



**HAL**  
open science

**A generalized  $\alpha$ -approach for a kernel estimator of conditional quantile with functional regressors: Consistency and asymptotic normality**

Ali Laksaci, Mohamed Lemdani, Elias Ould-Saïd

► **To cite this version:**

Ali Laksaci, Mohamed Lemdani, Elias Ould-Saïd. A generalized  $\alpha$ -approach for a kernel estimator of conditional quantile with functional regressors: Consistency and asymptotic normality. *Statistics and Probability Letters*, 2009, 79 (8), pp.1065-1073. 10.1016/j.spl.2008.12.016 . hal-00516884

**HAL Id: hal-00516884**

**<https://hal.science/hal-00516884>**

Submitted on 13 Sep 2010

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Accepted Manuscript

A generalized  $L^1$ -approach for a kernel estimator of conditional quantile with functional regressors: Consistency and asymptotic normality

Ali Laksaci, Mohamed Lemdani, Elias Ould-Saïd

PII: S0167-7152(08)00576-2  
DOI: 10.1016/j.spl.2008.12.016  
Reference: STAPRO 5310

To appear in: *Statistics and Probability Letters*

Received date: 30 September 2008  
Revised date: 15 December 2008  
Accepted date: 15 December 2008

Please cite this article as: Laksaci, A., Lemdani, M., Ould-Saïd, E., A generalized  $L^1$ -approach for a kernel estimator of conditional quantile with functional regressors: Consistency and asymptotic normality. *Statistics and Probability Letters* (2008), doi:10.1016/j.spl.2008.12.016

This is a PDF file of an unedited manuscript that has been accepted for publication. As a service to our customers we are providing this early version of the manuscript. The manuscript will undergo copyediting, typesetting, and review of the resulting proof before it is published in its final form. Please note that during the production process errors may be discovered which could affect the content, and all legal disclaimers that apply to the journal pertain.



A generalized  $L^1$ -approach for a kernel estimator of  
conditional quantile with functional regressors: consistency  
and asymptotic normality

ALI LAKSACI

Dép. de Mathématiques

Univ. Djillali Liabès

BP 89, S. B. A. 22000, Algeria

e-mail: alilak@yahoo.fr

MOHAMED LEMDANI \*

Lab. de Biomathématiques

Univ. de Lille 2. Fac. de Pharmacie

3, rue du Pr. Laguesse, 59006 Lille, France

e-mail: mohamed.lemdani@univ-lille2.fr

ELIAS OULD-SAÏD

L.M.P.A. J. Liouville, Univ. du Littoral Côte d'Opale

BP 699, 62228 Calais, France

e-mail: ouldsaid@lmpa.univ-littoral.fr

December 15, 2008

**Abstract:** A kernel estimator of the conditional quantile is defined for a scalar response variable given a covariate taking values in a semi-metric space. The approach generalizes the median's  $L^1$ -norm estimator. The almost complete consistency and asymptotic normality are stated.

**Keywords** Asymptotic distribution · functional data · kernel estimate · robust estimation · small balls probability.

**AMS Subject classification** Primary, 62G20 ; secondary 62G08 · 62G35 · 62E20.

---

\*Corresponding author, Tel: (33) 320 964 933, Fax: (33) 320 964 704.

## 1 Introduction

Let  $(X, Y)$  be a pair of random variables (rv) in  $\mathcal{F} \times \mathbb{R}$ , where the space  $\mathcal{F}$  is endowed with a semi-metric  $d(\cdot, \cdot)$  (this covers the case of semi-normed spaces of possibly infinite dimension). In this context,  $X$  can be a functional random variable. In the following, we fix a point  $x \in \mathcal{F}$  and we consider a given neighborhood  $N_x$  of  $x$ . We assume that the regular version  $F^z$  of the conditional distribution function of  $Y$  given  $X = z$  exists for any  $z \in N_x$ . Moreover we suppose that  $F^x$  has a continuous density  $f^x$  with respect to (w.r.t.) Lebesgue's measure over  $\mathbb{R}$ .

Now for  $p \in (0, 1)$ , we consider the conditional quantile of order  $p$  of  $F^x$

$$t_p(x) = \inf \{t \in \mathbb{R} : F^x(t) \geq p\}. \quad (1)$$

Alternatively, the  $p^{\text{th}}$  conditional quantile  $t_p(x)$  can be defined as the unique solution wrt  $t$  of the optimization problem

$$\min_{t \in \mathbb{R}} \Psi_p(x, t) \quad (2)$$

where

$$\Psi_p(x, t) = \mathbb{E}[\psi_p(Y - t) | X = x]$$

with  $\psi_p(t) = (2p - 1)t + |t|$ .

During the last decade, thanks to progress in computer tools, it has become easier to deal with increasingly bulky data. Therefore statistical problems related to the modelization of functional random variables have received an increasing interest in the recent literature (see, *e.g.*, Bosq, 2000, Ramsay and Silverman, 2002 and 2005 for the linear model and Ferraty and Vieu (2006) in the nonparametric case). It is well known that traditional statistical methods fail when dealing with functional data. Then, the goal of this paper is to study a nonparametric estimator of  $t_p(x)$  when the explanatory variable  $X$  is functional.

Conditional quantiles are widely studied when the explanatory variable lies within a finite dimensional space and there are many references on this topic (see, *e.g.*, Samanta (1989) for previous results and Gannoun *et al.* (2003) for recent advances and references). Let us also mention Zhou and Liang (2000) and Lin and Li (2007) where the asymptotic normality is considered in the case of the conditional median ( $p = 1/2$ ) under  $\alpha$ -mixing and association conditions, respectively. Note here that in our approach we get, for the median, the well-known  $L^1$ -norm estimator.

