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# Dirac Particles in a Gravitational Field

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The semiclassical approximation for the Hamiltonian of Dirac particles interacting with an arbitrary gravitational field is investigated. The time dependence of the metrics leads to new contributions to the in-band energy operator in comparison to previous works on the static case. In particular we find a new coupling term between the linear momentum and the spin, as well as couplings which contribute to the breaking of the particle - antiparticle symmetry.

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## INTRODUCTION

In this paper we consider the theory of Dirac fermions in an arbitrary curved space-time in the Hamiltonian formulation. To reveal the physical content of the theory it is necessary to perform the diagonalization of the Hamiltonian uncoupling the positive and the negative energy states. For a fermion interacting with an electromagnetic field the Foldy-Wouthuysen (FW) transformation based on an approximate scheme valid in the non relativistic limit is often used [1]. This same method was also applied in all the previous studies of Dirac fermions in a gravitational field [2]. Here instead, we will consider the fully relativistic regime but in a semiclassical approximation for which the de Broglie wave length of the fermion must be much smaller than the characteristic size of the inhomogeneities of the external field. A recent semiclassical FW-like transformation used for Dirac particle in a strong electromagnetic field could be adapted to the gravitational problem [3]. But instead we will use another method developed by the authors which essentially differs from the FW. This method allows us to find the diagonal representation of any kind of matrix valued quantum Hamiltonian as a series expansion in the Planck constant. Here we will directly apply the general formula obtained at the semiclassical (first order in the Planck constant) limit to the case of Dirac fermions in an arbitrary curved spacetime. This is an extension of previous papers where massless and massive particles in a static gravitational fields were treated. The extension to time dependent metrics turns out to be non-trivial and leads to new coupling terms in the in-band energy operator which break the particle-antiparticle symmetry.

## ELECTRON IN A GRAVITATIONAL FIELD

A one half spinning particle of mass  $m$  coupled to an arbitrary gravitational field is described by 4-spinor field  $\psi$  satisfying the covariant Dirac equation

$$(i\hbar\gamma^\alpha D_\alpha - m)\psi = 0 \quad (1)$$

where we use  $c = 1$ , but keep explicit the Planck constant  $\hbar$  and  $\alpha = 0, 1, 2, 3$ .

The covariant spinor derivative is defined as  $D_\alpha = h_\alpha^i D_i$  with  $D_i = \partial_i - \frac{1}{4} [\gamma^\alpha, \gamma^\beta] \Gamma_i^{\alpha\beta}$ . The matrices  $\gamma^\alpha$  are the usual Dirac matrices,  $h_\alpha^i$  are the orthonormal vierbein and  $\Gamma_i^{\alpha\beta}$  the spin connection components.

Rewriting Eq. (1) under the Schrödinger form

$$i\hbar \frac{\partial \psi}{\partial t} = H \psi$$

we obtain the following Hamiltonian

$$H = g_{00} h_\beta^0 \gamma^\beta (\gamma^\alpha \bar{P}_\alpha + m) + \frac{\hbar}{4} \varepsilon_{\rho\beta\gamma} \Gamma_0^{\rho\beta} \Sigma^\gamma + i \frac{\hbar}{4} \Gamma_0^{0\beta} \alpha_\beta \quad (2)$$

where we introduced the notation for the pseudo-momentum  $\bar{P}_\alpha = h_\alpha^i (P_i + \frac{\hbar}{4} \varepsilon_{\rho\beta\gamma} \Gamma_i^{\rho\beta} \Sigma^\gamma)$  with the spin matrix  $\Sigma^\gamma$ , satisfying the relation  $\varepsilon^{\gamma\alpha\beta} \Sigma_\gamma = \frac{i}{8} (\gamma^\alpha \gamma^\beta - \gamma^\beta \gamma^\alpha)$ . We use the conventions of Bjorken and Drell [4] for the Dirac matrices  $\gamma^\alpha$ , and  $\alpha^\beta$ .

Surprisingly, Eq. (2) turns out to be non-hermitian for a time dependent metric. We then follow the approach of Leclerc [5] who showed that one must add the term  $\frac{i}{2}\partial_t \ln(-gg^{00})$  to make it Hermitian. It will be shown later on, that the presence of this term is also necessary for the diagonalization procedure to work.

Therefore the Hamiltonian considered in the following will be

$$\hat{H} = g_{00}h_{\beta}^0\gamma^{\beta}(\gamma^{\alpha}\bar{P}_{\alpha} + m) + \frac{\hbar}{4}\varepsilon_{\rho\beta\gamma}\Gamma_0^{\rho\beta}\Sigma^{\gamma} + i\frac{\hbar}{4}\Gamma_0^{0\beta}\alpha^{\beta} + \frac{i}{2}\partial_t \ln(-gg^{00}) \quad (3)$$

The goal of this paper is therefore to diagonalise Eq. (3) to first order in  $\hbar$ . But before embarking into this, we need first to discuss the definitions and properties of the scalar product in a curved space-time.

### Scalar product.

As said before the Hamiltonian  $\hat{H}$  is Hermitian. However, the notion of hermiticity is here defined with respect to a scalar product in curved space [5], namely :

$$\langle\psi_1(t) | \psi_2(t)\rangle_{U(t)} = \int \psi_1^+ \sqrt{-g}h_{\beta}^0\gamma^{\beta}\gamma^0\psi_2 d^3x = \int \psi_1^+ U \psi_2 d^3x \quad (4)$$

where we introduced the notation  $U = \sqrt{-g}h_{\mu}^0\gamma^{\mu}\gamma^0$  and  $\psi_1, \psi_2$  two spinors. Therefore an operator  $O$  is hermitian for  $\langle|\rangle_{U(t)}$  defined in Eq. (4), if

$$\int \psi_1^+ \sqrt{-g}h_{\beta}^0\gamma^{\beta}\gamma^0 (O\psi_2) = \int (O\psi_1)^+ \sqrt{-g}h_{\beta}^0\gamma^{\beta}\gamma^0 (\psi_2) \quad (5)$$

It means that matricially

$$O^+U = UO \quad (6)$$

where "+" denotes from now, the usual Hermitic conjugation (transposition and complex conjugation), that is, the hermitic conjugate with respect to the scalar product in flat space denoted  $\langle|\rangle$  and defined by

$$\langle\psi_1(t) | \psi_2(t)\rangle = \int \psi_1^+ \psi_2 d^3x \quad (7)$$

Unfortunately the definition Eq. (5) turns out to be untractable for practical computations. Actually, for the sake of the diagonalization procedure, we aim at working with matrices which are Hermitian with respect to the usual transpose and complex conjugate operation Eq. (7), so that the diagonalization can be performed through a unitary matrix in the usual sense. To do so, notice that if  $O$  is Hermitian for Eq. (4), then Eq. (6) implies that  $U^{\frac{1}{2}}OU^{-\frac{1}{2}}$  is Hermitian for Eq. (7). Thus, starting with the Hamiltonian  $\hat{H}$  defined in Eq. (3),  $U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}}$  is Hermitian in the usual sense and can be diagonalized through a standard unitary matrix (that is unitary for (7)).

The Hamiltonian of interest for us will thus be  $U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}}$ . It's hermiticity in the usual sense allows us to write:

$$\begin{aligned} U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}} &= \frac{1}{2}U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}} + \frac{1}{2}\left(U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}}\right)^+ \\ &= \frac{1}{2}\left(\hat{H} + \hat{H}^+\right) + \frac{1}{2}\left[U^{\frac{1}{2}}, \hat{H}\right]U^{-\frac{1}{2}} + \frac{1}{2}U^{-\frac{1}{2}}\left[\hat{H}, U^{\frac{1}{2}}\right] \end{aligned} \quad (8)$$

The non unitarity of the transformation  $U$  is not problematic here, since it is precisely used to change the metric from curved to flat scalar product, and moreover  $\hat{H}$  and  $U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}}$  have the same spectrum. In the case of a static metric (time independent) and satisfying  $h_{\mu}^0 = f(\mathbf{R})\delta_{\mu}^0$ , for a certain position dependent function  $f(\mathbf{R})$ , the transformation  $U^{\frac{1}{2}}$  reduces to the multiplication by a function of  $\mathbf{R}$ . Then, using Eq. (3) for  $\hat{H}$ , Eq. (8) simplifies easily to:

$$U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}} = \frac{1}{2}U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}} + \frac{1}{2}\left(U^{\frac{1}{2}}\hat{H}U^{-\frac{1}{2}}\right)^+ = \frac{1}{2}\left(\hat{H} + \hat{H}^+\right)$$

It is this form that was considered in [6][7]. If in addition the metrics is diagonal, one recovers the transformation studied in [2].

Independently of the practical advantages of the flat scalar product, there is an other and deeper reason to transform the Hamiltonian to the flat space. Actually, when diagonalizing the Hamiltonian with respect to Eq. (7) the diagonal subspaces of up and down spinors will appear to be obviously orthogonal. This is of course not the case for the scalar product Eq. (4) which mixes both subspaces. As a consequence diagonalizing with respect to the curved space scalar product does not lead to a clear separation between particles and antiparticles.

### Unitarity

We will end up this section by stressing the fact that for a non-static metric the Hamiltonian and the time evolution operator do not coincide in the flat representation, and in addition, the time evolution operator cannot be made Hermitian. To make this point clearer, consider the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi = H \Psi$$

Applying the transformation  $U^{\frac{1}{2}}$  yields,

$$\frac{\partial}{\partial t} \Psi' = \left( U^{\frac{1}{2}}(t) H U^{-\frac{1}{2}}(t) - i\hbar U^{\frac{1}{2}}(t) \frac{\partial}{\partial t} U^{-\frac{1}{2}}(t) \right) \Psi'$$

with  $\Psi' = U^{\frac{1}{2}} \Psi$ . As a consequence the evolution for  $\Psi'$ , is given by the operator

$$H^e = U^{\frac{1}{2}}(t) H U^{-\frac{1}{2}}(t) - i\hbar U^{\frac{1}{2}}(t) \frac{\partial}{\partial t} U^{-\frac{1}{2}}(t) \quad (9)$$

One would like  $H^e$  to be hermitian for the scalar product Eq. (7). However, due to the non unitarity of  $U^{\frac{1}{2}}$ , the contribution  $i\hbar U^{\frac{1}{2}}(t) \frac{\partial}{\partial t} U^{-\frac{1}{2}}(t)$  is not. The reason for this non unitarity tracks back to the dependence in  $t$  of the scalar product Eq. (4), so that the norm of a wave function is not preserved in time. Actually starting with an initial condition  $\Psi'(t_0)$ , the solution for the Schroedinger equation is

$$\Psi'(t_1) = T \exp \left( \int_{t_0}^{t_1} H^e(t) dt \right) \Psi'(t_0) = U^{\frac{1}{2}}(t_1) T \exp \left( \int_{t_0}^{t_1} H(t) dt \right) U^{-\frac{1}{2}}(t_0) \Psi'(t_0)$$

where  $T$  is the time ordered exponential. Assuming the norm  $\Psi'(t_0)$  to be equal to 1,  $\Psi'(t_1)$  is easily seen to have a norm (respectively to Eq. (4)) different from one. That can be checked easily on an infinitesimal timeslice,  $t_1 = t_0 + \Delta t$ . Indeed

$$\begin{aligned} \langle \Psi'(t_1) | \Psi'(t_1) \rangle &= \left\langle U^{\frac{1}{2}}(t_1) \exp(-iH(t_0)\Delta t) \Psi(t_0) \mid U^{\frac{1}{2}}(t_1) \exp(iH(t_0)\Delta t) \Psi(t_0) \right\rangle \\ &= \langle \exp(-iH(t_0)\Delta t) \Psi(t_0) \mid U(t_1) \exp(iH(t_0)\Delta t) \Psi(t_0) \rangle \\ &= \langle \exp(-iH(t_0)\Delta t) \Psi(t_0) \mid \exp(iH(t_0)\Delta t) \Psi(t_0) \rangle_{U(t_1)} \end{aligned} \quad (10)$$

