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A new bound for parsimonious edge-colouring of graphs with maximum degree three

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Abstract

In a graph G of maximum degree 3, let $\gamma(G)$ denote the largest fraction of edges that can be 3 edge-coloured. Rizzi [9] showed that $\gamma(G) \geq 1 - \frac{2}{3g_{\text{odd}}(G)}$ where $g_{\text{odd}}(G)$ is the odd girth of G , when G is triangle-free. In [3] we extended that result to graphs with maximum degree 3. We show here that $\gamma(G) \geq 1 - \frac{2}{3g_{\text{odd}}(G)+2}$, which leads to $\gamma(G) \geq \frac{15}{17}$ when considering graphs with odd girth at least 5, distinct from the Petersen graph.

Keywords: Cubic graph; Edge-colouring;

1. Introduction

Throughout this paper, we shall be concerned with connected graphs with maximum degree 3. Staton [10] (and independently Locke [8]) showed that whenever G is a cubic graph distinct from K_4 then G contains a bipartite subgraph (and hence a 3-edge colourable graph, by König's theorem [7]) with at least $\frac{7}{9}$ of the edges of G . Bondy and Locke [2] obtained $\frac{4}{5}$ when considering graphs with maximum degree at most 3.

In [1] Albertson and Haas showed that whenever G is a cubic graph, we have $\gamma(G) \geq \frac{13}{15}$ (where $\gamma(G)$ denote the largest fraction of edges of G that can be 3 edge-coloured) while for graphs with maximum degree 3 they obtained $\gamma(G) \geq \frac{26}{31}$. Steffen [11] proved that the only cubic bridgeless graph with $\gamma(G) = \frac{13}{15}$ is the Petersen graph. In [3], we extended this result to graphs with maximum degree 3 where bridges are allowed. With the exception of G_5 (a C_5 with two chords), the graph P' obtained from two copies of G_5 by joining by an edge the two vertices of degree 2 and the Petersen graph, every graph

G is such that $\gamma(G) > \frac{13}{15}$. Rizzi [9] showed that $\gamma(G) \geq 1 - \frac{2}{3g_{\text{odd}}(G)}$ where $g_{\text{odd}}(G)$ is the odd girth of G , when G is triangle-free. In [3] we extended that result to graph with maximum degree 3 (triangles are allowed). We show here that $\gamma(G) \geq 1 - \frac{2}{3g_{\text{odd}}(G)+2}$, which leads to $\gamma \geq \frac{15}{17}$ when considering graphs with odd girth at least 5, distinct from the Petersen graph.

Theorem 1.1. *let G be a graph with maximum degree 3 distinct from the Petersen graph.*

Then $\gamma(G) \geq 1 - \frac{2}{3g_{\text{odd}}(G)+2}$

2. Technical lemmas

Let $\phi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-colouring of G . It is often of interest to try to use one colour (say δ) as few as possible. When an edge colouring is optimal, following this constraint, we shall say that ϕ is δ -*minimum*. Since any two δ -minimum edge-colouring of G have the same number of edges coloured δ we shall denote by $s(G)$ this number. For $x, y \in \{\alpha, \beta, \gamma, \delta\}$, $x \neq y$, $\phi(x, y)$ is the partial subgraph of G spanned by these two colours (this subgraph being a union of paths and even cycles where the colours x and y alternate) and $E_\phi(x)$ is the set of edges coloured with x .

In [5] we gave without proof (in French, see [4] for a translation) results on δ -minimum edge-colourings of cubic graphs

Lemma 2.1. [5, 6, 4] *Let ϕ be a δ -minimum edge-colouring of G . For any edge $e = uv$ coloured with δ there are two colours x and y in $\{\alpha, \beta, \gamma\}$ such that the connected component of $\phi(x, y)$ containing the two ends of e is an even path joining these two ends. Moreover e has one end of degree 2 and the other of degree 3 or the two ends of degree 3*