However the number of papers dealing with this model, in the case of functional  $X$ , is fairly limited. For instance, a B-spline approach is used in Cardot *et al.* (2004) to study the linear model of regression on quantiles with explanatory variable taking values in a Hilbert space. The authors obtained the estimator's  $L^2$ -convergence. In the nonparametric context, the almost complete (*a.co.*)<sup>1</sup> convergence of the conditional quantile's kernel estimator is established in Ferraty *et al.* (2006) when the observations are independent and identically distributed (iid) whereas the dependent case is considered in Ferraty *et al.* (2005) with an application to climatologic data. The asymptotic normality of this estimator was studied in both cases (iid and strong mixing) by Ezzahrioui and Ould-Saïd (2008). Recently, Dabo-Niang and Laksaci (2008) stated the convergence in  $L^p$ -norm under less restrictive conditions. They considered the concentration property on small balls of the probability measure of the underlying explanatory variable, in the iid case. All the works mentioned above use an estimation procedure based on the double-kernel method.

In this paper, we estimate the conditional quantile nonparametrically, by adapting the  $L^1$ -norm method which enjoys some robustness properties. This is done by replacing the absolute value function by  $\psi_p(\cdot)$  and is motivated by the fact that  $L^1$  is the natural space where the rvs live. To our knowledge, this kind of result has not so far been addressed. We establish, under mild assumptions (which include the concentration property), the almost complete consistency and asymptotic normality of this estimator. These results complete those obtained in Ferraty *et al.* (2006) where consistency with rate (but not asymptotic normality) is given for a smoothed estimator. On the other hand, using a robust approach allows us to avoid the classical small-departure-related problems which may occur for the double-kernel method. Moreover we do not have to care, in our case, for a second smoothing parameter.

The paper is organized as follows. We present our estimation procedure in Section 2 before giving the hypotheses and stating the main results in Section 3. The proofs of the auxiliary results are relegated to Section 4.

---

<sup>1</sup>We say that a sequence  $Z_n$  converges *a.co.* to  $Z$  if and only if, for any  $\epsilon > 0$ ,  $\sum_n \mathbb{P}(|Z_n - Z| > \epsilon) < \infty$ .

## 2 Nonparametric estimator of the conditional quantile

Let  $(X_1, Y_1), \dots, (X_n, Y_n)$  be  $n$  independent random pairs in  $\mathcal{F} \times \mathbb{R}$  which are identically distributed as  $(X, Y)$ . Based on this  $n$ -sample, a kernel estimate of  $\Psi_p(x, t)$  is given by

$$\widehat{\Psi}_p(x, t) = \sum_{i=1}^n W_{ni}(x) \psi_p(Y_i - t), \quad \forall t \in \mathbb{R}$$

where  $W_{ni}(x) = \frac{K(h^{-1}d(x, X_i))}{\sum_{j=1}^n K(h^{-1}d(x, X_j))}^2$ ,  $K$  is a kernel function and  $h := h_n$  is a sequence of positive real numbers which goes to zero as  $n$  goes to infinity. A natural estimator of  $t_p(x)$  is then

$$\widehat{t}_p(x) = \arg \min_{t \in \mathbb{R}} \widehat{\Psi}_p(x, t). \quad (3)$$

Note here that  $\widehat{\Psi}_p(x, \cdot)$  being convex with limit  $+\infty$  at both  $-\infty$  and  $+\infty$  (with great probability), it has at least one minimizing value over  $\mathbb{R}$ . Note also that  $\widehat{\Psi}_p(x, \cdot)$  is piecewise linear (with increasing slope) and may have a flat (zero-slope) part where it takes its minimum value. In that case, we take  $\widehat{t}_p(x)$  as the smallest minimizing value (thus the lowest abscissa of the flat part). Otherwise  $\widehat{t}_p(x)$  is unique and locates the turning point from negative to positive slope. It can be shown that (see the proof in the appendix)

$$\widehat{t}_p(x) = \widehat{\widehat{t}}_p(x) := \inf\{t \in \mathbb{R} : \widehat{F}^x(t) \geq p\} \quad (4)$$

where  $\widehat{F}^x(\cdot)$  is the estimator of  $F^x(\cdot)$  defined by

$$\widehat{F}^x(\cdot) = \sum_{i=1}^n W_{ni}(x) \mathbb{I}_{\{Y_i \leq \cdot\}} \quad (5)$$

with  $\mathbb{I}_{\{\cdot\}}$  being the indicator function.

## 3 Main results

In the following, the ball of center  $x$  and radius  $r > 0$  is denoted  $B(x, r)$ . Moreover, for  $i = 1, \dots, n$ , let  $K_i(x) = K(h^{-1}d(x, X_i))$  and, in view of (5), put

$$\widehat{F}^x(t) = \frac{\widehat{F}_N^x(t)}{\widehat{F}_D^x} \quad (6)$$

<sup>2</sup>In the case where the denominator of  $W_{ni}(x)$  is zero, we use the convention  $0/0 = 0$ . We here point out that this event has small probability under suitable condition (see Lemma 3.4 below).

where

$$\widehat{F}_N^x(t) = \frac{1}{n \mathbb{E}[K_1(x)]} \sum_{i=1}^n K_i(x) \mathbb{1}_{\{Y_i \leq t\}} \quad \text{and} \quad \widehat{F}_D^x = \frac{1}{n \mathbb{E}[K_1(x)]} \sum_{i=1}^n K_i(x). \quad (7)$$

We first study the consistency of our estimator before addressing the asymptotic normality issue.

