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► **To cite this version:**

Allan Gut, Ulrich Stadtmüller. An asymmetric Marcinkiewicz-Zygmund LLN for random fields. Statistics and Probability Letters, 2009, 79 (8), pp.1016. 10.1016/j.spl.2008.12.006 . hal-00516881

**HAL Id: hal-00516881**

**<https://hal.science/hal-00516881>**

Submitted on 13 Sep 2010

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## Accepted Manuscript

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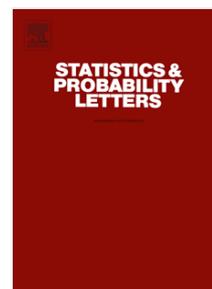
PII: S0167-7152(08)00569-5  
DOI: 10.1016/j.spl.2008.12.006  
Reference: STAPRO 5303

To appear in: *Statistics and Probability Letters*

Received date: 2 October 2008  
Revised date: 8 December 2008  
Accepted date: 9 December 2008

Please cite this article as: Gut, A., Stadtmüller, U., An asymmetric Marcinkiewicz-Zygmund LLN for random fields. *Statistics and Probability Letters* (2008), doi:10.1016/j.spl.2008.12.006

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# An asymmetric Marcinkiewicz-Zygmund LLN for random fields

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## Abstract

The classical Marcinkiewicz-Zygmund law for iid random variables has been generalized by Gut (1978) to random fields. Therein all indices have the same power in the normalization. Looking into some weighted means of random fields, such as Cesàro summation, it is of interest to generalize these laws to the case where different indices have different powers in the normalization. In this paper we give precise moment conditions for such laws.

## 1 Introduction

Let  $X, \{X_k, k \geq 1\}$  be i.i.d. random variables with and partial sums  $\{S_n, n \geq 1\}$ . The classical Marcinkiewicz-Zygmund strong law of large numbers [6] (see also [3], Theorem 6.7.1) reads as follows.

**Theorem 1.1** *Let  $0 < r < 2$ , and suppose that  $X, X_1, X_2, \dots$  are independent, identically distributed random variables. If  $E|X|^r < \infty$ , and  $EX = 0$  when  $1 \leq r < 2$ , then*

$$\frac{S_n}{n^{1/r}} \xrightarrow{a.s.} 0 \quad \text{as } n \rightarrow \infty.$$

*Conversely, if almost sure convergence holds as stated, then  $E|X|^r < \infty$ , and  $EX = 0$  when  $1 \leq r < 2$ .*

Now, let, for  $d \geq 2$ ,  $\mathbb{Z}^d$ , be the positive integer  $d$ -dimensional lattice with coordinate-wise partial ordering  $\leq$ . The following multiindex version of the Marcinkiewicz-Zygmund strong law was given in [2].

**Theorem 1.2** *Let  $0 < r < 2$ , and suppose that  $X, \{X_{\mathbf{k}}, \mathbf{k} \in \mathbb{Z}^d\}$  be i.i.d. random variables with partial sums  $S_{\mathbf{n}} = \sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}$ ,  $\mathbf{n} \in \mathbb{Z}^d$ . If  $E|X|^r (\log^+ |X|)^{d-1} < \infty$ , and  $EX = 0$  when  $1 \leq r < 2$ , then*

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} \xrightarrow{a.s.} 0 \quad \text{as } \mathbf{n} \rightarrow \infty.$$

*Conversely, if almost sure convergence holds as stated, then  $E|X|^r (\log^+ |X|)^{d-1} < \infty$ , and  $EX = 0$  when  $1 \leq r < 2$ .*

Here  $|\mathbf{n}| = \prod_{k=1}^d n_k$  and  $\mathbf{n} \rightarrow \infty$  means  $\min_{1 \leq k \leq d} n_k \rightarrow \infty$ , that is, all coordinates tend to infinity. Also, throughout,  $\log^+ x = \max\{1, \log x\}$ .

We wish to generalize the multiindex version so that instead of normalizing with  $|\mathbf{n}|^{1/r}$  we normalize with different powers for different coordinates. Looking into Cesàro means for random fields (see [4]) it is of interest to control these sums. The main results are presented in the following section followed by a section with proofs. For convenience we will use the parametrization  $\alpha = 1/r$ , i.e.,  $1 \leq r < 2$  is translated into  $1/2 < \alpha \leq 1$ .

