Distribution of the systolic volume of homology classes

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DISTRIBUTION OF THE SYSTOLIC VOLUME OF HOMOLOGY CLASSES

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ABSTRACT. Given a pair $(G, \mathbf{a})$ where G is a finitely presentable group and $\mathbf{a}$ is an integer homology class of this group, Gromov defined in [Gro83] a new numerical invariant associated to this pair called systolic volume. Our goal is to propose a systematic study of systolic volume as a function of the two variables G and $\mathbf{a}$. In particular we focus on the distribution of the values of the systolic volume on the real line.

CONTENTS

1. Introduction

Let G be a finitely presentable group, and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a non trivial homology class of dimension $m \geq 1$. A *geometric cycle* (X, f) *representing a class* $\mathbf{a}$ is a pair (X, f) consisting of an orientable pseudomanifold X of dimension m and a continuous map $f : X \to K(G, 1)$ such that $f_*[X] = \mathbf{a}$ where $K(G, 1)$ denotes the Eilenberg-MacLane space[1]. The representation is said to be *normal* if in addition the induced map $f_\sharp : \pi_1(X) \to G$ is an epimorphism.

Given a geometric cycle (X, f), we can consider for any polyhedral metric g on X (see [Bab06]) the *relative homotopic systole* denoted by $\mathrm{sys}_f(X, g)$ and defined as the least length of a loop γ of X whose image under f is not contractible. The *systolic volume* of the geometric cycle (X, f) is then the value

$$\mathfrak{S}_f(X) := \inf_g \frac{\mathrm{vol}(X, g)}{\mathrm{sys}_f(X, g)^m},$$

where the infimum is taken over all polyhedral metrics g on X and $\mathrm{vol}(X, g)$ denotes the m-dimensional volume of X. In the case where $f : X \to K(\pi_1(X), 1)$ is the classifying map (induced by an isomorphism between the fundamental groups), we simply denote by $\mathfrak{S}(X)$ the systolic volume of the pair (X, f). From [Gro83, Section 6], we have for any $m \geq 1$ that

$$\sigma_m := \inf_{(X, f)} \mathfrak{S}_f(X) > 0,$$

the infimum being taken over all geometric cycles (X, f) representing a non trivial homology class of dimension m. The following notion was introduced by Gromov (see [Gro83, Section 6]) :

Definition 1.1. *The systolic volume of the pair* $(G, \mathbf{a})$ *is defined as the number*

$$\mathfrak{S}(G, \mathbf{a}) := \inf_{(X, f)} \mathfrak{S}_f(X),$$

where the infimum is taken over all geometric cycles (X, f) *representing the class* $\mathbf{a}$.

Any integer class is representable by a geometric cycle. The systolic volume of $(G, \mathbf{a})$ is thus well defined and satisfies $\mathfrak{S}(G, \mathbf{a}) \geq \sigma_m$. But this definition does not give any information on the structure of a geometric cycle that might achieve this infimum. In the case where the homology class $\mathbf{a}$ is representable by a manifold, we know that this infimum is reached and coincides with the systolic volume of any normal representation of $\mathbf{a}$ by a manifold, see [Bab06, Bab08, Bru08]. A manifold is an example of *admissible* pseudomanifold, that is a pseudomanifold for which any element of the fundamental group can be represented by a curve not going through the singular locus of X. In section 2 we prove that any integer class $\mathbf{a}$ admits a normal representation by an admissible geometric cycle - a geometric cycle whose pseudomanifold is admissible - and we show the following

Theorem 1.2. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a homology class of dimension $m \geq 3$. For any normal representation of $\mathbf{a}$ by an admissible geometric cycle (X, f),*

$$\mathfrak{S}(G, \mathbf{a}) = \mathfrak{S}_f(X).$$

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[1]This definition slightly differs from the original one in [Gro83] where in addition a geometric cycle is provided with a polyhedral metric on X.

In particular an admissible geometric cycle (X, f) minimizes the systolic volume over all representations of $f_*[X]$. As a consequence, the systolic volume of an admissible orientable pseudomanifold X depends only on the image of its fundamental class $f_*[X] \in H_m(\pi_1(X), \mathbb{Z})$. In section 7 we exhibit an example showing that the condition of normalization (that is, $f_\sharp$ is an epimorphism between fundamental groups) can not be relaxed in our theorem.

In order to relate systolic volume with topological properties of the pair $(G, \mathbf{a})$, Gromov used quasi-extremal *regular* geometric cycles, see [Gro83, Theorem 6.4.A]. They are geometric cycles representing the class $\mathbf{a}$ endowed with a polyhedral metric whose systolic volume is arbitrarily close to the systolic volume of the pair $(G, \mathbf{a})$, and which are regular (that is the area of balls of radius less than half the systole is roughly at least the area of a ball of same radius in Euclidian space). These quasi-extremal regular cycles allowed Gromov to derive inequalities between systolic volume and two important topological invariants of $(G, \mathbf{a})$, see [Gro96, subsection 3.C.3]:

- the simplicial height $h(\mathbf{a})$ of the class $\mathbf{a} \in H_m(G, \mathbb{Z})$ which is the minimum number of simplices of any dimension of a geometric cycle representing $\mathbf{a}$,
- the simplicial volume $\|\mathbf{a}\|_\Delta$ defined as the infimum of the sums $\sum_i |r_i|$ over all representations of $\mathbf{a}$ by singular cycles $\sum_i r_i \sigma_i$ with real coefficients.

Gromov proved in [Gro83, Theorem 6.4.C" and Theorem 6.4.D'] (see also [Gro96, subsection 3.C.3]) the following:

Theorem 1.3 (Gromov). *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a homology class of dimension $m \geq 2$.*
(1) There exists positive constants C_m and C'_m depending only on the dimension m such that

$$\mathfrak{S}(G, \mathbf{a}) \geq C_m \cdot \frac{h(\mathbf{a})}{\exp(C'_m \sqrt{\ln h(\mathbf{a})})};$$

(2) There exists a positive constant C''_m depending only on the dimension m such that

$$\mathfrak{S}(G, \mathbf{a}) \geq C''_m \cdot \frac{\|\mathbf{a}\|_\Delta}{(\ln(2 + \|\mathbf{a}\|_\Delta))^m}.$$

These two lower bounds will be used in the sequel.

In this article, we first focus on general properties of the distribution of systolic volume. In section 3 we prove the relative density of the set of values taken by systolic volume:

Theorem 1.4. *Let $m \geq 3$. For any interval $I \subset \mathbb{R}^+$ of length at least σ_m, there exists a pair $(G, \mathbf{a})$ consisting of a finitely presentable group and a homology class of dimension m such that $\mathfrak{S}(G, \mathbf{a}) \in I$.*

We also prove the relative density of the systolic volume of essential manifolds of dimension m (see Theorem 3.2).

In section 4, we will prove that the finiteness of systolic volume does not hold for dimension $m \geq 3$. In order to give this statement content, we introduce the following definition. A class $\mathbf{a} \in H_m(G, \mathbb{Z})$ is said *reducible* if there exists a proper

subgroup $H \subset G$ and a class $\mathbf{b} \in H_m(H, \mathbb{Z})$ such that $i_*(\mathbf{b}) = \mathbf{a}$ where i denotes the canonical inclusion. Otherwise the class will be said *irreducible*. Given a positive constant T and a positive integer m, we denote by $\mathcal{F}(m, T)$ the set of finitely presentable groups G such that there exists an irreducible class $\mathbf{a} \in H_m(G, \mathbb{Z})$ with $\mathfrak{S}(G, \mathbf{a}) \leq T$. We will prove the following result:

Theorem 1.5. *The set $\mathcal{F}(m, 1)$ is infinite for any dimension $m \geq 3$.*

A natural question on the distribution of systolic volume is the following: given a finitely presentable group G and a homology class $\mathbf{a}$ of dimension m, how does the function $\mathfrak{S}(G, k\mathbf{a})$ behave in terms of the integer k ? In section 5 we show the following:

Theorem 1.6. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ where $m \geq 3$. There exists a positive constant $C(G, \mathbf{a})$ depending only on the pair $(G, \mathbf{a})$ such that*

$$\mathfrak{S}(G, k\mathbf{a}) \leq C(G, \mathbf{a}) \cdot \frac{k}{\ln(1 + k)}$$

for any integer $k \geq 1$. In particular,

$$\lim_{k \to \infty} \frac{\mathfrak{S}(G, k\mathbf{a})}{k} = 0.$$

This theorem is a considerable improvement of the main result in [BB05]. We may ask in which cases the function $\mathfrak{S}(G, k\mathbf{a})$ goes to infinity, and in which cases it does not. When the group is $\mathbb{Z}^n$, we obtain the following result:

Proposition 1.7. *Let $1 \leq m \leq n$ be two integers. Every class $\mathbf{a} \in H_m(\mathbb{Z}^n, \mathbb{Z})$ satisfies the inequality*

$$\mathfrak{S}(\mathbb{Z}^n, \mathbf{a}) \leq C_n^m \cdot \mathfrak{S}(\mathbb{T}^m)$$

where C_n^m denotes the binomial coefficient.

In the opposite direction, the systolic volume of the multiples of any homology class $\mathbf{a}$ whose simplicial volume $\|\mathbf{a}\|_\Delta$ is positive goes to infinity. More precisely, our Theorem 1.6 coupled with Theorem 1.3 of Gromov gives rise to the following statement:

Corollary 1.8. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ where $m \geq 3$ such that $\|\mathbf{a}\|_\Delta > 0$. Then there exists two positive constants $C(G, \mathbf{a})$ and $\tilde{C}(G, \mathbf{a})$ depending only on $(G, \mathbf{a})$ such that for k large enough we have*

$$\tilde{C}(G, \mathbf{a}) \cdot \frac{k}{(\ln(1 + k))^m} \leq \mathfrak{S}(G, k\mathbf{a}) \leq C(G, \mathbf{a}) \cdot \frac{k}{\ln(1 + k)}.$$

This double inequality implies that if $\|\mathbf{a}\|_\Delta > 0$ no linear recurrent equation is satisfied by the sequence $\mathfrak{S}(G, k\mathbf{a})$, see section 5.

The dependence of systolic volume on torsion is another natural question. Gromov mentions in [Gro96] that it may be possible to use the torsion of $H_*(\pi_1(M), \mathbb{Z})$ to bound from below the systolic volume of a manifold M. Given a finitely presentable group G and a homology class $\mathbf{a}$ of dimension m, we define the 1-*torsion of the class $\mathbf{a}$* as the integer

$$t_1(\mathbf{a}) := \min_{(X, f)} |\mathrm{Tors}\, H_1(X, \mathbb{Z})|,$$

where the minimum is taken over the set of geometric cycles (X, f) representing the class $\mathbf{a}$ and $|\mathrm{Tors}\, H_1(X, \mathbb{Z})|$ denotes the number of torsions elements in the first homology group of X. We now state the main result of section 6 :

Theorem 1.9. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ where $m \geq 2$. Then*

$$\mathfrak{S}(G, \mathbf{a}) \geq C_m \frac{\ln t_1(\mathbf{a})}{\exp(C'_m \sqrt{\ln(\ln t_1(\mathbf{a}))})},$$

where C_m and C'_m are two positive constants depending on m.
In particular, for any $\varepsilon > 0$

$$\mathfrak{S}(G, \mathbf{a}) \geq (\ln t_1(\mathbf{a}))^{1-\varepsilon}$$

if $t_1(\mathbf{a})$ is large enough.

The proof of this theorem involves two ingredients: Theorem 1.3 of Gromov, and a bound of the height of $\mathbf{a}$ by its 1-torsion. It is important to remark (see section 6) that for any dimension m there exists a sequence of groups G_n and of homology classes $\mathbf{a}_n \in H_m(G_n, \mathbb{Z})$ such that

$$\lim_{n \to \infty} \frac{\mathfrak{S}(G_n, \mathbf{a}_n)}{\ln t_1(\mathbf{a}_n)} = 0.$$

In general the 1-torsion of a class is difficult to compute. In the case of $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$, we can bound from below the 1-torsion of any generator by the number n (see Lemma 6.10). In particular, the fundamental classes of lens spaces $L_m(n)$ realize exactly the generators of the group $H_{2m+1}(\mathbb{Z}_n, \mathbb{Z})$ and we obtain

$$\mathfrak{S}(L_m(n)) \geq (\ln n)^{1-\varepsilon}$$

for any $\varepsilon > 0$ if n is large enough.

Theorem 1.9 also allows us to derive the following result.

Theorem 1.10. *There exists two positive constants a and b such that, for any manifold M of dimension 3 with finite fundamental group,*

$$\mathfrak{S}(M) \geq a \frac{\ln |\pi_1(M)|}{\exp(b \sqrt{\ln(\ln |\pi_1(M)|)})},$$

where $|\pi_1(M)|$ denotes the cardinal of $\pi_1(M)$.

In section 7, we explore the case of nilmanifolds, and more specifically the case of the Heisenberg group of dimension 3. We obtain a new illustration of the possible behaviour of the systolic volume of cyclic coverings. The study of the systolic volume of cyclic coverings in terms of the number of sheets has been suggested in [Gro96], and the first result in this direction can be found in [BB05].

The Heisenberg group $\mathcal{H}$ of dimension 3 is the group of triangular matrices

$$\left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\}.$$

The subset $\mathcal{H}(\mathbb{Z})$ of $\mathcal{H}$ composed of matrices with integer coefficients (*i.e.* matrices for which $x, y, z \in \mathbb{Z}$) is a lattice, and we will denote by $M_{\mathcal{H}} = \mathcal{H}/\mathcal{H}(\mathbb{Z})$ the corresponding quotient space. First of all, we obtain the following explicit upper bound for the systolic volume of multiples of the fundamental class $M_{\mathcal{H}}$.

Theorem 1.11. *Let* $\mathbf{a} = [M_\mathcal{H}] \in H_3(\mathcal{H}(\mathbb{Z}), \mathbb{Z})$ *be the fundamental class of* $M_\mathcal{H}$. *Then*

$$\mathfrak{S}(\mathcal{H}(\mathbb{Z}), k\mathbf{a}) \leq 19 \cdot \mathfrak{S}(\mathcal{H}(\mathbb{Z}), \mathbf{a})$$

for any integer $k \geq 1$.

The constant appearing here is the one involved in the resolution of the classical Waring problem (see [BDD86]): any integer number decomposes into a sum of at most 19 fourth powers. The idea of using the solution of the Waring problem in order to bound from above the function $\mathfrak{S}(G, k\mathbf{a})$ when $(G, \mathbf{a}) = (\mathcal{H}(\mathbb{Z}), [M_\mathcal{H}])$ carries over to any pair $(G, \mathbf{a})$ where G is a nilpotent graded group without torsion and $\mathbf{a}$ denotes the fundamental class of the corresponding nilmanifold, see Theorem 7.2.

Now consider the sequence of lattices $\{\mathcal{H}_n(\mathbb{Z})\}_{n=1}^\infty$ of $\mathcal{H}$, where $\mathcal{H}_n(\mathbb{Z})$ denotes the subset of matrices whose integer coefficients satisfy $x \in n\mathbb{Z}$ and $y, z \in \mathbb{Z}$. Denote by $M_{\mathcal{H}_n} = \mathcal{H}/\mathcal{H}_n(\mathbb{Z})$ the corresponding nilmanifolds. The manifold $M_{\mathcal{H}_n}$ is a cyclic covering with n sheets of $M_\mathcal{H}$, and the techniques involved in the proof of Theorem 1.6 implies that

$$\mathfrak{S}(M_{\mathcal{H}_n}) \leq C \cdot \frac{n}{\ln(1+n)}.$$

The fact that the function $\mathfrak{S}(M_{\mathcal{H}_n})$ goes to infinity is a consequence of Theorem 1.9.