### 3.1 Consistency

In this subsection, we establish the almost complete convergence of  $\widehat{t}_p(x)$  to  $t_p(x)$ . To do that, we consider the following hypotheses (hereafter  $C_1, C_2, \dots$  denote positive constants). Recall that we assume that  $F^z$  exists for any  $z \in \mathcal{N}_x$  and that  $F^x$  admits a continuous density  $f^x$ .

(H1)  $\mathbb{P}(X \in B(x, r)) =: \phi_x(r) > 0$  for all  $r > 0$  and  $\lim_{r \rightarrow 0} \phi_x(r) = 0$ .

(H2)  $\exists \delta > 0, \forall (t_1, t_2) \in [t_p(x) - \delta, t_p(x) + \delta]^2, \forall (x_1, x_2) \in \mathcal{N}_x^2$ ,

$$|F^{x_1}(t_1) - F^{x_2}(t_2)| \leq C_1 (d^b(x_1, x_2) + |t_1 - t_2|^k), \quad \text{for } b, k > 0.$$

(H3)  $K$  is a measurable function with support  $[0, 1]$  and satisfies  $0 < C_2 \leq K(\cdot) \leq C_3 < \infty$ .

(H4)  $n\phi_x(h)/\log n \rightarrow \infty$  as  $n \rightarrow \infty$ .

*Comments on the hypotheses*

Hypothesis (H1) characterizes the classical concentration property (see Ferraty and Vieu, 2006). A mild regularity condition (H2) is assumed for the distribution function. Hypotheses (H3)-(H4) are technical conditions imposed for the brevity of proofs.

Now, in order to state the *a.co.* convergence of our estimate, we need the following result.

**Proposition 3.1** *Under Hypotheses (H1)-(H4), we have,*

$$\sup_{t \in [t_p(x) - \delta, t_p(x) + \delta]} |\widehat{F}^x(t) - F^x(t)| = O(h^b) + O\left(\left(\frac{\log n}{n \phi_x(h)}\right)^{1/2}\right) \quad \text{a.co.}$$

**Theorem 3.1** *Under the hypotheses of Proposition 3.1, if  $f^x(t_p(x)) > 0$ , then*

$$|\widehat{t}_p(x) - t_p(x)| = O(h^b) + O\left(\left(\frac{\log n}{n \phi_x(h)}\right)^{1/2}\right) \quad \text{a.co.}$$

**Proof of Theorem 3.1.** As  $F^x$  is continuously differentiable with  $f^x(t_p(x)) > 0$ , we have for any small enough  $\epsilon > 0$

$$\sum_n \mathbb{P}(\hat{t}_p(x) > t_p(x) + \epsilon) \leq \sum_n \mathbb{P}\left(\sup_{t \in [t_p(x), t_p(x) + \epsilon]} |\hat{F}^x(t) - F^x(t)| \geq \epsilon f^x(\xi_1(\epsilon))\right) \quad (8)$$

and

$$\sum_n P(\hat{t}_p(x) < t_p(x) - \epsilon) \leq \sum_n \mathbb{P}\left(\sup_{t \in [t_p(x) - \epsilon, t_p(x)]} |\hat{F}^x(t) - F^x(t)| \geq \epsilon f^x(\xi_2(\epsilon))\right) \quad (9)$$

where  $\xi_1(\epsilon)$  (resp.  $\xi_2(\epsilon)$ ) is between  $t_p(x)$  and  $t_p(x) + \epsilon$  (resp.  $t_p(x) - \epsilon$  and  $t_p(x)$ ). Moreover, there exists  $\delta_0 \in ]0, \delta]$  such that

$$\inf_{t \in [t_p(x) - \delta_0, t_p(x) + \delta_0]} f^x(t) \geq C_4 > 0.$$

Then applying (8) and (9) with  $\epsilon = \left(h^b + \left(\frac{\log n}{n \phi_x(h)}\right)^{1/2}\right)$ , we get for a large enough  $n_0$

$$\sum_{n \geq n_0} P(|\hat{t}_p(x) - t_p(x)| > \epsilon) \leq \sum_{n \geq n_0} \mathbb{P}\left(\sup_{t \in [t_p(x) - \delta_0, t_p(x) + \delta_0]} |\hat{F}^x(t) - F^x(t)| \geq C_4 \epsilon\right) < \infty.$$

Then Theorem 3.1 is an easy consequence of Proposition 3.1. ■

### Proof of Proposition 3.1.

Using (6) and (7) and remarking that  $\mathbb{E}[\hat{F}_D^x] = 1$ , we write

$$\hat{F}^x(t) - F^x(t) = \frac{1}{\hat{F}_D^x} \left[ \left( \hat{F}_N^x(t) - \mathbb{E}[\hat{F}_N^x(t)] \right) - \left( F^x(t) - \mathbb{E}[\hat{F}_N^x(t)] \right) \right] - \frac{F^x(t)}{\hat{F}_D^x} \left[ \hat{F}_D^x - \mathbb{E}[\hat{F}_D^x] \right]$$

for all  $t \in \mathbb{R}$ . The proof is achieved through the following lemmas.

