] \leq -2c'Ah^d \log(n). \end{aligned} \quad (5.21)$$

Now, using (5.10), (5.13) and (5.21) for A large enough, we have

$$\sum_{k < \bar{k}} P(T^* = k) \leq \frac{2}{n^2}. \quad (5.22)$$

Borel-Cantelli's lemma yields then the inner control of Theorem 1.3.

Step 1. We invoke Theorem 4.1, with $n = r_k$, and $\Delta_n = h_k$ (the hypotheses $h_k = O(r_k^{1/3})$ and h_k large enough hold here, as seen in the first paragraph of section 3.1). We have for some positive constants κ' , K , and for n large enough

$$\begin{aligned} E[M_k(\mathcal{T}) - L_k(\mathcal{T})] &= E[M(|\mathbb{B}_n| - |\mathbb{B}_{r_k - h_k}|)\mathbf{1}_0, r_k, \mathcal{T}] \\ &\quad + E[M(|\mathbb{B}_{r_k - h_k}|)\mathbf{1}_0, r_k, \mathcal{T}] - E[M(\mathbb{B}_{r_k - h_k}, r_k, \mathcal{T})] \\ &\geq (|\mathbb{B}_n| - |\mathbb{B}_{r_k - h_k}|) \mathbb{P}_0(S(H_k) \in \mathcal{T}) - Kh_k^{d-1} \\ &\geq 2\kappa'(n^d - (r_k - h_k)^d) \frac{h_k^{d-1}}{r_k^{d-1}} - Kh_k^{d-1} \\ &\geq \kappa'(n^d - (r_k - h_k)^d) \frac{h_k^{d-1}}{r_k^{d-1}}, \end{aligned} \quad (5.23)$$

for $r_k \leq n$ and h_0 large enough.

Step 2. We recall that Lemma 4.5 gives a constant κ_G such that, for $y \in \mathbb{B}(0, r_k - h_k)$, and $z \in \tilde{\Sigma}_k$

$$P_y(S(H(\Sigma_k)) \in \mathcal{T}(z)) \leq \frac{\kappa_G |\mathcal{T}(z)|}{|z - y|^{d-1}}. \quad (5.24)$$

Therefore

$$\sum_{y \in \mathbb{B}(0, r_k - h_k)} P_y^2(S(H(\Sigma_k)) \in \mathcal{T}(z)) \leq \sum_{k: h_k \leq k \leq 2r_k} \sum_{y: k \leq |z - y| < k+1} \frac{\kappa_G^2 |\mathcal{T}(z)|^2}{k^{2(d-1)}}. \quad (5.25)$$

For a constant C_d , we bound $|\{y : k \leq |z - y| < k + 1\}| \leq C_d k^{d-1}$. Thus,

$$\begin{aligned} \sum_{y \in \mathbb{B}(0, r_k - h_k)} P_y^2(S(H(\Sigma_k)) \in \mathcal{T}(z)) &\leq \sum_{k: h_k \leq k \leq 2r_k} \frac{C_d \kappa_G^2 |\mathcal{T}(z)|^2}{k^{d-1}} \\ &\leq C' |\mathcal{T}(z)|^2 \left(\mathbb{1}_{d=2} \log(n) + \mathbb{1}_{d>2} \frac{1}{h_k^{d-2}} \right). \end{aligned} \quad (5.26)$$

Since $|\mathcal{T}(z)|$ is of order h_k^{d-1} , (5.20) holds.

5.3 The outer ball

This section follows closely [6]. The features of the flashing process allow for some simplification. We keep the notation of the previous section. There, we proved that for some integer k^*

$$P(T^* > k^*) = 1 - \epsilon^\delta(n), \quad \text{with} \quad \sum_{n \geq 1} \epsilon^\delta(n) < +\infty.$$

The integer k^* is the largest such that $r_{k^*} \leq n - 2Ah \log(n)/\kappa_1$, for a large constant A . As consequence, the following conditional law can be seen as a slight modification of P .

$$P^\delta(\cdot) = P(\cdot | \{T^* > k^*\}) \quad (5.27)$$

We begin by proving that under P^δ the probability to find some k with $n \leq r_k < 2n$ and some tile \mathcal{T} in \mathcal{T}_k with $W_k(\mathcal{T})$ larger than or equal to $\xi' = 2A'h^d \log n$ for a large enough A' decreases faster than any given power of n . First, note that on $\{T^* > k^*\}$

$$W_k(\mathcal{T}) + L_k^\delta(\mathcal{T}) \leq M_k(\mathcal{T}), \quad \text{with} \quad L_k^\delta = M(\mathbb{B}(0, r_{k^*} - h_{k^*}), r_k, \mathcal{T}). \quad (5.28)$$

Our key observation is that $W_k(\mathcal{T}) \mathbb{1}_{\{T^* > k^*\}}$ is independent of L_k^δ . Thus, for any $t > 0$,

$$\begin{aligned} P(W_k(\mathcal{T}) \geq \xi', T^* > k^*) &\leq e^{-t\xi} E \left[e^{tW_k(\mathcal{T})} \mathbb{1}_{\{T^* > k^*\}} \right] \\ &\leq e^{-t\xi} \frac{E \left[\exp(t(W_k(\mathcal{T}) + L_k^\delta)) \mathbb{1}_{\{T^* > k^*\}} \right]}{E \left[e^{tL_k^\delta} \right]} \\ &\leq e^{-t\xi} \frac{E \left[e^{tM_k(\mathcal{T})} \right]}{E \left[e^{tL_k^\delta} \right]} \\ &\leq \exp(-t(\xi - E[M_k(\mathcal{T}) - L_k^\delta])) \times \frac{E \left[e^{t\bar{M}_k(\mathcal{T})} \right]}{E \left[e^{tL_k^\delta} \right]}. \end{aligned} \quad (5.29)$$

By Lemma 2.2(a), we have

$$\begin{aligned} \log P(W_k(\mathcal{T}) \geq \xi', T^* > k^*) &\leq -t(\xi' - E[M_k(\mathcal{T}) - L_k^\delta]) + f(t) \times E[M_k(\mathcal{T}) - L_k^\delta] \\ &\quad + g(t) \times \sum_{y \in \mathbb{B}(0, r_{k^*} - h_{k^*})} P_y^2(S(H(\Sigma_k)) \in \mathcal{T}). \end{aligned} \quad (5.30)$$

The steps are now similar to the previous proof. We first estimate $E[M_k(\mathcal{T}) - L_k^\delta]$. By Theorem 4.1, for some positive constant K' and for n large enough,

$$\begin{aligned} E[M_k(\mathcal{T}) - L_k^\delta(\mathcal{T})] &\leq K' (n^d - (r_{k^*})^d) \frac{h_k^{d-1}}{r_k^{d-1}} + O(h_k^{d-1}) \\ &\leq K' dn^{d-1} (n - r_{k^*}) \frac{h_k^{d-1}}{r_k^{d-1}} + O(h_k^{d-1}). \end{aligned} \quad (5.31)$$