Remark 2.2. An edge coloured with δ by the δ -minimum edge-colouring ϕ is in A_ϕ when its ends can be connected by a path of $\phi(\alpha, \beta)$, B_ϕ by a path of $\phi(\beta, \gamma)$ and C_ϕ by a path of $\phi(\alpha, \gamma)$. It is clear that A_ϕ , B_ϕ and C_ϕ are not necessarily pairwise disjoint since an edge coloured with δ with one end of degree 2 is contained in 2 such sets. Assume indeed that $e = uv$ is coloured with δ while $d(u) = 3$ and $d(v) = 2$ then, if u is incident

to α and β and v is incident to γ we have an alternating path whose ends are u and v in $\phi(\alpha, \gamma)$ as well as in $\phi(\beta, \gamma)$. Hence e is in $A_\phi \cap B_\phi$. When $e \in A_\phi$ we can associate to e the odd cycle $C_{A_\phi}(e)$ obtained by considering the path of $\phi(\alpha, \beta)$ together with e . We define in the same way $C_{B_\phi}(e)$ and $C_{C_\phi}(e)$ when e is in B_ϕ or C_ϕ . In the following lemma we consider an edge in A_ϕ , an analogous result holds true whenever we consider edges in B_ϕ or C_ϕ as well.

Lemma 2.3. [5, 6, 4] *Let ϕ be a δ -minimum edge-colouring of G and let e be an edge in A_ϕ then for any edge $e' \in C_{A_\phi}(e)$ there is a δ -minimum edge-colouring ϕ' such that $E_{\phi'}(\delta) = E_\phi(\delta) - \{e\} \cup \{e'\}$, $e' \in A_{\phi'}$ and $C_{A_\phi}(e) = C_{A_{\phi'}}(e')$. Moreover, each edge outside $C_{A_\phi}(e)$ but incident with this cycle is coloured γ , ϕ and ϕ' only differ on the edges of $C_{A_\phi}(e)$.*

For each edge $e \in E_\phi(\delta)$ (where ϕ is a δ -minimum edge-colouring of G) we can associate one or two odd cycles following the fact that e is in one or two sets among A_ϕ , B_ϕ or C_ϕ . Let \mathcal{C} be the set of odd cycles associated to edges in $E_\phi(\delta)$.

Lemma 2.4. [5, 6, 4] *Let $e_1, e_2 \in E_\phi(\delta)$ and let $C_1, C_2 \in \mathcal{C}$ be such that $C_1 \neq C_2$, $e_1 \in E(C_1)$ and $e_2 \in E(C_2)$ then C_1 and C_2 are (vertex) disjoint.*

Lemma 2.5. [5, 6, 4] *Let $e_1 = uv_1$ be an edge of $E_\phi(\delta)$ such that v_1 has degree 2 in G . Then v_1 is the only vertex in $N(u)$ of degree 2 and for any edge $e_2 = u_2v_2 \in E_\phi(\delta)$, $\{e_1, e_2\}$ induces a $2K_2$.*

Lemma 2.6. [5, 6, 4] *Let e_1 and e_2 be two edges of $E_\phi(\delta)$. If e_1 and e_2 are contained in two distinct sets of A_ϕ, B_ϕ or C_ϕ then $\{e_1, e_2\}$ induces a $2K_2$ otherwise e_1, e_2 are joined by at most one edge.*

Lemma 2.7. [5, 6, 4] *Let e_1, e_2 and e_3 be three distinct edges of $E_\phi(\delta)$ contained in the same set A_ϕ, B_ϕ or C_ϕ . Then $\{e_1, e_2, e_3\}$ induces a subgraph with at most four edges.*

Lemma 2.8. [3] *Let G be a graph with maximum degree 3 then $\gamma(G) = 1 - \frac{s(G)}{m}$.*

A cubic graph G on n vertices is called a *permutation graph* if G has a perfect matching M such $G - M$ is the union of two chordless cycles A and B of equal length $\frac{n}{2}$. When we delete one edge of the perfect matching M in the above permutation graph G , we shall say that the graph obtained is a *near-permutation graph*.

Lemma 2.9. *Let G be a permutation graph or a near-permutation graph with n vertices and odd girth $\frac{n}{2}$. Suppose that G is not 3-edge colourable. Then G is the Petersen graph or the Petersen graph minus one edge.*

Proof Obviously, since G is not 3-edge colourable, $\frac{n}{2}$ is certainly odd. Let $A = a_0a_1 \dots a_{2k}$ and $B = b_0b_1 \dots b_{2k}$ be the two chordless cycles of length $\frac{n}{2} = 2k + 1$ which partition $V(G)$. When $2k + 1 = 3$, it can be easily verified that G is 3-edge colourable and when $2k + 1 = 5$, G is the Petersen graph or the Petersen graph minus one edge. Assume thus that $2k + 1 \geq 7$.