Corollary 1.12. *The function* $\mathfrak{S}(M_{\mathcal{H}_n})$ *satisfies the following inequality:*

$$\mathfrak{S}(M_{\mathcal{H}_n}) \geq a\frac{\ln n}{\exp(b\sqrt{\ln(\ln n)})},$$

where a *an* b *are two positive constants. In particular,*

$$\lim_{n \to +\infty} \mathfrak{S}(M_{\mathcal{H}_n}) = +\infty.$$

Note that in this case $\|M_{\mathcal{H}_n}\|_\Delta = 0$ and the second bound of Theorem 1.3 does not apply. For any integer n the manifold $M_{\mathcal{H}_n}$ gives a non normal realization of the class $n[M_\mathcal{H}]$. So normalization condition in Theorem 1.2 cannot be relaxed.

Finally, the last section is devoted to the description of some results for stable systolic volume.

2. SINGULAR MANIFOLDS AS EXTREMAL OBJECTS OF SYSTOLIC VOLUME

Definition 1.1 introduces the systolic volume of a homology class as an infimum. But it does not give any information on the structure of a pseudomanifold that might achieve this infimum. If a homology class $\mathbf{a}$ can be realized by a manifold, we know that the infimum is reached and that this value coincides with the systolic volume of any normal representation of $\mathbf{a}$ by a manifold, see [Bab06, Bab08, Bru08]. In the case where the class $\mathbf{a}$ does not admit a representation by a manifold, it is not even clear if the value $\mathfrak{S}(G, \mathbf{a})$ can be achieved by the systolic volume of some pseudomanifold.

Let X be a pseudomanifold of dimension m. The *singular locus* of X is by definition the set $\Sigma(X)$ of points of X which are not contained in a neighbourhood homeomorphic to a m-dimensional ball. By definition of a pseudomanifold, $\Sigma(X)$ is a simplicial subcomplex of codimension at least 2.

Definition 2.1. *A pseudomanifold X is said admissible if the natural inclusion $X \setminus \Sigma(X) \subset X$ induces an epimorphism of fundamental groups.*

That is, a pseudomanifold is admissible if any element of the fundamental group can be represented by a loop of $X \setminus \Sigma(X)$. A geometric cycle (X, f) representing some homology class **a** will be called admissible if the pseudomanifold X is admissible.

Example 2.2. Let M be a triangulated manifold and $N \subset M$ be a simplicial subcomplex of codimension ≥ 2. Denote by $\sharp N$ the set of connected components of N. The simplicial complex $M/\sharp N$ obtained from M by contraction of the connected components of N into distinct points is an admissible pseudomanifold. The singular locus $\Sigma(M/\sharp N)$ consists of the points corresponding to the connected components of N.

Example 2.3. Let M be a manifold with boundary and suppose that $\partial M = A \times P$, where A is a manifold and P a connected manifold. The result of the fibred contraction of P is an admissible pseudomanifold $\overline{M}$ homeomorphic to the space

$$M \underset{\partial M}{\cup} A \times CP$$

where CP stands for the cone over P, the singular locus $\Sigma(\overline{M})$ being homeomorphic to A. Remark that if P is simply connected , $\pi_1(\overline{M}) = \pi_1(M)$. The pseudomanifold $\overline{M}$ obtained that way is a particular example of manifold *of singularity type P*, see [Baas73] and subsection 2.3 for the general construction.

Remark that an admissible pseudomanifold of dimension 2 is a surface. In particular it does not possess any singularity. The interest of this class of pseudomanifold is underlined by the following two results.

Theorem 2.4. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ be a homology class of dimension $m \geq 3$. Suppose that there exists a normal representation of the class $\mathbf{a}$ by an admissible geometric cycle (X, f). Then*

$$\mathfrak{S}(G, \mathbf{a}) = \mathfrak{S}_f(X).$$

The condition of normalisation saying that $f_{\sharp}$ *is an epimorphism* can not be dropped, see section 7 and the example of the Heisenberg group.

The following proposition, together with the previous result, shows that for any pair $(G, \mathbf{a})$ the systolic volume $\mathfrak{S}(G, \mathbf{a})$ is reached by the systolic volume of some pseudomanifold.

Proposition 2.5. *Let K be a CW-complex whose fundamental group is finitely presentable. Any homology class $\mathbf{a} \in H_*(K, \mathbb{Z})$ admits a normal representation by an admissible geometric cycle (X, f).*

Before proving Theorem 2.4 and its topological support contained in Proposition 2.5, we need some technical results.

2.1. **Technical lemmas.** Hopf's trick perfectly adapts to the setting of admissible pseudomanifolds. Consider a map

$$f : (X, X_1) \longrightarrow (Y, Y_1)$$

between two relative manifolds of the same dimension $m \geq 3$. Suppose that f is transversal at $y \in Y \setminus Y_1$, *i.e.* there exists an embedded m-disk D such that

- $y \in D \subset Y \setminus Y_1$;
- $f^{-1}(D) = \cup_{i=1}^n D_i$ is a disjoint union of m-disks embedded in $X \setminus X_1$;
- the restriction of f to $f^{-1}(D)$ is a covering map with base space D and n sheets.

Set $x_i = D_i \cap f^{-1}(y)$. The technical trick of Hopf is essentially contained in the the following lemma.

Lemma 2.6. *Suppose that there exists a m-disk embedded in $D' \subset X \setminus X_1$ with the following properties:*

1. $D_1, D_2 \subset D'$ *and* $D' \cap D_i = \emptyset$ *if* $i > 2$;
2. *for any path γ from x_1 to x_2 in D', the loop $f(\gamma)$ is contractible in Y* ;
3. *for any orientation of D, the orientations on D_1 and D_2 induced by f are never coherent in D'.*

Then there exists a homotopy $\{f_t\}_{0 \le t \le 1}$ of $f_0 = f$ which is constant on $X \setminus D'$ and such that

$$f_1^{-1}(D) = \underset{i>2}{\cup} D_i,$$

the last union being empty if $n = 2$.

We refer to [Eps66, p.378-380] for a proof as the construction of the homotopy f_t occurs in D' and so carries over to our context. We state the corresponding version of Hopf's Theorem in the orientable case.

Lemma 2.7. *Let X be an admissible orientable connected pseudomanifold and (Y, Y_1) an orientable relative manifold of the same dimension $m \ge 3$. Suppose that $f : X \longrightarrow Y$ is a map of degree k inducing an epimorphism between fundamental groups.*

Then there exists a homotopy $\{f_t\}_{0 \le t \le 1}$ of $f_0 = f$ such that $f_1^{-1}(Y \setminus Y_1)$ is homeomorphic to the disjoint union of k disks and the restriction of f_1 to the union of these disks is a covering map with base space $Y \setminus Y_1$ and k sheets.

The degree of f stands here for the absolute value of the multiple defined by the induced map $f_* : H_m(X; \mathbb{Z}) \longrightarrow H_m(Y, Y_1; \mathbb{Z})$. A corresponding version of Lemma 2.7 also holds in the non-orientable context with the notion of absolute degree.

Proof. Consider a point $y \in Y \setminus Y_1$. We can assume that $y \notin f(\Sigma(X))$ and the map f to be transversal at y. Let $D \subset Y \setminus Y_1$ be a disk containing the point y such that $f^{-1}(D) = \cup_{i=1}^n D_i$ is a disjoint union of m-disks embedded in $X \setminus \Sigma(X)$ and such that the restriction of f to $\cup_{i=1}^n D_i$ is a covering map with n sheets and base space D. We have $n \ge k$ and suppose that $n > k$. We can choose generators of $H_m(X; \mathbb{Z})$ and $H_m(Y, Y_1; \mathbb{Z})$ such that the map f_* induced on m-dimensional homology is simply the multiplication by k. This induces an orientation both on X and $Y \setminus Y_1$, and also on disks $\{D_i\}_{i=1}^n$ and D. As $n > k$, there exists two disks say D_1 and D_2 such that $f|_{D_1}$ reverses the orientation and $f|_{D_2}$ preserves it. We now follow step by step the proof of [Eps66, Theorem 4.1]. Join the two points $x_i = f^{-1}(y) \cap D_i$, $i = 1, 2$ by a simple curve $\gamma \subset X \setminus \Sigma(X)$. Because f induces an epimorphism between fundamental groups, there exists a loop α based at x_1 such that $f(\alpha)$ and $f(\gamma)$ are homotopic as loops based at y. As X is admissible, we can furthermore choose α in $X \setminus \Sigma(X)$. The concatenation $\alpha^{-1} \star \gamma$ and this evident modification in a neighborhood of the concatenation defines a simple curve $\beta \subset X \setminus \Sigma(X)$ joining x_1 and x_2. The loop $f(\beta)$ is contractible in Y relatively to y.

We then define the disk C as a small enough neighborhood of β, and apply Lemma 2.6. Remark that the choice of C implies a possible diminution of the size of the disks $\{D_i\}_{i=1}^n$ and D. The end of the proof is straightforward, see [Eps66] for the missing details. $\qquad\square$

2.2. **Admissible geometric cycles are minima of the systolic volume.** In this subsection, we prove Theorem 2.4. For this we use in a decisive way the comparaison and extension techniques elaborated in [Bab06, Bab08], as well as the ideas contained in those articles.

Let (X_1, f_1) be a geometric cycle representing the class $\mathbf{a}$. The pseudomanifold X_1 admits a cell decomposition with only one m-cell (see for example [Sab06]). This allows us to describe X_1 as a relative m-manifold (Y, Y'), where Y denotes the m-cell and Y' lies in the $(m-1)$-skeleton. We construct an extension of X_1 as follows (compare with [Bab06]). By adding to X_1 cells of dimension between 1 and m, we first construct a new CW-complex $X_1(m)$ which is the m-skeleton of the Eilenberg-MacLane space $K(G, 1)$. More precisely, we start by adding 1- and 2-cells to X_1 such that the resulting CW-complex $X_1(2)$ has fundamental group G. Then, for each dimension k going from 3 to m, we add k-cells to $X_1(k-1)$ such that the new CW-complex $X_1(k)$ thus obtained satisfies $\pi_i(X_1(k)) = 0$ for $1 < i < k-1$. At the end we get a CW-complex $X_1(m)$ which is m-aspherical and with fundamental group G. By adding cells of dimension higher than m, we can realize the Eilenberg-MacLane space $K(G, 1)$ as an extension of $X_1(m)$ (see [Bab06]). The original CW-complex X_1 is naturally embedded in $K(G, 1)$ by a map denoted $i : X_1 \hookrightarrow X_1(m) \subset K(G, 1)$. Remark that $Y \setminus Y'$ is a m-cell of $X_1(m)$.

By hypothesis, there exists an admissible pseudomanifold X and a map $f : X \to K(G, 1)$ giving a realization of the same class $\mathbf{a} \in H_m(G, \mathbb{Z})$ such that $f_\sharp : \pi_1(X) \to G$ is an epimorphism. The map f induces a map

$$(2.1) \qquad\qquad g : X \to X_1(m).$$

This last map g is not uniquely determined up to homotopy, but Lemma 3.10 of [Bab06] applies in this context, so we can choose g such that

$$(2.2) \qquad\qquad g_*[X] = i_*[X_1]$$

in $H_m(X_1(m), \mathbb{Z})$. Let $\{Y\} \cup \{Y_i\}_{i \in I}$ denote the m-cells of $X_1(m)$ (this list can be finite or infinite). To each m-cell Y_i or Y is associated the relative manifold $(Y_i, \widehat{Y}_i)$ or $(Y, \widehat{Y})$, where $\widehat{Y}_*$ denotes the closure of the union of all the other cells of Y_*. The map (2.1) induces maps

$$\widetilde{g} : X \to (Y, \widehat{Y}) \ \text{ and } \ \widetilde{g}_i : X \to (Y_i, \widehat{Y}_i), \ \ \forall i \in I.$$

From equation (2.2), we deduce that the degree of $\widetilde{g}$ is equal to 1, and that the degree of each $\widetilde{g}_i$ is zero. As X is compact, $g(X)$ intersects only a finite number of m-cells. We then apply Lemma 2.7 to each relative manifold $(Y_i, \widehat{Y}_i)$ such that $Y_i \bigcap g(X) \neq \emptyset$. In this way we obtain a map

$$g_1 : X \to Z \subset X_1(m-1)$$

homotopic to g, where Z denotes the subcomplex of $X_1(m-1)$ obtained from $X_1(2)$ by adding the cells (in finite number) of dimension between 3 and $m-1$

which intersect $g_1(X)$. Remark that $\pi_1(Z) = G$ and that the inclusion

$$i : X_1 \hookrightarrow Z \subset X_1(m-1) \subset X_1(m)$$

satisfies $\ker i_\sharp = \ker(f_1)_\sharp$, where $i_\sharp : \pi_1(X_1) \to \pi_1(Z)$ denotes the map induced by i on fundamental groups. The comparison principle (see [Bab06]) then implies

$$(2.3) \qquad \qquad \mathfrak{S}_f(X) \le \mathfrak{S}(Z) = \mathfrak{S}_{f_1}(X_1).$$

As we started with any representation (X_1, f_1) of $\mathbf{a}$, we get from (2.3)

$$\mathfrak{S}_f(X) = \mathfrak{S}(G, \mathbf{a}).$$

2.3. Singular manifolds of prescribed singularity type according to Baas.

Before proving Proposition 2.5, we briefly recall the construction of some models of singular manifolds, see [Baas73] for more details. Recall that a manifold with general corners (also called *variété aux angles*, see [Cerf61]) is a manifold whose boundary admits singularities similar to those of the closed cube.

Definition 2.8. *A manifold M with general corners is said decomposed if there exist submanifolds with general corners $\partial_0 M, \partial_1 M, ..., \partial_n M$ such that*

$$\partial M = \partial_0 M \cup \partial_1 M \cup ... \cup \partial_n M,$$

where union means identification along a common part of the boundary.

If M is a decomposed manifold, by setting

$$(2.4) \qquad \begin{cases} \partial_j(\partial_i M) & = & \partial_j M \cap \partial_i M & \text{if } j \ne i, \\ \partial_i(\partial_i M) & = & \emptyset & \text{if not,} \end{cases}$$

we get

$$\partial(\partial_i M) = \overset{n}{\underset{j=0}{\cup}} \partial_j(\partial_i M).$$

So each $\partial_i M$ is again a decomposed manifold.

Example 2.9. If M denotes the m-dimensional cube, its boundary is naturally decomposed into $(m-1)$-faces :

$$\partial M = \partial_0 M \cup \partial_1 M \cup ... \cup \partial_{2m-1} M.$$

Remark 2.10. A manifold with general corners can be smoothed through a process which has been well studied (see [Cerf61] for instance).

We consider a finite sequence of closed manifolds $S = \{P_0, P_1, ..., P_n\}$ ordered by increasing dimension. If $S \ne \emptyset$, we will always assume that $P_0 = *$ is a point.

Definition 2.11. *A manifold M with general corners is said of singularity type S if*
1) For any subset $\omega \subset \{0, 1, ..., n\}$, there exists a decomposed manifold $M(\omega)$ such that
(a) $M(\emptyset) = M$,
(b) $\partial M(\omega) = \cup_{i \notin \omega} \partial_i M(\omega)$.
(c) For every $i \in \{1, ..., n\} \setminus \omega$, there exists a diffeomorphism

$$(2.5) \qquad \qquad \beta(\omega, i) : \partial_i M(\omega) \simeq M(\omega, i) \times P_i,$$

where $(\omega, i) = \omega \cup \{i\}$,

2) For any subset $\omega \subset \{0, 1, ..., n\}$, and for all $i, j \in \{1, \ldots, n\} \setminus \omega$, the following diagram is commutative:

$$\partial_j \partial_i M(\omega) \quad \overset{\beta(\omega, i)}{\longrightarrow} \quad \partial_j M(\omega, i) \times P_i \quad \overset{\beta(\omega, i, j) \times id}{\longrightarrow} \quad M(\omega, i, j) \times P_j \times P_i$$

$$\|$$

$$\partial_j M(\omega) \cap \partial_i M(\omega) \qquad\qquad\qquad\qquad\qquad \Big\downarrow id \times T$$

$$\|$$

$$\partial_i \partial_j M(\omega) \quad \overset{\beta(\omega, j)}{\longrightarrow} \quad \partial_j M(\omega, i) \times P_j \quad \overset{\beta(\omega, j, i) \times id}{\longrightarrow} \quad M(\omega, i, j) \times P_i \times P_j$$

where T denotes the transposition.