Note that since $r_k \leq 2n$, we have $r_k^{d-1} = o(n^{d-1}(n - r_{k^*}))$ so that $O(h_k^{d-1})$ is small compared to the first term in (5.31). Since $k \mapsto h_k/r_k$ is decreasing, we have for some constant K

$$E [M_k(\mathcal{T}) - L_k^\delta(\mathcal{T})] \leq Kh^d \log n.$$

Secondly, we estimate the sum of $P_y^2(S(H(\Sigma_k)) \in \mathcal{T})$ which appears on (5.30). We use (5.26) to obtain for a constant C

$$\sum_{y \in \mathbb{B}(0, r_{k^*} - h_{k^*})} P_y^2(S(H(\Sigma_k)) \in \mathcal{T}) \leq Ch_k^{2(d-1)} \left(\mathbb{1}_{d=2} \log(n) + \mathbb{1}_{d>2} \frac{1}{(r_k - (r_{k^*} - h_{k^*}))^{d-2}} \right). \quad (5.32)$$

Note that $r_k - (r_{k^*} - h_{k^*}) \geq h_k$, and, since $k \mapsto h_k/r_k$ is decreasing, we have, for n large enough, $h_k \leq h_{k^*}(r_k/r_{k^*}) \leq h \times (2n)/(n/2)$. Thus,

$$\sum_{y \in \mathbb{B}(0, r_{k^*} - h_{k^*})} P_y^2(S(H(\Sigma_k)) \in \mathcal{T}) \leq C \left(\mathbb{1}_{d=2} h^2 \log(n) + \mathbb{1}_{d>2} h^d \right) \leq Ch^d \log(n). \quad (5.33)$$

We choose $A' = 2K$ to obtain

$$\log P(W_k(\mathcal{T}) \geq \xi', T^* > k^*) \leq -(Kt - Kf(t) - Cg(t)) h^d \log(n). \quad (5.34)$$

Since we have $P(T^* > k^*) \geq 1/2$ for A large enough and K can be taken as large as we want, we have that $P^\delta(\{W_k(\mathcal{T}) \geq \xi'\})$ decreases faster than any given power of n .

Now, let F_k denote the event that no tile \mathcal{T} in Σ_k contains more than $\xi' = 2A'h^d \log n$ unsettled explorers after the k -th exploration wave. We recall that $\mathcal{G}_k = \sigma(\xi_0, \dots, \xi_k)$, and note that F_k and $\{\mathbb{B}(0, n - \delta h^d \log(n)) \subset \mathcal{A}^*\}$ are \mathcal{G}_k -measurable.

For any tile $\mathcal{T} \in \mathcal{T}_k$, let $z_k \in \tilde{\Sigma}_k$ be such that $\mathcal{T} = \mathcal{T}(z_k)$ and denote by $\tilde{\mathcal{C}} = \tilde{\mathcal{C}}(z_k)$. We are entitled, by Proposition 3.1, to use a coupon-collector estimate on the number of settled explorers during the $k+1$ -th exploration wave. On $F_k \cap \{\mathbb{B}(0, n - \delta h^d \log(n)) \subset \mathcal{A}^*\}$, and for some positive constant K_1

$$\begin{aligned} E \left[|\mathcal{A}_{k+1}^* \cap \tilde{\mathcal{C}}| \middle| \mathcal{G}_k \right] &\geq |\tilde{\mathcal{C}}| \left(1 - \left(1 - \frac{\alpha_1}{h_k^d} \right)^{W_k(\mathcal{T})} \right) \\ &\geq |\tilde{\mathcal{C}}| \left(1 - \exp \left\{ -\alpha_1 \frac{W_k(\mathcal{T})}{h_k^d} \right\} \right) \\ &\geq \frac{|\tilde{\mathcal{C}}|}{h_k^d} W_k(\mathcal{T}) \frac{h_k^d}{W_k(\mathcal{T})} \left(1 - \exp \left\{ -\alpha_1 \frac{W_k(\mathcal{T})}{h_k^d} \right\} \right) \\ &\geq K_1 W_k(\mathcal{T}) \inf_{x \leq 2A' \log n} \frac{1 - e^{-\alpha_1 x}}{x}. \end{aligned} \quad (5.35)$$

We now write for some positive constant K_2

$$\begin{aligned} \inf_{x \leq 2A' \log n} \frac{1 - e^{-\alpha_1 x}}{x} &\geq \frac{1}{2A' \log n} \inf_{x \leq 2A' \log n} \frac{1 - e^{-\alpha_1 x/2A' \log n}}{x/2A' \log n} \\ &\geq \frac{1}{2A' \log n} \inf_{x \leq 1} \frac{1 - e^{-\alpha_1 x}}{x} \geq \frac{K_2}{\log n}. \end{aligned}$$

We conclude that on $F_k \cap \{\mathbb{B}(0, n - \delta h^d \log(n)) \subset \mathcal{A}^*\}$,

$$E \left[\mathcal{A}_{k+1}^* \cap \tilde{\mathcal{C}} \middle| \mathcal{G}_k \right] \geq K_1 K_2 \frac{W_k(\mathcal{T})}{\log n}. \quad (5.36)$$

Recall now that property (5.2) implies that $K_F |\mathcal{A}_{k+1}^* \cap \mathcal{S}_k| \geq \sum_{z \in \tilde{\Sigma}_k} |\mathcal{A}_{k+1}^* \cap \tilde{\mathcal{C}}(z)|$. Thus, summing over $\mathcal{C}(z)$ with $z \in \tilde{\Sigma}_k$ in (5.36), we obtain on $F_k \cap \{\mathbb{B}(0, n - \delta h^d \log(n)) \subset \mathcal{A}^*\}$

$$E \left[|\mathcal{A}_{k+1}^* \cap \mathcal{S}_k| \middle| \mathcal{G}_k \right] \geq K \frac{W_k(\mathcal{S}_k)}{\log n}, \quad \text{where} \quad K = \frac{K_1 K_2}{K_F}.$$

Also, since $W_k(\mathcal{S}_k) \leq |\mathbb{B}(0, n)|$,

$$E \left[\mathbf{1}_{F_k \cap \{\mathbb{B}(0, n - \delta h^d \log(n)) \subset \mathcal{A}^*\}} |\mathcal{A}_{k+1}^* \cap \mathcal{S}_k| \right] \geq K \frac{E[\mathbf{1}_{\{\mathbb{B}(0, n - \delta h^d \log(n)) \subset \mathcal{A}^*\}} W_k(\mathcal{S}_k)]}{\log n} - n^d P(F_k^c).$$