AMS 2000 subject classifications. Primary 60F15, 60G50, 60G60; Secondary 60F05.

Keywords and phrases. Sums of i.i.d. random variables, random fields, convergence in probability, almost sure convergence, Marcinkiewicz-Zygmund strong law of large numbers.

Abbreviated title. Asymmetric Marcinkiewicz-Zygmund LLN.

Date. December 8, 2008

## 2 Main results

In order to formulate our main results let  $|\mathbf{n}^\alpha| = \prod_{k=1}^d n_k^{\alpha_k}$ . More precisely, we wish to examine the asymptotics for

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}^\alpha|} = \frac{\sum_{\mathbf{k} \leq \mathbf{n}} X_{\mathbf{k}}}{|\mathbf{n}^\alpha|}, \quad \text{where w.l.o.g. } 1/2 < \alpha_1 = \dots = \alpha_p < \alpha_{p+1} \leq \dots \leq \alpha_d \leq 1, \quad (2.1)$$

and where  $p$  is some integer between 1 and  $d$ . In case  $p = d$  the situation reduces to that of Theorem 1.2. Our main result is as follows.

**Theorem 2.1** *Assume the parameter constellation (2.1). Then  $E|X|^{1/\alpha_1}(\log^+ |X|)^{p-1} < \infty$ , and  $EX = 0$  imply that*

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}^\alpha|} \xrightarrow{\text{a.s.}} 0 \quad \text{as } \mathbf{n} \rightarrow \infty. \quad (2.2)$$

*Conversely, (2.2) implies that  $E|X|^{1/\alpha_1}(\log^+ |X|)^{p-1} < \infty$  and that  $EX = 0$ .*

Secondly we give a result on convergence in probability, which we need for e.g. desymmetrization, where a weaker moment assumption should suffice. Here only the number of indices but not the order of the index set is important. The following result is a slight extension of a result of Le Van Thanh in [5].

**Theorem 2.2** *Suppose that  $1/2 \leq \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_d \leq 1$  with  $\alpha_d > 1/2$ . If  $E|X|^{1/\alpha_1} < \infty$  and  $EX = 0$ , then*

$$\frac{|S_{\mathbf{n}}|}{|\mathbf{n}^\alpha|} \leq \frac{|S_{\mathbf{n}}|}{|\mathbf{n}^{\alpha_1}|} \rightarrow 0 \quad \text{in } L^{1/\alpha_1} \text{ and in probability} \quad \text{as } \mathbf{n} \rightarrow \infty. \quad (2.3)$$

*Actually it suffices here that  $\max_{\{j:\alpha_j > 1/2\}} \{n_j\} \rightarrow \infty$ .*

For the case  $1/2 < \alpha_1 < 1$  we have the following somewhat stronger result.

**Theorem 2.3** *Suppose that  $1/2 < \alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_d < 1$ . If*

$$nP(|X| > n^{\alpha_1}) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (2.4)$$

*then  $E|X| < \infty$ , and if  $EX = 0$ , then*

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}^\alpha|} \xrightarrow{p} 0 \quad \text{as } \max_j \{n_j\} \rightarrow \infty.$$

**REMARK 2.1** Note that the Feller-type condition (2.4) is somewhat weaker than demanding that  $E(|X|^{1/\alpha_1}) < \infty$ .

**REMARK 2.2** The case  $\max_j \{n_j\} \rightarrow \infty$  is not relevant in Theorem 2.1 since the result there depends on the structure of the index set.

The results so far show that the case when one or several  $\alpha$ :s are equal to  $1/2$  are special. Indeed, in the most extreme case when  $\alpha_1 = \dots = \alpha_d = 1/2$  there is obviously no convergence in probability in view of the CLT. Let us therefore discuss the boundary cases w.r.t. a.s. convergence in dimension  $d = 2$  in more detail. Again, if  $\alpha_1 = \alpha_2 = 1/2$  then we are in the domain of the CLT and the pointwise sequences are a.s. unbounded. What happens if only  $\alpha_1 = 1/2$ ? Here the following situation occurs.