and this is different from 1 since,

$$\begin{aligned} &\langle \exp(-iH(t_0)\Delta t) \Psi(t_0) \mid \exp(iH(t_0)\Delta t) \Psi(t_0) \rangle_{U(t_1)} \\ &= \langle \exp(-iH(t_0)\Delta t) \Psi(t_0) \mid U(t_0) \exp(iH(t_0)\Delta t) \Psi(t_0) \rangle + \left\langle \Psi(t_0) \mid \frac{\partial}{\partial t} U(t_0) \Delta t \Psi(t_0) \right\rangle \end{aligned} \quad (11)$$

$$= \langle \exp(-iH(t_0)\Delta t) \Psi(t_0) \mid \exp(iH(t_0)\Delta t) \Psi(t_0) \rangle_{U(t_0)} + \left\langle \Psi(t_0) \mid \frac{\partial}{\partial t} U(t_0) \Delta t \Psi(t_0) \right\rangle \quad (12)$$

the first term is equal to 1, actually  $\Psi(t_0)$  is of norm 1 for  $\langle \mid \rangle_{U(t_0)}$  and  $\exp(iH(t_0)\Delta t)$  is unitary for this scalar product. As a consequence,  $\langle \Psi'(t_1) | \Psi'(t_1) \rangle$  differs from one and  $H^e$  is non unitary. The reason is clear from Eq. (10): a vector of norm 1 for  $\langle \mid \rangle_{U(t_0)}$  is transported to a vector, that has no more norm 1 for  $\langle \mid \rangle_{U(t_1)}$ . During the evolution, the matrix  $U$  defining the scalar product has changed too, and the non hermitian connexion term  $-i\hbar U^{\frac{1}{2}}(t) \frac{\partial}{\partial t} U^{-\frac{1}{2}}(t)$  tracks the change of metric between  $t$  and  $t + \Delta t$ .

Therefore, the time evolution of the state is non-unitary. In the rest of the paper we focus on the diagonalization of the energy operator  $U^{\frac{1}{2}}(t) H U^{-\frac{1}{2}}(t)$  this one being Hermitian, although the diagonalization of  $-i\hbar U^{\frac{1}{2}}(t) \frac{\partial}{\partial t} U^{-\frac{1}{2}}(t)$  is provided for the sake of completeness in appendix B.

### Transformation to the flat space

We thus now focus on the Hamiltonian Eq. (3), and compute explicitly  $U^{\frac{1}{2}} \hat{H} U^{-\frac{1}{2}}$  which as shown is hermitian in the usual sense. The transformation  $U$  is given by

$$U = \sqrt{-g} (h_0^0 + h_\beta^0 \alpha^\beta) = \sqrt{-g} h_0^0 (1 + \alpha^\beta h_\beta^0 / h_0^0)$$

One can then deduce

$$U^{\frac{1}{2}} = f (1 + u_\beta \alpha^\beta)$$

and thus

$$U^{-\frac{1}{2}} = \frac{1}{f(1-u^2)} (1 - u_\beta \alpha^\beta)$$

where  $u^2 = u_\beta u^\beta$  with

$$u_\beta = \frac{h_\beta^0/h_0^0}{\sqrt{-g}h_0^0 \left(1 + \sqrt{1 - \frac{h_\beta^0 h^{0\beta}}{(h_0^0)^2}}\right)}.$$

and

$$f = \left( \frac{1 + \sqrt{1 - \frac{h_\beta^0 h^{0\beta}}{(h_0^0)^2}}}{\sqrt{-g}h_0^0} \right)^{1/2}$$

The greek (lorentzian) indices are assumed now to run only from 1 to 3. As a consequence the Hamiltonian  $H = U^{\frac{1}{2}} \hat{H} U^{-\frac{1}{2}}$  given by Eq. (8) reads

$$H = \frac{1}{2} (H + H^+) + \frac{1}{2} f (1 + u_\beta \alpha^\beta) \left[ H, \frac{1}{f(1-u^2)} (1 - u_\beta \alpha^\beta) \right] - \frac{1}{2} \left[ H^+, (1 - u_\beta \alpha^\beta) \frac{1}{f(1-u^2)} \right] (1 + u_\beta \alpha^\beta) f$$

which can be rewritten as

$$\begin{aligned} H &= \frac{1}{2} (H + H^+) - \frac{1}{2} \frac{(1 + u_\beta \alpha^\beta)}{(1 - u^2)} [H, u_\beta \alpha^\beta] + \frac{1}{2} [H^+, u_\beta \alpha^\beta] \frac{(1 + u_\beta \alpha^\beta)}{(1 - u^2)} \\ &\quad - i \frac{\hbar}{2} (1 + u_\beta \alpha^\beta) (\nabla_{\mathbf{P}} H) \cdot \nabla_{\mathbf{R}} \left( \frac{1}{f(1-u^2)} \right) (1 - u_\beta \alpha^\beta) \\ &\quad + i \frac{\hbar}{2} (1 - u_\beta \alpha^\beta) (\nabla_{\mathbf{P}} H^+) \cdot \nabla_{\mathbf{R}} \left( \frac{1}{f(1-u^2)} \right) (1 + u_\beta \alpha^\beta) \end{aligned}$$

where  $\mathbf{P}$  and  $\mathbf{R}$  are the canonical momentum and position operator satisfying  $[R^i, P_j] = i\hbar \delta_j^i$ . The Hamiltonian can also be written as

$$\begin{aligned} H &= \frac{1}{2} \left( \frac{1}{(1-u^2)} H + H^+ \frac{1}{(1-u^2)} \right) - \frac{1}{2} \frac{1}{(1-u^2)} [H, u_\beta \alpha^\beta] + \frac{1}{2} [H^+, u_\beta \alpha^\beta] \frac{1}{(1-u^2)} \\ &\quad - \frac{1}{2} u_\beta \alpha^\beta \left( \frac{1}{(1-u^2)} H + H^+ \frac{1}{(1-u^2)} \right) u_\beta \alpha^\beta - i \frac{\hbar}{2} \left[ u_\beta \alpha^\beta, (\nabla_{\mathbf{P}} H) \cdot \nabla_{\mathbf{R}} \left( \frac{1}{f(1-u^2)} \right) \right] \end{aligned} \quad (13)$$

At this level, the computation of the last expression turns out to be quite technical is fully developed in Appendix A. The result is given by the following expression

$$\begin{aligned} H &= \frac{1}{2} \alpha^\beta \tilde{H}_\beta^i \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\tilde{\Gamma}_i^{\rho\beta}}{4} \Sigma^\gamma \right) + \frac{1}{2} \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\tilde{\Gamma}_i^{\rho\beta}}{4} \Sigma^\gamma \right) \tilde{H}_\beta^i \alpha^\beta \\ &\quad + \beta \tilde{m} + \frac{1}{2} g^i P_i + P_i g^i + \hbar \left( \tilde{\Gamma}^0 + \Gamma^e \right) \cdot \Sigma + \hbar \frac{(g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) - \mathbf{u} \times \mathbf{H}^i)}{(1-u^2)} \cdot (\nabla_i \mathbf{u}) J \end{aligned} \quad (14)$$

where from now on, all indices  $i, \beta, \rho, \dots$  are only spatial and run from 1 to 3, but roman indices are raised and lowered by the metric  $g_{ij}$  and greek indices by the lorentzian metrics  $\eta_{\alpha\beta}$ . The several notations introduced above are given by

the following expressions

$$\begin{aligned}
\tilde{H}_\beta^i &= H_\beta^i + \frac{2g_{00} \left( h_\delta^0 h_\beta^i - h_\beta^0 h_\delta^i \right) u_\delta}{(1-u^2)} \\
\hat{\Gamma}_i^{\rho\beta} &= \Gamma_i^{\rho\beta} - \frac{1}{4} \varepsilon^{\rho\beta}{}_\gamma (H^{-1})_i{}^\kappa g_{00} h_\delta^0 h_\eta^i \varepsilon^{\delta\eta}{}_\kappa \Gamma_j^{0\gamma} \\
\tilde{\Gamma}_i^{\rho\beta} &= \left( \tilde{H}^{-1} \right)_i{}^\eta H_\eta^j \hat{\Gamma}_j^{\rho\beta} \\
\hat{\Gamma}_\gamma^0 &= \frac{1}{4} \left( \varepsilon_{\rho\beta\gamma} \Gamma_0^{\rho\beta} + g_{00} g^{0i} \varepsilon_{\rho\beta\gamma} \Gamma_i^{\rho\beta} + \varepsilon^\beta{}_{\nu\gamma} H_\beta^i \Gamma_i^{0\nu} \right) \\
\tilde{\Gamma}_\gamma^0 &= \frac{(1+u^2) \delta_\gamma^\eta - u^\eta u_\gamma}{(1-u^2)} \hat{\Gamma}_\eta^0 + \frac{1}{2} \left( \left( 2H_\delta^i \hat{\Gamma}_i^\delta + \Gamma_0^0 - g_{00} g^{0i} \Gamma_i^0 \right) \times \mathbf{u} \right)_\gamma \\
\Gamma_\gamma^e &= - \left( \mathbf{u} \times \mathbf{H}^i \right)_\gamma \left( \frac{\nabla_{R_i} (f(1-u^2))}{f^2 (1-u^2)^3} \right) + \frac{(\mathbf{u} \times \nabla_i (\mathbf{H}^i))_\gamma}{(1-u^2)} - \frac{(1+u^2) \delta_\gamma^\eta - u^\eta u_\gamma}{(1-u^2)} \nabla_{R_i} \left( g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)_\eta \right)
\end{aligned}$$

where we used vectorial notations  $(\mathbf{h}^0)_\alpha = h_\alpha^0$ ,  $(\mathbf{h}^i)_\alpha = h_\alpha^i$ , as well as the vectors  $(\mathbf{H}^i)_\beta \equiv H_\beta^i = g_{00} h_0^0 \left( h_\beta^i - \frac{h_\beta^0 h_0^i}{h_0^0} \right)$ , the following various vectors  $(\tilde{\Gamma}_i^\delta)^\beta = \tilde{\Gamma}_i^{\delta\beta}$ ,  $(\hat{\Gamma}_i^\delta)^\beta = \hat{\Gamma}_i^{\delta\beta}$ ,  $(\Gamma_i^0)^\beta = \Gamma_i^{0\beta}$ , and  $(\Gamma_0^0)^\beta = \Gamma_0^{0\beta}$ . The matrix  $J$  is given by  $J = \begin{pmatrix} 0 & I_{2 \times 2} \\ I_{2 \times 2} & 0 \end{pmatrix}$  and the effective mass by  $\tilde{m} = m g_{00} h_0^0 \left( \frac{1+u^2}{1-u^2} \right)$  as well as  $g^i = g_{00} g^{0i}$ . The expression Eq.(14) for the energy operator  $H$  will be the one to diagonalize in the next section.

### SEMI-CLASSICAL ENERGY

The semiclassical diagonalization the Hamiltonian Eq. (14) is expected to lead to an effective Hamiltonian with gauge fields resulting from the back reaction of the spin degree of freedom (fast) on the translational momentum which can be treated semiclassically for slowly varying enough inhomogeneities. Indeed, the emergence of gauge fields is a general feature of systems providing fast and slow degrees of freedom. The purpose of ref [9], was to investigate the origin of quantum gauge fields and forces by considering the diagonalization of an arbitrary matrix valued quantum Hamiltonian. To be precise, by diagonalization we mean the derivation of an effective in-band Hamiltonian made of block-diagonal energy subspaces. This approach, based on a new differential calculus on a non-commutative space where  $\hbar$  plays the role of running parameter, leads to an in-band energy operator that can be obtained systematically up to arbitrary order in  $\hbar$ . Particularly important for our purpose, it has been possible, for an arbitrary Hamiltonian  $H(\mathbf{R}, \mathbf{P})$  with the canonical coordinates and momentum  $[R_i, P^j] = i\hbar \delta_i^j$ , to obtain the corresponding diagonal representation  $\varepsilon(\mathbf{r}, \mathbf{p})$  to order  $\hbar^2$ , in terms of non-canonical coordinates and momentum  $(\mathbf{r}, \mathbf{p})$  defined later and commutators between gauge fields. The method is quite involved, so that in the present paper we restrict ourself to the semiclassical approximation (order  $\hbar$ ).