**LEMMA 3.1** (see Ferraty et al., 2006) Under Hypotheses (H1), (H3) and (H4), we have

$$\hat{F}_D^x - \mathbb{E}[\hat{F}_D^x] = O\left(\left(\frac{\log n}{n \phi_x(h)}\right)^{1/2}\right) \quad a.co.$$

Moreover

$$\sum_n \mathbb{P}\left(\hat{F}_D^x < \frac{1}{2}\right) < \infty.$$

**LEMMA 3.2** Under Hypotheses (H1)-(H3), we have,

$$\sup_{t \in [t_p(x) - \delta, t_p(x) + \delta]} \left| F^x(t) - \mathbb{E}[\hat{F}_N^x(t)] \right| = O(h^b).$$



LEMMA 3.3 Under Hypotheses (H1), (H3) and (H4), we have,

$$\sup_{t \in [t_p(x) - \delta, t_p(x) + \delta]} \left| \widehat{F}_N^x(t) - \mathbb{E} \left[ \widehat{F}_N^x(t) \right] \right| = O \left( \left( \frac{\log n}{n \phi_x(h)} \right)^{1/2} \right), \quad a.co.$$

### 3.2 Asymptotic normality

Now, we study the asymptotic normality of  $\widehat{t}_p(x)$ . We replace (H1), (H3) and (H4) by the following hypotheses, respectively.

(H1') The concentration property (H1) holds. Moreover, there exists a function  $\beta_x(\cdot)$  such that

$$\forall s \in [0, 1], \quad \lim_{r \rightarrow 0} \phi_x(sr) / \phi_x(r) = \beta_x(s).$$

(H3') The kernel  $K$  satisfies (H3) and is a differentiable function on  $]0, 1[$  with derivative  $K'$  such that  $-\infty < C_5 < K'(\cdot) < C_6 < 0$ .

(H4')  $n\phi_x(h) / \log n \rightarrow \infty$  and  $nh^{2b}\phi_x(h) \rightarrow 0$  as  $n \rightarrow \infty$ .

In what follows, we assume that  $x$  is in  $\mathcal{A} = \{z \in \mathcal{F}, f^z(t_p(z)) \neq 0 \text{ and } \beta_z(\cdot) \text{ is not identically zero}\}$ .

**Remark 3.1** - The function  $\beta_x(\cdot)$  defined in (H1') plays a fundamental role in the asymptotic normality result. It permits to give the variance term explicitly. Noting that (H1') is fulfilled by several small ball probability functions, we quote the following cases (which can be found in Ferraty et al. (2007)):

i)  $\phi_x(h) = C_x h^\gamma$  for some  $\gamma > 0$  with  $\beta_x(u) = u^\gamma$ ,

ii)  $\phi_x(h) = C_x h^\gamma \exp\{-Ch^{-p}\}$  for some  $\gamma > 0$  and  $p > 0$  with  $\beta_x(u) = \delta_1(u)$  where  $\delta_1(\cdot)$  is Dirac's function,

iii)  $\phi_x(h) = C_x |\ln h|^{-1}$  with  $\beta_x(u) = \mathbf{1}_{]0,1]}(u)$ .

- The first condition defining the set  $\mathcal{A}$  is classical and related to a nonvanishing conditional density. The second condition means that a small amount a concentration is needed in order to ensure asymptotic normality.

- Finally we point out that the second rate in our Hypothesis (H4') is the same as the one in Masry (2005, Corollary 2) for infinite-dimension spaces. More precisely, this assumption is used to remove

the bias term. In other words, asymptotic normality could be established without this assumption but, in that case, the result would be perturbed by the presence of the bias term.

Now we are in position to give our main result.

**Theorem 3.2** *Under Hypotheses (H1'), (H2), (H3') and (H4'), we have*

$$\left(\frac{n\phi_x(h)}{\sigma^2(x)}\right)^{1/2} (\hat{t}_p(x) - t_p(x)) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as } n \rightarrow \infty$$

where  $\xrightarrow{\mathcal{D}}$  denotes the convergence in distribution,  $\sigma^2(x) = \frac{p(1-p)a_2(x)}{(f^x(t_p(x)))^2 a_1^2(x)}$  and

$$a_j(x) = K^j(1) - \int_0^1 (K^j)'(s) \beta_x(s) ds \quad \text{for } j = 1, 2. \quad (10)$$

### Proof of Theorem 3.2

For  $u \in \mathbb{R}$ , put  $\tau_p(u, x) := t_p(x) + u [n\phi_x(h)]^{-1/2} \sigma(x)$  and  $\hat{\Phi}(u, x) := p\hat{F}_D^x - \hat{F}_N^x(\tau_p(u, x))$ . Then

$$\begin{aligned} \mathbb{P} \left\{ \sqrt{n\phi_x(h)} \sigma^{-1}(x) (\hat{t}_p(x) - t_p(x)) \leq u \right\} &= \mathbb{P} \left\{ \hat{t}_p(x) \leq \tau_p(u, x) \right\} \\ &= \mathbb{P} \left\{ p \leq \hat{F}^x(\tau_p(u, x)) \right\} + \mathbb{P} \left\{ (\hat{F}_D^x = 0) \cap (\hat{t}_p(x) \leq \tau_p(u, x)) \right\} \\ &= \mathbb{P} \left\{ \hat{\Phi}(u, x) - \mathbb{E} [\hat{\Phi}(u, x)] \leq \mathbb{E} [-\hat{\Phi}(u, x)] \right\} \\ &\quad + \mathbb{P} \left\{ (\hat{F}_D^x = 0) \cap (\hat{t}_p(x) \leq \tau_p(u, x)) \right\}. \end{aligned}$$

So the proof is a consequence of the following lemmas.