Since $P(\{\mathbb{B}(0, n - \delta h^d \log(n)) \subset \mathcal{A}^*\}) \geq 1/2$,

$$E^\delta [|\mathcal{A}_{k+1}^* \cap \mathcal{S}_k|] \geq K \frac{E^\delta[W_k(\mathcal{S}_k)]}{\log n} - 2n^d P(F_k^c). \quad (5.37)$$

In other words, noting that $|\mathcal{A}_{k+1}^* \cap \mathcal{S}_k| = W_k(\mathcal{S}_k) - W_{k+1}(\mathcal{S}_{k+1})$

$$E^\delta [W_{k+1}(\mathcal{S}_{k+1})] \leq \left(1 - \frac{K}{\log n}\right) E^\delta [W_k(\mathcal{S}_k)] + 2n^d P(F_k^c). \quad (5.38)$$

By iterating (5.38), and using our previous estimate on $P^\delta(W_k(\mathcal{T}) \geq \xi')$, we obtain that for a large enough ϵ , $E^\delta[W_{l+\epsilon \log^2 n}(\mathcal{S}_{l+\epsilon \log^2 n})]$, is summable, when l is the lowest index for which $r_l \geq n$. Also, the probability (under P !) of seeing at least one explorer reaching the shell $\mathcal{S}_{l+\epsilon \log^2 n}$ is summable. Using Borel-Cantelli lemma, this yields the proof of Theorem 1.3.

5.4 Lower bound for the deviations

5.4.1 Proof of Theorem 1.4: the outer deviation

The lower bound for the outer error is trivial and based on two features of the flashing processes: (i) there is a positive ϵ_0 such that the probability of not flashing in any given shell is larger than ϵ_0 , (ii) the flashing times of the different explorers are independent.

Let us denote by \tilde{k} the largest index such that $\mathcal{S}_{\tilde{k}} \subset \mathbb{B}(0, n)$. Note that at time \tilde{k} , there are at least $|\mathcal{S}_{\tilde{k}}|$ explorers stopped on $\Sigma_{\tilde{k}}$. We need to show that the probability that all of the $W_{\tilde{k}}(\Sigma_{\tilde{k}})$ explorers settle in one of the shells $\{\mathcal{S}_j, \tilde{k} \leq j < \tilde{k} + b \log(n)\}$ is summable. Indeed, the latter event is implied by $A^*(N) \subset \cup_{j < \tilde{k} + b \log(n)} \mathcal{S}_j$, and the former probability is less than

$$\begin{aligned} & \left(1 - P \left(\text{one explorer starting on } \Sigma_{\tilde{k}} \text{ is unsettled at time } \tilde{k} + b \log(n) \right)\right)^{|\mathcal{S}_{\tilde{k}}|} \\ & \leq \left(1 - \epsilon_0^{b \log(n)}\right)^{|\mathcal{S}_{\tilde{k}}|}. \end{aligned} \quad (5.39)$$

When choosing b small enough, we see that the right hand side of (5.39) is the general term of a convergent series.

5.4.2 Proof of Theorem 1.4: the inner deviation

We now obtain a lower bound for the inner error. To simplify the computations, let us first define three positive constants c_1, c_2 and c_3 such that for any k with $n/2 \leq r_k \leq n$, we have

$$|\mathcal{S}_k| \leq c_1 h_k n^{d-1}, \quad \frac{h_k^d}{|\mathbb{B}(0, 6h_k) \cap \Sigma_k|} |\Sigma_k| \geq c_2 n^{d-1} h_k, \quad \text{and} \quad \forall z \in \Sigma_k, \quad |\tilde{\mathcal{C}}(z)| \geq c_3 h_k^d. \quad (5.40)$$

For α_2 given in (3.1), we choose

$$a_n = \frac{1}{4\alpha_2} \log(h_{\bar{k}}), \quad \text{and} \quad A_n = \left\lceil \frac{c_2 c_3}{2c_1} a_n \right\rceil. \quad (5.41)$$

Now we assume h_0 large enough to have $A_n > 0$.

Let time k^* be such that $r_{k^*+A_n} \leq n < r_{k^*+A_n+1}$. We show that $P(T^* > k^* + 1)$ decays faster than any polynomial in n . Since $T^* \leq k^* + 1$ implies that $\cup_{j \leq k^*} \mathcal{S}_j \not\subset A^*(N)$, this imply the desired lower bound.

On the event $\{T^* > k^* + 1\}$, we have $\mathbb{B}(0, r_{k^*} - h_{k^*}) = A_{k^*}^*(N)$ after the k^* -th wave, so that, for some constant K ,

$$W_{k^*}(\Sigma_{k^*}) = |\mathbb{B}(0, n)| - |\mathbb{B}(0, r_{k^*} - h_{k^*})| \leq c_1 A_n n^{d-1} \times h_{k^*}. \quad (5.42)$$

A key feature of the flashing process is that explorers stopped, at time k^* , outside $\mathbb{B}(z, 3h_{k^*}) \cap \Sigma_{k^*}$ cannot settle in $\tilde{\mathcal{C}}(z)$. In other words, knowing \mathcal{G}_k , the covering of a family of cells $\{\tilde{\mathcal{C}}(z_j), j = 1, \dots, \mathcal{N}\}$ are independent events if $\|z_i - z_j\| \geq 6h_{k^*}$ for $i \neq j$. Now, there is an integer \mathcal{N} and sites $\{z_j, j = 1, \dots, \mathcal{N}\}$ satisfying the former property. Since we necessarily have $\mathcal{N} |\mathbb{B}(0, 6h_{k^*}) \cap \Sigma_{k^*}| \geq |\Sigma_{k^*}|$, we get with (5.40)

$$\mathcal{N} h_{k^*}^d \geq c_2 h_{k^*} n^{d-1}. \quad (5.43)$$

Let

$$\Gamma = \{j \in [1, \mathcal{N}] : W_{k^*}(B(z_j, 3h_{k^*}) \cap \Sigma_{k^*}) \leq c_3 a_n h_{k^*}^d\}, \quad \text{and} \quad \Gamma^c = [1, \mathcal{N}] \setminus \Gamma.$$

On $\{T^* > k^* + 1\}$,

$$c_1 A_n n^{d-1} \times h_{k^*} \geq W_{k^*}(\Sigma_{k^*}) \geq \sum_{j \in \Gamma^c} W_{k^*}(B(z_j, 3h_{k^*}) \cap \Sigma_{k^*}) \geq |\Gamma^c| \times (c_3 a_n h_{k^*}^d). \quad (5.44)$$

Thus,

$$|\Gamma^c| \leq \frac{c_1 A_n h_{k^*} n^{d-1}}{c_3 a_n h_{k^*}^d} \leq \frac{c_1 c_2 c_3 a_n \mathcal{N}}{2c_3 c_1 a_n c_2} = \frac{\mathcal{N}}{2}. \quad (5.45)$$