The mathematical difficulty in performing the diagonalization of  $H$  comes from the intricate entanglement of noncommuting operators due to the canonical relation  $[R_i, P^j] = i\hbar \delta_i^j$ . In [8] starting with a very general but time independent  $H(\mathbf{R}, \mathbf{P})$  and by considering  $\hbar$  as a running parameter, we related the in-band Hamiltonian  $VHV^+ = \varepsilon(\mathbf{X})$  and the unitary transforming matrix  $V(\mathbf{X})$  (where  $\mathbf{X} \equiv (\mathbf{R}, \mathbf{P})$ ) to their classical expressions through integro-differential operators, i.e.  $\varepsilon(\mathbf{X}) = \hat{O}(\varepsilon(\mathbf{X}_0))$  and  $V(\mathbf{X}) = \hat{N}(V(\mathbf{X}_0))$ , where in the matrices  $\varepsilon(\mathbf{X}_0)$  and  $V(\mathbf{X}_0)$ , the dynamical operators  $\mathbf{X}$  are replaced by classical commuting variables  $\mathbf{X}_0 = (\mathbf{R}_0, \mathbf{P}_0)$ .

The only requirement of the method is therefore the knowledge of  $V(\mathbf{X}_0)$  which gives the diagonal form  $\varepsilon(\mathbf{X}_0)$ . Generally, these equations do not allow to find directly  $\varepsilon(\mathbf{X})$ ,  $V(\mathbf{X})$ , however, they allow us to produce the solution recursively in a series expansion in  $\hbar$ . With this assumption that both  $\varepsilon$  and  $V$  can be expanded in power series of  $\hbar$ , we determined in [8], the explicit  $n$ -th in band energy to order  $\hbar$  for an arbitrary given Hamiltonian

$$\varepsilon_n(\mathbf{r}, \mathbf{p}) = \varepsilon_{0,n}(\mathbf{r}, \mathbf{p}) + \frac{i\hbar}{2} \mathcal{P}_n \left\{ [\varepsilon_0(\mathbf{r}, \mathbf{p}), \mathcal{A}^{R_i}] \mathcal{A}^{P_i} - [\varepsilon_0(\mathbf{r}, \mathbf{p}), \mathcal{A}^{P_i}] \mathcal{A}^{R_i} \right\} + O(\hbar^2) \quad (15)$$

where  $\mathcal{A}_R = i\hbar V \nabla_{\mathbf{P}} V^+$  and  $\mathcal{A}_P = -i\hbar V \nabla_{\mathbf{R}} V^+$  with  $V$  the diagonalizing matrix. The operator  $\mathcal{P}_n$  has the meaning of the projection on the  $n$ -th energy subspace. The new non-canonical dynamical operators  $\mathbf{r}$  and  $\mathbf{p}$  depend on gauge fields  $\mathbf{A}_R = \mathcal{P}_n(\mathcal{A}_R)$  and  $\mathbf{A}_P = \mathcal{P}_n(\mathcal{A}_P)$  similarly to electromagnetism, as we have  $\mathbf{r} = i\hbar \partial_{\mathbf{p}} + \mathbf{A}_R$  and  $\mathbf{p} = \mathbf{P} + \mathbf{A}_P$ .

These gauge invariant quantities not only are emerging naturally but are also necessary to have a gauge invariant energy Eq. (15). The operator  $\varepsilon_0(\mathbf{r}, \mathbf{p})$  is the diagonal energy obtained at zero order ( $\hbar^0$ ) in which the classical variable  $\mathbf{R}_0$  and  $\mathbf{P}_0$  are replaced by new non-commuting operators  $\mathbf{r}, \mathbf{p}$ . Therefore the first step consists in finding the matrix  $V(\mathbf{X}_0)$  which diagonalizes the "classical" Hamiltonian  $H(\mathbf{X}_0)$ .

### Zero Order Diagonalization.

For practical purpose we introduce the three dimensional effective metric  $G^{ij} = \tilde{H}_\alpha^i \tilde{H}_\beta^j \delta^{\alpha\beta}$  as well as the gravity coupled momentum  $\tilde{P}_\alpha = \tilde{H}_\alpha^i (P_i + \frac{\hbar}{4} \varepsilon_{\rho\beta\gamma} \tilde{\Gamma}_i^{\rho\beta} \sigma^\gamma)$ .

As shown in [7], the classical block-diagonalization of the Hamiltonian Eq. (14), but without the last term (corresponding to the static case), can be performed by the following unitary FW-like matrix (denoted  $F_0$  for Foldy-Wouthuysen)

$$F_0(\tilde{\mathbf{P}}) = D \left( E_0 + \tilde{m} + c \frac{1}{2} \beta \left( \alpha \cdot \tilde{\mathbf{P}} + \tilde{\mathbf{P}}^+ \cdot \alpha \right) + N \right) / \sqrt{2E_0(E_0 + \tilde{m})}$$

$$\text{with } E_0 = \sqrt{\left( \frac{\alpha \cdot \tilde{\mathbf{P}} + \tilde{\mathbf{P}}^+ \cdot \alpha}{2} \right)^2 + \tilde{m}^2}, N = \frac{\hbar}{4} \frac{i\alpha \cdot (\tilde{\mathbf{P}} \times (\tilde{\mathbf{r}}^0 + \hbar \mathbf{\Gamma}^e))}{E_0}, \text{ and } D = 1 + \frac{\hbar}{4} \beta \frac{(\tilde{\mathbf{P}} \times (\tilde{\mathbf{r}}^0 + \hbar \mathbf{\Gamma}^e)) \cdot \tilde{\mathbf{P}}}{2E_0^2(E_0 + \tilde{m})}.$$

Indeed one can easily check that

$$\begin{aligned} F_0 H F_0^+ &= \beta \sqrt{P_i G^{ij} P_j + \hbar \varepsilon_{\alpha\beta\gamma} \tilde{\Gamma}_j^{\alpha\beta} \Sigma^\gamma G^{ij} P_i + \tilde{m}^2} \\ &+ \frac{\hbar}{4E_0} \left( \tilde{\mathbf{\Gamma}}^0 + \mathbf{\Gamma}^e \right) \cdot \left( \tilde{m} \mathbf{\Sigma} + \frac{(\mathbf{\Sigma} \cdot \tilde{\mathbf{H}}^i P_i) \tilde{\mathbf{H}}^i P_i}{(E_0 + \tilde{m})} \right) + \frac{1}{2} g^i P_i + \frac{1}{2} P_i g^i \\ &+ F_0 \left( \frac{(g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) - \mathbf{u} \times \mathbf{H}^i)}{(1 - \mathbf{u}^2)} \cdot (\nabla_i \mathbf{u}) J \right) F_0^+ \end{aligned} \quad (16)$$

with  $\tilde{\Gamma}_{i\gamma} = \varepsilon_{\rho\beta\gamma} \tilde{\Gamma}_i^{\rho\beta}$ . All contributions are block-diagonal except the last term which was not present before in the case of a static metrics [7]. The proof of this block diagonalization relies on the simple fact that for classical variables  $\mathbf{X}_0$ , the matrices  $\tilde{H}_\alpha^i$  and  $\tilde{\Gamma}_i^{\alpha\beta}$  are independent of both the momentum and position,  $\beta$  and  $\alpha \cdot \tilde{\mathbf{P}}$  anticommute and in the Taylor expansion of  $E_0$  all terms commute with  $\beta$  and  $\alpha \cdot \tilde{\mathbf{P}} + \tilde{\mathbf{P}}^+ \cdot \alpha$ .

As said before, the last term in Eq.(16) is non diagonal and must be treated specifically. Actually, one can apply a second unitary transformation that will cancel the non diagonal contributions of  $F_0 \left( \frac{(g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) - \mathbf{u} \times \mathbf{H}^i)}{(1 - \mathbf{u}^2)} \cdot (\nabla_i \mathbf{u}) J \right) F_0^+$  without affecting the rest of the diagonalized Hamiltonian to the first order in  $\hbar$ . The explicit form of this transformation is :

$$F'_0 = 1 - \mathcal{P}_- \left( F_0 \frac{(g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) - \mathbf{u} \times \mathbf{H}^i)}{(1 - \mathbf{u}^2)} \cdot (\nabla_i \mathbf{u}) J F_0^+ \right) \frac{\beta}{2\sqrt{P_i G^{ij} P_j + \tilde{m}^2}}$$

$\mathcal{P}_-$  being the projection outside the diagonal. One can check that  $F'_0 - 1$  is antihermitian so that  $F'_0$  is unitary to the first order. As a consequence, the composition of the two unitary transformations yields the following diagonal energy operator :

$$\begin{aligned} \varepsilon_0(\mathbf{R}, \mathbf{P}, t) &= F'_0 F_0 \hat{H}_0 F_0^+ F_0'^+ = \beta \sqrt{P_i G^{ij} P_j + \hbar \varepsilon_{\alpha\beta\gamma} \tilde{\Gamma}_j^{\alpha\beta} \Sigma^\gamma G^{ij} P_i + \tilde{m}^2} \\ &+ \frac{\hbar}{4E_0} \left( \tilde{\mathbf{\Gamma}}^0 + \mathbf{\Gamma}^e \right) \cdot \left( \tilde{m} \mathbf{\Sigma} + \frac{(\mathbf{\Sigma} \cdot \tilde{\mathbf{H}}^i P_i) \tilde{\mathbf{H}}^i P_i}{(E_0 + \tilde{m})} \right) + \frac{1}{2} g^i P_i + \frac{1}{2} P_i g^i \\ &+ \mathcal{P}_+ \left( F_0 \left( \frac{(g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) - \mathbf{u} \times \mathbf{H}^i)}{(1 - \mathbf{u}^2)} \cdot (\nabla_i \mathbf{u}) J \right) F_0^+ \right) \end{aligned} \quad (17)$$

The last term is explicitly given by:

$$\begin{aligned}
& \mathcal{P}_+ \left( F_0 \left( \frac{(g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) - \mathbf{u} \times \mathbf{H}^i)}{(1 - \mathbf{u}^2)} \cdot (\nabla_i \mathbf{u}) J \right) F_0^+ \right) \\
&= D\mathcal{P}_+ \left( E_0 + \tilde{m} + c\beta (\alpha \cdot \tilde{\mathbf{P}}) \right) \left( (E_0 + V(\mathbf{r})m) J + c\beta (\boldsymbol{\Sigma} \cdot \tilde{\mathbf{P}}) \right) \frac{\hbar (g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) - \mathbf{u} \times \mathbf{H}^i) \cdot (\nabla_i \mathbf{u})}{2E_0 (E_0 + \tilde{m})} D^+ \\
&= \hbar c\beta \frac{(g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) - \mathbf{u} \times \mathbf{H}^i) \cdot (\nabla_i \mathbf{u})}{2E_0 (1 - \mathbf{u}^2)} \tilde{\mathbf{P}} \cdot \boldsymbol{\Sigma} = \hbar c\beta \frac{((f-1)g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \cdot (\nabla_i \mathbf{u})}{2fE_0 (1 - \mathbf{u}^2)} \tilde{\mathbf{P}} \cdot \boldsymbol{\Sigma}
\end{aligned}$$

Ultimately, the diagonalization process yields:

$$\begin{aligned}
\varepsilon_0(\mathbf{R}, \mathbf{P}, t) &= F_0' F_0 \hat{H}_0 F_0^+ F_0'^+ \\
&= \beta \sqrt{P_i G^{ij} P_j + \hbar \varepsilon_{\alpha\beta\gamma} \tilde{\Gamma}_j^{\alpha\beta} \Sigma^\gamma G^{ij} P_i + \tilde{m}^2} \\
&\quad + \hbar \left( \tilde{\Gamma}^0 + \Gamma^e \right) \cdot \left( \tilde{m} \boldsymbol{\Sigma} + \frac{(\boldsymbol{\Sigma} \cdot \tilde{\mathbf{H}}^i P_i) \tilde{\mathbf{H}}^i P_i}{(E_0 + \tilde{m})} \right) + \frac{1}{2} g^i P_i + \frac{1}{2} P_i g^i + \hbar c\beta \frac{((f-1)g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \cdot (\nabla_i \mathbf{u})}{2fE_0 (1 - \mathbf{u}^2)} \tilde{\mathbf{P}} \cdot \boldsymbol{\Sigma}
\end{aligned} \tag{18}$$

where for the moment  $\mathbf{R}$  and  $\mathbf{P}$  are treated as classical commuting quantities.