**Theorem 2.4** *If  $1/2 = \alpha_1 < \alpha_2 \leq 1$  and  $EX^2 < \infty$  and  $EX = 0$ , then the following holds:*

$$\frac{S_{m,n}}{m^{1/2}n^{\alpha_2}} \xrightarrow{p} 0 \quad \text{as } m, n \rightarrow \infty, \quad (2.5)$$

$$\limsup_{m,n \rightarrow \infty} \frac{S_{m,n}}{m^{1/2}n^{\alpha_2}} = \infty \quad \text{a.s.}, \quad (2.6)$$

$$\frac{S_{m,n}}{\sqrt{m \log \log m} n^{\alpha_2}} \xrightarrow{\text{a.s.}} 0 \quad \text{as } m, n \rightarrow \infty, \quad (2.7)$$

where for the last result we assume the slightly stronger moment condition  $E(X^2 \frac{\log^+ |X|}{\log^+ \log^+ |X|}) < \infty$ .

REMARK 2.3 In (2.7) we replace  $\log \log m$  by a function  $f(m) = o(\log \log m)$  then the random field is again a.s. unboundedly oscillating. For the proof arguments similar to those for (2.6) can be used.

### 3 Proofs

*Proof of Theorem 2.1:* For the following we define the random variables  $Y_{\mathbf{n}} = X_{\mathbf{n}} \mathbb{1}\{|X_{\mathbf{n}}| \leq |\mathbf{n}^\alpha|\}$ . Then,

$$\begin{aligned} \sum_{\mathbf{n}} P(X_{\mathbf{n}} \neq Y_{\mathbf{n}}) &= \sum_{\mathbf{n}} P(|X_{\mathbf{n}}| > |\mathbf{n}^\alpha|) = \sum_{\nu=1}^{\infty} P(|X_{\mathbf{n}}| > \nu^{\alpha_1}) \sum_{n_1 \dots n_p n_{p+1}^{\alpha_1} \dots n_d^{\alpha_1} = \nu} 1 \\ &= \sum_{\nu=1}^{\infty} P(|X_{\mathbf{n}}| > \nu^{\alpha_1}) \Delta g(\nu) \leq \sum_{\nu=0}^{\infty} g(\nu) P(\nu^{\alpha_1} < |X| \leq (\nu+1)^{\alpha_1}), \end{aligned}$$

where

$$g(\nu) := \sum_{n_1 \dots n_p n_{p+1}^{\alpha_1} \dots n_d^{\alpha_1} \leq \nu} 1 \sim c \frac{\nu(\log \nu)^{p-1}}{(\nu-1)!} \quad \text{as } \nu \rightarrow \infty$$

with a suitable constant  $c > 0$  (see Lemma 3 in [9]) and  $\Delta g(\nu) = g(\nu) - g(\nu-1)$ . Now standard arguments show that the sum is finite iff  $E(|X|^{1/\alpha_1} (\log^+ |X|)^{p-1}) < \infty$ .

Hence we restrict our attention to the random variables  $Y_{\mathbf{n}}$ . Now, with  $\beta_\ell = \alpha_\ell / \alpha_1 > 1$  for  $p+1 \leq \ell \leq d$ , the function

$$f(\nu) := \sum_{n_1 \dots n_p \leq \nu} 1 \sim \frac{\nu(\log \nu)^{p-1}}{(\nu-1)!} \quad \text{as } \nu \rightarrow \infty$$

(see again Lemma 3 in [9]), with differences  $\Delta f(\nu) = f(\nu) - f(\nu-1) = \sum_{n_1 \dots n_p = \nu} 1$ , we find that

$$\begin{aligned} \sum_{\mathbf{n}} \frac{\text{Var}(Y_{\mathbf{n}})}{|\mathbf{n}^\alpha|^2} &= \sum_{\nu, n_{p+1}, \dots, n_d=1}^{\infty} \Delta f(\nu) \frac{1}{\nu^{2\alpha_1} \cdot n_{p+1}^{2\alpha_{p+1}} \dots n_d^{2\alpha_d}} \sum_{j=1}^{\nu^{\alpha_1} n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d}} E(X^2 \mathbb{1}\{j-1 < |X| \leq j\}) \\ &\leq \sum_{\nu, n_{p+1}, \dots, n_d=1}^{\infty} \frac{1}{\nu^{2\alpha_1} \cdot n_{p+1}^{2\alpha_{p+1}} \dots n_d^{2\alpha_d}} \Delta f(\nu) \sum_{j=1}^{\nu^{\alpha_1} n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d}} j^2 P(j-1 < |X| \leq j) \end{aligned}$$