**LEMMA 3.4** *Under Hypotheses (H1), (H3) and (H4'), we have*

$$\mathbb{P} \left\{ (\hat{F}_D^x = 0) \cap (\hat{t}_p(x) \leq \tau_p(u, x)) \right\} \leq \exp \{-n\phi_x(h)\}.$$

**LEMMA 3.5** *Under the hypotheses of Theorem 3.2, we have, for all  $u \in \mathbb{R}$ ,*

$$\sqrt{n\phi_x(h)} [f^x(t_p(x))\sigma(x)]^{-1} \mathbb{E} [-\hat{\Phi}(u, x)] = u + o(1) \quad \text{as } n \rightarrow \infty.$$

**LEMMA 3.6** *Under the hypotheses of Theorem 3.2, we have, for all  $u \in \mathbb{R}$ ,*

$$\sqrt{n\phi_x(h)} [f^x(t_p(x))\sigma(x)]^{-1} \left( \hat{\Phi}(u, x) - \mathbb{E} [\hat{\Phi}(u, x)] \right) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 1) \quad \text{as } n \rightarrow \infty.$$

**Remark 3.2** In Lemma 3.5, Hypothesis ( $H_4'$ ) allows to get a result with no asymptotic bias term. Weaking this hypothesis into  $n\phi_x(h) \rightarrow \infty$  and proceeding as in Ferraty et al. (2007) we consider the function  $\zeta(r) = \mathbb{E} [\{F^X(t_p(x)) - p\} | d(x, X) = r]$  and suppose it to be differentiable at zero. Moreover we assume that  $F^z$  is absolutely continuous for  $z \in \mathcal{N}_x$  with density  $f^z$ . In that case we get the bias term

$$\zeta'(0) \frac{a_0(x)}{a_1(x)} h$$

where

$$a_0(x) = K(1) - \int_0^1 (sK(s))' \beta_x(s) ds.$$

## 4 Auxiliary results and proofs

### Proof of (4)

From (3) it is easy to see that  $\hat{t}_p(x)$  is the smallest minimizer of  $\sum_{i=1}^n W_{ni}(x) |Y_i - t| - (2p - 1)t$ . Writing  $|Y_i - t| = (Y_i - t) \mathbb{I}_{\{Y_i > t\}} + (t - Y_i) \mathbb{I}_{\{Y_i \leq t\}}$ , we get the equivalent minimization problem (wrt  $t$ ) of

$$D(t) = t(\hat{F}^x(t) - p) - \sum_{i=1}^n W_{ni}(x) Y_i \mathbb{I}_{\{Y_i \leq t\}}.$$

Then, by simple calculations, we get, for any  $t \geq s$

$$(t - s)(\hat{F}^x(s) - p) \leq D(t) - D(s) \leq (t - s)(\hat{F}^x(t) - p). \quad (11)$$

Since  $\hat{t}_p(x)$  minimizes  $D(\cdot)$  then for any  $t \geq \hat{t}_p(x)$  we have  $D(t) - D(\hat{t}_p(x)) \geq 0$  which by (11) implies  $(t - \hat{t}_p(x))(\hat{F}^x(t) - p) \geq 0$  and thus

$$\forall t > \hat{t}_p(x) \quad \hat{F}^x(t) \geq p. \quad (12)$$

In the same way, as  $\hat{t}_p(x)$  is the smallest among all minimizers, then for any  $s < \hat{t}_p(x)$  we have  $D(\hat{t}_p(x)) - D(s) < 0$ . Using the other part of (11) we obtain

$$\forall s < \hat{t}_p(x) \quad \hat{F}^x(s) < p. \quad (13)$$

The equivalence between (3) and (4) is then a direct consequence of (12) and (13).  $\blacksquare$

### Proof of Lemma 3.2

First note that  $K_1(x) = K_1(x)\mathbb{1}_{B(x,h)}(X_1)$ , under (H3). We deduce from (7) and (H1)

$$F^x(t) - \mathbb{E} \left[ \widehat{F}_N^x(t) \right] = \frac{1}{\mathbb{E} [K_1(x)]} \mathbb{E} \left[ K_1(x) \mathbb{1}_{B(x,h)}(X_1) (F^x(t) - F^{X_1}(t)) \right].$$

Then by (H2) we have

$$\mathbb{1}_{B(x,h)}(X_1) |F^x(t) - F^{X_1}(t)| \leq C_1 h^b,$$

this last inequality completes the proof, since  $C_1$  does not depend on  $t \in [t_p(x) - \delta, t_p(x) + \delta]$ . ■

### Proof of Lemma 3.3

Using the compactness of  $[t_p(x) - \delta, t_p(x) + \delta]$ , we can write

$$[t_p(x) - \delta, t_p(x) + \delta] \subset \bigcup_{j=1}^{d_n} ]y_j - l_n, y_j + l_n[ \quad (14)$$

with  $l_n = n^{-1/2k}$  and  $d_n = O(n^{1/2k})$ . The monotony of  $\mathbb{E}[\widehat{F}_N^x(\cdot)]$  and  $\widehat{F}_N^x(\cdot)$  gives, for  $1 \leq j \leq d_n$

$$\begin{aligned} \mathbb{E} \widehat{F}_N^x(y_j - l_n) &\leq \sup_{t \in ]y_j - l_n, y_j + l_n[} \mathbb{E} \widehat{F}_N^x(t) \leq \mathbb{E} \widehat{F}_N^x(y_j + l_n) \\ \widehat{F}_N^x(y_j - l_n) &\leq \sup_{t \in ]y_j - l_n, y_j + l_n[} \widehat{F}_N^x(t) \leq \widehat{F}_N^x(y_j + l_n). \end{aligned} \quad (15)$$