The first part of the definition describes the local structure of product on the boundary of the decomposed manifold M. The diagram describes how the boundary components are glued together. We now define a particular class of singular manifolds.

Definition 2.12. *To any manifold with general corners M of singularity type S, we associate the singular manifold M_S defined as the quotient space $M(\emptyset)/\sim$ where*

$$a \sim b$$

if

$$a, b \in \partial_{i_1} ... \partial_{i_k} M, \quad i_j \geq 1, \quad k \geq 1,$$

and

$$(2.6) \qquad \begin{aligned} pr \circ \beta(i_1, ..., i_k) \circ .... \circ \beta(i_1, i_2) \circ \beta(\emptyset, i_1)(a) = \\ pr \circ \beta(i_1, ..., i_k) \circ .... \circ \beta(i_1, i_2) \circ \beta(\emptyset, i_1)(b). \end{aligned}$$

Here

$$(2.7) \qquad pr : M(i_1, ..., i_k) \times P_{i_1} \times ... \times P_{i_k} \longrightarrow M(i_1, ..., i_k)$$

denotes projection on the first factor.

The singular manifold M_S is then said of singularity type S or S-singular manifold.

If the elements of S are connected manifolds, then every S-singular manifold M is an admissible pseudomanifold. If not, the following remark will be of fundamental importance in the next section.

Remark 2.13. For each $i = 1, \ldots, n$, we decompose the manifold P_i into connected components Q_{ij} and set

$$T = \{Q_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq k_i\}.$$

Given a singular S-manifold M_S modeled on M, the local S-structure (2.5) on ∂M defines a local T-structure. The commutative diagram of Definition 2.11 and the equivalence (2.6) allow us to define an equivalence relation on M denoted by $\sim_T$ and such that the projections (2.7) only occur along the factors of type Q_{ij}. This gives rise to a T-singular manifold M_T defined as the quotient $M(\emptyset)/\sim_T$. A class for the relation $\sim_T$ being a subclass of the relation $\sim_S$, we get a canonical map of degree 1

$$(2.8) \qquad\qquad\qquad\qquad q : M_T \to M_S.$$

2.4. **Realization of homology classes by admissible geometric cycles.** We now prove Proposition 2.5. Following Milnor [Mil60] and Novikov [Nov62], the complex cobordism ring Ω_*^U is isomorphic to the ring of integer polynomials $\Omega_*^U = \mathbb{Z}[x_1, x_2, ...]$ where each generator x_k is of degree $2k$ and can be represented by a manifold P_k. Each representant P_k can be chosen as a complex algebraic manifold, see [Sto68] for instance. But the connectivity of this complex manifold is not clear in general (if $k = p^s - 1$ where p is a prime number, P_k can be choosen as $\mathbb{C}P^k$). Define the following sequence of singularities

(2.9) $S = \{P_1, P_2, ...\}.$

Given a homology class $\mathbf{a} \in H_m(X, \mathbb{Z})$, there exists according to Bass' Theorem [Baas73, Corollaire 5.1] a S-singular manifold M_S of dimension m and a map $f : M_S \to X$ such that $f_*[M_S] = \mathbf{a}$. The elements in S are not necessarily connected manifolds. So we proceed as in remark 2.13, and obtain in this way a new singular manifold M' representing $\mathbf{a}$ which is now an admissible pseudomanifold. Finally, we add if necessarily 1-handles to M' and extend the map $f' = q \circ f$ (where q denotes the canonical map from M' to M_S of degree 1) in such a way that $f'_\sharp$ becomes an epimorphism between fundamental groups. This concludes the proof.

Remark 2.14. The admissible pseudomanifold M' which realizes $\mathbf{a}$ can be chosen as a singular manifold whose singularities are more specifics. In $\mathbb{C}P^m \times \mathbb{C}P^n$ with $m \leq n$, we consider the hypersurface of degree $(1, 1)$

$$H_{m,n} = \{z_0 w_0 + z_1 w_1 + ... + z_m w_m = 0\},$$

where $(z_0, z_1, ..., z_m)$ and $(w_0, w_1, ..., z_n)$ denote the homogeneous coordinates in $\mathbb{C}P^m$ and $\mathbb{C}P^n$ respectively. The manifolds $H_{m,n}$ are known as *Milnor's manifolds*. The cobordism classes of the $\{H_{m,n}\}_{m \leq n}$ together with the familly of classes $\{\mathbb{C}P^s\}_{s \geq 1}$ give rise to a spanning familly of Ω_*^U (see [Hirz58] and [Nov62]). The classes in Ω_*^U are thus linear combinations (with integer coefficients) of cobordim classes $\{H_{m,n}\}_{m \leq n}$ and $\{\mathbb{C}P^s\}_{s \geq 1}$. So we can choose the $\{P_k\}_{k \geq 1}$ as a disjoint union of some of these manifolds endowed with an adequate orientation. Taking into account remark 2.13, an admissible pseudomanifold which represents the class $\mathbf{a} \in H_*(K, \mathbb{Z})$ can be choosen as a singular manifold of singularity type $\{\mathbb{C}P^s, H_{m,n}\}$.

3. Relative density of the values of systolic volume

The aim of this section is to show that the set of values of systolic volume over the set of homology classes (resp. over the set of orientable manifolds) of fixed dimension is a relatively dense set in the following sense.

Definition 3.1. *Given a subset $A \subset \mathbb{R}^+$ and d a positive constant, A is said d-dense in $\mathbb{R}^+$ if, for any interval $I \subset \mathbb{R}^+$ of length $|I| > d$, the intersection $I \cap A$ is not empty.*

For a fixed dimension $m \geq 3$, define

$\Sigma_m := \{\mathfrak{S}(G, \mathbf{a}) \mid G \text{ finitely presentable group and } \mathbf{a} \neq 0 \in H_m(G, \mathbb{Z})\},$

and

$$\sigma_m := \inf_{(G, \mathbf{a}) \in \Sigma_m} \mathfrak{S}(G, \mathbf{a}).$$

Similarly, set

$$\Omega_m^+ := \{\mathfrak{S}(M) \mid M \text{ orientable essential manifold of dimension } m\},$$

and

$$\omega_m^+ := \inf_{M \in \Omega_m^+} \mathfrak{S}(M).$$

Recall that $\sigma_m > 0$ by [Gro83] and that an orientable manifold M is said *essential* if the image of its fundamental class under its classifying map is not zero. In particular, $\omega_m^+ \geq \sigma_m$. The main result of this section is the following.

Theorem 3.2. *For any dimension $m \geq 3$, the set Σ_m (resp. Ω_m^+) is σ_m-dense (resp. ω_m^+-dense) in $\mathbb{R}^+$.*

In order to prove this result, we will study the behaviour of systolic volume under the operation of connected sum. Part of these results will also be useful in the next section.

Remark 3.3. Remark that the set Ω_m of values taken by the systolic volume over all manifolds of the same dimension m (not necessarily orientable) contains the subset Ω_m^+ : it is also a relatively dense set of $\mathbb{R}^+$ with density ω_m^+. It is not clear if this density can be decreased and the answer may depend on the parity of the dimension.

Fix a morphism of groups $\pi : G \to G'$ and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a homology class.

Proposition 3.4. *If $m \geq 3$,*

$$\mathfrak{S}(G', \pi_*\mathbf{a}) \leq \mathfrak{S}(G, \mathbf{a}).$$

Proof. According to Proposition 2.5, fix an admissible geometric cycle (X, f) representing normally the class $\mathbf{a}$. The admissible geometric cycle $(X, \pi \circ f)$ is a normal representation of the class $\pi_*(\mathbf{a})$, so

$$\mathfrak{S}(G', \pi_*(\mathbf{a})) = \mathfrak{S}_{\pi \circ f}(X) \leq \mathfrak{S}_f(X) = \mathfrak{S}(G, \mathbf{a})$$

by Theorem 2.4. $\qquad\square$

Corollary 3.5. *Let X_1 and X_2 be two orientable admissible pseudomanifolds of dimension $m \geq 3$. Then*

$$\max\{\mathfrak{S}(X_1), \mathfrak{S}(X_2)\} \leq \mathfrak{S}(X_1 \# X_2),$$

where $X_1 \# X_2$ denotes the connected sum of X_1 and X_2.

Proof. Denote by $f_j : X_j \to K(\pi_1(X_j), 1)$ the classifying map for $j = 1, 2$. Observe that $\pi_1(X_1 \# X_2) = \pi_1(X_1) * \pi_1(X_2)$. We have a natural monomorphism $i_j : \pi_1(X_j) \hookrightarrow \pi_1(X_1) * \pi_1(X_2)$ and a natural epimorphism $s_j : \pi_1(X_1) * \pi_1(X_2) \to \pi_1(X_j)$ such that $s_j \circ i_j = id_{\pi_1(X_j)}$, $s_2 \circ i_1 = 0$ and $s_1 \circ i_2 = 0$. From Proposition 3.4,

$$
\begin{aligned}
\mathfrak{S}(X_1 \# X_2) &= \mathfrak{S}(\pi_1(X_1) * \pi_1(X_2), (i_1 \circ f_1)_*[X_1] + (i_2 \circ f_2)_*[X_2]) \\
&\geq \mathfrak{S}(s_j(\pi_1(X_1) * \pi_1(X_2)), (s_j)_* \circ ((i_1 \circ f_1)_*[X_1] + (i_2 \circ f_2)_*[X_2])) \\
&\geq \mathfrak{S}(\pi_1(X_j), (f_j)_*[X_j])\} = \mathfrak{S}(X_j)
\end{aligned}
$$

for $j = 1, 2$. $\qquad\square$

Furthermore, we have the following comparaison result:

Proposition 3.6. *Let X_1 and X_2 be two orientable pseudomanifolds of dimension $m \geq 3$. Then*

$$\mathfrak{S}(X_1 \# X_2) \leq \mathfrak{S}(X_1) + \mathfrak{S}(X_2).$$

Proof. The contraction of the gluing sphere into a point gives rise to a natural projection map

$$p : X_1 \# X_2 \to X_1 \vee X_2$$

which induces an isomorphism between fundamental groups if $m \geq 3$. Applying the comparaison principle (see [Bab06]), we get

$$\mathfrak{S}(X_1 \# X_2) \leq \mathfrak{S}(X_1 \vee X_2) = \mathfrak{S}(X_1) + \mathfrak{S}(X_2).$$

$\square$

With this two comparaison results, we can now prove Theorem 3.2.

We first prove the σ_m-density of the set Σ_m. Consider any interval $I \subset \mathbb{R}^+$ such that $|I| > \sigma_m$. Choose a pair $(G, \mathbf{a})$ composed of a finitely presentable group and a m-class of its homology such that $\sigma_m \leq \mathfrak{S}(G, \mathbf{a}) < |I|$. According to Proposition 2.5, we can choose an admissible geometric cycle (X, f) representing normally the class $\mathbf{a} \in H_m(G, \mathbb{Z})$ which satisfies $\mathfrak{S}_f(X) = \mathfrak{S}(G, \mathbf{a})$ by Theorem 2.4. If the map $f_\#$ is not a monomorphism, we can directly contract some loops in X and get a new admissible geometric cycle (X', f') representing $\mathbf{a}$ such that $f'_\#$ is an isomorphism. So

$$\mathfrak{S}_f(X) = \mathfrak{S}(X').$$

The sequence $\{a_n = \mathfrak{S}(\#_n X')\}_{n=1}^\infty$ is increasing by Corollary 3.5 and satisfies $a_{n+1} - a_n \leq \mathfrak{S}(X') < |I|$ by Proposition 3.6. By [Sab07], we have

$$\mathfrak{S}(\#_n X) \geq C_m \frac{n}{\exp(C'_m \sqrt{\log n})}$$

where C_m and C'_m are two positive constants depending only on the dimension m (strictly speaking, this inequality is proved for manifolds, but carries over perfectly to pseudomanifolds). So the sequence $\{a_n\}$ is not bounded and does intersect I.

The proof of the ω_m^+-density of Ω_m^+ is similar: for any interval $I \subset \mathbb{R}^+$ with $|I| > \omega_m^+$, we can argue as previously starting with the sequence $\{a_n = \mathfrak{S}(\#_n M)\}_{n=1}^\infty$ where M is an orientable essential manifold of dimension m with $\omega_m^+ \leq \mathfrak{S}(M) < |I|$.

4. Non-finiteness of irreducible homology classes

This section deals with the following natural question:

Question 4.1. Given a positive constant T, how many finitely presentable groups G exist such that any essential (orientable) manifold M of dimension m with fundamental group G satisfies $\mathfrak{S}(M) \leq T$?

In dimension 2, this number is bounded from below by $c \cdot T (\ln T)^2$ and from above by $C \cdot T (\ln T)^2$ for some universal positive constants c and C, see [BS94] and [Gro96] . In fact the situation is quite rigid in dimension 2. Even the finiteness of the systolic volume over the set of finite simplicial complexes of dimension 2 holds. More precisely, recall that

- the *systolic area* of a finitely presentable group G is defined as

$$\mathfrak{S}(G) = \inf_{\pi_1(P)=G} \mathfrak{S}(P),$$

  where the infimum is taken over all finite simplicial complexes P of dimension 2 with fundamental group G ;
- a finitely presentable group G is said *of zero Grushko free index* if G can not be written as a free product $H * F_n$ for some $n > 0$, compare [RS08].

Then the number of finitely presentable groups G of zero Grushko free index such that $\mathfrak{S}(G) \leq T$ does not exceed K^{T^3} where K is a universal explicit positive constant, see [RS08].

The situation is rather different in higher dimensions. Let M be an essential manifold of dimension $m \geq 4$ whose fundamental group is of zero Grushko free index and N a non-essential manifold of the same dimension with fundamental group of zero Grushko free index. The fundamental group $\pi_1(M) * \pi_1(N)$ of the connected sum $M\#N$ is still of zero Grushko free index. By Proposition 3.6 we get

$$\mathfrak{S}(M\#N) \leq \mathfrak{S}(M).$$

That is, while staying in the class of groups of zero Grushko free index we can considerably modify the fundamental group of a manifold without increasing the systolic volume. So there is no hope to obtain finiteness results in this context. This is why we introduce the following.

Definition 4.2. *Let G be a finitely presentable group. A class $\mathbf{a} \in H_m(G, \mathbb{Z})$ is said reducible if there exists a proper subgroup $H \subset G$ and a class $\mathbf{b} \in H_m(H, \mathbb{Z})$ such that $i_*(\mathbf{b}) = \mathbf{a}$ where i denotes the canonical inclusion. In the contrary case, the class $\mathbf{a}$ will be said irreducible.*

Furthermore let say that a manifold M is reducible (resp. irreducible) if the image of its fundamental class $[M]$ (under the classifying map) in $H_m(\pi_1(M), \mathbb{Z})$ is a reducible (resp. irreducible) class.

Example 4.3. Let M be an aspherical manifold of dimension m (that is $\pi_k(M) = 0$ for $k > 1$). Then M is irreducible.

Example 4.4. Let G be a finite group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a class of order $|G|$. Then $\mathbf{a}$ is irreducible.

This last example shows that the fundamental class of a lenticular manifold is irreducible.