$$\begin{aligned}
&\leq c \sum_{\nu, n_{p+1}, \dots, n_d=1}^{\infty} \frac{1}{\nu^{2\alpha_1} \cdot n_{p+1}^{2\alpha_{p+1}} \dots n_d^{2\alpha_d}} \Delta f(\nu) \sum_{j=1}^{\nu^{\alpha_1} n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d}} \left( \sum_{i=1}^j i \right) P(j-1 < |X| \leq j) \\
&\leq c \sum_{\nu, n_{p+1}, \dots, n_d=1}^{\infty} \frac{1}{\nu^{2\alpha_1} \cdot n_{p+1}^{2\alpha_{p+1}} \dots n_d^{2\alpha_d}} \Delta f(\nu) \sum_{i=1}^{\nu^{\alpha_1} n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d}} i P(|X| > i) \\
&\leq c \sum_{n_{p+1}, \dots, n_d=1}^{\infty} \frac{1}{n_{p+1}^{2\alpha_{p+1}} \dots n_d^{2\alpha_d}} \left( \sum_{i=1}^{n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d}} i^{1/\alpha_1 - 1} i^{2-1/\alpha_1} P(|X| > i) \sum_{\nu=1}^{\infty} \nu^{-2\alpha_1} \Delta f(\nu) \right. \\
&\quad \left. + \sum_{i=n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d} + 1}^{\infty} i P(|X| > i) \sum_{\nu=(i/n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d})^{1/\alpha_1}}^{\infty} \nu^{-2\alpha_1} \Delta f(\nu) \right) \\
&\leq c \sum_{n_{p+1}, \dots, n_d=1}^{\infty} \frac{1}{n_{p+1}^{2\alpha_{p+1}} \dots n_d^{2\alpha_d}} \left( (n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d})^{2-1/\alpha_1} \sum_{i=1}^{n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d}} i^{1/\alpha_1 - 1} P(|X| > i) \right. \\
&\quad \left. + \sum_{i=n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d} + 1}^{\infty} i P(|X| > i) \sum_{\nu=(i/n_{p+1}^{\alpha_{p+1}} \dots n_d^{\alpha_d})^{1/\alpha_1}}^{\infty} (\nu^{-2\alpha_1} - (\nu+1)^{-2\alpha_1}) f(\nu) \right) \\
&\leq c \sum_{n_{p+1}, \dots, n_d=1}^{\infty} \frac{1}{n_{p+1}^{\beta_{p+1}} \dots n_d^{\beta_d}} \sum_{i=1}^{\infty} i^{1/\alpha_1 - 1} (\log i)^{p-1} P(|X| > i),
\end{aligned}$$

which is finite iff  $E(|X|^{1/\alpha_1} (\log^+ |X|)^{p-1}) < \infty$ , once again by Lemma 3 in [9], since  $\beta_k > 1$  for  $p+1 \leq k \leq d$ .

In order to apply the multiindex Kolmogorov's convergence criterion (see e.g [1]) we next show that

$$\begin{aligned}
\left| \sum_{\mathbf{n}} \frac{E(Y_{\mathbf{n}})}{|\mathbf{n}^{\alpha}|} \right| &= \left| \sum_{\mathbf{n}} \frac{E(X \mathbb{1}\{|X| > |\mathbf{n}^{\alpha}|\})}{|\mathbf{n}^{\alpha}|} \right| \\
&\leq \sum_{\nu=1}^{\infty} \frac{1}{\nu^{\alpha_1}} \Delta g(\nu) \sum_{j=\nu}^{\infty} j^{\alpha_1} P((j-1)^{\alpha_1} < |X| \leq j^{\alpha_1}) \\
&\leq \sum_{j=1}^{\infty} j^{\alpha_1} P((j-1)^{\alpha_1} < |X| \leq j^{\alpha_1}) \sum_{\nu=1}^j \frac{1}{\nu^{\alpha_1}} \Delta g(\nu),
\end{aligned}$$

which is again finite by our moment assumption. Hence Kolmogorov's convergence criterion applies and  $\sum_{\mathbf{n}} \frac{Y_{\mathbf{n}}}{|\mathbf{n}^{\alpha}|}$  and thus  $\sum_{\mathbf{n}} \frac{X_{\mathbf{n}}}{|\mathbf{n}^{\alpha}|}$  converge almost surely.