In other words, we have that $|\Gamma| \geq \mathcal{N}/2$. Now, as already noticed, knowing \mathcal{G}_{k^*} , for any subset $I \subset [1, \mathcal{N}]$, the events $\{\tilde{\mathcal{C}}(z_j) \subset \mathcal{A}_{k^*+1}^*(N), j \in I\}$ are independent. By conditioning

on \mathcal{G}_{k^*} , we obtain

$$\begin{aligned}
 P(\{T^* > k^* + 1\}) &= E \left[\sum_{I \subset [1, \mathcal{N}], |I| \geq \mathcal{N}/2} \mathbb{1}_{\Gamma=I} \times P\left(\forall i \in I, \tilde{\mathcal{C}}(z_j) \subset \mathcal{A}_{k^*+1}^*(N) | \mathcal{G}_k\right) \right] \\
 &\leq E \left[\sum_{I \subset [1, \mathcal{N}], |I| \geq \mathcal{N}/2} \mathbb{1}_{\Gamma=I} \times \prod_{i \in I} P\left(\tilde{\mathcal{C}}(z_j) \subset \mathcal{A}_{k^*+1}^*(N) | \mathcal{G}_k\right) \right] \\
 &\leq \sup_{z_j \in \Sigma_{k^*}} P\left(\mathcal{A}_{k^*+1}^*(N) \supset \tilde{\mathcal{C}}(z_j), W_{k^*}(B(z_j, 3h_{k^*}) \cap \Sigma_{k^*}) \leq c_3 a_n h_{k^*}^d\right)^{\mathcal{N}/2}.
 \end{aligned} \tag{5.46}$$

Considering the probability appearing on the right hand side of (5.46), we can think of a coupon-collector problem, where an album of size $|\tilde{\mathcal{C}}(z)|$ has to be filled when we collect no more than $c_3 a_n h_{k^*}^d$ coupons. Using Lemma 5.1 below, we show that the probability of $\{T^* > k^* + 1\}$ is bounded by

$$\exp\left(-\frac{\alpha_1}{4} a_n^2 \frac{c_2}{2} h_{k^*}^{1-d} n^{d-1}\right).$$

This concludes the proof.

The result about filling an album, that we just mentioned, is based on the following simple coupon-collector lemma (together with Proposition 3.1), which we did not find in the vast literature on such problems.

Lemma 5.1 *Consider an album of L items for which are bought independent random coupons, each of them covering one (or possibly none) of the possible L items. If Y_i is the item associated with the i -th coupons, we assume that for positive constants α_1, α_2 , such that for any $j = 1, \dots, L$,*

$$\frac{\alpha_1}{L} \leq P(Y_i = j) \leq \frac{\alpha_2}{L}. \tag{5.47}$$

Let τ_L be the number of coupons needed to complete the album. Then, for any $0 < A < \frac{1}{4\alpha_2} \log(L)$, we have

$$P(\tau_L < AL) \leq \exp\left(-\frac{\alpha_1^2 A^2 e^{-2\alpha_2 A}}{4} \sqrt{L}\right) \leq \exp\left(-\frac{\alpha_1^2 A^2}{4}\right). \tag{5.48}$$

Proof. We denote by σ_i the time needed to collect the i -th distinct item after having collected $i - 1$ distinct items. The sequence $\{\sigma_1, \sigma_2, \dots, \sigma_L\}$ is not independent, but if $\mathcal{Y}_k = \sigma(\{Y_1, \dots, Y_k\})$, and $\tau(k) = \sigma_1 + \dots + \sigma_k$, then for $i = 1, \dots, L$

$$\left(1 - \frac{\alpha_1(L - i + 1)}{L}\right)^k \geq P(\sigma_i > k | \mathcal{Y}_{\tau(i-1)}) \geq \left(1 - \frac{\alpha_2(L - i + 1)}{L}\right)^k. \tag{5.49}$$

Indeed, calling $\mathcal{E}(i - 1)$ the set of the first $i - 1$ collected items,

$$\begin{aligned}
 P(\sigma_i > k | \mathcal{Y}_{\tau(i-1)}) &= P(\{Y_{\tau(i-1)+1}, \dots, Y_{\tau(i-1)+k}\} \subset \mathcal{E}(i-1) | \mathcal{Y}_{\tau(i-1)}) \\
 &= (P(Y \in \mathcal{E}(i-1) | \mathcal{Y}_{\tau(i-1)}))^k \\
 &= (1 - P(Y \notin \mathcal{E}(i-1) | \mathcal{Y}_{\tau(i-1)}))^k.
 \end{aligned} \tag{5.50}$$

Using (5.47) we deduce (5.49) from (5.50). Formula (5.49) gives that

$$\frac{L}{\alpha_1(L-i+1)} \geq E[\sigma_i | \mathcal{Y}_{\tau(i-1)}] \geq \frac{L}{\alpha_2(L-i+1)}, \quad (5.51)$$

as well as

$$E[\sigma_i^2 | \mathcal{Y}_{\tau(i-1)}] \leq 2 \frac{L^2}{\alpha_1^2(L-i+1)^2}. \quad (5.52)$$

Now, we look for $B \leq \sqrt{L}$ such that

$$\sum_{i=\sqrt{L}}^{B\sqrt{L}} E[\sigma_{L-i}] \geq 2AL. \quad (5.53)$$

Note that

$$\sum_{i=\sqrt{L}}^{B\sqrt{L}} E[\sigma_{L-i}] \geq \frac{L}{\alpha_2} \sum_{i=\sqrt{L}}^{B\sqrt{L}} \frac{1}{i+1} \geq \frac{L}{\alpha_2} \log(B).$$

Thus, condition (5.53) holds for $B \geq \exp(2\alpha_2 A)$, but recall that $B \leq \sqrt{L}$ also, and this gives a bound on A . Finally, note that

$$\max \left\{ E[\sigma_{L-i} | \mathcal{Y}_{\tau(L-i-1)}], i = \sqrt{L}, \dots, B\sqrt{L} \right\} \leq \frac{\sqrt{L}}{\alpha_1},$$

and set

$$X_i = \frac{E[\sigma_{L-i} | \mathcal{Y}_{\tau(L-i-1)}] - \sigma_{L-i}}{(\sqrt{L}/\alpha_1)} \leq 1.$$