### First order in $\hbar$ diagonalization

From expression Eq. (18) we can deduce the diagonal energy operator for, let say, the particle subspace  $\varepsilon_{0,+}$ . The semiclassical energy is given by Eq. (15), where  $\varepsilon_{0,+}$  corresponds to the positive energy subspace Eq. (18) in which the classical variables  $\mathbf{R}, \mathbf{P}$  are replaced by the quantum covariant ones  $\mathbf{r} = i\hbar\partial_{\mathbf{P}} + \hbar\mathbf{A}_R$  and  $\mathbf{p} = \mathbf{P} + \hbar\mathbf{A}_P$ . The explicit computation for the Berry connections  $\mathbf{A}_R = \mathcal{P}_+(\mathcal{A}_R)$  and  $\mathbf{A}_P = \mathcal{P}_+(\mathcal{A}_P)$  with  $\mathcal{A}_R = \mathcal{P}_+(i\hbar(F_0'F_0)\nabla_{\mathbf{P}}(F_0'F_0)^+)$  and  $\mathcal{A}_P = \mathcal{P}_+(i\hbar(F_0'F_0)\nabla_{\mathbf{R}}(F_0'F_0)^+)$  gives the components

$$A_{R_k} = c^2 \frac{\varepsilon^{\alpha\beta\gamma} \tilde{H}_\gamma^i \tilde{P}_\alpha \Sigma_\beta}{2E(E + \tilde{m})} g_{ik} + o(\hbar) \tag{19}$$

$$A_{P_k} = -c^2 \frac{\varepsilon^{\alpha\beta\gamma} \tilde{P}_\alpha \Sigma_\beta (\nabla_{R_k} \tilde{P}_\gamma)}{2E(E + \tilde{m})} + o(\hbar) \tag{20}$$

where  $E$  is the same as  $E_0$  above, but now  $\mathbf{R}$  is an operator and  $\tilde{H}_\gamma^k$  is the inverse matrix of  $\tilde{H}_\beta^i$ .

We also denote, for the rest of the paper,  $\tilde{\mathbf{p}}$  to be the same expression as  $\tilde{\mathbf{P}}$  in which  $\mathbf{R}$  and  $\mathbf{P}$  have been replaced by  $\mathbf{r}$  and  $\mathbf{p}$ , namely:

$$\tilde{p}_\alpha = \tilde{H}_\alpha^i (p_i + \frac{\hbar}{4} \varepsilon_{\rho\beta\gamma} \tilde{\Gamma}_i^{\rho\beta} \sigma^\gamma)$$

To complete the diagonalization we need to evaluate the quantity

$$M_+ = \frac{i}{2} \mathcal{P}_+ \left\{ [\varepsilon_0(\mathbf{X}), \mathcal{A}_0^{R_i}] \mathcal{A}_0^{P_i} - [\varepsilon_0(\mathbf{X}), \mathcal{A}_0^{P_i}] \mathcal{A}_0^{R_i} \right\} + O(\hbar^2)$$

which being on all point similar to the one given in [6] or [7] is simply stated :

$$M_+ = \frac{1}{E} \left( \left( \frac{1}{2} \boldsymbol{\Sigma} - (\mathbf{A}_R \times \tilde{\mathbf{p}}) \right) \cdot \mathbf{B} - \frac{1}{2E} \nabla \tilde{m}(\mathbf{r}) \cdot (\tilde{\mathbf{p}} \times \boldsymbol{\Sigma}) \right)$$

where the "magnetotorsion field"  $\mathbf{B}$  is defined through a three dimensional effective torsion tensor  $B_\gamma = -\frac{1}{2} P_\delta T^{\delta\alpha\beta} \varepsilon_{\alpha\beta\gamma}$  where the effective torsion  $T^{\delta\alpha\beta}$  is defined as

$$T^{\alpha\beta\delta} = \tilde{H}_k^\delta \left( \tilde{H}^{l\alpha} \partial_l \tilde{H}^{k\beta} - \tilde{H}^{l\beta} \partial_l \tilde{H}^{k\alpha} \right) + \tilde{H}^{l\alpha} \tilde{\Gamma}_l^{\beta\delta} - \tilde{H}^{l\beta} \tilde{\Gamma}_l^{\alpha\delta}$$



The physical origin of the term  $M$  has been discussed in [6] and [7] for the static case. Note that for the static case ( $g^{0i} = 0$ ) we retrieve the true torsion  $T^{\alpha\beta\delta} = H_k^\delta (H^{l\alpha} \partial_l H^{k\beta} - H^{l\beta} \partial_l H^{k\alpha}) + H^{l\alpha} \Gamma_l^{\beta\delta} - H^{l\beta} \Gamma_l^{\alpha\delta}$ .

Considering similarly the anti-particle subspace, it turns out that the coordinate operators are identical with the particle ones and that  $M_- = -M_+$  therefore the full energy operator with both particles and anti particles can be cast in the form

$$\varepsilon(\mathbf{p}, \mathbf{r}) = \beta\tilde{\varepsilon} + \frac{1}{2}g^i p_i + \frac{1}{2}p_i g^i + \frac{\hbar}{4E} (\tilde{\Gamma}^0 + \Gamma^e) \cdot \left( \tilde{m}\Sigma + \frac{(\Sigma \cdot \tilde{H}^i P_i) \tilde{H}^i P_i}{(E + \tilde{m})} \right) + \beta\hbar c \frac{((f-1)g_{00}(\mathbf{h}^0 \times \mathbf{h}^i)) \cdot (\nabla_i \mathbf{u})}{2fE(1-u^2)} \tilde{\mathbf{p}} \cdot \Sigma + \beta\hbar M \quad (21)$$

where  $M = M_+$  and

$$\tilde{\varepsilon} = c \sqrt{\left( p_i + \frac{\hbar}{4E} \tilde{\Gamma}_i \cdot \left( \tilde{m}\Sigma + \frac{(\Sigma \cdot \tilde{\mathbf{p}}) \tilde{\mathbf{p}}}{(E + \tilde{m})} \right) \right)^2 G^{ij} \left( p_i + \frac{\hbar}{4E} \tilde{\Gamma}_i \cdot \left( \tilde{m}\Sigma + \frac{(\Sigma \cdot \tilde{\mathbf{p}}) \tilde{\mathbf{p}}}{(E + \tilde{m})} \right) \right)^2 + \tilde{m}^2}$$

with the vector  $\tilde{\Gamma}_i$  defined in terms of its components as  $(\tilde{\Gamma}_i)_\gamma \equiv \varepsilon_{\alpha\beta\gamma} \tilde{\Gamma}_i^{\alpha\beta}(\mathbf{r})$ . Equation (21) is the main result of this paper.

The coordinates operators in Eq. (21) satisfy a non commutative algebra as

$$[r_i, r_j] = i\hbar^2 \Theta_{ij}^{rr} \quad (22)$$

$$[p_i, p_j] = i\hbar^2 \Theta_{ij}^{pp} \quad (23)$$

$$[p_i, r_j] = -i\hbar g_{ij} + i\hbar^2 \Theta_{ij}^{pr} \quad (24)$$

where  $\Theta_{ij}^{\alpha\beta} = \partial_{\alpha_i} \mathbf{A}_{\beta_j} - \partial_{\beta_j} \mathbf{A}_{\alpha_i} + [\mathbf{A}_{\alpha_i}, \mathbf{A}_{\beta_j}]$  the so-called Berry curvature. An explicit computation gives :

$$\begin{aligned} \Theta_{kl}^{rr} &= \frac{-1}{2E^3} \left( \tilde{m}\Sigma_\gamma + \frac{(\Sigma^\delta \tilde{P}_\delta) \tilde{P}_\gamma}{(E + \tilde{m})} \right) \varepsilon^{\alpha\beta\gamma} \tilde{H}_\alpha^i \tilde{H}_\beta^j g_{ik} g_{jl} \\ \Theta_{ij}^{pp} &= \frac{-1}{2E^3} \left( \tilde{m}\Sigma_\gamma + \frac{(\Sigma^\delta \tilde{P}_\delta) \tilde{P}_\gamma}{(E + \tilde{m})} \right) \nabla_{r_i} \tilde{P}_\alpha \nabla_{r_j} \tilde{P}_\beta \varepsilon^{\alpha\beta\gamma} + \frac{1}{2E^3} \left[ \nabla_{r_i} \tilde{m} \left( (\Sigma \times \tilde{\mathbf{P}}) \cdot \nabla_{r_j} \tilde{\mathbf{P}} \right) - \nabla_{r_j} V(\mathbf{r}) \left( (\Sigma \times \tilde{\mathbf{P}}) \cdot \nabla_{r_i} \tilde{\mathbf{P}} \right) \right] \\ \Theta_{ij}^{pr} &= \frac{1}{2E^3} \left( \tilde{m}\Sigma_\gamma + \frac{(\Sigma^\delta \tilde{P}_\delta) \tilde{P}_\gamma}{(E + \tilde{m})} \right) \varepsilon^{\alpha\beta\gamma} \nabla_{r_i} \tilde{P}_\alpha H_\beta^l g_{jl} - \frac{1}{2E^3} \nabla_{r_i} \tilde{m} (\Sigma \times \tilde{\mathbf{P}})_j \end{aligned} \quad (25)$$

There are also other Berry curvature mixing coordinates and spin

$$\begin{aligned} \Theta_{ij}^{r\Sigma} &= [r_i, \Sigma_j] = ic^2 \frac{-p_j \Sigma_i + \mathbf{p} \cdot \Sigma \delta_{ij}}{E(E + \tilde{m})} \\ \Theta_{ij}^{p\Sigma} &= [p_i, \Sigma_j] = -ic^2 \frac{-p_j \Sigma_i + \mathbf{p} \cdot \Sigma \delta_{ij}}{E(E + \tilde{m})} \tilde{H}_i^\gamma \nabla_{r_i} \tilde{p}_\gamma \end{aligned} \quad (26)$$

Interestingly Eq. (21) can be rewritten as

$$\begin{aligned} \varepsilon &= c\beta \sqrt{\left( p_i - \frac{\hbar}{2} \tilde{\Gamma}_i \cdot \tilde{\Theta}^{rr} \right)^2 G^{ij} \left( p_j - \frac{\hbar}{2} \tilde{\Gamma}_j \cdot \tilde{\Theta}^{rr} \right)^2 + \tilde{m}^2(\mathbf{r}) + \frac{1}{2}g^i p_i + \frac{1}{2}p_i g^i - \frac{\hbar}{2} (\tilde{\Gamma}^0 + \Gamma^e) \cdot \tilde{\Theta}^{rr}} \\ &+ \beta\hbar c \frac{((f-1)g_{00}(\mathbf{h}^0 \times \mathbf{h}^i)) \cdot (\nabla_i \mathbf{u})}{2fE(1-u^2)} \tilde{\mathbf{p}} \cdot \Sigma + \beta\hbar M \end{aligned} \quad (27)$$

where  $\tilde{\Theta}_\gamma^{rr} = \frac{-1}{2E} \left( \tilde{m}\Sigma_\gamma + \frac{(\Sigma^\delta \tilde{P}_\delta) \tilde{P}_\gamma}{(E + \tilde{m})} \right)$  is the "rescaled" Berry curvature. This formula clearly shows that the spin connection couples only to the Berry curvature.