Finally, by the multiindex Kronecker lemma (cf. [7] for the necessary multiindex partial summation formula) we conclude that  $S_{\mathbf{n}}/|\mathbf{n}^{\alpha}| \rightarrow 0$  a.s.

For the converse, we note that

$$\frac{S_{\mathbf{n}}}{|\mathbf{n}^{\alpha}|} \xrightarrow{a.s.} 0 \quad \text{as } \mathbf{n} \rightarrow \infty \implies X_{\mathbf{n}}/|\mathbf{n}^{\alpha}| \rightarrow 0 \quad \text{as } \mathbf{n} \rightarrow \infty \implies \sum_{\mathbf{n}} P(|X| > |\mathbf{n}^{\alpha}|) < \infty,$$

yielding the desired moment condition  $E(|X|^{1/\alpha_1}) < \infty$  (cf. [9]) and, obviously, that  $EX = 0$ .  $\square$

*Proof of Theorem 2.2:* Since the inequality in the statement is trivial we only have to prove convergence of  $\frac{|S_{\mathbf{n}}|}{|\mathbf{n}^{\alpha}|}$ .

In case  $\alpha_1 = 1/2$  we may simply compute the variance which is bounded by

$$E(X^2)/(n_{p+1}^{2\alpha_{p+1}-1} \cdots n_d^{2\alpha_d-1}),$$

and tends to zero as  $\max\{n_{p+1}, \dots, n_d\} \rightarrow \infty$ . For  $\alpha_1 > 1/2$  the Pyke-Root-inequality (see [8]) tells us that

$$E\left(\frac{|S_{\mathbf{n}}|}{|\mathbf{n}|^{\alpha_1}}\right)^{1/\alpha_1} \leq \frac{o(|\mathbf{n}|)}{|\mathbf{n}|} \rightarrow 0 \quad \text{as } |\mathbf{n}| \rightarrow \infty,$$

which, in turn, establishes  $L^{1/\alpha_1}$ -convergence, and, hence, in particular, also convergence in probability.  $\square$

*Proof of Theorem 2.3:* Define  $Y_{\mathbf{k}}^{\mathbf{n}} = X_{\mathbf{k}} \mathbb{1}\{|X_{\mathbf{k}}| \leq |\mathbf{n}^{\alpha}| \}$  and  $\mu_{\mathbf{n}} = |\mathbf{n}|E(Y_{\mathbf{k}}^{\mathbf{n}})$ . Then, by the truncated Chebyshev inequality, we have

$$P\left(\left|\frac{S_{\mathbf{n}} - \mu_{\mathbf{n}}}{|\mathbf{n}^{\alpha}|}\right| > \varepsilon\right) \leq \frac{1}{|\mathbf{n}^{\alpha}|^2 \varepsilon^2} \sum_{\mathbf{k}=1}^{\mathbf{n}} \text{Var}(Y_{\mathbf{k}}^{\mathbf{n}}) + |\mathbf{n}|P(|X| > |\mathbf{n}^{\alpha}|) = I_{\mathbf{n}} + II_{\mathbf{n}}.$$

Note that  $II_{\mathbf{n}} \leq |\mathbf{n}|P(|X| > |\mathbf{n}^{\alpha}|) \rightarrow 0$  as  $\mathbf{n} \rightarrow \infty$  by assumption. Next,