Now, from (H2) and (7) we have, for any  $t_1, t_2 \in [t_p(x) - \delta, t_p(x) + \delta]$

$$\left| \mathbb{E} \widehat{F}_N^x(t_1) - \mathbb{E} \widehat{F}_N^x(t_2) \right| \leq C_1 |t_1 - t_2|^k. \quad (16)$$

So, we deduce from (14)–(16) that

$$\sup_{t \in [t_p(x) - \delta, t_p(x) + \delta]} \left| \widehat{F}_N^x(t) - \mathbb{E} \widehat{F}_N^x(t) \right| \leq \max_{1 \leq j \leq d_n} \max_{z \in \{y_j - l_n, y_j + l_n\}} \left| \widehat{F}_N^x(z) - \mathbb{E} \widehat{F}_N^x(z) \right| + 2^k C_1 l_n^k.$$

Since  $l_n = n^{-1/2k}$ , it follows that, under (H4),  $l_n^k = o\left(\sqrt{\frac{\log n}{n \phi_x(h)}}\right)$ . Thus, it remains to show that

$$\max_{1 \leq j \leq d_n} \max_{z \in \{y_j - l_n, y_j + l_n\}} \left| \widehat{F}_N^x(z) - \mathbb{E} \widehat{F}_N^x(z) \right| = O\left(\sqrt{\frac{\log n}{n \phi_x(h)}}\right), \quad a.co. \quad (17)$$

For this, using  $\mathbb{P}(\cup_i \Omega_i) \leq \sum_i \mathbb{P}(\Omega_i)$ , we have for any  $\eta > 0$

$$\begin{aligned} &\mathbb{P} \left( \max_{1 \leq j \leq d_n} \max_{z \in \{y_j - l_n, y_j + l_n\}} \left| \widehat{F}_N^x(z) - \mathbb{E} \widehat{F}_N^x(z) \right| > \eta \sqrt{\frac{\log n}{n \phi_x(h)}} \right) \\ &\leq 2d_n \max_{1 \leq j \leq d_n} \max_{z \in \{y_j - l_n, y_j + l_n\}} \mathbb{P} \left( \left| \widehat{F}_N^x(z) - \mathbb{E} \widehat{F}_N^x(z) \right| > \eta \sqrt{\frac{\log n}{n \phi_x(h)}} \right). \end{aligned}$$

Set

$$\Lambda_i(z) = \frac{[K_i(x)\mathbb{1}_{\{Y_i \leq z\}} - \mathbb{E}[K_1(x)\mathbb{1}_{\{Y_1 \leq z\}}]]}{\mathbb{E}[K_1(x)]}.$$

From (H1), (H3) and (H4) we get

$$|\Lambda_i(z)| \leq C_3/C_2\phi_x(h) \text{ and } Var[\Lambda_i(z)] \leq C_3/C_2^2\phi_x(h).$$

Applying Bernstein's exponential inequality we get

$$\mathbb{P}\left(\left|\widehat{F}_N^x(z) - \mathbb{E}\widehat{F}_N^x(z)\right| > \eta\sqrt{\frac{\log n}{n\phi_x(h)}}\right) \leq 2\exp\{-C_7\eta^2 \log n\}, \quad z \in \{y_j - l_n, y_j + l_n\}, \quad 1 \leq j \leq d_n.$$

Therefore, using  $d_n = O(n^{1/2k})$  and choosing  $\eta$  such that  $C_7\eta^2 = 1 + 1/k$ , the right-hand side of the last inequality is the general term of a convergent series which, in view of (17), completes the proof. ■

#### Proof of Lemma 3.4.

Since the  $X_i$ 's are independent we get, under (H1), (H3) and (H4')

$$\begin{aligned} \mathbb{P}\left\{\left(\widehat{F}_D^x = 0\right) \cap \left(\widehat{t}_p(x) < \tau_p(u, x)\right)\right\} &\leq \mathbb{P}\left\{\widehat{F}_D^x = 0\right\} \\ &= \mathbb{P}\left(\bigcap_{i=1}^n \{\omega \in \Omega \text{ such that } X_i(\omega) \notin B(x, h)\}\right) \\ &= (1 - \phi_x(h))^n \leq \exp(-n\phi_x(h)). \end{aligned} \quad \blacksquare$$

#### Proof of Lemma 3.5.

First recall that  $\tau_p(u, x) = t_p(x) + u[n\phi_x(h)]^{-1/2}\sigma(x)$  and  $\widehat{\Phi}(u, x) = p\widehat{F}_D^x - \widehat{F}_N^x(\tau_p(u, x))$ . Since  $p\mathbb{E}[\widehat{F}_D^x] = p = F^x(t_p(x))$ , we have

$$\begin{aligned} \mathbb{E}\left[-\widehat{\Phi}(u, x)\right] &= \frac{1}{\mathbb{E}[K_1(x)]}\mathbb{E}\left\{K_1(x)\left[F^{X_1}(\tau_p(u, x)) - F^x(t_p(x))\right]\right\} \\ &= \frac{\mathbb{E}\left\{K_1(x)\left[F^{X_1}(\tau_p(u, x)) - F^x(\tau_p(u, x))\right]\right\}}{\mathbb{E}[K_1(x)]} + F^x(\tau_p(u, x)) - F^x(t_p(x)) \\ &=: I_1(x) + I_2(x). \end{aligned} \quad (18)$$