We note in Eq. (27) the presence of the term  $-\frac{\hbar}{2} (\tilde{\Gamma}^0 + \Gamma^e) \cdot \tilde{\Theta}^{rr} + \frac{1}{2}g^i p_i + \frac{1}{2}p_i g^i$  not proportional to  $\beta$  and independant of the particle charge, that is discriminating between particles and anti particles. These terms give

different energy levels for the Dirac particles and antiparticles as  $\varepsilon_+ \neq -\varepsilon_-$ . The coupling  $\frac{-\hbar}{2} (\tilde{\Gamma}^0 + \Gamma^e) \cdot \tilde{\Theta}^{rr}$  proportional to the spin will survive in the case of the non-diagonal static gravitational field but cancels for a diagonal metrics as studied in [2]. On the other hand the term  $\frac{1}{2}g^i p_i + \frac{1}{2}p_i g^i$  vanishes for a static metrics since in this case  $g^{0i} = \partial^0 g^{ij} = 0$  so that  $g^i = 0$ . Therefore the symmetry between particle and antiparticle is reestablished only for static diagonal metrics.

### THE STATIC GRAVITATIONAL FIELD

This case is characterized by the following time independent metric:  $g_{ij} \equiv g_{ij}(\mathbf{R})$ ,  $g_{00} \equiv g_{00}(\mathbf{R})$ ,  $g_{i0} = 0$ . In that case expressions simplify greatly. Actually, the transformation matrix  $U$  is  $U = \sqrt{-gh_0^0}$  and  $U^{\frac{1}{2}} = \sqrt{\sqrt{-gh_0^0}} = f$ . The effective quantities reduce to:

$$\begin{aligned} (\tilde{\mathbf{H}}^i)_\beta &= \tilde{H}_\beta^i = H_\beta^i = g_{00} h_0^0 h_\beta^i = \sqrt{g_{00}} h_\beta^i \\ \tilde{\Gamma}_i^{\rho\beta} &= \hat{\Gamma}_i^{\rho\beta} = \Gamma_i^{\rho\beta} \\ \hat{\Gamma}_\gamma^0 &= \tilde{\Gamma}_\gamma^0 = \frac{1}{4} \left\{ \varepsilon_{\rho\beta\gamma} \Gamma_0^{\rho\beta} + \varepsilon_{\nu\gamma}^\beta H_\beta^i \Gamma_i^{0\nu} \right\} = -\frac{1}{4} g_{00} (h_0^0)^2 \varepsilon_{\nu\gamma}^\beta h_\beta^i h_l^\nu h_i^\lambda \Gamma_\lambda^{kl} \\ \Gamma^e &= 0 \\ G^{ij} &= g_{00} g^{ij} \\ \tilde{m} &= m g_{00} h_0^0 \text{ as } u^\beta = 0 \end{aligned}$$

where  $\Gamma_\lambda^{kl}$  stands for the Christoffel symbol. In this case Eq. (21) reduces to

$$\begin{aligned} \varepsilon &\simeq c\beta \sqrt{\left( p_i + \frac{\hbar}{4E} \Gamma_i(\mathbf{r}) \cdot \left( \tilde{m} \boldsymbol{\Sigma} + \frac{(\boldsymbol{\Sigma} \cdot \tilde{\mathbf{p}}) \tilde{\mathbf{p}}}{(E + \tilde{m})} \right) \right) G^{ij}(\mathbf{r}) \left( p_j + \frac{\hbar}{4E} \Gamma_j(\mathbf{r}) \cdot \left( \tilde{m} \boldsymbol{\Sigma} + \frac{(\boldsymbol{\Sigma} \cdot \tilde{\mathbf{p}}) \tilde{\mathbf{p}}}{(E + \tilde{m})} \right) \right) + \tilde{m}^2 + \hbar\beta M} \\ &+ \frac{\hbar}{4E} \tilde{\Gamma}_\gamma^0 \left( \tilde{m} \Sigma^\gamma + \frac{(\boldsymbol{\Sigma} \cdot \tilde{\mathbf{p}}) \tilde{p}^\gamma}{(E + \tilde{m})} \right) \end{aligned} \quad (28)$$

with  $\tilde{p}_\alpha = \sqrt{g_{00}} h_\alpha^i (p_i + \frac{\hbar}{4} \varepsilon_{\rho\beta\gamma} \Gamma_i^{\rho\beta} \sigma^\gamma)$  and the vector  $\Gamma_i$  defined in terms of its components as  $(\Gamma_i)_\gamma \equiv \varepsilon_{\alpha\beta\gamma} \Gamma_i^{\alpha\beta}(\mathbf{r})$ . Then even for this case, particles and antiparticles avec a different in-band energy operator because of term  $\frac{\hbar}{4E} \tilde{\Gamma}_\gamma^0 \left( \tilde{m} \Sigma^\gamma + \frac{(\boldsymbol{\Sigma} \cdot \tilde{\mathbf{p}}) \tilde{p}^\gamma}{(E + \tilde{m})} \right)$  breaking the symmetry  $\varepsilon_+ \neq -\varepsilon_-$ . Although this term was already in previous studies [6][7] this essential point has not been pointed out. For a diagonal metric  $\tilde{\Gamma}_0 = 0$ , and the symmetry particles/anti-particles is recovered.

### ULTRARELATIVISTIC LIMIT

It is interesting to look at the ultrarelativistic limit  $mc^2 \rightarrow 0$ . One readily obtain

$$\varepsilon \simeq \beta \tilde{\varepsilon} + c\beta \frac{((f-1) g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \cdot (\nabla_i \mathbf{u})}{2f(1-\mathbf{u}^2)} \tilde{\lambda} + \beta \frac{\tilde{\lambda} g_{00} \mathbf{B} \cdot \tilde{\mathbf{p}}}{2E \tilde{p}} + \frac{\tilde{\lambda} \tilde{p}^\gamma (\tilde{\Gamma}_\gamma^0 + \Gamma_\gamma^e)}{4 \tilde{p}} + \frac{1}{2} g^i p_i + \frac{1}{2} p_i g^i \quad (29)$$

with  $\tilde{\varepsilon} = c \sqrt{\left( p_i + \frac{\tilde{\lambda} \tilde{\Gamma}_i(\mathbf{r}) \cdot \mathbf{p}}{p} \right) G^{ij} \left( p_j + \frac{\tilde{\lambda} \tilde{\Gamma}_j(\mathbf{r}) \cdot \mathbf{p}}{p} \right)}$  and  $\tilde{\lambda} = \hbar \tilde{\mathbf{p}} \cdot \boldsymbol{\Sigma} / \tilde{p}$  a biased helicity, that is not projected on the momentum  $\mathbf{p}$  but rather on  $\tilde{\mathbf{p}}$ . This fact is not astonishing. Actually, as we saw in the diagonalization process, the particle is submitted to the action of an effective gravitationnal field, which differs slightly from the initial field. This effective metric is responsible for considering the momentum  $\tilde{\mathbf{p}}$  rather than as a dynamical variable  $\mathbf{P}$ . Nicely this energy can be expressed in terms of the helicity which is the relevant variable for massless particles and not in term of  $\Sigma$ . As shown in [6], Eq. (29) is also valid for photon with the one-half spin matrix  $\boldsymbol{\Sigma}$  replaced with spin one matrix  $\mathbf{S}$ .

Here also we see that photons and anti-photons do not have the same energy spectrum. The symmetry is again only restored for a static diagonal metric.

**THE TIME DEPENDENT SYMMETRIC GRAVITATIONAL FIELD**

A typical example of such a metric is the Schwarzschild space-time in isotropic coordinates. This case, studied in a different manner in [2] and [3] for a time independent metric, received a full independent treatment within our formalism in [6][7]. We now present the equivalent results for a time dependent metric, completed with the spin matrix dynamics. For a symmetric metric, the semiclassical Hamiltonian has the following form [3]

$$H_0 = \frac{1}{2} (\alpha.PF(\mathbf{R}, t) + F(\mathbf{R}, t)\alpha.P) + \beta mV(\mathbf{R}, t) \quad (30)$$

corresponding to the metric  $g_{ij} = \delta_{ij} \left( \frac{V(\mathbf{R}, t)}{F(\mathbf{R})} \right)^2$ ,  $g_{i0} = 0$  and  $g_{00} = V^2(\mathbf{R}, t)$ . We will define  $\phi = \frac{V}{F}$ . In that context the relevant quantities for the diagonalization appear to be :

$$h_i^\beta = \phi \delta_i^\beta, \quad h_\beta^i = \frac{1}{\phi} \delta_\beta^i$$

$$h_0^\beta = V(\mathbf{R}, t) \delta_0^\beta, \quad h_\beta^0 = \frac{1}{V(\mathbf{R}, t)} \delta_\beta^0$$

and

$$\begin{aligned} (\mathbf{H}^i)_\beta &= H_\beta^i = g_{00} h_0^i h_\beta^i = V h_\beta^i \\ (\tilde{\mathbf{H}}^i)_\beta &= \tilde{H}_\beta^i = H_\beta^i \\ \tilde{\Gamma}_i^{\rho\beta} &= \hat{\Gamma}_i^{\rho\beta} = \Gamma_i^{\rho\beta} = \frac{(\partial^\rho \phi h_i^\beta - \partial^\beta \phi h_i^\rho)}{\phi^2} \\ \Gamma_i^{0\nu} &= \hbar \frac{\partial^0 \phi h_i^\nu}{\phi V} \\ \tilde{\Gamma}_\gamma^0 &= \hat{\Gamma}_\gamma^0 = \frac{\hbar}{4} \varepsilon_{\rho\beta\gamma} \frac{F(\mathbf{R})}{V^2(\mathbf{R})} \partial^0 g^{\rho\beta} + \hbar g_{00} h_0^0 \frac{F(\mathbf{R})}{V^2(\mathbf{R})} \left( \partial^0 \left( \frac{V(\mathbf{R})}{F(\mathbf{R})} \right) \right) \varepsilon^\beta_{\nu\gamma} h_\beta^i h_i^\nu = 0 \\ \mathbf{\Gamma}^e &= 0 \\ G^{ij} &= V^2 g^{ij} \end{aligned}$$

Similar computations to the ones performed in the previous section lead to the following expressions for the dynamical variables and the diagonalized Hamiltonian

$$\mathbf{r} = \mathbf{R} - \hbar \frac{F^2(\mathbf{R}, t) \boldsymbol{\Sigma} \times \mathbf{P}}{2E(E + mV(\mathbf{R}))}, \quad \mathbf{p} = \mathbf{P} \quad (31)$$

and the diagonal energy becomes:

$$\varepsilon = \beta \sqrt{F^2(\mathbf{r}, t) \mathbf{P}^2 + \mathbf{P}^2 F^2(\mathbf{r}, t) + mV^2(\mathbf{r}, t)} - \frac{F^3(\mathbf{r}, t)}{2E^2} m \hbar \beta \nabla \phi(\mathbf{r}, t) \cdot (\mathbf{P} \times \boldsymbol{\Sigma}) \quad (32)$$

The Berry curvatures are given by:

$$\Theta_{ij}^{rr} = - \frac{\hbar F^3(\mathbf{r}, t) \varepsilon^{ijk}}{2E^3} \left( m \phi(\mathbf{r}, t) \boldsymbol{\Sigma}_k + \frac{F(\mathbf{r}, t) (\boldsymbol{\Sigma} \cdot \mathbf{P}) \mathbf{P}_k}{E + mV(\mathbf{r})} \right) \quad (33)$$

$$\Theta_{ij}^{pr} = - \frac{\hbar F^3(\mathbf{r}, t)}{2E^3} m \nabla_i \phi(\mathbf{r}) (\boldsymbol{\Sigma} \times \mathbf{P})_j \quad (34)$$

$$\Theta_{ij}^{pp} = 0 \quad (35)$$

and  $\Theta_{ij}^{r\Sigma}$  being unchanged,  $\Theta_{ij}^{p\Sigma} = 0$ , and  $\tilde{\varepsilon} = \sqrt{F^2(\mathbf{r}, t) \mathbf{P}^2 + \mathbf{P}^2 F^2(\mathbf{r}, t) + mV^2(\mathbf{r}, t)}$ .