Remark that it is possible that

- any multiple of an irreducible classe is irreducible as in the case $G = \mathbb{Z}_p$ for p a prime number,
- each multiple of an irreducible classe is reducible as in the case of tori $\mathbb{T}^m$.

On the other hand, there exists classes $\mathbf{a} \in H_m(G, \mathbb{Z})$ which are *completely reducible* in the following sense: $\mathbf{a}$ is reducible, and any class $\mathbf{b} \in H_m(H, \mathbb{Z})$ where H is a proper subgroup $H \subset G$ and such that $i_*(\mathbf{b}) = \mathbf{a}$ is also reducible.

Given a positive constant T and a positive integer m, we denote by $\mathcal{F}(m, T)$ the set of finitely presentable groups G such that there exists an irreducible class $\mathbf{a} \in H_m(G, \mathbb{Z})$ with $\mathfrak{S}(\mathbf{a}) \leq T$.

Theorem 4.5. *The set $\mathcal{F}(m, 1)$ is infinite for any dimension $m \geq 3$.*

Proof. Let p be a prime number and set

$$G(p, m) := \underbrace{\mathbb{Z}_p \oplus \ldots \oplus \mathbb{Z}_p}_{m}.$$

Denote by $\phi_p : \pi_1(\mathbb{T}^m) \longrightarrow G(p, m)$ the natural projection and set

$$\mathbf{a}(p, m) := (\phi_p)_*[\mathbb{T}^m] \in H_m(G(p, m), \mathbb{Z}).$$

In order to prove that $\mathbf{a}(p, m) \neq 0$ in $H_m(G(p, m), \mathbb{Z})$, we will show that the reduction modulo p of $\mathbf{a}(p, m)$ is not null in $H_m(G(p, m), \mathbb{Z}_p)$. Consider the generators $v_1, \ldots, v_m$ of $H^1(G(p, m), \mathbb{Z}_p)$ corresponding to the natural projections of $G(p, m)$ on each factor. The elements $u_i := (\phi_p)^*(v_i)$, $i = 1, 2, ..., m$ generate the group $H^1(\mathbb{T}^m, \mathbb{Z}_p)$, and $u := u_1 \cup \ldots \cup u_m$ generates the group $H^m(\mathbb{T}^m, \mathbb{Z}_p)$. So $u \cap [\mathbb{T}^m]_p = 1$ where $[\mathbb{T}^m]_p$ denotes the reduction modulo p of the fundamental class $[\mathbb{T}^m]$. This implies

$$(v_1 \cup \ldots \cup v_m) \cap (\phi_p)_*[\mathbb{T}^m]_p \quad = \quad (\phi_p)^*(v_1) \cup \ldots \cup (\phi_p)^*(v_m) \cap [\mathbb{T}^m]_p$$
$$= \quad 1.$$

This proves the non-triviality of $(\phi_p)_*[\mathbb{T}^m]_p$, and so of $(\phi_p)_*[\mathbb{T}^m]$.

We now prove the irreducibility of $\mathbf{a}(p, m)$. Let suppose the contrary. Any proper subgroup H of $G(p, m)$ is also a $\mathbb{Z}_p$-vector subspace of $G(p, m)$ of dimension $k < m$. Associated to some complementary of H in $G(p, m)$, we construct a projection map

$$\pi : G(p, m) \longrightarrow H$$

which is the identity on H. Fix a basis of the free $\mathbb{Z}$-module $\pi_1(\mathbb{T}^m)$ such that the composition

$$\pi \circ \phi_p : \pi_1(\mathbb{T}^m) \longrightarrow H$$

decomposes as

$$\pi \circ \phi_p = \psi \circ \rho_\sharp,$$

where $\rho_\sharp$ is induced by some projection $\rho : \mathbb{T}^m \longrightarrow \mathbb{T}^k$ and $\psi : \pi_1(\mathbb{T}^k) \longrightarrow H$ corresponds to the reduction modulo p. Now assume that $\mathbf{a}(p, m) = i_*(\mathbf{b})$, where $\mathbf{b} \in H_m(H, \mathbb{Z})$ and $i : H \longrightarrow G(p, m)$ denotes the inclusion. Then

$$\mathbf{b} = \pi_*(\mathbf{a}(p, m)) = \pi_* \circ (\phi_p)_*[\mathbb{T}^m] = \psi_* \circ \rho_*[\mathbb{T}^m] = \psi_*(0) = 0$$

as $\mathbf{b} = \pi_* \circ i_*(\mathbf{b})$. This gives a contradiction the class $\mathbf{a}(p, m)$ being non trivial.

In order to conclude the proof, remark that

$$\mathfrak{S}(\mathbf{a}(p, m)) = \mathfrak{S}_{\phi_p}(\mathbb{T}^m) \leq \mathfrak{S}(\mathbb{T}^m) \leq 1,$$

for any $m \geq 3$ and any prime p. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

This theorem implies the following unexpected result in dimensions $m \geq 4$.

Corollary 4.6. *For any dimension $m \geq 4$, there exists an infinite number of irreducible orientable manifolds M of dimension m with pairewise non-isomorphic fundamental groups such that $\mathfrak{S}(M) \leq 1$.*

Proof. By construction, every class $\mathbf{a}(p, m)$ is representable by a manifold. If $m \geq 4$, such a manifold can be modify by surgery in order to get a new manifold denoted by $M(p, m)$ such that $\pi_1(M(p, m)) = G(p, m)$ and $\Phi_*[M(p, m)] = \mathbf{a}(p, m)$, where $\Phi : M(p, m) \longrightarrow K(G(p, m), 1)$ denotes the classifying map, see [Bab06]. The infinite sequence of manifolds $\{M(p, m)\}$ where p runs over all prime numbers gives an infinite sequence of irreducible orientable manifolds M of dimension m with pairewise non-isomorphic fundamental groups such that $\mathfrak{S}(M) \leq 1$. $\qquad\square$

The following natural question remains open :

Question 4.7. Consider the systolic volume $\mathfrak{S}(\cdot)$ as a function over all irreducible orientable manifolds of dimension m. Does there exist a positive constant C such that the number of distinct values of the function $\mathfrak{S}(\cdot)$ less than C is infinite ?

5. Systolic volume of multiple classes

Given a finitely presentable group G and a homology class $\mathbf{a}$ of dimenson m, how does the function $\mathfrak{S}(G, k\mathbf{a})$ behave in term of the integer variable k ? In this section, we obtain the following result:

Theorem 5.1. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ where $m \geq 3$. Then there exists a positive constant $C(G, \mathbf{a})$ depending only on the pair $(G, \mathbf{a})$ such that*

$$(5.1) \qquad \mathfrak{S}(G, k\mathbf{a}) \leq C(G, \mathbf{a}) \cdot \frac{k}{\ln(1 + k)}$$

for any positive integer k.

Before proving this result, we put the problem in a more general context.

5.1. **Systolic volume of the sum of homology classes.** Let G_1 and G_2 be two finitely presentable groups and for $j = 1, 2$ denote by $i_j : G_j \hookrightarrow G_1 * G_2$ the natural monomorphism. Fix two integer homology classes $\mathbf{a}_1 \in H_m(G_1, \mathbb{Z})$ and $\mathbf{a}_2 \in H_m(G_2, \mathbb{Z})$. The natural isomorphism

$$H_m(G_1 * G_2, \mathbb{Z}) \simeq H_m(G_1, \mathbb{Z}) \oplus H_m(G_2, \mathbb{Z})$$

allows us to identify the class $(i_1)_*(\mathbf{a}_1) + (i_2)_*(\mathbf{a}_2)$ with $\mathbf{a}_1 + \mathbf{a}_2$.

Proposition 5.2. *If $m \geq 3$,*

$$\mathfrak{S}(G_1 * G_2, \mathbf{a}_1 + \mathbf{a}_2) \leq \mathfrak{S}(G_1, \mathbf{a}_1) + \mathfrak{S}(G_2, \mathbf{a}_2).$$

Proof. For any $\varepsilon > 0$, we choose for $j = 1, 2$ a geometric cycle (X_j, f_j) of dimension m representing $\mathbf{a}_j$ and satisfying

$$\mathfrak{S}_{f_j}(X_j) \leq \mathfrak{S}(G_j, \mathbf{a}_j) + \frac{\varepsilon}{2}.$$

The geometric cycle $(X_1 \# X_2, f_1 \# f_2)$ obtained as the connected sum of (X_1, f_1) and (X_2, f_2) represents the class $\mathbf{a}_1 + \mathbf{a}_2$. By the comparison principle (see [Bab06]),

$$\mathfrak{S}_{f_1 \# f_2}(X_1 \# X_2) \leq \mathfrak{S}_{f_1 \vee f_2}(X_1 \vee X_2) = \mathfrak{S}_{f_1}(X_1) + \mathfrak{S}_{f_2}(X_2)$$

where $(X_1 \vee X_2, f_1 \vee f_2)$ denotes the wedge of (X_1, f_1) and (X_2, f_2). As ε can be chosen arbitrarily small, we get the result. $\qquad\square$

If $\mathbf{a}_1$ and $\mathbf{a}_2$ are two homology classes of dimension m of the same group G, we deduce the following subadditivity property of the systolic volume.

Corollary 5.3. *Let* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *be two classes of* $H_m(G, \mathbb{Z})$, $m \geq 3$. *Then*

$$\mathfrak{S}(G, \mathbf{a}_1 + \mathbf{a}_2) \leq \mathfrak{S}(G, \mathbf{a}_1) + \mathfrak{S}(G, \mathbf{a}_2).$$

Proof. Indeed, if we denote by $\pi : G * G \to G$ the epimorphism defined by $\pi \circ i_j = id_G$, then $\pi_*(\mathbf{a}_1 + \mathbf{a}_2) = \mathbf{a}_1 + \mathbf{a}_2$. By Proposition 3.4, we get the result. $\square$

As a direct consequence of this corollary, we get

$$\mathfrak{S}(G, k\mathbf{a}) \leq k \cdot \mathfrak{S}(G, \mathbf{a})$$

for any $\mathbf{a} \in H_m(G, \mathbb{Z})$ with $m \geq 3$ and any integer k, and that the limit

$$\lim_{k \to \infty} \frac{\mathfrak{S}(G, k\mathbf{a})}{k}$$

exists. Theorem 5.1 permits us to conclude that the value of this limit is always zero.

5.2. **Sublinear upper bound for systolic volume of multiple classes.** Theorem 5.1 is related to the behaviour of systolic volume under connected sum operation and is a direct consequence of the following result.

Theorem 5.4. *Let* X *be a connected pseudomanifold of dimension* $m \geq 3$. *There exists a constant* $C(X)$ *depending only on the topology of* X *such that*

(5.2) $$\mathfrak{S}(\#_k X) \leq C(X) \cdot \frac{k}{\ln(1 + k)}$$

for any positive integer k. *If* k *is large enough, this last inequality (5.2) is satisfied for*

$$C(X) = m \cdot c(X) \cdot \ln c(X)$$

where $c(X)$ *stands for the number of cubes of a minimal decomposition of* X *by cubes.*

This theorem is a subsequent improvement of [BB05, Theorem A]. But the proof is entirely based on the techniques and ideas of [BB05]. A better upper bound is known for surfaces, see [Gro96]. Futhermore the upper bound (5.2) also holds for a sequence of cyclic covering. The details are similar to those considered in the sequel, see [BB05] on this subject.

Proof. We consider a minimal decomposition Θ of X by cubes and let $c := c(X)$ be the number of cubes of such a decomposition. We endow each m-dimensional cube of Θ with the flat euclidian metric g_0 such that the length of the edges of the cube is 1. Now cut each cube $\mathcal{C} \in \Theta$ with a strictly smaller concentric cube $\mathcal{C}' \subset \mathcal{C}$. On each *sleeve*

$$\overline{\mathcal{C}} = \overline{\mathcal{C} \setminus \mathcal{C}'} \simeq \partial \mathcal{C} \times [0, 1],$$

we consider the product metric $g_{0|\partial\mathcal{C}} \times \varepsilon \, dt$. The parameter ε will be chosen in the sequel. Denote by (X', g_0') the riemannian polyhedron thus obtained and remark that g_0' coincides with g_0 on the $(m-1)$-skeleton $\Theta^{(m-1)}$ of Θ.

By construction,

- The complex X' is homeomorphic to X minus c disjoint m-disks, so

$$\pi_1(X') = \pi_1(X);$$

- the volume of X' is given by the following formula:

$$\text{vol}(X', g'_0) = c \cdot 2m \cdot \varepsilon;$$

- the obvious contraction of X' on $\Theta^{(m-1)}$ decreases distances.

Moreover,

Lemma 5.5. *Let $\mathcal{C} \in \Theta$. If a relative curve γ of $(X', \partial\mathcal{C})$ is not entirely included in the open star $st(\mathcal{C})$ of $\mathcal{C}$, then its length satisfies $l_{g'_0}(\gamma) \geq 2$.*

This lemma can be proved *mutatis mutandis* as in [BB05, Lemma 1].

We define the graph $\mathcal{A}_c$ composed of c edges starting from one vertex s_0, see [BB05, Formula (3.9)]. The PL-map

(5.3) $$f : X' \longrightarrow \mathcal{A}_c$$

which contracts $\Theta^{(m-1)}$ into s_0 and projects in the evident way each sleeve $\overline{\mathcal{C}} \simeq \partial\mathcal{C} \times [0,1]$ on an edge of $\mathcal{A}_c$ is analog to that one defined in formula (3.10) of [BB05]. In order to adapt this formula to our case, we need to replace the term $\frac{1}{2k^2}$ by ε. If $\mathcal{A}_c$ is endowed with the linear metric for which each edge has length ε, the map f contracts distances (see [BB05, Lemma 3]).

We now adapt the construction of [BB05, Subsection 3.4]. We replace $M(k)$ by X', k^2 by $\frac{1}{2\varepsilon}$ and the valency D by c. This gives rise to a pseudomanifold $X(2n, \varepsilon)$ homeomorphic to

$$(\underset{2n}{\#} X) \# (\underset{n(c-2)+1}{\#} S^1 \times S^{m-1}),$$

and a graph Γ with $2n$ vertices of valency c where

$$n \geq 2 \sum_{t=1}^{[\frac{1}{2\varepsilon}]-1} (c-1)^t.$$

The graph Γ is endowed with the linear metric h for which each edge has length 2ε. The combinatorial systole of Γ being bigger than $\frac{1}{2\varepsilon}$, the metric systole $\text{sys}(\Gamma, h)$ is thus bigger than 1. Each piece $X' \subset X(2n, \varepsilon)$ is endowed with the metric g'_0 and this defines a metric on $X(2n, , \varepsilon)$ denoted by g_1. The map

(5.4) $$F : (X(2n, \varepsilon), g_1) \longrightarrow (\Gamma, h)$$

is then distance-decreasing. The restriction of F to any element $X' \subset X(2n, \varepsilon)$ coincides with the map f defined in 5.3 : it thus contracts the $(m-1)$-skeleton of Θ into the corresponding vertex of Γ and it projects each sleeve on a half-edge of Γ starting from this vertex.

In order to estimate $\mathfrak{S}(\#_{2n}X) = \mathfrak{S}(X(2n, \varepsilon))$ (the addition of 1-handles does not change the value of the systolic volume, see [BB05]), it remains to prove

Lemma 5.6.

(5.5) $$sys(X(2n, \varepsilon), g_1) \geq 1.$$

Proof. If $\gamma : [0,1] \to X(2n, \varepsilon)$ is a closed curve whose image $F(\gamma)$ is not contractible in Γ, we have

$$l_{g_1}(\gamma) \geq l_h(F(\gamma)) \geq \text{sys}(\Gamma, h) \geq 1.$$

Now assume that γ is not contractible in $X(2n, \varepsilon)$ but that $F(\gamma)$ is contractible in Γ. We will show that $l_{g_1}(\gamma) \geq 2$.