Since at most one vertex of A and one vertex of B have degree 2, we can suppose that a_0, a_1, a_2 are joined to 3 distinct vertices of B . Without loss of generality we suppose that $a_0b_0 \in E(G)$. Since G is not 3-edge colourable a_1b_1 and a_1b_{2k} are not edges of G and since the odd girth is at least 7 we do not have the edges a_1b_2 and a_1b_{2k-1} . Henceforth let b_i ($2 < i \leq 2k - 2$) the neighbour of a_1 . One of the two paths determined by b_0 and b_i on B must have odd length. Suppose, without loss of generality, that $b_0b_1 \dots b_i$ has odd length then we have an odd cycle $a_0b_0b_1 \dots b_i a_1$ whose length is at least $2k + 1$, which leads to $i = 2k - 2$.

The vertex a_2 is not joined to b_{2k-1} or b_{2k-3} , otherwise G is 3-edge colourable, neither to b_{2k}, b_{2k-4} or b_1 , otherwise the odd girth is 5. Henceforth a_2 is joined to some vertex b_j with $2 \leq j \leq 2k - 5$. If j is odd then $b_0b_1 \dots b_j a_2 a_1 a_0$ is an odd cycle of length at most $2k - 1$, impossible. If j is even then $b_j \dots b_{2k-3} b_{2k-2} a_1 a_2$ an odd cycle of length at most $2k - 1$, impossible. \square

3. Proof of Theorem 1.1

Proof

Let ϕ be a δ -minimum edge-colouring of G and $E_\phi(\delta) = \{e_1, e_2 \dots e_{s(G)}\}$. \mathcal{C} being the set of odd cycles associated to edges in $E_\phi(\delta)$, we write $\mathcal{C} = \{C_1, C_2 \dots C_{s(G)}\}$ and suppose that for $i = 1, 2 \dots s(G)$, e_i is an edge of C_i . We know by Lemma 2.4 that the cycles of \mathcal{C} are vertex-disjoint.

Let $l(\mathcal{C}) = \sum_{C \in \mathcal{C}} l(C)$ (where $l(C)$ is the length of the cycle C) and assume that ϕ has been chosen in such a way that $l(\mathcal{C})$ is maximum.

Let us write $\mathcal{C} = \mathcal{C}_2 \cup \mathcal{C}_3$, where \mathcal{C}_2 denotes the set of odd cycles of \mathcal{C} which have a vertex of degree 2, while \mathcal{C}_3 is for the set of cycles in \mathcal{C} whose all vertices have degree 3. Let $k = |\mathcal{C}_2|$, obviously we have $0 \leq k \leq s(G)$ and $\mathcal{C}_2 \cap \mathcal{C}_3 = \emptyset$.

If $C_i \in \mathcal{C}_2$, we may suppose that e_i has a vertex of degree 2 (see Lemma 2.3) and we can associate to e_i another odd cycle say C'_i (Remark 2.2) whose edges distinct from e_i form an even path P_i using at least $\frac{g_{\text{odd}}(G)}{2}$ edges which are not edges of C_i . When $l(C_i) = g_{\text{odd}}(G)$, C_i has no chord and it is an easy task to find a supplementary edge of P_i not belonging to C_i . When $l(C_i) \geq g_{\text{odd}}(G) + 2$ (recall that C_i has odd length), we can choose an edge of C_i as a *private* edge. In both cases $C_i \cup C'_i$ contains at least $\frac{3}{2}g_{\text{odd}}(G) + 1$ edges. Consequently there are at least $k \times (\frac{3}{2}g_{\text{odd}}(G) + 1)$ edges in $\bigcup_{C_i \in \mathcal{C}_2} (C_i \cup C'_i)$.

When $C_i \in \mathcal{C}_3$, C_i contains at least $g_{\text{odd}}(G)$ edges, moreover, each vertex of C_i being of degree 3, there are $\frac{s(G)-k}{2} \times g_{\text{odd}}(G)$ additional edges which are incident to a vertex of $\bigcup_{C_i \in \mathcal{C}_3} C_i$. Let us remark that each above additional edge is counted as $\frac{1}{2}$ whatever are these edges. In order to refine our counting, we need to introduce the following notion of *free* edge. An edge will be said to be *free* when at most one end belongs to some $C_i \in \mathcal{C}_3$.