For $x \leq 1$, note that $e^x \leq 1 + x + x^2$ to obtain for $0 \leq \lambda \leq 1$, by successive conditioning

$$\begin{aligned} P \left(\sum_{i=\sqrt{L}}^{B\sqrt{L}} \sigma_{L-i} \leq AL \right) &\leq P \left(\sum_{i=\sqrt{L}}^{B\sqrt{L}} X_i \geq \alpha_1 A \sqrt{L} \right) \\ &\leq e^{-\lambda \alpha_1 A \sqrt{L}} \prod_{i=\sqrt{L}}^{B\sqrt{L}} (1 + \lambda^2 \sup E[X_i^2 | \mathcal{Y}_{\tau(L-i-1)}]) \\ &\leq \exp \left(-\lambda \alpha_1 A \sqrt{L} + \lambda^2 \sum_i \sup E[X_i^2 | \mathcal{Y}_{\tau(L-i-1)}] \right). \end{aligned} \quad (5.54)$$

Finally, we have, using (5.52),

$$\sum_{i=\sqrt{L}}^{B\sqrt{L}} \sup E[X_i^2 | \mathcal{Y}_{\tau(L-i-1)}] \leq \sum_{i=\sqrt{L}}^{B\sqrt{L}} \alpha_1^2 \sup \frac{E[\sigma_{L-i}^2 | \mathcal{Y}_{\tau(L-i-1)}]}{L} \leq 2B\sqrt{L}. \quad (5.55)$$

The results follows as we optimize on $\lambda \leq 1$ in the upper bound in (5.54). ■

A Time spent in an annulus (By S.Blachère)

This appendix is devoted to an asymptotic expansion of the expected time spent in an annulus $\mathcal{A}(r_n, n)$ for $r_n < n$, when the random walk is started at some point z within the annulus, and before it exits the outer shell.

Proposition A.1 *Consider a sequence $\{\Delta_n, n \in \mathbb{N}\}$ with $K_0 \leq \Delta_n \leq Kn^{1/3}$ for some constants K_0, K . Let $r_n = n - \Delta_n$, and $z \in \mathcal{A}(r_n, n)$. There is a constant K_b , independent on z and n , such that*

$$\left| \sum_{y \in \mathcal{A}(r_n, n)} G_n(z, y) - (2d\Delta_n\alpha_0(z) - d(n - \|z\|)^2) \right| \leq K_b((n - \|z\|) \vee 1), \quad (\text{A.1})$$

with

$$\alpha_0(z) = \mathbb{E}_z [\|S(H_n)\| - \|z\| \mathbf{1}_{H(B^c(0, n)) < H(B(0, r_n))}].$$

Proof. Our strategy is to decompose a path into successive strands lying entirely in the annulus. The first strand is special since the starting point is any $z \in \mathcal{A}(r_n, n)$. The other strands, if any, start all on $\partial\mathbb{B}(0, r_n)$. We estimate the time spent inside the annulus for each strand. Let us remark that we make use of three facts: (i) precise asymptotics for Green's function, (ii) $(G(0, S(n)), n \in \mathbb{N})$ is a martingale, and (iii) $(\|S(n)\|^2 - n, n \in \mathbb{N})$ is a martingale.

Choose $z \in \mathcal{A}(r_n, n)$. We define the following stopping times $(D_i, U_i, i \geq 0)$, corresponding to the i^{th} downward and upward crossings of the sphere of radius r_n . Let $\theta(n)$ act on trajectories by time-translation of n -units. Let $\tau = H(B_{r_n}) \wedge H_n$, $D_0 = U_0 = 0$, and

$$D_1 = \tau \mathbf{1}_{H(B_{r_n}) < H_n} + \infty \mathbf{1}_{H_n < H(B_{r_n})}.$$

If $D_1 < \infty$, then $U_1 = H_{r_n} \circ \theta(D_1) + D_1$, whereas if $D_1 = \infty$, then we set $U_1 = \infty$. We now proceed by induction, and assume D_i, U_i are defined. If $D_i = \infty$, then $D_{i+1} = \infty$, whereas if $D_i < \infty$, (and necessarily $U_i < \infty$) then

$$D_{i+1} = U_i + (\tau \mathbf{1}_{\tau = H(B_{r_n})} + \infty \mathbf{1}_{\tau = H_n}) \circ \theta(U_i), \quad \text{and} \quad U_{i+1} = D_{i+1} + H_{r_n} \circ \theta(D_{i+1}).$$

With this notation, we can write

$$\begin{aligned} \sum_{y \in \mathcal{A}(r_n, n)} G_n(z, y) &= \mathbb{E}_z[\tau] + \sum_{i=1}^{\infty} \mathbb{E}_z[\tau \circ \theta(U_i) \mathbf{1}_{D_i < \infty}] \\ &= \mathbb{E}_z[\tau] + \mathbb{P}_z(D_1 < \infty) \times \mathfrak{l}(z), \end{aligned} \quad (\text{A.2})$$

where

$$\mathfrak{l}(z) = \sum_{i=1}^{\infty} \mathbb{E}_z[\tau \circ \theta(U_i) \mid D_i < \infty] \prod_{j=1}^{i-1} (1 - \mathbb{P}_z(D_{j+1} = \infty \mid D_j < \infty)). \quad (\text{A.3})$$

Now, we compute each term of the right hand side of (A.2).

We have divided the proof in three steps.

Step 1: First, we show that there is a positive constant K , (independent of z and n) such that when $z \in \mathcal{A}(r_n, n)$, then

$$\left| \mathbb{P}_z(D_1 < \infty) - \frac{\alpha_0(z)}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2} ((n - \|z\|) \vee 1). \quad (\text{A.4})$$

Note that when $z \in \mathbb{B}(0, n)$, and $n - \|z\| \leq 1$, (A.4) yields

$$\left| \mathbb{P}_z(D_1 < \infty) - \frac{\mathbb{E}_z[\|S(\tau)\| - \|z\| | D_1 = \infty]}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}, \quad (\text{A.5})$$

Secondly, we show that for $z \in \mathcal{A}(r_n, n)$, and $i \geq 1$

$$\left| \mathbb{P}_z(D_{i+1} = \infty | D_i < \infty) - \frac{\mathbb{E}_z[(\|S(U_i)\| - \|S(D_{i+1})\|) \mathbf{1}_{D_1 \circ \theta(U_i) < \infty} | D_i < \infty]}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}. \quad (\text{A.6})$$

Our starting point is the classical Gambler's ruin estimate, which in dimension 2 reads with the potential kernel instead of Green's function,

$$\mathbb{P}_z(D_1 < \infty) = \frac{G(0, z) - \mathbb{E}_z[G(0, S(\tau)) | D_1 = \infty]}{\mathbb{E}_z[G(0, S(\tau)) | D_1 < \infty] - \mathbb{E}_z[G(0, S(\tau)) | D_1 = \infty]}. \quad (\text{A.7})$$