For  $I_1(x)$  we write, under (H2) and (H3'),

$$K_1(x) \left|F^{X_1}(\tau_p(u, x)) - F^x(\tau_p(u, x))\right| \leq C_1 h^b K_1(x)$$

which yields

$$I_1(x) = O(h^b).$$

Then using a Taylor expansion we get, under (H1')

$$I_2(x) = u [n\phi_x(h)]^{-1/2} \sigma(x) f^x(t_p(x)) + o\left([n\phi_x(h)]^{-1/2}\right).$$

The result is then a consequence of (18), under (H4'). ■

### Proof of Lemma 3.6.

We use Liapounov's theorem (see Loève (1963, p. 275)). In the first step we consider the asymptotics of the variance term

$$\begin{aligned} n\phi_x(h) \text{Var} \left[ \widehat{\Phi}(u, x) \right] &= \frac{\phi_x(h)}{\mathbb{E}^2[K_1(x)]} \text{Var} \left\{ K_1(x) \left[ \mathbb{1}_{\{Y_1 \leq \tau_p(u, x)\}} - p \right] \right\} \\ &= \phi_x(h) \frac{\mathbb{E}[K_1^2(x)]}{\mathbb{E}^2[K_1(x)]} \mathbb{E} \left[ \frac{K_1^2(x) (\mathbb{1}_{\{Y_1 \leq \tau_p(u, x)\}} - p)^2}{\mathbb{E}[K_1^2(x)]} \right] - \phi_x(h) \mathbb{E}^2 \left[ \frac{K_1(x) (\mathbb{1}_{\{Y_1 \leq \tau_p(u, x)\}} - p)}{\mathbb{E}[K_1(x)]} \right] \\ &=: \phi_x(h) \frac{\mathbb{E}[K_1^2(x)]}{\mathbb{E}^2[K_1(x)]} J_1(x) + J_2(x). \end{aligned} \quad (19)$$

Conditioning wrt  $X_1$ , we prove using the triangle inequality that, under (H2)

$$\begin{aligned} \mathbb{1}_{\{d(x, X_1) \leq h\}} \left| \mathbb{E} \left[ \mathbb{1}_{\{Y_1 \leq \tau_p(u, x)\}} \middle| X_1 \right] - p \right| &= \mathbb{1}_{\{d(x, X_1) \leq h\}} \left| F^{X_1}(\tau_p(u, x)) - F^x(t_p(x)) \right| \\ &\leq C_1 h^b + u [n\phi_x(h)]^{-1/2} \sigma(x) f^x(t_p^*(x)) \end{aligned} \quad (20)$$

where  $t_p^*(x)$  is between  $t_p(x)$  and  $\tau_p(u, x)$ . Since  $f^x$  is continuous, we deduce from Theorem 3.1 that, under (H1') and (H4')

$$J_2(x) = o(1). \quad (21)$$

On the other hand, simple calculations lead to

$$J_1(x) = (1 - 2p) \frac{\mathbb{E} \left[ K_1^2(x) (\mathbb{1}_{\{Y_1 \leq \tau_p(u, x)\}} - p) \right]}{\mathbb{E}[K_1^2(x)]} + (p - p^2).$$

Using the same analytic arguments as for (20) allows us to obtain

$$J_1(x) \longrightarrow p(1 - p) \quad \text{as } n \rightarrow \infty. \quad (22)$$

Next, by integrating wrt the distribution of the real rv  $Z := d(x, X_1)$ , we can show that, under (H1') and (H3') (see Ferraty and Vieu, 2006, p. 44),

$$\mathbb{E} \left[ K_1^j(x) \right] = K^j(1) \phi_x(h) - \int_0^1 (K^j)'(s) \phi_x(sh) ds + o(\phi_x(h)), \quad j = 1, 2. \quad (23)$$

From (10) it follows that

$$\frac{\phi_x(h)\mathbb{E}[K_1^2(x)]}{\mathbb{E}^2[K_1(x)]} \longrightarrow \frac{a_2(x)}{a_1^2(x)} \quad \text{as } n \rightarrow \infty. \quad (24)$$

Then, using (19), (21), (22) and (24), we get

$$n\phi_x(h)\text{Var}[\widehat{\Phi}(u, x)] \longrightarrow p(1-p)\frac{a_2(x)}{a_1^2(x)} = [f^x(t_p(x))\sigma(x)]^2 \quad \text{as } n \rightarrow \infty. \quad (25)$$

In the second step we focus on the asymptotic normality. For this, let

$$M_i(x) = \frac{1}{n\mathbb{E}[K_1(x)]} K_i(x) [\mathbb{1}_{\{Y_i \leq \tau_p(u, x)\}} - p].$$

It is enough to show that for some  $\delta > 0$

$$\frac{\sum_{i=1}^n \mathbb{E}[|M_i(x) - \mathbb{E}[M_i(x)]|^{2+\delta}]}{\left(\text{Var}\left(\sum_{i=1}^n M_i(x)\right)\right)^{(2+\delta)/2}} \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (26)$$