First of all, we can assume that γ is minimizing in its own homotopy class. The metric g_1 is piecewise flat, so γ is piecewise linear. The contractibility of $F(\gamma)$ implies the existence of a *cusp* in the following sense: there exists an edge $[s_i, s_j]$ of Γ joining two vertices s_i and s_j, a point $v \in]s_i, s_j]$ and a triplet (t_1, t_*, t_2) such that $0 \leq t_1 < t_* < t_2 \leq 1$ satisfying

- $v = F(\gamma(t_*))$;
- $F(\gamma([t_1, t_2]) \subset [s_i, v]$;
- $F(\gamma(t_1)) = F(\gamma(t_2)) = s_i$.

If $v \in]s_i, s_j[$, we contract the part of the curve $\{\gamma(t)\}_{t \in [t_1, t_2]}$ in $F^{-1}([s_i, v[)$. This contraction strictly decreases the length of γ which is in contradiction with its minimality. So the cusp v coincides with s_j. The restriction of γ to the interval $[t_1, t_2]$ is thus the concatenation $\gamma_1 \star \gamma_2 \star \gamma_3$ of three curves with $\gamma_1, \gamma_3 \subset F^{-1}([s_i, s_j[)$ and $\gamma_2 \subset F^{-1}(\{s_j\}) = \Theta_j^{m-1}$. If m_{ij} denotes the middle of the edge $[s_i, s_j]$, we set

$$\overline{\mathcal{C}}_j = \overline{F^{-1}([m_{ij}, s_j[)}.$$

We have two cases to consider.

(1) Either γ_2 is not entirely contained in the star $\mathrm{st}(\overline{\mathcal{C}}_j)$ of $\overline{\mathcal{C}}_j$ and

$$l_{g_1}(\gamma) \geq l_{g_0'}(\gamma_2) \geq 2$$

by Lemma 5.5 ;

(2) Or γ_2 is entirely contained in the star $\mathrm{st}(\overline{\mathcal{C}}_j)$ of $\overline{\mathcal{C}}_j$. The orthogonal projections of $\mathrm{st}(\overline{\mathcal{C}}_j)$ on each face of $\partial \mathcal{C}_j$ are correctly defined and coherents. So we project γ_2 orthogonally on $\partial \mathcal{C}_j$. This projection does not change the homotopy class of γ, and if $\gamma_2 \not\subset \partial \mathcal{C}_j$, it strictly decreases the length. We still denote by γ_2 the part of the curve contained in $\overline{\mathcal{C}}_j$. Now we contract γ_2 in $\overline{\mathcal{C}}_j \setminus \partial \mathcal{C}_j$. This contraction strictly decreases the length of γ which is in contradiction with the minimality of γ.

This concludes the proof of the lemma.   □

So we have shown that

$$(5.6) \qquad \mathfrak{S}(\#_{2n} X) \leq \frac{\mathrm{vol}(X(2n, \varepsilon), g_1)}{(\mathrm{sys}(X(2n, \varepsilon), g_1)^m} \leq 4m \cdot n \cdot c \cdot \varepsilon.$$

The graph Γ is c-regular and satisfies the statement of [BB05, Theorem 2]. The number of vertices can be chosen to satisfy the relation

$$2n \geq \frac{4}{c-2}[(c-1)^l - (c-1)]$$

where $l = \frac{1}{2\varepsilon}$. As $c \geq 2m + 1 \geq 7$, we can always choose n such that

$$(5.7) \qquad\qquad\qquad 2n \leq (c-1)^l.$$

We deduce from 5.6 and 5.7 that

$$\mathfrak{S}(\#_{2n} X) \leq m \, c \, \ln(c-1) \frac{2n}{\ln(2n)}.$$

So

$$
\begin{aligned}
\mathfrak{S}(\#_{2n+1}X) &\leq \mathfrak{S}(\#_{2n}X) + \mathfrak{S}(X) \\
&\leq m \cdot c \cdot \ln(c-1)\frac{2n}{\ln(2n)} + \mathfrak{S}(X) \\
&\leq m \cdot c \cdot \ln(c-1)\frac{2n+1}{\ln(2n+1)} + \mathfrak{S}(X).
\end{aligned}
$$

For any large enough k, we thus get the universal upper bound (5.2) for $d = m \cdot c \cdot \ln c$. This concludes the proof. $\square$

5.3. **Homology classes with positive simplicial volume.** Recall the following definition (see [Gro82]).

Definition 5.7. *Let X be a pseudomanifold of dimension m. Its simplicial volume is the quantity*

$$
\|X\|_\Delta = inf\{\sum_i |r_i| \mid [X] = \sum_i r_i\sigma_i^m\},
$$

where the infimum is taken over the set of representations of the fundamental class $[X]$ by singular simplicial chains with real coefficients.

   If G denotes a finitely presentable group and $\mathbf{a}$ a homology class of dimension m, the simplicial volume of $\mathbf{a}$ is then the number

$$
\|\mathbf{a}\|_\Delta = inf\{\|X\| \mid X \text{ representing } \mathbf{a}\}.
$$

For homology classes whose simplicial volume is positive, the function $\mathfrak{S}(G, k\mathbf{a})$ goes to infinity and the following result precises its asymptotic behaviour.

Corollary 5.8. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ be a homology class of dimension $m \geq 3$ such that $\|\mathbf{a}\|_\Delta > 0$. Then there exists two positive constants $C(G, \mathbf{a})$ and $\tilde{C}(G, \mathbf{a})$ depending only the pair $(G, \mathbf{a})$ such that*

$$
\tilde{C}(G, \mathbf{a}) \cdot \frac{k}{(\ln(1+k))^m} \leq \mathfrak{S}(G, k\mathbf{a}) \leq C(G, \mathbf{a}) \cdot \frac{k}{\ln(1+k)}
$$

for any positive integer k.

Proof. The lower bound is a direct consequence of the following inequality of Gromov (see [Gro83, Theorem 6.4.D']): any pseudomanifold X of dimension m satisfies the inequality

$$
C'\frac{\|X\|_\Delta}{(\ln(2+\|X\|_\Delta))^m} \leq \mathfrak{S}(X),
$$

where C' is a positive constant depending only on the dimension m. It remains to remark that, if X represents the class ka, then $\|X\|_\Delta = \|k\mathbf{a}\|_\Delta = k\|\mathbf{a}\|_\Delta$. The upper bound then follows by Theorem 5.1. $\square$

5.4. **Large oscillations of systolic volume.** The following example shows that the function $k \mapsto \mathfrak{S}(G, k\mathbf{a})$ can be to some extent irregular.

   Let $m = 2l + 1 \geq 3$ be an odd integer and $q \geq 2$ an integer. Let X be an essential manifold of dimension m (for example aspherical) and $f : X \to K(\pi_1(X), 1)$ denotes its classifying map. If X is not aspherical, we assume that the image of its fundamental class $f_*[X]$ is an element of infinite order in $H_m(\pi_1(X), \mathbb{Z})$. Set $a = f_*[X]$. Fix a generator $\mathbf{b} \in H_m(\mathbb{Z}_q, \mathbb{Z}) = \mathbb{Z}_q$. For each $1 \leq l \leq q-1$, we choose

a normal representation of $l\mathbf{b}$ by a manifold Y_l with $\pi_1(Y_l) = \mathbb{Z}_q$. For l prime with q, the corresponding lenticular space can be chosen to be Y_l. Set

$$D = \max_{1 \leq k \leq q} \mathfrak{S}(\pi_1(X), k\mathbf{a})$$

and fix any positive constant C. Consider the free product

$$G_n = \pi_1(X) * \underbrace{\mathbb{Z}_q * \ldots * \mathbb{Z}_q}_{n},$$

and pick in

$$H_m(G_n, \mathbb{Z}) = H_m(\pi_1(X), \mathbb{Z}) \oplus H_m(\mathbb{Z}_q, \mathbb{Z}) \oplus H_m(\mathbb{Z}_q, \mathbb{Z}) \oplus \ldots \oplus H_m(\mathbb{Z}_q, \mathbb{Z})$$

the class $\mathbf{c} = \mathbf{a} \oplus \underbrace{\mathbf{b} \oplus \ldots \oplus \mathbf{b}}_{n}$. If X_k is a manifold representing the class $k\mathbf{a}$, then

$$X_k \# \underbrace{Y_k \# \ldots \# Y_k}_{n}$$

represents the class $k\mathbf{c}$. By Corollary 3.5,

$$\mathfrak{S}(G_n, k\mathbf{c}) \geq \mathfrak{S}(\underbrace{Y_k \# \ldots \# Y_k}_{n}).$$

Now Theorem **A** of [Sab07] implies that if n is chosen large enough, we have $\mathfrak{S}(G_n, k\mathbf{c}) > C$ for any $1 \leq k \leq q - 1$. Besides $q\mathbf{c} = q\mathbf{a}$ in $H_m(G_n, \mathbb{Z})$, and so $\mathfrak{S}(G_n, q\mathbf{c}) \leq D$.

5.5. **Systolic generating function.** If G is a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ is a homology class of infinite order, the study of the sequence $\{\mathfrak{S}(G, k\mathbf{a})\}_{k=1}^{\infty}$ is equivalent to the study of analytic properties of the following generating function:

$$\sigma_{G,\mathbf{a}}(z) = \sum_{k=1}^{\infty} \mathfrak{S}(G, k\mathbf{a}) \cdot z^k.$$

If the group G is clearly identified by the context, we simplify the notation into $\sigma_{\mathbf{a}}(z)$. The upper bound (5.1) implies that $\sigma_{\mathbf{a}}(z)$ is an analytic function on the disk $|z| < 1$. Furthermore the complex point $z = 1$ is a singular point of this function as $\mathfrak{S}(G, k\mathbf{a}) \geq \sigma_m > 0$.

There is no hope in general for $\sigma_{G,\mathbf{a}}(z)$ to be a rational function. Indeed, if $\mathbf{a}$ is class with positive simplicial volume, Corollary 5.8 teachs us that $z = 1$ is not a pole of the corresponding systolic generating function: so $\sigma_{\mathbf{a}}(z)$ is not rational. It is well known that the rationality of the generating function of a numerical sequence is equivalent to the recurrence of this sequence. We deduce the following

Proposition 5.9. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a homology class of infinite order. If the simplicial volume of $\mathbf{a}$ is positive, the sequence of systolic volumes $\{\mathfrak{S}(G, k\mathbf{a})\}_{k=1}^{\infty}$ does not satisfy any linear recurrent equation.*

Nevertheless the rationality of $\sigma_{\mathbf{a}}(z)$ seems plausible for classes $\mathbf{a}$ with a bounded sequence of systolic volume $\{\mathfrak{S}(G, k\mathbf{a})\}_{k=1}^{\infty}$. Tori give a model of this type of behaviour for multiple classes.

Proposition 5.10. *Let $1 \leq m \leq n$ be two integers. Any class $\mathbf{a} \in H_m(\mathbb{Z}^n, \mathbb{Z})$ satisfies the inequality*

$$\mathfrak{S}(\mathbb{Z}^n, \mathbf{a}) \leq C_n^m \cdot \mathfrak{S}(\mathbb{T}^m),$$

where C_n^m denotes the binomial coefficient.

Proof. Fix a basis of $H_m(\mathbb{Z}^n, \mathbb{Z})$ composed of embedded m-torus, and write the class $\mathbf{a}$ in this basis:

$$\mathbf{a} = \sum_{i=1}^{C_b^m} k_i[\mathbb{T}_i^m]$$

where $k_i \in \mathbb{Z}$ for $i = 1, \ldots, C_b^m$. By Corollary 5.3,

$$(5.8) \qquad \mathfrak{S}(G, \mathbf{a}) \leq \sum_{i=1}^{C_b^m} \mathfrak{S}(G, k_i[\mathbb{T}_i^m]).$$

Observe that

$$(5.9) \qquad \mathfrak{S}(\mathbb{Z}^n, k[\mathbb{T}_i^m]) \leq \mathfrak{S}(\mathbb{T}^m)$$

for any integer k. In fact, if $f : \mathbb{T}^m \to \mathbb{T}_i^m$ denotes a map of degree k, the geometric cycle $(\mathbb{T}^m, f)$ represents the class $k[\mathbb{T}_i^m]$. By adding 1-handles, we can normalize this representation into a geometric cycle $(\mathbb{T}^m \# (S^1 \times S^{m-1}) \# \ldots \# (S^1 \times S^{m-1}), \tilde{f})$. We get

$$\begin{aligned} \mathfrak{S}(\mathbb{Z}^n, k[\mathbb{T}_i^m]) &= \mathfrak{S}_{\tilde{f}}(\mathbb{T}^m \# (S^1 \times S^{m-1}) \# \ldots \# (S^1 \times S^{m-1})) \\ &\leq \mathfrak{S}(\mathbb{T}^m \# (S^1 \times S^{m-1}) \# \ldots \# (S^1 \times S^{m-1})) = \mathfrak{S}(\mathbb{T}^m). \end{aligned}$$

Now we deduce the result by combining inequalities (5.8) and (5.9). $\qquad \square$

We close this chapter with the following

Conjecture 5.11. *If $\mathbf{a} = [\mathbb{T}^m] \in H_m(\mathbb{Z}^m, \mathbb{Z})$, then the associated systolic generating function is*

$$\sigma_{\mathbf{a}}(z) = \mathfrak{S}(\mathbb{T}^m) \cdot \frac{z}{1 - z}.$$

6. Torsion, simplicial complexity of groups and systolic volume

Let G be a finitely presentable group and $\mathbf{a}$ a homology class of dimension m. We define the 1-*torsion of the class* $\mathbf{a}$ as the number

$$t_1(\mathbf{a}) = \inf_{(X,f)} |\text{Tors } H_1(X, \mathbb{Z})|.$$

Here the infimum is taken over the set of geometric cycles (X, f) representing the class $\mathbf{a}$ and $|\text{Tors } H_1(X, \mathbb{Z})|$ denotes the number of torsion elements in the first integer homology group of X. We present the main result of this section:

Theorem 6.1. *Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$. Then*

$$\mathfrak{S}(G, \mathbf{a}) \geq C_m \frac{\ln t_1(\mathbf{a})}{\exp(C_m' \sqrt{\ln(\ln t_1(\mathbf{a}))})},$$

where C_m and C_m' are two positive constants depending only on m.

In particular,

$$\mathfrak{S}(G, \mathbf{a}) \geq (\ln t_1(\mathbf{a}))^{1-\varepsilon}$$

for any $\varepsilon > 0$ if $t_1(\mathbf{a})$ is large enough. It is important to remark that there is no hope in any dimension to prove a universal lower bound of the type

$$\mathfrak{S}(G, \mathbf{a}) \geq C \ln t_1(\mathbf{a})$$

for some positive constant C. Indeed, for any positive integer n, the Eilenberg-MacLane space of the group

$$G_n := \underbrace{\mathbb{Z}_2 * \ldots * \mathbb{Z}_2}_{n}$$

is the complex $\vee_{i=1}^n \mathbb{R}P_i^\infty$. If $\mathbb{R}P_i^{2m+1} \subset \mathbb{R}P_i^\infty$ denotes the skeleton of some odd dimension $2m+1$ of the i-th component, we consider the sequence of homology classes

$$\mathbf{a}_n = \sum_{i=1}^n [\mathbb{R}P_i^{2m+1}] \in H_{2m+1}(G_n, \mathbb{Z}).$$

We can see that $|\mathrm{Tors}\, H_1(X, \mathbb{Z})| \geq 2^n$ for any representation (X, f) of $\mathbf{a}$ and so, by application of Theorem 5.4,

$$\mathfrak{S}(G_n, \mathbf{a}_n) \leq \mathfrak{S}(\#_{i=1}^n \mathbb{R}P_i^{2m+1}) \leq C \cdot \frac{\ln t_1(\mathbf{a}_n)}{\ln \ln t_1(\mathbf{a}_n)}$$

for some positive constant $C > 0$. For even dimension, we consider the sequence of classes $\tilde{\mathbf{a}}_n = [S^1] \times \mathbf{a}_n \in H_{2m+2}(\mathbb{Z} \times G_n, \mathbb{Z})$ for which the same estimates hold.