Suppose that we can associate to each $C_i \in \mathcal{C}_3$ one private edge or two private free edges. Since $C_i \cap C_j = \emptyset$ and $C'_i \cap C_j = \emptyset$ ($1 \leq i, j \leq s(G)$, $i \neq j$), we would have

$$m \geq k \times (\frac{3}{2}g_{\text{odd}}(G) + 1) + (s(G) - k) \times (g_{\text{odd}}(G) + 1) + \frac{s(G) - k}{2} \times (g_{\text{odd}}(G))$$

and

$$m \geq s(G) \times (\frac{3}{2} \times g_{\text{odd}}(G) + 1).$$

Consequently $\gamma(G) = 1 - \frac{s(G)}{m} \geq 1 - \frac{2}{3g_{\text{odd}}(G) + 2}$, as claimed.

Our goal now is to associate to each $C_i \in \mathcal{C}_3$ one private edge or two private free edges.

When C_i is incident to at least two free edges, let us choose any two such free edges as the private free edges associate to C_i . By definition, these two free edges are not incident to any $C_j \in \mathcal{C}_3$ with $j \neq i$, insuring thus that they cannot be associated to C_j . When $l(C_i) \geq g_{\text{odd}}(G) + 2$, we choose any edge of C_i as a private edge.

Assume thus that $l(C_i) = g_{\text{odd}}(G)$. Hence $C_i = x_0x_1 \dots x_{g_{\text{odd}}(G)-1}$ is chordless. Suppose that C_i is incident to at most one free edge. Without loss of generality, we can

consider that x_0 is the only possible vertex of C_i incident to some free edge. Since the edge incident to x_1 , not belonging to C_i , is not free, let $C_j \in \mathcal{C}_3$, $i \neq j$, such that x_1 is adjacent to $y_1 \in C_j$. In the same way, the edge incident to x_2 , not belonging to C_i , is not free. Let $C_{j'} \in \mathcal{C}_3$, $i \neq j'$, such that x_2 is adjacent to $z_2 \in C_{j'}$. Suppose that $j \neq j'$, then by Lemma 2.3 we can consider that x_1x_2 is coloured δ by ϕ as well as one of the edges of C_j incident with the vertex y_1 and one of the edges of $C_{j'}$ incident with z_2 , a contradiction with Lemma 2.7 or Lemma 2.6. Henceforth, x_2 is adjacent to some vertex of C_j . In the same way every vertex of C_i , distinct from x_0 , is adjacent to some vertex of C_j .

In this situation, let us say that C_i is *extremal for* C_j . Assume that we have a set of $p \geq 2$ distinct extremal cycles of \mathcal{C}_3 for C_j . Since two distinct extremal cycles have no consecutive neighbours in C_j (otherwise we get a contradiction with Lemmas 2.7 or 2.6 as above), C_j has length at least $p \times (g_{\text{odd}}(G) - 1) + 2$. But $p \times (g_{\text{odd}}(G) - 1) + 2 \geq g_{\text{odd}}(G) + p + 1$ as soon as $g_{\text{odd}}(G) \geq 3$. hence, when $p \geq 2$, we can associate to each extremal cycle for C_j a private edge belonging to C_j as well as a private edge for C_j itself, since $l(C_j) \geq g_{\text{odd}}(G) + p + 1$.

Assume thus that C_i is the only extremal cycle for C_j . If $l(C_j) \geq g_{\text{odd}}(G) + 2$, we can associate any edge of C_j as a private edge of C_i and any other edge of C_j as a private edge of C_j . It remains thus to consider the case where $l(C_j) = g_{\text{odd}}(G)$. In that situation, C_i is an extremal cycle for C_j , as well as C_j is an extremal cycle for C_i and the subgraph induced by $C_i \cup C_j$ is a permutation graph or a near permutation graph with odd girth $g_{\text{odd}}(G) = \frac{|C_i \cup C_j|}{2}$.

By Lemma 2.9, G itself is the Petersen graph or $C_i \cup C_j$ induces a Petersen graph minus one edge (by the way, $g_{\text{odd}}(G) = 5$). By hypothesis, the first case is excluded. Assume thus that $C_i \cup C_j$ induces a Petersen graph minus one edge. In the last part of this proof, we show that this situation is not possible.