From (25) it is clear that  $n\phi_x(h)\text{Var}(\sum_{i=1}^n M_i(x))$  converges to  $[f^x(t_p(x))\sigma(x)]^2$  as  $n$  tends to infinity. Therefore, to conclude the proof, it is enough to show that, after normalization, the numerator in (26) converges to 0. Since the observations are iid, using the  $C_r$ -inequality (see Loève (1963, p. 155)), we get

$$\begin{aligned} (n\phi_x(h))^{(2+\delta)/2} \sum_{i=1}^n \mathbb{E}\left[|M_i(x) - \mathbb{E}[M_i(x)]|^{2+\delta}\right] &\leq 2^{1+\delta} n (n\phi_x(h))^{(2+\delta)/2} \{\mathbb{E}[|M_1(x)|^{2+\delta}] + |\mathbb{E}[M_1(x)]|^{2+\delta}\} \\ &=: N_1(x) + N_2(x). \end{aligned} \quad (27)$$

Observe that, according to (23), we have, for  $j = 1, 2$ , under (H1'),  $\mathbb{E}[K_1^j(x)] = O(\phi_x(h))$ . Then,

$$\begin{aligned} N_1(x) &= 2^{1+\delta} n^{-\delta/2} (\phi_x(h))^{-1-\delta/2} \left(\frac{\phi_x(h)}{\mathbb{E}[K_1(x)]}\right)^{2+\delta} \mathbb{E}\left[K_1^{2+\delta}(x) |\mathbb{1}_{\{Y_1 \leq \tau_p(u, x)\}} - p|^{2+\delta}\right] \\ &\leq 2^{1+\delta} (n\phi_x(h))^{-\delta/2} \left(\frac{\phi_x(h)}{\mathbb{E}[K_1(x)]}\right)^{2+\delta} \left(\mathbb{E}[K_1^{2+\delta}(x)] / \phi_x(h)\right) \longrightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

under (H1'), (H3') and (H4'). Similarly,

$$\begin{aligned} N_2(x) &= 2^{1+\delta} n^{-\delta/2} (\phi_x(h))^{(2+\delta)/2} \mathbb{E}^{-2-\delta}[K_1(x)] \mathbb{E}^{2+\delta}[K_1(x) |\mathbb{1}_{\{Y \leq \tau_p(u, x)\}} - p|] \\ &\leq 2^{1+\delta} n^{-\delta/2} (\phi_x(h))^{(2+\delta)/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

which concludes the proof. ■

**Acknowledgment.** The authors are grateful to an anonymous referee whose careful reading and appropriate remarks gave us the opportunity to improve the quality of the paper and add some theoretical results.

## References

- [1] Bosq, D. 2000. *Linear Processes in Function Spaces: Theory and Applications*, Lecture Notes in Statistics, 149, Springer-Verlag, New York.
- [2] Cardot, H., Crambes, H., Sarda, P. 2004. Estimation spline de quantiles conditionnels pour variables explicatives fonctionnelles, *C. R. Math., Paris* 339, 141–144.
- [3] Dabo-Niang, S., Laksaci, A. 2008. Nonparametric estimation of conditional quantile when the regressor is valued in a semi-metric space, (Submitted).
- [4] Ezzahrioui, M., Ould-Saïd, E. 2008. Asymptotic results of a nonparametric conditional quantile estimator for functional time series data, *Comm in Statist. Theory & Methods* 37 2335–2759.
- [5] Ferraty, F., Rabhi, A., Vieu, P. 2005. Conditional quantiles for functional dependent data with application to the climatic El Niño phenomenon, *Sankhyā* 67 378–398.
- [6] Ferraty, F., Laksaci, A., Vieu, P. 2006. Estimating some characteristics of the conditional distribution in nonparametric functional models, *Statist. Inf. for Stoch. Proc.* 9 47–76.
- [7] Ferraty, F., Vieu, P. 2006. *Nonparametric Functional Data Analysis. Theory and Practice*, Springer-Verlag, New York.
- [8] Ferraty, F., Mas, A., Vieu, P. 2007. Nonparametric regression on functional data: inference and practical aspects. *Aust. and New Zeal. J. of Statist.* 49 267–286.
- [9] Gannoun, A., Saracco, J., Yu, K. 2003. Nonparametric prediction by conditional median and quantiles, *J. Statist. Plann. Inference* 117 207–223.



- [10] Gasser, T., Hall, P., Presnell, B. 1998. Nonparametric estimation of the mode of a distribution of random curves. *J. R. Stat. Soc., Ser. B*, 60 681–691.
- [11] Lin, Z., Li, D. 2007. Asymptotic normality for  $L_1$ -norm kernel estimator of conditional median under association dependence, *J. Multivariate Anal.* 98 1214–1230.
- [12] Loève, M. 1963. *Probability Theory*, Third Edition, Van Nostrand, Princeton.
- [13] Masry, E. 2005. Nonparametric regression estimation for dependent functional data: Asymptotic normality, *Stoch. Proc. and their Appl.* 115 155–177.
- [14] Ramsay, J.O., Silverman, B.W. 2002. *Applied Functional Data Analysis: Methods and Case Studies*, Springer, New York.
- [15] Ramsay, J.O., Silverman, B.W. 2005. *Functional Data Analysis*, Second Edition, Springer, New York.
- [16] Samanta, M. 1989. Non-parametric estimation of conditional quantiles, *Statist. Probab. Lett.* 7 407–412.
- [17] Zhou, Y., Liang, H. 2000. Asymptotic normality for  $L_1$ -norm kernel estimator of conditional median under  $\alpha$ -mixing dependence, *J. Multivariate Anal.* 73 136–154.