We now expand Green's function (resp. the potential kernel) using asymptotics (2.8) (resp. (2.9)). For this purpose, it is convenient to define a random variable

$$X(z) = \frac{1}{\|z\|} (\|S(\tau)\|^2 - \|z\|^2).$$

Note that for any $z \in \mathcal{A}(r_n, n)$, $X(z)/\|z\|$ is small. Indeed,

$$\frac{X(z)}{\|z\|} = \frac{(\|S(\tau)\| - \|z\|)(\|S(\tau)\| + \|z\|)}{\|z\|^2} \quad (\text{A.8})$$

Since $\Delta_n = n - r_n = O(n^{1/3})$, we have for n large enough

$$\frac{X(z)}{\|z\|} \leq \frac{2(n+1)\Delta_n}{(n-\Delta_n)^2} \leq \frac{8\Delta_n}{n}, \quad \text{and} \quad \sup_{z \in \mathcal{A}(r_n, n)} \left(\frac{|X(z)|}{\|z\|} \right)^3 \leq \frac{8^3 \Delta_n^3}{n} \times \frac{1}{n^2}. \quad (\text{A.9})$$

More precisely, $X(z)$ is of order $2(\|S(\tau)\| - \|z\|)$. Indeed, $\Delta_n^3 \leq K'n$ for some $K' > 0$, and (A.8) yields

$$X(z) = 2(\|S(\tau)\| - \|z\|) + \left(\frac{(\|S(\tau)\| - \|z\|)^2}{\|z\|} \right) \implies |X(z) - 2(\|S(\tau)\| - \|z\|)| \leq \frac{K'}{\Delta_n}. \quad (\text{A.10})$$

When dimension $d > 2$, we set $\eta(d) = \frac{d-2}{2}$. In order to use Green's function asymptotics (2.8), we express $S(\tau)$ in terms of $X(z)$ as follows

$$\frac{1}{\|S(\tau)\|^{d-2}} = \frac{1}{\|z\|^{d-2}} \left(1 + \frac{X(z)}{\|z\|} \right)^{-\eta(d)}. \quad (\text{A.11})$$

We have a constant K_d such that

$$\left| \left(1 + \frac{X(z)}{\|z\|} \right)^{-\eta(d)} - \left(1 - \eta(d) \frac{X(z)}{\|z\|} + \eta(d) \frac{\eta(d)+1}{2} \left(\frac{X(z)}{\|z\|} \right)^2 \right) \right| \leq \frac{K_d}{n^2}. \quad (\text{A.12})$$

For $d > 2$ and any $z \neq 0$, (2.8), (A.11) and (A.12) yields

$$\left| G(0, S(\tau)) - G(0, z) - \eta(d) C_d \left(-\frac{X(z)}{\|z\|^{d-1}} + \frac{\eta(d)+1}{2} \frac{X(z)^2}{\|z\|^d} \right) \right| \leq \frac{K_d}{n^d}. \quad (\text{A.13})$$

In dimension 2, the potential kernel asymptotic yields for $K_2 > 0$

$$\left| a(0, S(\tau)) - a(0, z) - \frac{1}{\pi} \left(\frac{X(z)}{\|z\|} + \frac{1}{2} \frac{X(z)^2}{\|z\|^2} \right) \right| \leq \frac{K_2}{n^2}. \quad (\text{A.14})$$

In view of (A.14), we assume henceforth that (A.13) holds, but in $d = 2$ we think of $\eta(d)C_d = 1/\pi$, and $\frac{\eta(d)+1}{2} = 1/2$.

Using (A.7) and (A.13), we obtain

$$\mathbb{P}_z(D_1 < \infty) = \frac{\mathbb{E}_z[X(z)|D_1 = \infty] - \bar{C}(z) + O(\frac{1}{n})}{\mathbb{E}_z[X(z)|D_1 = \infty] - \mathbb{E}_z[X(z)|D_1 < \infty] + \underline{C}(z) - \bar{C}(z) + O(\frac{1}{n})}, \quad (\text{A.15})$$

where

$$\bar{C}(z) = \frac{\eta(d)+1}{2} \mathbb{E}_z \left[\frac{X^2(z)}{\|z\|} | D_1 = \infty \right], \quad \text{and} \quad \underline{C}(z) = \frac{\eta(d)+1}{2} \mathbb{E}_z \left[\frac{X^2(z)}{\|z\|} | D_1 < \infty \right]. \quad (\text{A.16})$$

Using (A.9), we have some rough estimates on \bar{C} and \underline{C} . For any $z \in \mathcal{A}(r_n, n)$,

$$\bar{C}(z) = O\left(\frac{\Delta_n^2}{n}\right) = O\left(\frac{1}{\Delta_n}\right), \quad \text{and} \quad \underline{C}(z) = O\left(\frac{\Delta_n^2}{n}\right) = O\left(\frac{1}{\Delta_n}\right). \quad (\text{A.17})$$

Using (A.10), we have better estimates for \bar{C} and \underline{C} .

$$\bar{C}(z) = d \frac{(n - \|z\|)^2}{\|z\|} + O\left(\frac{(n - \|z\|) \vee 1}{n}\right), \quad \underline{C}(z) = d \frac{(\|z\| - r_n)^2}{\|z\|} + O\left(\frac{(\|z\| - r_n) \vee 1}{n}\right). \quad (\text{A.18})$$

The rough estimates (A.17) together with (A.10) allow us to derive from (A.15) an estimate for $\mathbb{P}_z(D_1 < \infty)$, for any $z \in \mathcal{A}(r_n, n)$.

$$\begin{aligned} \mathbb{P}_z(D_1 < \infty) &= \frac{\mathbb{E}_z[\|S(\tau)\| - \|z\| | D_1 = \infty] + O\left(\frac{1}{\Delta_n}\right)}{\mathbb{E}_z[\|S(\tau)\| - \|z\| | D_1 = \infty] - \mathbb{E}_z[\|S(\tau)\| - \|z\| | D_1 < \infty] + O\left(\frac{1}{\Delta_n}\right)} \\ &= \frac{\alpha_0(z) + O\left(\frac{1}{\Delta_n}\right)}{\Delta_n(1 + O\left(\frac{1}{\Delta_n}\right))}. \end{aligned} \quad (\text{A.19})$$

This yields (A.4) since $\alpha_0(z) \leq 1 + (n - \|z\|) \vee 1 \leq 2(n - \|z\|) \vee 1$.

Case where $z \in \partial B(0, r_n)$.