Note here that the magnetotorsion field  $B = 0$ . From Appendix B, we can also get the non Hermitian contributions the time evolution operator which in this case reads  $i\hbar \frac{\partial}{\partial t} \ln f = \frac{1}{2} \frac{\partial}{\partial t} \ln \sqrt{-g} h_0^0$ .

One can check, after developing  $\mathbf{r}$  as a function of  $\mathbf{R}$  and the Berry phase, that our Hamiltonian coincides, in the weak field approximation, with the one given in [3] when considering the semiclassical limit (order  $\hbar$ ). This also confirms the validity of the Foldy Wouthuysen approach asserted in [3].

### ONLY TIME DEPENDENT METRIC

The metric tensor and the vierbein only depend on time. Therefore we have

$$\begin{aligned}
(\mathbf{H}^i)_\beta &= H_\beta^i = g_{00} h_0^0 \left( h_\beta^i - \frac{h_\beta^0 h_0^i}{h_0^0} \right) \\
(\tilde{\mathbf{H}}^i)_\beta &= \tilde{H}_\beta^i = H_\beta^i + \frac{2g_{00} \left( h_\delta^0 h_\beta^i - h_\beta^0 h_\delta^i \right) u_\delta}{(1 - \mathbf{u}^2)} \\
\hat{\Gamma}_i^{\rho\beta} &= \Gamma_i^{\rho\beta} - \hbar \varepsilon^{\rho\beta}{}_\gamma (H^{-1})_i^\kappa g_{00} h_\delta^0 h_\eta^i \varepsilon^{\delta\eta}{}_\kappa \frac{\Gamma_j^{0\gamma}}{4} \\
\tilde{\Gamma}_i^{\rho\beta} &= \left( \tilde{H}^{-1} \right)_i^\eta H_\eta^j \hat{\Gamma}_j^{\rho\beta} \\
\hat{\Gamma}_\gamma^0 &= \frac{1}{4} \left( \varepsilon_{\rho\beta\gamma} \Gamma_0^{\rho\beta} + g_{00} g^{0i} \varepsilon_{\rho\beta\gamma} \Gamma_i^{\rho\beta} + \varepsilon_{\nu\gamma}^\beta H_\beta^i \Gamma_i^{0\nu} \right) \\
\tilde{\Gamma}_\gamma^0 &= \frac{(1 + \mathbf{u}^2) \delta_\gamma^\eta - u^\eta u_\gamma}{(1 - \mathbf{u}^2)} \hat{\Gamma}_\eta^0 + \left( \left( H_\delta^i \hat{\Gamma}_i^\delta + \frac{\Gamma_0^0}{2} - g_{00} g^{0i} \frac{\Gamma_i^0}{2} \right) \times \mathbf{u} \right)_\gamma \\
\Gamma_\gamma^e &= 0
\end{aligned}$$

so that the energy becomes

$$\varepsilon(\mathbf{p}, \mathbf{r}, t) = \beta \tilde{\varepsilon} + \frac{\hbar}{4E} \tilde{\Gamma}^0 \cdot \left( \tilde{m} \boldsymbol{\Sigma} + \frac{(\boldsymbol{\Sigma} \cdot \tilde{\mathbf{H}}^i p_i) \tilde{\mathbf{H}}^i p_i}{(E + \tilde{m})} \right) + \frac{1}{2} g^i(t) p_i + \frac{1}{2} p_i g^i(t) + \beta \hbar M \quad (36)$$

with

$$\tilde{\varepsilon} = c \sqrt{\left( p_i + \frac{\hbar}{4E} \Gamma_i(\mathbf{r}, t) \cdot \left( \tilde{m} \boldsymbol{\Sigma} + \frac{(\boldsymbol{\Sigma} \cdot \tilde{\mathbf{p}}) \tilde{\mathbf{p}}}{(E + \tilde{m})} \right) \right)^2 G^{ij}(\mathbf{r}, t) \left( p_i + \frac{\hbar}{4E} \Gamma_i(\mathbf{r}, t) \cdot \left( \tilde{m} \boldsymbol{\Sigma} + \frac{(\boldsymbol{\Sigma} \cdot \tilde{\mathbf{p}}) \tilde{\mathbf{p}}}{(E + \tilde{m})} \right) \right)^2 + \tilde{m}^2}$$

which show that particles and antiparticles have a different energy spectrum in this gravitational field.

### CONCLUSION

The semiclassical limit for Dirac particles interacting with a fully general gravitational field was investigated through a first order in  $\hbar$  diagonalization of the Dirac Hamiltonian. This work extends previous ones where only static metrics were considered. The time dependence of the metrics leads to new contributions of the in-band energy operator. In particular we found a coupling term between the linear momentum and the spin, and terms which in general will break the particle - antiparticle symmetry.

As already found by other authors, the time dependence leads also to special features like the non-unitarity of the evolution operator, whose origin can be tracked back to the notion of scalar product in the Hilbert space of wave functions for a time dependent metric. This non-unitarity is unavoidable but we could nevertheless diagonalize the full evolution operator, even though our main focus was to obtain the block-diagonal form of the energy, this one turning out to be Hermitian. In addition, to the very general semiclassical diagonal energy operator, we provided several physically relevant examples.

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## APPENDIX.

### Derivation of Eq. (14).

Let us start with the following development for  $\hat{H}$ :

$$\begin{aligned}
 \hat{H} &= g_{00}h_{\beta}^0\gamma^{\beta}\gamma^{\alpha}\hat{P}_{\alpha} + \frac{\hbar}{4}\varepsilon_{\rho\beta\gamma}\Gamma_0^{\rho\beta}\Sigma^{\gamma} + i\frac{\hbar}{4}\Gamma_0^{0\beta}\alpha^{\beta} + g_{00}h_{\beta}^0\gamma^{\beta}m + \frac{i}{2}\partial_t \ln(-gg^{00}) \\
 &= g_{00}g^{0i}(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}) \\
 &\quad + ig_{00}(h_{\delta}^0h_{\eta}^i\varepsilon^{\delta\eta\kappa})\Sigma_{\kappa}(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}) \\
 &\quad + g_{00}h_0^0\alpha^{\beta}h_{\beta}^i(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}) \\
 &\quad - g_{00}\alpha^{\beta}h_{\beta}^0h_0^i(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}) \\
 &\quad + \frac{\hbar}{4}\varepsilon_{\rho\beta\gamma}\Gamma_0^{\rho\beta}\Sigma^{\gamma} + i\frac{\hbar}{4}\Gamma_0^{0\beta}\alpha^{\beta} + g_{00}h_{\beta}^0\gamma^{\beta}m + \frac{i}{2}\partial_t \ln(-gg^{00})
 \end{aligned}$$

and remark that we can rewrite the four first terms in the following form :

$$\begin{aligned}
 &g_{00}h_0^0\alpha^{\beta}\left(h_{\beta}^i - \frac{h_{\beta}^0h_0^i}{h_0^0}\right)(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}) \\
 &\quad + g_{00}g^{0i}(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}) \\
 &\quad + ig_{00}(h_{\delta}^0h_{\eta}^i\varepsilon^{\delta\eta\kappa})\Sigma_{\kappa}(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma}) \\
 &\quad - \hbar g_{00}(h_{\delta}^0h_{\eta}^i\varepsilon^{\delta\eta\kappa})\alpha_{\kappa}\frac{\Gamma_i^{0\gamma}}{4}\Sigma^{\gamma} \\
 &= g_{00}h_0^0\alpha^{\beta}\left(h_{\beta}^i - \frac{h_{\beta}^0h_0^i}{h_0^0}\right) \\
 &\quad \times \left(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} - \left(\left(h_{\beta}^i - \frac{h_{\beta}^0h_0^i}{h_0^0}\right)^{-1}\right)_{\kappa}^{\kappa} \left(\frac{\hbar}{h_0^0}(h_{\delta}^0h_{\eta}^j\varepsilon^{\delta\eta\kappa})\right) \frac{\Gamma_j^{0\gamma}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}\right) \\
 &\quad + g_{00}g^{0i}(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}) \\
 &\quad + ig_{00}(h_{\delta}^0h_{\eta}^i\varepsilon^{\delta\eta\kappa})\Sigma_{\kappa}(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma}) \\
 &= \alpha^{\beta}H_{\beta}^i\left(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\tilde{\Gamma}_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}\right) + g_{00}g^{0i}(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma} + i\hbar\frac{\Gamma_i^{0\gamma}}{4}\alpha^{\gamma}) \\
 &\quad + ig_{00}(h_{\delta}^0h_{\eta}^i\varepsilon^{\delta\eta\kappa})\Sigma_{\kappa}(P_i + \hbar\varepsilon_{\rho\beta\gamma}\frac{\Gamma_i^{\rho\beta}}{4}\Sigma^{\gamma})
 \end{aligned}$$

where the effective dreibein and spin connection :

$$H_{\beta}^i = g_{00} h_0^0 \left( h_{\beta}^i - \frac{h_{\beta}^0 h_0^i}{h_0^0} \right)$$

$$\hat{\Gamma}_i^{\rho\beta} = \Gamma_i^{\rho\beta} - \hbar \varepsilon^{\rho\beta}_{\gamma} (H^{-1})_i^{\kappa} g_{00} (h_{\delta}^0 h_{\eta}^j \varepsilon^{\delta\eta}_{\kappa}) \frac{\Gamma_j^{\rho\gamma}}{4}$$

have the form claimed in the text.