In general the 1-torsion of a class is difficult to compute. In the case of $\mathbb{Z}_n$, we can estimate from below the 1-torsion of any generator by the number n. As the fundamental class of a lenticular manifold $L_m(n)$ of dimension $2m+1$ with fundamental group $\mathbb{Z}_n$ realizes a generator $\mathbf{a}$ of the homology group $H_{2m+1}(\mathbb{Z}_n, \mathbb{Z})$, we obtain the following result:

Theorem 6.2. *For any lenticular manifold $L_m(n)$,*

$$\mathfrak{S}(L_m(n)) \geq C_m \frac{\ln n}{\exp(C_m' \sqrt{\ln(\ln n)})}$$

where C_m and C_m' are two positive constants depending only on m.

In order to prove Theorem 6.1, we are going to estimate the minimal number of 2-simplices of any representation $\mathbf{a}$ by a geometric cycle. This number of simplices turns out to be estimated by the torsion of the first homology group. Remark that the possibility to use the torsion of $H_*(\pi_1(M), \mathbb{Z})$ to estimate from below $\mathfrak{S}(M)$ has been mentionned by Gromov in [Gro96]. This principle will be also used for the computation of some estimations in the next section. Indirectly, we also derive from this result the following.

Theorem 6.3. *There exists two positive constants a and b such that any manifold M of dimension 3 with finite fundamental group satisfies*

$$\mathfrak{S}(M) \geq a \frac{\ln |\pi_1(M)|}{\exp(b\sqrt{\ln(\ln |\pi_1(M)|)})},$$

where $|\pi_1(M)|$ denotes the cardinal of $\pi_1(M)$.

Remark that finite fundamental groups of 3-manifolds can have a large number of elements but a very small torsion in $H_1(\pi_1(M), \mathbb{Z})$. A direct estimate of $\mathfrak{S}(M)$ by the torsion of $H_1(\pi_1(M), \mathbb{Z})$ is interesting only if the manifold M is a lenticular space.

6.1. **Simplicial complexity of a group.** First of all, we introduce a definition which appears useful independently of the study of the systolic volume. Given a finite simplicial complex P, we denote by $s_k(P)$ the number of its k-simplices.

Definition 6.4. *Let G be a finitely presentable group. We define the simplicial complexity of G by the following formula:*

$$\kappa(G) := \inf_{\pi_1(P)=G} s_2(P),$$

the infimum being taken over all finite simplicial 2-complexes P with fundamental group G. A 2-complex P is said minimal for G if $\pi_1(P) = G$ and $s_2(P) = \kappa(G)$.

This quantity posseses the following properties:

1. For any finitely presentable group G, $\kappa(G) = 0$ if and only if G is a free group.

2. The free product of two finitely presentable groups G_1 and G_2 satisfies

$$\kappa(G_1 * G_2) \leq \kappa(G_1) + \kappa(G_2).$$

If $\kappa(G_1)$ and $\kappa(G_2)$ are both positive, this last inequality can be strengthened in the following way:

$$\kappa(G_1 * G_2) \leq \kappa(G_1) + \kappa(G_2) - 2.$$

3. For a simplicial complex X, its *simplicial height $h(X)$* (total number of its simplices, notion introduced by Gromov, see [Gro96]) is obviously bounded from below by the simplicial complexity of its fundamental group:

$$h(X) \geq \kappa(\pi_1(X)).$$

Remark 6.5. Even for groups whose structure is simple, the exact value of κ seems hard to compute. We can show that $\kappa(\mathbb{Z}_2) = 10$: this value is realized on a minimal triangulation of $\mathbb{R}P^2$. It seems reasonable but not obvious that the minimal value of κ over non free groups will be reached for the group $\mathbb{Z}_2$.

Remark 6.6. For a group of surface of large genus, the exact computation of the complexity reminds an open problem. We can nevertheless give some bound of the complexity in term of the genus.

Let π_l be the fundamental group of an orientable surface of genus $l \geq 1$. By elementary algebraic and combinatorial considerations,

$$(6.1) \qquad \qquad \frac{4}{3}l \leq \kappa(\pi_l).$$

Besides, by a result of Jungerman & Ringel [JR80],

$$(6.2) \qquad \qquad \kappa(\pi_l) \leq 4(l-1) + 2\left\{\frac{7 + \sqrt{1 + 48l}}{2}\right\}.$$

Here $\{a\}$ denotes the integer part of $a + 1$ if a is not an integer and a for integers. Strictly speaking, the upper bound is only available for $l \neq 2$. For $l = 2$ we have to replace the upper bound by 24, see [JR80]. We thus have

$$\kappa(\mathbb{Z} \oplus \mathbb{Z}) \leq 14$$

from which we derive easily that the abelian group A_n of rank n satisfies

$$\frac{1}{2}n(n-1) \leq \kappa(A_n) \leq 7n(n-1).$$

More precisely, the lower bound is given by the second Betti number and the upper bound is obtained by recurrence using previous inequality. The precise computation of $\kappa(A_n)$ remains open.

Let P be a connected simplicial 2-complex. An edge r of P will be said *free* if the edge is not incident to a 2-simplex of P. An elementary analysis of the simplicial structure of P tells us that there exists a new connected simplicial 2-complex Q satisfying the following properties:

- $s_2(P) = s_2(Q)$;
- Q does not possess free edges ;
- $\pi_1(P) = \pi_1(Q) * F_n$, where F_n is the free group with n generators.

Proposition 6.7. *For any positive constant K, the set of finitely presentable groups G of zero Grushko free index satisfying $\kappa(G) \leq K$ is finite and can be bounded by $2^{\frac{K^3}{14}}$.*

Proof. Let G be an group of zero Grushko free index with $\kappa(G) \leq K$. By the preceding remark, we can find a minimal 2-complex P for G without free edges. Remark also that the minimality of P implies that any edge is incident to at least two 2-simplexes, and that any vertex of P is incident to at least four 2-simplexes. In particular the number of vertices of P is bounded from above by $\frac{3}{4}\kappa(G) \leq M := \frac{3}{4}K$. Thus P is a subcomplex of the $(M-1)$-dimensional simplex Δ_M.

The number of 2-simplices of Δ_M is equal to C_M^3. So for each integer $0 \leq s \leq K$, the number of 2-complexes of Δ_M without free edges and composed of a number s of 2-simplices is equal to $C_{C_M^3}^s$. We deduce that the number of groups G of zero Grushko free index with $\kappa(G) \leq K$ is bounded by

$$\sum_{s=0}^{K} C_{C_M^3}^s \leq 2^{C_M^3} = 2^{\frac{K(3K-4)(3K-8)}{128}} \leq 2^{\frac{K^3}{14}}.$$

$\square$

Remark 6.8. The bound of Proposition 6.7 seems far to be optimal.

6.2. **Simplicial complexity and 1-torsion.** The simplicial complexity $\kappa(G)$ is quite sensitive to the number of torsion elements in $H_1(G, \mathbb{Z})$.

Proposition 6.9. *Let X be a simplicial complex of dimension 2. Then*

$$s_2(X) \geq 2\, log_3 |Tors\, H_1(X, \mathbb{Z})|.$$

In particular, any finitely presentable group G satisfies the inequality

$$\kappa(G) \geq 2\, log_3 |Tors\, H_1(G, \mathbb{Z})|.$$

Proof. Consider the complex of simplicial cochains

$$(6.3) \qquad\qquad C^1(X, \mathbb{Z}) \xrightarrow{d^1} C^2(X, \mathbb{Z}) \xrightarrow{d^2} 0.$$

The universal coefficient theorem implies a duality between homology torsion and cohomology torsion, and we have (see [Hat02, Corollary 3.3] for instance)

$$(6.4) \qquad \operatorname{Tors} H_1(X, \mathbb{Z}) \simeq \operatorname{Tors} H^2(X, \mathbb{Z}).$$

This implies that $|\operatorname{Tors} H_1(X, \mathbb{Z})| = |\operatorname{Tors}\big(C^2(X, \mathbb{Z})/\operatorname{Im} d^1\big)|$.

We endow $C^i(X, \mathbb{Z})$ with the basis dual to the simplicial basis of $C_i(X, \mathbb{Z})$. Let D denotes the matrix of d^1 with respect to these bases. The matrix D has $s_2(X)$ rows, and each row has exactly three non zero elements whose value is either 1 or -1. It follows that every row vector of D has euclidian length $\sqrt{3}$. If we interpret the determinant of a square matrix V of order k as the volume of the parallelotope generated by its row vectors, we see that for any smaller square matrix V of D of order k

$$(6.5) \qquad |\det V| \leq (\sqrt{3})^k.$$

Assume that the rank of D is equal to d, and denote by $t(D)$ the greatest common divisor of all minors of order d of D. By (6.5),

$$(6.6) \qquad t(D) \leq (\sqrt{3})^d \leq (\sqrt{3})^{s_2(X)}.$$

Furthermore, it is obvious that $t(D)$ is invariant under change of basis of $C^1(X, \mathbb{Z})$ or of $C^2(X, \mathbb{Z})$. By a general result on free $\mathbb{Z}$-modules (see [VdW71]), there exist a basis $e^1, \ldots, e^{s_1(X)}$ of $C^1(X, \mathbb{Z})$ and a basis $f^1, \ldots, f^{s_2(X)}$ of $C^2(X, \mathbb{Z})$ such that $d^1(e^i) = m_i \cdot f^i$ for $1 \leq i \leq d$ and $d^1(e^i) = 0$ for $i > d$ (note that the m_i's can be chosen such that m_i divides m_{i+1} albeit we will not need this). On the one hand, we have

$$|\operatorname{Tors} H_1(X, \mathbb{Z})| = |\operatorname{Tors}\big(C^2(X, \mathbb{Z})/\operatorname{Im} d^1\big)| = \prod_{i=1}^{d} m_i.$$

On the other hand, the matrix of d^1 in the new bases $(e^i)_{i=1}^{s_1(X)}$ and $(f^j)_{j=1}^{s_2(X)}$ is the diagonal matrix $\operatorname{Diag}(m_1, \ldots, m_d, 0, \ldots, 0)$ and a straightforward computation of the minors of order d of this matrix gives

$$t(D) = \prod_{i=1}^{d} m_i.$$

Together with the inequality (6.6), this completes the proof. $\qquad\square$

Now we prove Theorem 6.1. Let G be a finitely presentable group and $\mathbf{a} \in H_m(G, \mathbb{Z})$ a homology class. Recall that the *simplicial height* $h(\mathbf{a})$ of a homology class $\mathbf{a}$ is the minimum number of simplexes (of any dimension) of a geometric cycle representing the class $\mathbf{a}$. By Proposition 6.9,

$$(6.7) \qquad h(\mathbf{a}) \geq 2 \log_3 t_1(\mathbf{a}).$$

It remains to apply the estimate of the systolic volume $\mathfrak{S}(G, \mathbf{a})$ by the simplicial height $h(\mathbf{a})$, see [Gro83, 6.4.C"] and [Gro96, 3.C.3].

6.3. **Application to lenticular spaces.** Given two integers $m \geq 0$ and $n \geq 2$, let $L_m(n)$ be a lenticular space of dimension $2m+1$ with fundamental group $\mathbb{Z}_n$. That is, there exist integers $q_1, \ldots, q_n$ coprime with n and an isometry A of order n of the form

$$A(z_1, \ldots, z_m) = (e^{2\pi i \frac{q_1}{n}} z_1, \ldots, e^{2\pi i \frac{q_m}{n}} z_m)$$

such that

$$L_m(n) := \{Z = (z_1, \ldots, z_m) \in \mathbb{C}^m \Big| \sum_{k=1}^m |z_k|^2 = 1\}/_{\sim A} \simeq S^{2m+1}/\mathbb{Z}_n,$$

where $Z \sim Z'$ if and only if $Z = A^k Z'$. Observe that the fundamental class of a lenticular space $L_m(n)$ realizes a generator $\mathbf{a}$ of the homology group $H_{2m+1}(\mathbb{Z}_n, \mathbb{Z})$.

Lemma 6.10. *Let* $\mathbf{a}$ *be a generator of* $H_{2m+1}(\mathbb{Z}_n, \mathbb{Z})$. *Then*

$$t_1(\mathbf{a}) \geq n.$$

Proof. Let (X, f) be a geometric cycle representing $\mathbf{a}$. As $\mathbf{a}$ is a generator of $H_{2m+1}(\mathbb{Z}_n, \mathbb{Z}) \simeq \mathbb{Z}_n$, the map f induces an isomorphism

(6.8) $$f^* : H^{2m+1}(\mathbb{Z}_n, \mathbb{Z}_n) \longrightarrow H^{2m+1}(X, \mathbb{Z}_n).$$

Let

$$\beta : H^1(\mathbb{Z}_n, \mathbb{Z}_n) \longrightarrow H^2(\mathbb{Z}_n, \mathbb{Z}).$$

denotes the Bockstein homomorphism and

$$j : H^2(\mathbb{Z}, \mathbb{Z}) \longrightarrow H^2(\mathbb{Z}, \mathbb{Z}_n)$$

the morphism of reduction modulo n. In our case, j is an isomorphism. A generator of $H^{2m+1}(\mathbb{Z}_n, \mathbb{Z}_n)$ (non necessarily dual to $\mathbf{a}$) can be choosen as $\mathbf{u} \cup (j \circ \beta(\mathbf{u}))^m$, where $\mathbf{u} \in H^1(\mathbb{Z}_n, \mathbb{Z}_n)$ is a generator. Now consider $f^*(\beta(\mathbf{u})) = \beta(f^*(\mathbf{u})) \in H^2(X, \mathbb{Z})$. Taking into account the isomorphism (6.8), $f^*(\mathbf{u}) \cup (f^*(j \circ \beta(\mathbf{u})))^m$ is an element of order n in $H^{2m+1}(X, \mathbb{Z}_n)$. This implies that the order of $f^*(j \circ \beta(\mathbf{u})) \in H^2(X, \mathbb{Z}_n)$ is n, and so the order of $\beta(f^*(\mathbf{u})) \in H^2(X, \mathbb{Z})$ is also n. By the duality (6.4), we get the result.   □

Remark that the statement of this lemma as well as its proof hold in the case of a simplicial complex X representing the class $\mathbf{a}$. In this more general case, we have to note that the map (6.8) is a monomorphism.

By combining Lemma 6.10 and Theorem 6.1, we derive Theorem 6.2.

6.4. **Application to 3-manifold.** Given a closed manifold M and a covering space M' with k sheets, the following inequality is obvious:

(6.9) $$\mathfrak{S}(M) \geq \frac{1}{k}\mathfrak{S}(M').$$

Let M be a manifold of dimension 3 with finite fundamental group. Its universal cover is the sphere S^3, and the action of the fundamental group $\pi_1(M)$ on S^3 is orthogonal. The list of finite groups which act orthogonally on S^3 can be found in [Mil60] for instance. An analyse of this list shows that $\pi_1(M)$ possesses a cyclic subgroup of index $k \leq 12$. Denote by M' the covering space corresponding to this subgroup. The manifold M' is a lenticular space $L_1(n)$ with $n \geq \frac{|\pi_1(M)|}{12}$. Theorem 6.3 now follows from Theorem 6.2 and the inequality (6.9) with $k = 12$.

6.5. **Simplicial complexity and systolic area of a group.** Recall that the *systolic area* of a finitely presentable group G is defined as

$$\mathfrak{S}(G) = \inf_{\pi_1(P)=G} \mathfrak{S}(P),$$

where the infimum is over all finite simplicial complex P of dimension 2 with fundamental group G.