On $\{D_1 = \infty\}$, we have

$$X(z) = 2(\|S(\tau)\| - \|z\|) + O\left(\frac{1}{\Delta_n}\right). \quad (\text{A.20})$$

On $\{D_1 < \infty\}$, we have

$$X(z) = 2(\|S(\tau)\| - \|z\|) + O\left(\frac{1}{n}\right). \quad (\text{A.21})$$

This implies

$$\bar{C}(z) = d\frac{\Delta_n^2}{\|z\|} + O\left(\frac{\Delta_n}{n}\right), \quad \text{and} \quad \underline{C}(z) = O\left(\frac{1}{n}\right). \quad (\text{A.22})$$

Thus,

$$\begin{aligned} \mathbb{P}_z(D_1 = \infty) &= \frac{2\mathbb{E}_z[\|z\| - \|S(\tau)\| | D_1 < \infty] + \underline{C}(z) + O\left(\frac{1}{n}\right)}{\mathbb{E}_z[X(z) | D_1 = \infty] - \mathbb{E}_z[X(z) | D_1 < \infty] + \underline{C}(z) - \bar{C}(z) + O\left(\frac{1}{n}\right)} \\ &= \frac{\mathbb{E}_z[\|z\| - \|S(\tau)\| | D_1 < \infty] + O\left(\frac{1}{n}\right)}{\Delta_n + O(1)} \\ &= \frac{\mathbb{E}_z[\|z\| - \|S(\tau)\| | D_1 < \infty]}{\Delta_n} + O\left(\frac{1}{\Delta_n^2}\right). \end{aligned} \quad (\text{A.23})$$

In order to obtain (A.6), we write (A.23) on $\{D_i < \infty\}$, and $z = S(U_i)$ as follows. There is a constant K such that on the event $\{D_i < \infty\}$,

$$\left| \mathbb{E}_{S(U_i)}[\mathbf{1}_{D_{i+1}=\infty}] - \frac{\mathbb{E}_{S(U_i)}[\|S(U_i)\| - \|S(\tau)\| \mathbf{1}_{D_1 \circ \theta(U_i) < \infty}]}{\Delta_n \times \mathbb{P}_{S(U_i)}(D_1 < \infty)} \right| \leq \frac{K}{\Delta_n^2}. \quad (\text{A.24})$$

Note that (A.23) implies that $\mathbb{P}_{S(U_i)}(D_1 < \infty) = 1 + O(1/\Delta_n)$, so that (A.24) reads as we integrate over $\{D_i < \infty\}$ with respect to \mathbb{E}_z

$$\left| \mathbb{P}_z(D_{i+1} = \infty, D_i < \infty) - \frac{\mathbb{E}_z[\mathbf{1}_{D_i < \infty} (\|S(U_i)\| - \|S(\tau)\|) \mathbf{1}_{D_1 \circ \theta(U_i) < \infty}]}{\Delta_n} \right| \leq \frac{K\mathbb{P}_z(D_i < \infty)}{\Delta_n^2}. \quad (\text{A.25})$$

We obtain (A.6) as we divide both sides of (A.25) by $\mathbb{P}_z(D_i < \infty)$.

Step 2: We show now that for any $z \in \mathcal{A}(r_n, n)$ we have

$$\left| \mathbb{E}_z[\tau] - (d\Delta_n\alpha_0(z) - d(n - \|z\|)^2) \right| \leq K((n - \|z\|) \vee 1). \quad (\text{A.26})$$

When $z \in B_n$ and $n - \|z\| \leq 1$, (A.26) reads

$$\left| \mathbb{E}_z[\tau] - (d\Delta_n\alpha_0(z) - d(n - \|z\|)^2) \right| \leq K. \quad (\text{A.27})$$

When $z \in \mathcal{A}(r_n, n)$, and $i \geq 1$, we show that

$$\left| \frac{\mathbb{E}_z[\tau \circ \theta(U_i) | D_i < \infty]}{d\Delta_n^2} - \frac{\mathbb{E}_z[(\|S(U_i)\| - \|S(D_{i+1})\|) \mathbf{1}_{D_1 \circ \theta(U_i) < \infty} | D_i < \infty]}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}. \quad (\text{A.28})$$

Using that $\{\|S(n)\|^2 - n, n \in \mathbb{N}\}$ is a martingale, and the optional sampling theorem (see Lemma 3 of [7])

$$\begin{aligned} \mathbb{E}_z[\tau] &= \mathbb{E}_z[\|S(\tau)\|^2] - \|z\|^2 = \|z\| \times \mathbb{E}_z[X(z)] \\ &= \|z\| \times (\mathbb{E}_z[X(z)|D_1 = \infty] \mathbb{P}_z(D_1 = \infty) + \mathbb{E}_z[X(z)|D_1 < \infty] \mathbb{P}_z(D_1 < \infty)). \end{aligned} \quad (\text{A.29})$$

Thus, using (A.15), simple algebra yields

$$\mathbb{E}_z[\tau] = \|z\| \times ((\underline{C}(z) - \bar{C}(z))\mathbb{P}_z(D_1 < \infty) + \bar{C}(z)) + O(1). \quad (\text{A.30})$$

By recalling (A.18) and (A.4)

$$\begin{aligned} \mathbb{E}_z[\tau] &= d \left(((\|z\| - r_n)^2 - (n - \|z\|)^2 + O(\Delta_n)) \left(\frac{\alpha_0(z)}{\Delta_n} + O\left(\frac{(n - \|z\|) \vee 1}{\Delta_n^2}\right) \right) \right) \\ &\quad + d(n - \|z\|)^2 + O((n - \|z\|) \vee 1) \\ &= d(2\|z\| - n - r_n)\alpha_0(z) + d(n - \|z\|)^2 + O((n - \|z\|) \vee 1) \\ &= d\Delta_n\alpha_0(z) - d(n - \|z\|)^2 + O((n - \|z\|) \vee 1) \end{aligned} \quad (\text{A.31})$$

This yields (A.26).

Assume now that $z \in \partial B(0, r_n)$. From (A.30), we have

$$\mathbb{E}_z[\tau] = \|z\| \times ((\bar{C}(z) - \underline{C}(z))\mathbb{P}_z(D_1 = \infty) + \underline{C}(z)) + O(1). \quad (\text{A.32})$$

We use (A.6) and (A.22) to obtain

$$\begin{aligned} \mathbb{E}_z[\tau] &= \|z\| \left((d\Delta_n^2 + O(\Delta_n)) \left(\frac{\mathbb{E}_z[\|z\| - \|S(\tau)\||D_1 < \infty]}{\Delta_n} + O\left(\frac{1}{\Delta_n^2}\right) \right) \right) + O(1) \\ &= d\Delta_n \mathbb{E}_z[\|z\| - \|S(\tau)\||D_1 < \infty] + O(1). \end{aligned} \quad (\text{A.33})$$

Now, write (A.33) as follows. There is a constant K such that for any $z \in \partial \mathbb{B}(0, r_n)$

$$\left| \frac{\mathbb{E}_z[\tau]}{d\Delta_n^2} - \frac{\mathbb{E}_z[\|z\| - \|S(\tau)\|\mathbf{1}_{D_1 < \infty}]}{\Delta_n \mathbb{P}_z(D_1 < \infty)} \right| \leq \frac{K}{\Delta_n^2}. \quad (\text{A.34})$$