Now, we compute  $\hat{H} + \hat{H}^+$ , which is the first contribution in Eq.(8) (up to the factors  $\frac{1}{(1-u_{\beta}u^{\beta})}$  that will be skipped constantly in this section for the sake of readability, and reintroduced ultimately) .

$$\begin{aligned} \frac{1}{2} (\hat{H} + \hat{H}^+) &= \frac{1}{2} \alpha^{\beta} H_{\beta}^i \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\hat{\Gamma}_i^{\rho\beta}}{4} \Sigma^{\gamma} \right) + \frac{1}{2} \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\hat{\Gamma}_i^{\rho\beta}}{4} \Sigma^{\gamma} \right) \alpha^{\beta} H_{\beta}^i \\ &+ \frac{1}{2} g_{00} g^{0i} P_i + \frac{1}{2} P_i g_{00} g^{0i} \\ &+ g_{00} g^{0i} (\hbar \varepsilon_{\rho\beta\gamma} \frac{\Gamma_i^{\rho\beta}}{4} \Sigma^{\gamma}) - \hbar \nabla_{R_i} (g_{00} (h_{\delta}^0 h_{\eta}^i \varepsilon^{\delta\eta\kappa})) \Sigma_{\kappa} - g_{00} (h_{\delta}^0 h_{\eta}^i \varepsilon^{\delta\eta\kappa}) \varepsilon_{\kappa}^{\gamma\nu} (\hbar \varepsilon_{\rho\beta\gamma} \frac{\Gamma_i^{\rho\beta}}{4}) \Sigma_{\nu} \\ &- \hbar \varepsilon_{\gamma\nu}^{\beta} H_{\beta}^i \frac{\Gamma_i^{\rho\gamma}}{4} \Sigma^{\nu} + \frac{\hbar}{4} \varepsilon_{\rho\beta\gamma} \Gamma_0^{\rho\beta} \Sigma^{\gamma} + g_{00} h_0^0 \beta m \\ &= \frac{1}{2} \alpha^{\beta} H_{\beta}^i \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\hat{\Gamma}_i^{\rho\beta}}{4} \Sigma^{\gamma} \right) + \frac{1}{2} \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\hat{\Gamma}_i^{\rho\beta}}{4} \Sigma^{\gamma} \right) \alpha^{\beta} H_{\beta}^i + g_{00} h_0^0 \beta m \\ &+ \left( \hat{\Gamma}^0 - \hbar \nabla_{R_i} (g_{00} (h_{\delta}^0 h_{\eta}^i \varepsilon^{\delta\eta}_{\gamma})) \right) \cdot \Sigma + \frac{1}{2} g_{00} g^{0i} P_i + \frac{1}{2} P_i g_{00} g^{0i} \end{aligned}$$

with :

$$\begin{aligned} (\hat{\Gamma}^0)_{\gamma} &= \frac{\hbar}{4} \varepsilon_{\rho\beta\gamma} \Gamma_0^{\rho\beta} + \hbar g_{00} g^{0i} \varepsilon_{\rho\beta\gamma} \frac{\Gamma_i^{\rho\beta}}{4} \\ &+ \hbar g_{00} (h_{\delta}^0 h_{\eta}^i \varepsilon^{\delta\eta\kappa}) \varepsilon_{\kappa\gamma}^{\nu} (\varepsilon_{\rho\beta\nu} \frac{\Gamma_i^{\rho\beta}}{4}) + \hbar \varepsilon_{\nu\gamma}^{\beta} H_{\beta}^i \frac{\Gamma_i^{\rho\nu}}{4} \end{aligned}$$

Introduce, as in the text  $(\mathbf{\Gamma}^0)_{\gamma} = \Gamma_{0\gamma} = \varepsilon_{\rho\beta\gamma} \Gamma_0^{\rho\beta}$ ,  $(\mathbf{\Gamma}^i)_{\gamma} = g_{00} g^{0i} \hbar \varepsilon_{\rho\beta\gamma} \Gamma_i^{\rho\beta}$ , and use the following expression for the spin connection

$$\Gamma_i^{\rho\beta} = h_{\mu}^{\rho} h_{\nu}^{\beta} \left( h_i^{\xi} \Gamma_i^{\mu\nu} - \nabla^{\mu} h_i^{\nu} \right)$$

to show that  $\hbar g_{00} (h_{\delta}^0 h_{\eta}^i \varepsilon^{\delta\eta\kappa}) \varepsilon_{\kappa\gamma}^{\nu} (\varepsilon_{\rho\beta\nu} \frac{\Gamma_i^{\rho\beta}}{4}) = 0$ . As a consequence, we are left with

$$\hat{\Gamma}^0 = \frac{\hbar}{4} (\mathbf{\Gamma}_0 + g_{00} g^{0i} \mathbf{\Gamma}_i) + \hbar \mathbf{H}^i \times \frac{\mathbf{\Gamma}_i^0}{4}$$

as announced.

To compute the other contributions to  $U^{\frac{1}{2}} \hat{H} U^{-\frac{1}{2}}$  in Eq. (8), we need to find the expressions for  $-u_{\beta} \alpha^{\beta} \frac{(H+H^+)}{2} u_{\beta} \alpha^{\beta}$ ,  $\frac{-[H, u_{\beta} \alpha^{\beta}] + [H^+, u_{\beta} \alpha^{\beta}]}{2}$ , and  $-\frac{i(1+u_{\beta} \alpha^{\beta}) \nabla_{\mathbf{P}} H \nabla_{\mathbf{R}} (u_{\beta} u^{\beta})}{2((1-u_{\beta} u^{\beta}))^2} (1-u_{\beta} \alpha^{\beta})$ . We start with the computation of  $u_{\beta} \alpha^{\beta} (\hat{H} + \hat{H}^+) u_{\beta} \alpha^{\beta}$  by decomposing  $(\hat{H} + \hat{H}^+)$  as:

$$\begin{aligned} \frac{(\hat{H} + \hat{H}^+)}{2} &= \frac{(\hat{H}' + \hat{H}'^+)}{2} + \left( \hat{\Gamma}_{\gamma}^0 - \hbar \nabla_{R_i} (g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \right) \Sigma^{\gamma} + g_{00} h_0^0 \beta m \\ &+ \frac{1}{2} g_{00} g^{0i} P_i + \frac{1}{2} P_i g_{00} g^{0i} \end{aligned}$$

where

$$\frac{(\hat{H}' + \hat{H}'^+)}{2} = \frac{1}{2}\alpha^\beta H_\beta^i \left( P_i + \hbar\varepsilon_{\rho\beta\gamma} \frac{\hat{\Gamma}_i^{\rho\beta}}{4} \Sigma^\gamma \right) + \frac{1}{2} \left( P_i + \hbar\varepsilon_{\rho\beta\gamma} \frac{\hat{\Gamma}_i^{\rho\beta}}{4} \Sigma^\gamma \right) \alpha^\beta H_\beta^i$$

For the sake of the computations, we will denote  $\hat{P}_\alpha = H_\alpha^i \left( P_i + \hbar\varepsilon_{\rho\beta\gamma} \frac{\hat{\Gamma}_i^{\rho\beta}}{4} \Sigma^\gamma \right) \equiv H_\alpha^i \hat{P}_i$ , so that

$$\begin{aligned} u_\beta \alpha^\beta \frac{(\hat{H}' + \hat{H}'^+)}{2} u_\beta \alpha^\beta &= \frac{1}{2} u_\beta \alpha^\beta \alpha^\beta H_\beta^i \hat{P}^i u_\beta \alpha^\beta + \frac{1}{2} u_\beta \alpha^\beta \hat{P}^i \alpha^\beta u_\beta \alpha^\beta H_\beta^i \\ &= \frac{1}{2} u_\gamma \alpha^\gamma \alpha^\beta H_\beta^i \hat{P}^i u_\gamma \alpha^\gamma + \frac{1}{2} u_\gamma \alpha^\gamma \hat{P}^i \alpha^\beta u_\gamma \alpha^\gamma H_\beta^i \\ &= \frac{1}{2} \sqrt{g_{00}} u_\gamma \alpha^\gamma \alpha^\beta u_\gamma \alpha^\gamma H_\beta^i \hat{P}^i + \frac{1}{2} \hat{P}^i H_\beta^i u_\gamma \alpha^\gamma \alpha^\beta u_\gamma \alpha^\gamma \\ &\quad + \frac{1}{2} u_\gamma \alpha^\gamma \alpha^\beta H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] + \frac{1}{2} \left[ u_\gamma \alpha^\gamma, \hat{P}^i \right] \alpha^\beta u_\gamma \alpha^\gamma H_\beta^i \sqrt{g_{00}} \\ &= \frac{1}{2} u_\gamma \alpha^\gamma \alpha^\beta u_\gamma \alpha^\gamma H_\beta^i \hat{P}^i + \frac{1}{2} \hat{P}^i H_\beta^i u_\gamma \alpha^\gamma \alpha^\beta u_\gamma \alpha^\gamma \\ &\quad + \frac{1}{2} i u_\gamma \varepsilon^{\gamma\beta\delta} \left( \left[ \Sigma^\delta H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] + H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] \Sigma^\delta \right] \right) \\ &= \frac{1}{2} \alpha^\beta \mathbf{u}^2 H_\beta^i \hat{P}^i + \frac{1}{2} \hat{P}^i H_\beta^i \mathbf{u}^2 \alpha^\beta \\ &\quad + i u_\gamma \varepsilon^{\gamma\beta\delta} \Sigma^\delta u_\gamma \alpha^\gamma H_\beta^i \hat{P}^i - H_\beta^i \hat{P}^i u_\gamma \alpha^\gamma i u_\gamma \varepsilon^{\gamma\beta\delta} \Sigma^\delta \\ &\quad + \frac{1}{2} i u_\gamma \varepsilon^{\gamma\beta\delta} \left( \left[ \Sigma^\delta H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] + H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] \Sigma^\delta \right] \right) \\ &= \frac{1}{2} \alpha^\beta \mathbf{u}^2 H_\beta^i \hat{P}^i + \frac{1}{2} \hat{P}^i H_\beta^i \mathbf{u}^2 \alpha^\beta \\ &\quad + \left[ (\mathbf{u}^2 \Sigma^\beta - (\mathbf{u} \Sigma) u^\beta) H_\beta^i \hat{P}^i - H_\beta^i \hat{P}^i (\mathbf{u}^2 \Sigma^\beta - (\mathbf{u} \Sigma) u^\beta) \right] \\ &\quad + \frac{1}{2} i u_\gamma \varepsilon^{\gamma\beta\delta} \left( \left[ \Sigma^\delta H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] + H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] \Sigma^\delta \right] \right) \end{aligned}$$

Now, given that:

$$\begin{aligned} &i u_\gamma \varepsilon^{\gamma\beta\delta} \left( \left[ \Sigma^\delta H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] + H_\beta^i \left[ \hat{P}^i, u_\gamma \alpha^\gamma \right] \Sigma^\delta \right] \right) \\ &= -\hbar u_\gamma \varepsilon^{\gamma\kappa\delta} \left( \left[ \Sigma^\delta H_\kappa^i \left( -\nabla_i u_\gamma \alpha^\gamma + \frac{\hat{\Gamma}_i^{\rho\beta}}{2} \Sigma^\beta u_\rho \right) + H_\kappa^i \left( -\nabla_i u_\gamma \alpha^\gamma + \frac{\hat{\Gamma}_i^{\rho\beta}}{2} \Sigma^\beta u_\rho \right) \Sigma^\delta \right] \right) \\ &= -\hbar u_\gamma \varepsilon^{\gamma\kappa\delta} H_\kappa^i \left( -2J \nabla_i u_\delta + \hat{\Gamma}_i^{\rho\delta} u_\rho \right) \end{aligned}$$

one thus has,

$$\begin{aligned} u_\beta \alpha^\beta \frac{(\hat{H}' + \hat{H}'^+)}{2} u_\beta \alpha^\beta &= \frac{1}{2} \alpha^\beta \mathbf{u}^2 \hat{P}_\beta + \frac{1}{2} \hat{P}_\beta \mathbf{u}^2 \alpha^\beta - \frac{1}{2} g_{00} h_0^0 \hbar u_\gamma \varepsilon^{\gamma\kappa\delta} H_\kappa^i \left( -2J \nabla_i u_\delta + \hat{\Gamma}_i^{\rho\delta} u_\rho \right) \\ &= \frac{1}{2} \alpha^\beta \mathbf{u}^2 \hat{P}_\beta + \frac{1}{2} \hat{P}_\beta \mathbf{u}^2 \alpha^\beta - \frac{1}{2} g_{00} h_0^0 \hbar u_\gamma \varepsilon^{\gamma\kappa\delta} H_\kappa^i \left( -2J \nabla_i u_\delta \right) \end{aligned}$$

since, by an argument already used,  $u_\gamma \varepsilon^{\gamma\kappa\delta} H_\kappa^i \hat{\Gamma}_i^{\rho\delta} u_\rho \propto (\mathbf{h}^0 \times \mathbf{h}^i)_\delta \hat{\Gamma}_i^{\rho\delta} u_\rho = 0$ .