$$\begin{aligned} I_{\mathbf{n}} &\leq \frac{|\mathbf{n}|}{|\mathbf{n}^{\alpha}|^2} \sum_{j=1}^{n_1 n_2^{\alpha_2/\alpha_1} \cdots n_d^{\alpha_d/\alpha_1}} E(X^2 \mathbb{1}\{(j-1)^{\alpha_1} < |X| \leq j^{\alpha_1}\}) \\ &\leq \frac{1}{|\mathbf{n}^{2\alpha-1}|} \sum_{j=1}^{n_1 n_2^{\alpha_2/\alpha_1} \cdots n_d^{\alpha_d/\alpha_1}} j^{2\alpha_1} P((j-1)^{\alpha_1} < |X| \leq j^{\alpha_1}) \\ &\leq \frac{c}{|\mathbf{n}^{2\alpha-1}|} \sum_{j=1}^{n_1 n_2^{\alpha_2/\alpha_1} \cdots n_d^{\alpha_d/\alpha_1}} \left(\sum_{i=1}^j i^{2\alpha_1-1}\right) P((j-1)^{\alpha_1} < |X| \leq j^{\alpha_1}) \\ &\leq \frac{c}{|\mathbf{n}^{2\alpha-1}|} \sum_{i=1}^{n_1 n_2^{\alpha_2/\alpha_1} \cdots n_d^{\alpha_d/\alpha_1}} i^{2\alpha_1-2} \cdot i P(|X| > i^{\alpha_1}) \\ &\leq \frac{c n_2^{1-\alpha_2/\alpha_1} \cdots n_d^{1-\alpha_d/\alpha_1}}{n_1^{2\alpha_1-1} n_2^{2\alpha_2-2\alpha_2/\alpha_1} \cdots n_d^{2\alpha_d-2\alpha_d/\alpha_1}} \sum_{i=1}^{n_1 n_2^{\alpha_2/\alpha_1} \cdots n_d^{\alpha_d/\alpha_1}} i^{2\alpha_1-2} \cdot i P(|X| > i^{\alpha_1}) \\ &\rightarrow 0 \quad \text{as } \max_j \{n_j\} \rightarrow \infty, \end{aligned}$$

since we apply — up to a bounded or asymptotically vanishing factor — a regular mean to a nullsequence (cf. [3], Lemma A.6.1).

If in addition  $\alpha_1 < 1$  then, condition (2.4) implies that  $E(|X|) < \infty$ , since

$$\sum_{n=1}^{\infty} P(|X| > n) = \sum_{n=1}^{\infty} \frac{1}{n^{1/\alpha_1}} \cdot n^{1/\alpha_1} P(|X| > n) < \infty,$$

and, moreover, that

$$\begin{aligned} |\mathbf{n}| |E(X \mathbb{1}\{|X| \leq |\mathbf{n}|\})|/|\mathbf{n}^{\alpha}| &= |\mathbf{n}| |E(X \mathbb{1}\{|X| > |\mathbf{n}|\})|/|\mathbf{n}^{\alpha}| \\ &\leq |\mathbf{n}| E(|X| \mathbb{1}\{|X| > |\mathbf{n}|\})/|\mathbf{n}^{\alpha}| \rightarrow 0 \quad \text{as } |\mathbf{n}| \rightarrow \infty. \quad \square \end{aligned}$$

*Proof of Theorem 2.4:* The first result was just shown. For the next one we consider the subsequences  $m_k = k$  and  $n_k = \log_3 k := \log \log \log k$ . Then, with the i.i.d. random variables  $Z_k \stackrel{d}{=} X$  and partial sums  $T_n = \sum_{k=1}^n Z_k$ , we may, equivalently, consider

$$\frac{T_{\ell_k}}{\sqrt{\ell_k \log_2 \ell_k}} \cdot \frac{(\log_2 \ell_k)^{1/2}}{(\log_3 \ell_k)^{\alpha_2-1/2}}$$

with  $\ell_k = k \log_3 k$ . This sequence oscillates unboundedly as  $k \rightarrow \infty$  by the law of iterated logarithm for any  $\alpha_2$ . Note that in the LIL the behaviour along the subsequence  $\ell_k$  is the same as that of the full sequence.

The third result follows from the LIL for random arrays by Wichura [11], which under the given moment assumption yields

$$\limsup_{m,n \rightarrow \infty} \frac{S_{mn}}{\sqrt{mn \log \log mn}} \stackrel{a.s.}{=} \text{Var}(X).$$

Now, since

$$\frac{\log \log m}{\log \log mn} n^{2\alpha_2 - 1} \rightarrow \infty \quad \text{as } m, n \rightarrow \infty,$$

the desired conclusion follows.  $\square$

### Acknowledgement

The work on this paper has been supported by Kungliga Vetenskapssamhället i Uppsala. Their support is gratefully acknowledged. In addition, the second author likes to thank his partner Allan Gut for the great hospitality during two wonderful and stimulating weeks at the University of Uppsala.

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