Note that by (A.23) $\Delta_n \mathbb{P}_z(D_1 < \infty) = \Delta_n + O(1)$ and $|\|z\| - \|S(\tau)\|\mathbf{1}_{D_1 < \infty}| \leq 1$, thus

$$\left| \frac{\mathbb{E}_z[\tau]}{d\Delta_n^2} - \frac{\mathbb{E}_z[\|z\| - \|S(\tau)\|\mathbf{1}_{D_1 < \infty}]}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}. \quad (\text{A.35})$$

We replace z by $S(U_i)$ in (A.35) under the event $\{D_i < \infty\}$ to obtain

$$\left| \frac{\mathbb{E}_{S(U_i)}[\tau]}{d\Delta_n^2} - \frac{\mathbb{E}_{S(U_i)}[(\|S(U_i)\| - \|S(D_1 \circ \theta(U_i))\|)\mathbf{1}_{D_1 \circ \theta(U_i) < \infty}]}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}, \quad (\text{A.36})$$

We multiply both sides of (A.36) by $\mathbf{1}_{D_i < \infty}$, take the expectation on both side of (A.36), and divide by $\mathbb{P}_z(D_i < \infty)$ to obtain (A.28).

Step 3: For $i \geq 1$, we show the following bounds

$$2 \geq \gamma_i \geq \frac{1}{4d\sqrt{d}}, \quad \text{where } \gamma_i = \mathbb{E}_z \left[(\|S(U_i)\| - \|S(D_{i+1})\|) \mathbf{1}_{D_{i+1} < \infty} \mid D_i < \infty \right]. \quad (\text{A.37})$$

The upper bound is obvious. For the lower bound, first we restrict to $\{D_i < \infty\}$, so that $U_i < \infty$. By Lemma 2.1, $S(U_i)$ has a nearest neighbor x , within $\mathbb{B}(0, r_n)$ such that $\|S(U_i)\| - \|x\| \geq 1/(2\sqrt{d})$, and (A.37) is immediate.

Step 4: We show (A.1) using (A.2). For p such that $1 \leq p \leq \infty$, let

$$\sigma_p = \sum_{i=1}^p \mathbb{E}_z [\tau \circ \theta(U_i) \mid D_i < \infty] \prod_{j=1}^{i-1} (1 - \mathbb{P}_z(D_{j+1} = \infty \mid D_j < \infty)). \quad (\text{A.38})$$

Now, (A.3) reads $l(z) = \lim_{p \rightarrow \infty} \sigma_p$ (this is the limit of an increasing sequence). We establish in this step that, for some constant \tilde{K} , any integer n

$$\lim_{p \rightarrow \infty} \left| 1 - \frac{\sigma_p}{d\Delta_n^2} \right| \leq \frac{\tilde{K}}{\Delta_n}. \quad (\text{A.39})$$

Once we prove (A.39), we have all the bounds to estimate the right hand side of (A.2). Indeed, using (A.26), (A.4) and (A.39), we have

$$\begin{aligned} \mathbb{E}_z[\tau] + \mathbb{P}_z(D_1 < \infty) \times l(z) &= d\Delta_n \alpha_0(z) - d(n - \|z\|)^2 + O((n - \|z\|) \vee 1) \\ &\quad + \left(\frac{\alpha_0(z)}{\Delta_n} + O\left(\frac{(n - \|z\|) \vee 1}{\Delta_n^2}\right) \right) \times (d\Delta_n^2 + O(\Delta_n)) \\ &= 2d\Delta_n \alpha_0(z) - 2d(n - \|z\|)^2 + O((n - \|z\|) \vee 1). \end{aligned} \quad (\text{A.40})$$

In order now to prove (A.39), we introduce first some shorthand notation. For p and j positive integers

$$a_p = 1 - \frac{\sigma_p}{d\Delta_n^2}, \quad \alpha_j = \mathbb{P}_z(D_{j+1} = \infty \mid D_j < \infty), \quad \text{and} \quad \beta_j = \frac{\mathbb{E}_z[\tau \circ \theta(U_j) \mid D_j < \infty]}{d\Delta_n^2}. \quad (\text{A.41})$$

With this notation (A.6) and (A.28) read as follows.

$$\left| \alpha_j - \frac{\gamma_j}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}, \quad \text{and} \quad \left| \beta_j - \frac{\gamma_j}{\Delta_n} \right| \leq \frac{K}{\Delta_n^2}, \quad \text{so that} \quad |\alpha_j - \beta_j| \leq \frac{2K}{\Delta_n^2}. \quad (\text{A.42})$$

Let us rewrite (A.38) as

$$a_1 = 1 - \beta_1 \quad \text{and} \quad a_p = a_{p-1} - \beta_p \prod_{j=1}^{p-1} (1 - \alpha_j) \quad \text{for } p > 1. \quad (\text{A.43})$$

In order to establish (A.39), we show by induction that

$$\left| a_p - \prod_{j=1}^p (1 - \alpha_j) \right| \leq \epsilon_p, \quad (\text{A.44})$$

with for $p > 1$

$$\epsilon_p = \epsilon_{p-1} + \frac{2K}{\Delta_n^2} \prod_{j=1}^{p-1} (1 - \alpha_j) \quad \text{and} \quad \epsilon_1 = \frac{2K}{\Delta_n^2}. \quad (\text{A.45})$$

Note that it is easy to estimate ϵ_p from (A.45). By (A.42) and for K_0 large enough there is a constant κ_S such that

$$\begin{aligned} \epsilon_p &\leq \frac{2K}{\Delta_n^2} \left(1 + \sum_{k=1}^p \exp \left(- \sum_{j=1}^k \alpha_j \right) \right) \\ &\leq \frac{2K}{\Delta_n^2} \left(1 + \sum_{k=1}^p \exp \left(- \sum_{j=1}^k \frac{\gamma_j}{2\Delta_n} \right) \right) \leq \frac{2K}{\Delta_n^2} \kappa_S \Delta_n = \frac{2K\kappa_S}{\Delta_n}. \end{aligned}$$

Now, by (A.42) (A.44) holds for $p = 1$, and we assume it holds for $p - 1$. Then

$$(1 - \beta_p) \prod_{j=1}^{p-1} (1 - \alpha_j) - \epsilon_{p-1} \leq a_p \leq (1 - \beta_p) \prod_{j=1}^{p-1} (1 - \alpha_j) + \epsilon_{p-1}. \quad (\text{A.46})$$

Then by (A.42), we have (A.44) with ϵ_p satisfying (A.45).

Now (A.39) follows as we notice that Step 3 implies, together with (A.42) and for K_0 large enough, that

$$\lim_{p \rightarrow \infty} \prod_{j=1}^p (1 - \alpha_j) = 0.$$

■

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