For finitely presentable groups G of zero Grushko free index, the quantity of metric nature $\mathfrak{S}(G)$ is relied to the the purely combinatorical quantity $\kappa(G)$ by the following two results.

Proposition 6.11. *Let G be an finitely presentable group of zero Grushko free index. Then*

$$\mathfrak{S}(G) \leq \frac{\kappa(G)}{2\pi}.$$

Proof. Consider a minimal simplicial complex P of dimension 2 with fundamental group G. Endow P with the metric h such that any edge is of length $\frac{2\pi}{3}$ and any face is the round hemisphere of radius 1. As P is minimal, $s_2(P) = \kappa(G)$ and so

$$(6.10) \qquad\qquad \text{area}(P,h) = 2\pi\kappa(G).$$

The definition of h implies that any systolic geodesic can be homotoped to the 1-skeleteon without increasing its length. Such a curve passes through at least three edges and thus $\text{sys}(P,h) \geq 2\pi$. This implies with (6.10) that

$$\mathfrak{S}(G) \leq \mathfrak{S}(P,h) = \frac{\kappa(G)}{2\pi}.$$

$\square$

Theorem 6.12. *Let G be a finitely presentable group of zero Grushko free index. Then*

$$\kappa(G) \leq C \cdot \mathfrak{S}(G) \exp\left(C'\sqrt{log\,\mathfrak{S}(G)}\right),$$

for some universal positive constants C and C'. In particular, for any $\theta > 0$ there exists a positive constant C_θ such that

$$\mathfrak{S}(G) \geq C_\theta \cdot \kappa(G)^{1-\theta}.$$

Proof. We argue as in [Gro83, 5.3.B]. In the sequel, if $B := B(p,R)$ denotes the metric ball centered at p of radius R, we denote by $|B|$ its area for the metric g, and nB the concentric ball $B(p,nR)$ for any positive integer n.

Set $R_0 = \frac{1}{25}$ and

$$(6.11) \qquad\qquad \alpha = 25\exp\left(\sqrt{\log(62500 \cdot \mathfrak{S}(G))}\right).$$

By [RS08, Theorem 1.4],

$$\mathfrak{S}(G) \geq \frac{\pi}{16},$$

so α is well defined ans satisfies $\alpha > 5$. Fix some positive ε small enough such that

$$(6.12) \qquad\qquad \log_5\frac{R_0}{5\varepsilon} \cdot \log\frac{\alpha}{5} \geq \log\left(62500 \cdot \kappa(G)\right).$$

By [RS08, Theorem 3.5 and Lemma 4.2], there exists a simplicial complex P of dimension 2 endowed with a metric g such that

- $\pi_1(P) = G$;
- $\mathrm{sys}(P, g) = 1$;
- $\mathfrak{S}(P, g) = \mathrm{area}(P, g) < \mathfrak{S}(G) + \varepsilon$;
- any ball $B(p, R) \subset P$ of radius $R \in [\varepsilon, \frac{1}{2}]$ centered at any point $p \in P$ satisfies the inequality

$$|B(p, R)| \geq \frac{1}{4} R^2.$$

Following Gromov, we introduce the following definition, see [Gro83, Theorem 5.3.B]. A ball $B(p, R)$ with $\varepsilon \leq R \leq R_0$ is said α-*admissible* if

- $|B(p, 5R)| \leq \alpha \cdot |B(p, R)|$;
- $\forall R' \in]R, R_0]$, $\alpha \cdot |B(p, R')| \leq |B(p, 5R')|$.

If there exists a point $p \in P$ such that for any $R \in [\varepsilon, R_0]$ the ball $B(p, R)$ is never α-admissible, then, if $r \in \mathbb{N}$ denotes the unique integer such that $\frac{R_0}{5^{r+1}} \leq \varepsilon < \frac{R_0}{5^r}$,

$$|B(p, R_0)| \geq \alpha^r |B(p, \frac{R_0}{5^r})| \geq \frac{1}{4} \alpha^r \varepsilon^2 \geq \frac{1}{100} \left(\frac{\alpha}{5}\right)^r R_0^2 \geq \frac{1}{62500} \left(\frac{\alpha}{5}\right)^{\log_5 \frac{R_0}{5\varepsilon}}.$$

Thus

$$\mathfrak{S}(P, g) \geq \kappa(G)$$

according to the inequality (6.12) and the result is proved in this case.

So we can assume that for any $p \in P$ there exists $R_p \in [\varepsilon, R_0]$ such that $B(p, R_p)$ is α-admissible. Denote by A the area of (P, g).

Lemma 6.13. *Let $B(p, R)$ be an α-admissible ball. Then*

$$|B(p, R)| \geq A(\alpha) := \frac{1}{100} \left(\frac{1}{25}\right)^{\frac{\log \frac{100 A}{R_0^2}}{\log \frac{\alpha}{25}}} R_0^2.$$

Proof of the lemma. Let $r \in \mathbb{N}$ be the unique integer such that $\frac{R_0}{5^{r+1}} \leq R < \frac{R_0}{5^r}$. We have

$$A = \mathrm{area}(P, g) \geq |B(p, R_0)| \geq \alpha^r |B(p, R)| \geq \frac{1}{4} \alpha^r R^2 \geq \frac{1}{100} \left(\frac{\alpha}{25}\right)^r R_0^2,$$

and so

$$r \leq r(\alpha) := \frac{\log \frac{100 A}{R_0^2}}{\log \frac{\alpha}{25}}.$$

This implies

$$|B(p, R)| \geq \frac{1}{4} R^2 \geq \frac{1}{100} \left(\frac{1}{25}\right)^r R_0^2 \geq A(\alpha).$$

$\square$

We now construct a family $\{B_i\}_{i=1}^N$ of α-admissible balls of P in the following way. We first choose an α-admissible ball $B_1 := B(p_1, R_1)$ with $R_1 := \max\{R_p \mid p \in P\}$. At each step $i \geq 2$, we construct B_i using the data of $\{B_j\}_{j<i}$ as follows. Let R_i be the maximal radius of an α-admissible ball centered at a point of the complementary of the union of the balls $\{2B_j\}_{j<i}$ and let $B_i := B(p_i, R_i)$ be such an α-admissible ball. By construction, B_i is disjoint from the other balls B_j as $R_i \leq R_j$. The process ends in a finite number N of steps when the balls $\{2B_i\}_{i=1}^N$

cover P, as $R_i \geq \varepsilon$ for every $i = 1, \ldots, N$.

Consider $\mathcal{N}$ the corresponding nerve of this cover. In a general way, if X is a paracompact topological space and $\mathcal{U}$ a locally finite cover of X, there exists a canonical map Φ from X to the nerve $\mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$ defined as follows. If $\{\phi_V\}_{V \in \mathcal{U}}$ denotes a partition of unity associated to $\mathcal{U}$,

$$
\begin{aligned}
\Phi : X &\rightarrow \mathcal{N}(\mathcal{U}) \\
x &\mapsto \sum_{V \in \mathcal{U}} \phi_V(x) V.
\end{aligned}
$$

This map is uniquely defined up to homotopy. In our case, $\mathcal{U} = \{2B_i\}_{i=1}^N$ and Φ associates the center of such a ball to the corresponding vertex of $\mathcal{N}$.

Lemma 6.14. *The map $\Phi : P \to \mathcal{N}$ induces an isomorphism of fundamental groups.*

Proof of the lemma. Denote by $\mathcal{N}^{(k)}$ the k-skeleton of $\mathcal{N}$. We will construct a map $\Psi : \mathcal{N}^{(2)} \to X$ such that the induced map

$$\Psi_\sharp : \pi_1(\mathcal{N}) \simeq \pi_1(\mathcal{N}^{(2)}) \to \pi_1(P)$$

is the inverse of $\Phi_\sharp : \pi_1(P) \to \pi_1(\mathcal{N})$.

We have denoted $\{p_i\}_{i=1}^N$ the set of centers of balls of the covering $\mathcal{U}$ and set $v_i = \Phi(p_i)$. We first define Ψ on $\mathcal{N}^{(0)}$ by

$$\Psi(v_i) = p_i.$$

If two vertices v_i and v_j are connected by an edge $[v_i, v_j]$, we join p_i and p_j in P by any minimizing geodesic denoted by $\gamma_{i,j}$. The map Ψ is then defined on the edge $[v_i, v_j]$ to the arc $\gamma_{i,j}$ in the obvious way

$$\Psi : [v_i, v_j] \longrightarrow \gamma_{i,j}.$$

This defines Ψ on the 1-skeleton $\mathcal{N}^{(1)}$. Remark that $l_g(\gamma_{i,j}) \leq 4 \cdot R_0$ (v_i and v_j are connected by an edge if and only if $2B_i \cap 2B_j \neq \emptyset$).

Next we consider any 2-simplex $\tau = [v_i, v_j, v_k]$ of $\mathcal{N}$. The concatenation $\gamma_{i,j} \star \gamma_{j,k} \star \gamma_{k,i}$ is a closed curve of P of length at most $12 \cdot R_0 < 1$. So it is contractible and any contraction of this curve into a point gives rise to an extension of the map Ψ to τ. We get this way a map

$$\Psi : \mathcal{N}^{(2)} \to P.$$

Observe that the restriction of Ψ to $\mathcal{N}^{(1)}$ is unique up to homotopy.

By construction, $\Phi(p_i) = v_i$ for any $i = 1, \ldots, N$, and if $[v_i, v_j]$ denotes an edge of $\mathcal{N}$ and p belongs to the corresponding geodesic γ_{ij}, $\Phi(p) \in \text{St}([v_i, v_j])$ where $\text{St}([v_i, v_j])$ denotes the star of $[v_i, v_j]$. This implies that $\Phi \circ \Psi : [v_i, v_j] \longrightarrow \mathcal{N}$ is homotopically equivalent to the identity relatively to $\{v_i, v_j\}$. So $\Phi \circ \Psi : \mathcal{N}^{(1)} \longrightarrow \mathcal{N}$ is homotopically equivalent to the identity relatively to $\mathcal{N}^{(0)}$. From this, we get that the induced morphism $\Phi_\sharp \circ \Psi_\sharp : \pi_1(\mathcal{N}) \to \pi_1(\mathcal{N})$ is the identity and so $\Psi_\sharp : \pi_1(\mathcal{N}) \to \pi_1(P)$ is into.

It remains to prove that $\Psi_\sharp$ is onto. Consider a geodesic loop α based at the center p_1 of the ball B_1 and whose length is minimal in its own homotopy class. We complete p_1 into a finite family $\{p_{i_j}\}_{j \in \mathbb{Z}_n}$ of points of P such that

- each p_{i_j} is the center of some ball B_{i_j} of $\mathcal{U}$;
- the family $\{2B_{i_j}\}_{j \in \mathbb{Z}_n}$ covers α ;
- $2B_{i_j} \cap 2B_{i_{j+1}} \neq \emptyset$.

For each $j \in \mathbb{Z}_n$, fix any point $x_j \in 2B_{i_j} \cap \alpha$ and denote by α_j the part of the loop α joining x_j and x_{j+1} and contained in $2B_{i_j} \cup 2B_{i_{j+1}}$. By construction, $l_g(\alpha_j) \leq 8 \cdot R_0$. Fix a minimizing geodesic β_j joining p_{i_j} and x_j. The concatenation

$$\gamma_{i_j, i_{j+1}} \star \beta_{j+1} \star (\alpha_j)^{-1} \star (\beta_j)^{-1}$$

is closed curve of length at most $24 \cdot R_0 < 1$ thus contractible. So α is homotopic to $\gamma_{1,2} \star \gamma_{2,3} \star ... \star \gamma_{n,1}$ with based point p_1 fixed. This proves the surjectivity of $\Psi_\sharp$ and completes the proof. $\qquad \square$

As $\pi_1(\mathcal{N}) \simeq G$, we deduce the lower bound

$$s_2(\mathcal{N}) \geq \kappa(G).$$

We now estimate the number $s_2(\mathcal{N})$ by the systolic volume of (P, g). First of all,

$$A = \text{area}(P, g) \geq \sum_{i=1}^{N} |B_i| \geq \frac{1}{\alpha} \sum_{i=1}^{N} |5B_i|,$$

as the balls B_i are pairwise disjoints and α-admissibles. If B_i belongs to exactly F_i distinct 2-simplexes of $\mathcal{N}$, the ball $5B_i$ contains at least F_i pairewise disjoints balls of $\{B_i\}_{i=1}^{N}$, and so

$$|5B_i| \geq F_i \cdot A(\alpha).$$

From the equality $\sum_{i=1}^{N} F_i = 3 \cdot s_2(\mathcal{N})$, we deduce that

$$\kappa(G) \leq \frac{\alpha A}{3A(\alpha)}.$$

As

$$A(\alpha) = \frac{1}{100} \left(\frac{1}{25} \right)^{\frac{\log \frac{100 A}{R_0^2}}{\log \frac{\alpha}{25}}} R_0^2,$$

we get

$$\kappa(G) \leq \frac{62500}{3} \cdot 25^{\frac{\log \frac{100 A}{R_0^2}}{\log \frac{\alpha}{25}}} \cdot \alpha A.$$

as $R_0 = \frac{1}{25}$. From the equality

$$\log \frac{\alpha}{25} = \sqrt{\log(62500 \cdot \mathfrak{S}(G))},$$

we then compute that

$$\kappa(G) \leq \frac{62500}{3} \cdot e^{\log 25 \cdot \frac{\log(62500 \cdot A)}{\sqrt{\log(62500 \cdot \mathfrak{S}(G))}}} \cdot 25 e^{\sqrt{\log(62500 \cdot \mathfrak{S}(G))}} \cdot A.$$

Now observe that $A = \text{area}(P, g) = \mathfrak{S}(P, g) < \mathfrak{S}(G) + \varepsilon$. This finally implies the result for $C = \frac{62500 \cdot 25}{3}$ and $C' = 1 + \log 25$. $\qquad \square$

Remark that for surface groups we have

$$c \, \frac{\kappa(\pi_l)}{(\log \kappa(\pi_l))^2} \leq \mathfrak{S}(\pi_l) \leq C \, \frac{\kappa(\pi_l)}{(\log \kappa(\pi_l))^2}$$

for some positive constants c and C. This can be deduced from [BPS10] using inequalities (6.1) and (6.2).

7. THE HEISENBERG GROUP, NILMANIFOLDS AND THE WARING PROBLEM

The nilpotent groups give particularily interesting examples: the systolic volume of multiples of certain homology classes are bounded, albeit certain of these multiples admit (non-normalized) representations by manifolds whose systolic volume is not bounded. This phenomena already appears in the simplest case of nilpotent non abelian group, that is the Heisenberg group.

7.1. **Nilmanifolds and the Waring problem.**
Consider a nilpotent group G of finite type without torsion. The classical result of Mal'cev [Malc49] implies that there exists a simply connected nilpotent Lie group $\mathcal{G}(G)$ such that G embeds in $\mathcal{G}(G)$ as a lattice, that is as a cocompact discret subgroup. Denote by $\mathcal{L}(G)$ the Lie algebra of $\mathcal{G}(G)$ and suppose that $\mathcal{L}(G)$ is graded in the following way:

$$(7.1) \qquad \mathcal{L}(G) = \overset{s}{\underset{k=1}{\oplus}} \mathcal{L}_k, \quad [\mathcal{L}_i, \mathcal{L}_j] \subset \mathcal{L}_{i+j},$$

where $\mathcal{L}_{i+j} = 0$ if $i + j > s$. We do not suppose here that

$$(7.2) \qquad \{\mathcal{L}_{(i)} = \overset{s}{\underset{k=i}{\oplus}} \mathcal{L}_k\}_{i=1}^s$$

is a lower central series, s being not in general the nilpotence degree of $\mathcal{L}(G)$. For any $t \in \mathbb{R}$, a natural homothety δ_t is associated to the decompostion (7.1) by the formula:

$$\delta_t(v) = t^k v \ \text{ if } \ v \in \mathcal{L}_k.$$

This homothety δ_t is an endomorphism of $\mathcal{L}(G)$ for any real parameter t. By the Baker-Campbell-Hausdorff formula, and taking into account the structure (7.1), the homothety δ_t generates a homothety Δ_t of $\mathcal{G}$.