In order to fix the situation let H be the subgraph of G not containing $C_i \cup C_j$. We suppose that C_j is the chordless cycle of length 5 $y_0y_3y_1y_4y_2$ while x_1y_1, x_2y_2, x_3y_3 and x_4y_4 are the edges joining C_i to C_j . Moreover x_0 is joined to some vertex $a \in V(H)$ and y_0 is joined to some vertex $b \in V(H)$. Without loss of generality, we can consider that ϕ colours alternately the edges of C_i (C_j respectively) with β and γ with the exception of

the edge x_0x_1 coloured with δ (y_0y_3 respectively).

The edges x_1y_1, x_2y_2, x_3y_3 and x_4y_4 are thus coloured with α as well as the edges x_0a and y_0b (let us remark that $a \neq b$). The final situation is depicted in Figure 1.

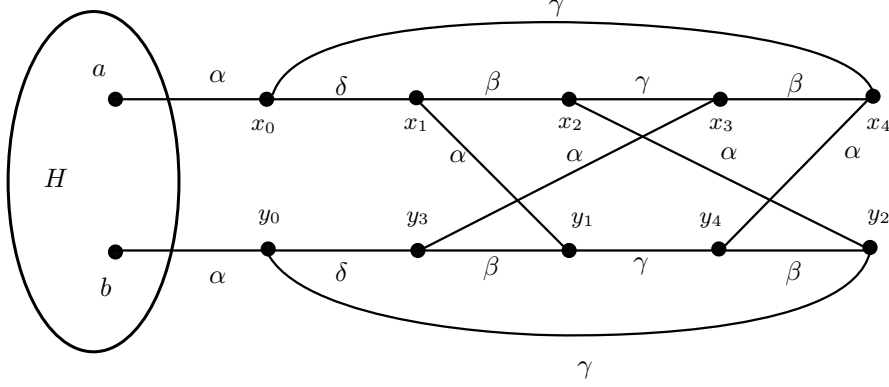


Figure 1: Final situation

Without changing any colour in H and keeping the colour α for the edges x_0a and y_0b , we can construct a new δ -minimum edge-colouring ϕ' in the following way (see Figure 2):

- x_0x_4 and x_1x_2 are coloured with δ
- x_4y_4, y_0y_2, x_2x_3 and y_1y_3 are coloured with β
- x_1y_1, x_3y_3 and y_4y_2 are coloured with α
- $x_0x_1, x_3x_4, y_0y_3, y_1y_4$ and x_2y_2 are coloured with γ .

Let \mathcal{C}' be the set of odd cycles associated to edges coloured with δ by ϕ' . Since we do not have change any colour in H , we have $\mathcal{C}' = \mathcal{C} - \{C_i, C_j\} + \{C'_i, C'_j\}$ where C'_i is the chordless cycle of length 5 $x_1x_2x_3y_3y_1$ and C'_j is the odd cycle obtained by concatenation of the path $ax_0x_4y_4y_2y_0b$ and the path coloured alternately β and α joining a to b in H (dashed line in Figure 2), whose existence comes from Lemma 2.1. Since the length of C'_j is at least 7, $l(\mathcal{C}') > l(\mathcal{C})$, a contradiction with the initial choice of ϕ .

□

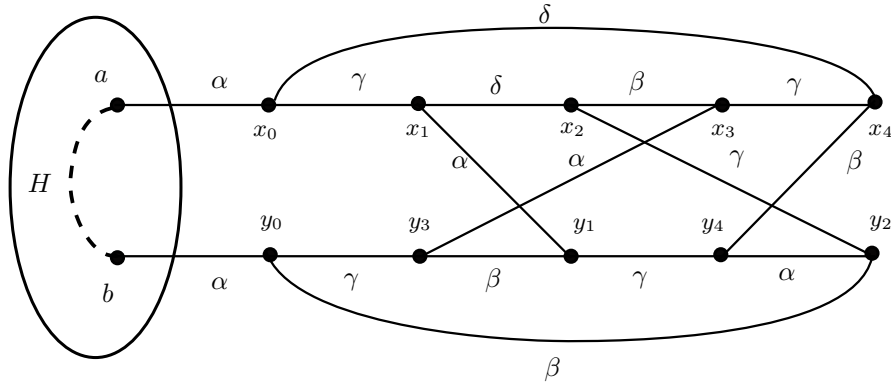


Figure 2: New colouring ϕ'

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