Definition 7.1. *The nilpotent group G is said graded if there exists a graduation (7.1) of the corresponding Lie algebra $\mathcal{L}(G)$ such that for integer parameters the corresponding homotheties of $\mathcal{G}(G)$ preserve the lattice $G \subset \mathcal{G}(G)$, that is*

$$\Delta_n(G) \subset G, \ \forall n \in \mathbb{Z}.$$

If G is a nilpotent graded group, let

$$d(G) = \sum_{k=1}^{s} k \dim \mathcal{L}_k$$

be the weighted dimension of its corresponding Lie algebra $\mathcal{L}(G)$. Remark that if the sequence of subalgebras (7.2) is the lower central series, $d(G)$ coincides with the degree of polynomial growth of G, see [Wolf68] and [Bass72].

Denote by $M = \mathcal{G}(G)/G$ the nilmanifold corresponding to the nilpotent group G. If G is graded, the homothety $\Delta_n : \mathcal{G}(G) \longrightarrow \mathcal{G}(G)$ defined using the graduation induces for every positive integer n a map

$$\widetilde{\Delta}_n : M \longrightarrow M$$

of degree n^d with $d = d(G)$. Let $\mathbf{a} = [M] \in H_m(G, \mathbb{Z})$ be the fundamental class of M and k be a positive integer. We can represent the class $k\mathbf{a}$ by the connected sum of a uniformily bounded number of copies of M as follows. By a result of Hilbert (see [Ellis71]), there exists an integer $\mathcal{K}(d)$ such that

$$(7.3) \qquad\qquad k = \sum_{i=1}^{s} a_i^d,$$

where each coefficient a_i is a positive integer and $s \leq \mathcal{K}(d)$. Now the class $k\mathbf{a}$ is represented by the geometric cycle $(\#_{i=1}^s M_i, f)$ where $M_i \simeq M$ for $i = 1, \dots, s$ and

$$f : \underset{i=1}{\overset{s}{\#}} M_i \longrightarrow \bigvee_{i=1}^{s} M_i \overset{\overset{s}{\underset{i=1}{\bigvee}} \widetilde{\Delta}_{a_i}}{\longrightarrow} M_{\mathcal{H}},$$

the first map being the contraction of the connected sum into a wedge. We easily compute that $\deg f = \sum_{i=1}^{s} a_i^d = k$ and so

$$\mathfrak{S}(G, k\mathbf{a}) \leq \mathfrak{S}(\#_{i=1}^s M).$$

Then we apply Proposition 3.6 in order to derive the following result.

Theorem 7.2. *Let G be a graded nilpotent group. If $\mathbf{a} = [\mathcal{G}(G)/G]$ denotes the fundamental class of the corresponding nilmanifold, then*

$$\mathfrak{S}(G, k\mathbf{a}) \leq \mathcal{K}(d(G)) \cdot \mathfrak{S}(G, \mathbf{a})$$

for any positive integer k.

7.2. **Family of lattices in the Heisenberg group.** Consider the Heisenberg group of dimension 3 composed of the following set of upper triangular matrices:

$$\mathcal{H} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, x, y, z \in \mathbb{R} \right\}.$$

The subset $\mathcal{H}(\mathbb{Z})$ of matrices of $\mathcal{H}$ with integer coefficients (*i.e.* for which $x, y, z \in \mathbb{Z}$) is a lattice, and we denote by $M_{\mathcal{H}} = \mathcal{H}/\mathcal{H}(\mathbb{Z})$ the corresponding nilmanifold. The fundamental group $\mathcal{H}(\mathbb{Z})$ of $M_{\mathcal{H}}$ satisfies the assumptions of Theorem 7.2. In fact the homotheties $\{\Delta_t\}_{t>0}$ are given by the formula

$$\Delta_t \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & tx & t^2 z \\ 0 & 1 & ty \\ 0 & 0 & 1 \end{pmatrix},$$

so $\Delta_n(\mathcal{H}(\mathbb{Z})) \subset \mathcal{H}(\mathbb{Z})$ for any integer $n \geq 1$. The map Δ_n factorizes through a map

$$\widetilde{\Delta}_n : M_{\mathcal{H}} \longrightarrow M_{\mathcal{H}}$$

for which $\deg(\widetilde{\Delta}_n) = n^4$. The resolution of the Waring problem for the sum of fourth powers (see [BDD86]) gives that any integer number decomposes into a sum of at most 19 fourth powers. That is, with the notation of Theorem 7.2, we have $d(\mathcal{H}(\mathbb{Z})) = 4$ and $\mathcal{K}(4) = 19$. This implies the following

Corollary 7.3. *Let* $\mathbf{a} = [M_{\mathcal{H}}] \in H_3(\mathcal{H}(\mathbb{Z}), \mathbb{Z})$ *be the fundamental class of* $M_{\mathcal{H}}$. *Then*

$$\mathfrak{S}(\mathcal{H}(\mathbb{Z}), k\mathbf{a}) \leq 19 \cdot \mathfrak{S}(\mathcal{H}(\mathbb{Z}), \mathbf{a})$$

for any positive integer k.

The different lattices of $\mathcal{H}$ give rise to nilmanifolds whose systolic behaviour is particularly interesting. Consider the sequence of lattices $\{\mathcal{H}_n(\mathbb{Z})\}_{n=1}^{\infty}$ of $\mathcal{H}$, where $\mathcal{H}_n(\mathbb{Z})$ is the subset of matrices of $\mathcal{H}$ such that $x \in n\mathbb{Z}$ and $y, z \in \mathbb{Z}$. Denote by $M_n = M_{\mathcal{H}_n} = \mathcal{H}/\mathcal{H}_n(\mathbb{Z})$ the corresponding nilmanifolds. The manifold M_n is a cyclic covering of $M_{\mathcal{H}}$ with n sheets, so

$$(7.4) \qquad\qquad \mathfrak{S}(M_n) \leq C\frac{n}{\ln n},$$

according to the version of Theorem 5.4 for cyclic coverings. The fact that the function $\mathfrak{S}(M_n)$ goes to infinity is not obvious. For instance the simplicial volume of these manifolds is zero, and thus the corresponding lower bound (see Corollary (5.8)) does not apply.

Proposition 7.4. *The function* $\mathfrak{S}(M_n)$ *satisfies the following inequality:*

$$\mathfrak{S}(M_n) \geq a\frac{\ln n}{\exp(b\sqrt{\ln(\ln n)}},$$

where a *and* b *are two positive constants. Especially*

$$\lim_{n \to +\infty} \mathfrak{S}(M_n) = +\infty.$$

Proof. The proof uses the results of section 6. First we have:

Lemma 7.5. *Let* X *be a pseudomanifold of dimension* 3 *which admits a map of degree* 1 *on the nilmanifold* M_n. *Then*

$$|\,Tors\,H_1(X, \mathbb{Z})| \geq n.$$

Proof of the lemma. By duality between homology and cohomology, the existence of n-torsion in the group $H_1(X, \mathbb{Z})$ is equivalent to the existence of the same torsion in $H^2(X, \mathbb{Z})$.

The cohomologies of M_n are well known:

$$H^2(M_n, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_n.$$

The generator of the torsion subgroup $Tors\,H^2(M_n, \mathbb{Z})$ is the class $b = \beta(a)$, where $a \in H^1(M_n, \mathbb{Z}_n)$ and β denotes the Bokstein homomorphism. Let

$$j : H^2(M_n, \mathbb{Z}) \longrightarrow H^2(M_n, \mathbb{Z}_n)$$

denotes the homomorphism of reduction modulo n. We have the following decomposition of the fundamental cohomology class in $H^*(M_n, \mathbb{Z}_n)$:

$$\mathbf{m} = a \cup \big(j \circ \beta(a)\big) = a \cup j(b).$$

If $f : X \longrightarrow M_n$ is a map of degree 1, then $f^*(\mathbf{m})$ is a generator of the cohomology group $H^3(X, \mathbb{Z}_n) \simeq \mathbb{Z}_n$. This implies that $f^*(j(b))$ is an element of order n in $H^2(X, \mathbb{Z}_n)$, and thus $f^*(b)$ is an element of order n in $H^2(X, \mathbb{Z})$. This ends the proof. $\square$

The lemma implies that any pseudomanifold representing the fundamental class $[M_n]$ satisfies $t_1(X) \geq n$, and so

$$t_1([M_n]) \geq n.$$

Proposition 7.4 is then a consequence of Theorem 6.1. $\qquad\square$

As the cover $M_n \longrightarrow M_{\mathcal{H}}$ has n sheets, the manifold M_n represents the class $n[M_{\mathcal{H}}] \in H_3(\mathcal{H}(\mathbb{Z}), \mathbb{Z})$ for any positive integer n. This representation is not normalized, and Corollary 7.3 together with Proposition 7.4 shows that the assumption of normalization can not be dropped in Theorem 2.4.

8. ON STABLE SYSTOLIC VOLUME

In this section, we explain how part of our approach adapts to the context of stable systolic volume.

Let $b \geq 1$ be an integer and fix a non trivial homology class $\mathbf{a} \in H_m(\mathbb{Z}^b, \mathbb{Z}) \simeq H_m(\mathbb{T}^b, \mathbb{Z})$ of dimension $1 \leq m \leq b$.

Let (X, f) be a geometric cycle representing $\mathbf{a}$ and endowed with a polyhedral metric g. The representation is said *normal* if the induced map

$$f_* : H_1(X, \mathbb{Z})/\mathrm{Tors}\, H_1(X, \mathbb{Z}) \to \mathbb{Z}^b$$

is an epimorphism. For any class $\mathbf{h} \in H_1(X, \mathbb{Z})$, set

$$\|\mathbf{h}\|_g = \lim_{n \to \infty} \frac{l_g(n \cdot \mathbf{h})}{n}$$

where $l_g(n \cdot \mathbf{h})$ denotes the least length of a loop of X representing the class $n \cdot \mathbf{h}$. For this semi-norm, $\|\mathbf{h}\|_g = 0$ if and only if $\mathbf{h} \in \mathrm{Tors}\, H_1(X, \mathbb{Z})$. The semi-norm $\|\cdot\|_g$ induces a norm on $H_1(X, \mathbb{R})$ called the *stable norm*. The *relative stable systole* denoted by $\mathrm{sys}_f^{st}(X, g)$ is defined as the infimum of $\|\cdot\|_g$ over the classes $\mathbf{h} \in H_1(X, \mathbb{Z})/\mathrm{Tors}\, H_1(X, \mathbb{Z})$ such that $f_*(\mathbf{h}) \neq 0$. The *stable systolic volume* of (X, f) is then the value

$$\mathfrak{S}_f^{st}(X) = \inf_g \frac{\mathrm{vol}(X, g)}{(\mathrm{sys}_f^{st}(X, g))^m},$$

where the infimum is taken over all polyhedral metrics on X. In the case where f is the Jacobi map $J : X \to \mathbb{T}^b$ where b denotes the first Betti number of X, we simply denote by $\mathfrak{S}^{st}(X)$ the stable systolic volume of the pair (X, f). This is a consequence of results of Gromov [Gro83, Theorem 7.4.C] and the first author [Bab92, Theorem 8.2.C] that $\mathfrak{S}_f^{st}(X) > 0$ if and only if the map $f : X \to \mathbb{T}^b = K(\mathbb{Z}^b, 1)$ satisfies $f_*[X] \neq 0 \in H_m(\mathbb{Z}^b, \mathbb{Z})$.

Definition 8.1. *The stable systolic volume of* $\mathbf{a}$ *is defined as the quantity*

$$\mathfrak{S}^{st}(\mathbf{a}) := \inf_{(X,f)} \mathfrak{S}_f^{st}(X),$$

the infimum being taken over all geometric cycles (X, f) *representing the class* $\mathbf{a}$.

Note that any homology class $\mathbf{a} \in H_m(\mathbb{Z}^b, \mathbb{Z})$ is representable by some manifold, and this representation can be normalized by surgery if $m \geq 3$. Thus the stable systolic volume of $\mathbf{a}$ is well defined.

The comparison result of [Bab06] holds for stable systolic volume. This implies the following version of Theorem 2.4.

Theorem 8.2. *If $m \geq 3$, any normal representation of a class $\mathbf{a} \in H_m(\mathbb{Z}^b, \mathbb{Z})$ by an admissible pseudomanifold M satisfies*

$$\mathfrak{S}^{st}(\mathbb{Z}^b, \mathbf{a}) = \mathfrak{S}^{st}(M).$$

Given an morphism of abelian groups $\pi : \mathbb{Z}^b \to \mathbb{Z}^{b'}$ and a homology class $\mathbf{a} \in H_m(\mathbb{Z}^b, \mathbb{Z})$ with $m \geq 3$, we have

$$\mathfrak{S}^{st}(\mathbb{Z}^{b'}, \pi_* \mathbf{a}) \leq \mathfrak{S}^{st}(\mathbb{Z}^b, \mathbf{a}).$$

We deduce that if X_1 and X_2 are two orientable admissible pseudomanifolds of dimension $m \geq 3$, then

$$\max\{\mathfrak{S}^{st}(X_1), \mathfrak{S}^{st}(X_2)\} \leq \mathfrak{S}^{st}(X_1 \# X_2).$$

For this it is sufficient to remark that if the torus $\mathbb{T}^{b_i}$ is the classifying space for the stable systole of X_i, then $\mathbb{T}^{b_1+b_2}$ is the classifying space of $X_1 \# X_2$.

Finally, the comparison principle of [Bab06] implies that

$$\mathfrak{S}^{st}(X_1 \# X_2) \leq \mathfrak{S}^{st}(X_1) + \mathfrak{S}^{st}(X_2).$$

As a consequence, we derive the following inequality:

$$\mathfrak{S}^{st}(\mathbb{Z}^b, \mathbf{a}_1 + \mathbf{a}_2) \leq \mathfrak{S}^{st}(\mathbb{Z}^b, \mathbf{a}_1) + \mathfrak{S}^{st}(\mathbb{Z}^b, \mathbf{a}_2).$$

We end this section with the following

Proposition 8.3. *Fix two integers $3 \leq m \leq b$. For any class $\mathbf{a} \in H_m(\mathbb{Z}^b, \mathbb{Z})$,*

$$\mathfrak{S}^{st}(\mathbb{Z}^b, \mathbf{a}) \leq C_b^m \cdot \mathfrak{S}^{st}(\mathbb{T}^m)$$

where C_b^m denotes the binomial coefficient.

Proof. Fix a basis $\{\mathbb{T}_1^m, \ldots, \mathbb{T}_{C_b^m}^m\}$ of $H_m(\mathbb{Z}^b, \mathbb{Z})$ composed of C_b^m embedded m-tori, and write the class $\mathbf{a}$ in this basis:

$$\mathbf{a} = \sum_{i=1}^{C_b^m} k_i [\mathbb{T}_i^m]$$

where $k_i \in \mathbb{Z}$ for $i = 1, \ldots, C_b^m$. We have

$$\mathfrak{S}^{st}(\mathbb{Z}^b, \mathbf{a}) \leq \sum_{i=1}^{C_b^m} \mathfrak{S}^{st}(\mathbb{Z}^b, k_i [\mathbb{T}_i^m]),$$

and the inequality

$$\mathfrak{S}^{st}(\mathbb{Z}^b, k_i [\mathbb{T}_i^m]) \leq \mathfrak{S}^{st}(\mathbb{T}^m)$$

for any $i = 1, \ldots, C_b^m$ - whose proof is analog to the proof of inequality (5.9) - completes the proof. $\square$

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