To complete the computation of  $u_\beta \alpha^\beta \frac{(H+H^+)}{2} u_\beta \alpha^\beta$ , we need to calculate the following contribution:

$$\begin{aligned}
& u_\gamma \alpha^\gamma \left( g_{00} h_0^0 \beta m + \frac{1}{2} g_{00} g^{0i} P_i + \frac{1}{2} P_i g_{00} g^{0i} + \left( \hat{\Gamma}_\gamma^0 - \hbar \nabla_{R_i} (g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \right)_\gamma \Sigma^\gamma \right) u_\gamma \alpha^\gamma \\
&= -g_{00} h_0^0 \beta \mathbf{u}^2 m + \frac{1}{2} g_{00} g^{0i} \mathbf{u}^2 P_i + \frac{1}{2} \mathbf{u}^2 P_i g_{00} g^{0i} + \mathbf{u}^2 \left( \hat{\Gamma}_\gamma^0 - \hbar \nabla_{R_i} (g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \right)_\gamma \Sigma^\gamma \\
&- 2\mathbf{u}^2 \left( \hat{\Gamma}_\gamma^0 - \hbar \nabla_{R_i} (g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \right)_\gamma \Sigma^\gamma + \mathbf{u} \cdot \Sigma \left( \hat{\Gamma}_\gamma^0 - \hbar \nabla_{R_i} (g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \right)_\gamma u_\gamma \\
&= -g_{00} h_0^0 \beta \mathbf{u}^2 m + \frac{1}{2} g_{00} g^{0i} \mathbf{u}^2 P_i + \frac{1}{2} \mathbf{u}^2 P_i g_{00} g^{0i} + \mathbf{u} \cdot \Sigma \left( \hat{\Gamma}_\gamma^0 - \hbar \nabla_{R_i} (g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \right)_\gamma u_\gamma \\
&- \mathbf{u}^2 \left( \hat{\Gamma}_\gamma^0 - \hbar \nabla_{R_i} (g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)) \right)_\gamma \Sigma^\gamma
\end{aligned}$$

Now, we turn our attention toward the second contribution in Eq. (8), namely  $\frac{-[\hat{H}, u_\beta \alpha^\beta] + [\hat{H}^+, u_\beta \alpha^\beta]}{2}$ . To do so, and since

$$-\frac{[\hat{H}, u_\beta \alpha^\beta] + [\hat{H}^+, u_\beta \alpha^\beta]}{2} = -\frac{1}{2} [\hat{H} - \hat{H}^+, u_\beta \alpha^\beta]$$

we only need to concentrate on the anti hermitian part of  $H$ . Given that,

$$\begin{aligned}
\hat{H} &= \alpha^\beta H_\beta^i \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\hat{\Gamma}_i^{\rho\beta}}{4} \Sigma^\gamma + i \hbar \frac{\Gamma_i^{0\gamma}}{4} \alpha^\gamma \right) + g_{00} g^{0i} i \hbar \frac{\Gamma_i^{0\gamma}}{4} \alpha^\gamma \\
&+ i g_{00} (h_\delta^0 h_\eta^i \varepsilon^{\delta\eta\kappa}) \Sigma_\kappa (P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\Gamma_i^{\rho\beta}}{4} \Sigma^\gamma) + i \frac{\hbar}{4} \Gamma_0^{0\beta} \alpha^\beta
\end{aligned}$$

we can write that

$$\begin{aligned}
\frac{\hat{H} - \hat{H}^+}{2} &= i \frac{\hbar}{2} \nabla_i (\alpha^\beta H_\beta^i) + i H_\beta^i \hbar \frac{\Gamma_i^{0\beta}}{4} + g_{00} g^{0i} i \hbar \frac{\Gamma_i^{0\beta}}{4} \alpha^\beta - i \frac{\hbar}{2} H_\rho^i \alpha_\beta \hat{\Gamma}_i^{\beta\rho} \\
&+ \frac{i}{2} g_{00} (h_\delta^0 h_\eta^i \varepsilon^{\delta\eta\kappa}) \Sigma_\kappa P_i + \frac{i}{2} P_i g_{00} (h_\delta^0 h_\eta^i \varepsilon^{\delta\eta\kappa}) \Sigma_\kappa + i g_{00} (h_\delta^0 h_\eta^i) \hbar \frac{\Gamma_i^{\delta\eta}}{2} + i \frac{\hbar}{4} \Gamma_0^{0\beta} \alpha^\beta
\end{aligned}$$

and

$$\begin{aligned}
-\frac{[\hat{H}, u_\beta \alpha^\beta] + [\hat{H}^+, u_\beta \alpha^\beta]}{2} &= -\hbar \left( \nabla_i (H_\beta^i) + g_{00} g^{0i} \frac{\Gamma_i^{0\beta}}{2} + \frac{\Gamma_0^{0\beta}}{2} - H_\rho^i \hat{\Gamma}_i^{\beta\rho} \right) \varepsilon^{\beta\gamma\kappa} u_\gamma \Sigma_\kappa \\
&+ g_{00} (h_\delta^0 h_\eta^i - h_\eta^0 h_\delta^i) u_\delta \alpha_\eta P_i + P_i g_{00} (h_\delta^0 h_\eta^i - h_\eta^0 h_\delta^i) u_\delta \alpha_\eta \\
&- g_{00} \hbar (h_\delta^0 h_\eta^i \varepsilon^{\delta\eta\kappa}) J \nabla_i u_\kappa
\end{aligned}$$

Ultimately, we need the third contribution to Eq. (8):

$$\begin{aligned}
& -i \frac{\hbar}{2} \left[ u_\beta \alpha^\beta, (\nabla_{\mathbf{P}} \hat{H}) \cdot \nabla_{\mathbf{R}} \left( \frac{1}{f(1 - u_\beta u^\beta)} \right) \right] \\
&= i \frac{\hbar}{2} \left[ u_\beta \alpha^\beta, \alpha^\beta H_\beta^i \cdot \left( \frac{\nabla_{R_i} (f(1 - u_\beta u^\beta))}{[f(1 - u_\beta u^\beta)]^2} \right) \right] \\
&= -\hbar \Sigma \cdot (\mathbf{u} \times \mathbf{H}^i) \left( \frac{\nabla_{R_i} (f(1 - \mathbf{u}^2))}{[f(1 - \mathbf{u}^2)]^2} \right)
\end{aligned}$$

We can now gather all these terms to obtain the expression of  $U^{\frac{1}{2}} \hat{H} U^{-\frac{1}{2}}$ .

Define, as in text, the vectors  $\tilde{\Gamma}_i^\delta, \Gamma_i^0$  by:

$$\left( \tilde{\Gamma}_i^\delta \right)^\beta = \tilde{\Gamma}_i^{\delta\beta}, \left( \Gamma_i^0 \right)^\beta = \Gamma_i^{0\beta}, \left( \Gamma_0^0 \right)^\beta = \Gamma_0^{0\beta}$$



and  $\tilde{H}_\beta^i$ :

$$\tilde{H}_\beta^i = H_\beta^i + \frac{2g_{00} \left( h_\delta^0 h_\beta^i - h_\beta^0 h_\delta^i \right) u_\delta}{(1 - \mathbf{u}^2)}$$

$$\tilde{\Gamma}_i^{\alpha\beta} = \left( \tilde{H}^{-1} \right)_i^\eta H_\eta^j \hat{\Gamma}_j^{\alpha\beta}$$

Ultimately, reintroducing the factors  $\frac{1}{(1-\mathbf{u}^2)}$  when needed in Eq. (8),  $U^{\frac{1}{2}} \hat{H} U^{-\frac{1}{2}}$  can be written in a compact form :

$$\begin{aligned} U^{\frac{1}{2}} \hat{H} U^{-\frac{1}{2}} &= \frac{1}{2} \alpha^\beta \tilde{H}_\beta^i \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\tilde{\Gamma}_i^{\rho\beta}}{4} \Sigma^\gamma \right) + \frac{1}{2} \left( P_i + \hbar \varepsilon_{\rho\beta\gamma} \frac{\tilde{\Gamma}_i^{\rho\beta}}{4} \Sigma^\gamma \right) \tilde{H}_\beta^i \alpha^\beta \\ &+ g_{00} h_0^0 \frac{(1 + \mathbf{u}^2)}{(1 - \mathbf{u}^2)} \beta m + \frac{1}{2} g_{00} g^{0i} P_i + \frac{1}{2} P_i g_{00} g^{0i} \\ &+ \left( \tilde{\Gamma}_\gamma^0 + \Gamma^e \right) \cdot \Sigma - \hbar \frac{(g_{00} (\mathbf{h}^0 \times \mathbf{h}^i) + g_{00} h_0^0 \mathbf{u} \times \mathbf{H}^i)}{(1 - \mathbf{u}^2)} \cdot (\nabla_i \mathbf{u}) J \end{aligned}$$

where we have defined  $\tilde{\Gamma}_\gamma^0$  by :

$$\tilde{\Gamma}_\gamma^0 = \frac{(1 + \mathbf{u}^2)}{(1 - \mathbf{u}^2)} \frac{\delta_\gamma^\eta - u^\eta u_\gamma}{f} \hat{\Gamma}_\eta^0 + \hbar \left( \left( H_\delta^i \hat{\Gamma}_i^\delta + \frac{\Gamma_0^0}{2} - g_{00} g^{0i} \frac{\Gamma_i^0}{2} \right) \times \mathbf{u} \right)_\gamma$$

$$\Gamma_\gamma^e = -\hbar (\mathbf{u} \times \mathbf{H}^i)_\gamma \left( \frac{\nabla_{R_i} (f (1 - \mathbf{u}^2))}{f^2 (1 - \mathbf{u}^2)^3} \right) + \hbar \frac{(\mathbf{u} \times \nabla_i (\mathbf{H}^i))_\gamma}{(1 - \mathbf{u}^2)} - \hbar \frac{(1 + \mathbf{u}^2)}{(1 - \mathbf{u}^2)} \frac{\delta_\gamma^\eta - u^\eta u_\gamma}{f} \nabla_{R_i} \left( g_{00} (\mathbf{h}^0 \times \mathbf{h}^i)_\eta \right)$$

#### Non Hermitian contributions of the time evolution operator

We compute here the contributions to the diagonalization due to the non-hermitian term  $-i\hbar U^{\frac{1}{2}}(t) \frac{\partial}{\partial t} U^{-\frac{1}{2}}(t)$ . First we obtain

$$-i\hbar U^{\frac{1}{2}}(t) \frac{\partial}{\partial t} U^{-\frac{1}{2}}(t) = i\hbar \frac{\frac{\partial}{\partial t} [f (1 - \mathbf{u}^2)]}{f (1 - \mathbf{u}^2)} + i\hbar \frac{(1 + u_\beta \frac{\partial}{\partial t} u_\beta)}{(1 - \mathbf{u}^2)} + i\hbar \frac{\frac{\partial}{\partial t} u_\beta \alpha^\beta}{(1 - \mathbf{u}^2)} - \hbar \frac{(\mathbf{u} \times \frac{\partial}{\partial t} \mathbf{u}) \cdot \Sigma}{(1 - \mathbf{u}^2)} \quad (37)$$

and then, the contributions of the terms in Eq.(37) to the diagonalization are computed by applying the transformation matrix and then projecting on the diagonal blocks. We are left with:

$$\begin{aligned} U \left( \tilde{\mathbf{P}} \right) \left[ -i\hbar U^{\frac{1}{2}}(t) \frac{\partial}{\partial t} U^{-\frac{1}{2}}(t) \right] U \left( \tilde{\mathbf{P}} \right)^+ &= i\hbar \frac{\frac{\partial}{\partial t} [f (1 - \mathbf{u}^2)]}{f (1 - \mathbf{u}^2)} + i\hbar \frac{(1 + \mathbf{u} \cdot \frac{\partial}{\partial t} \mathbf{u})}{(1 - \mathbf{u}^2)} + i\hbar c\beta \frac{\frac{\partial}{\partial t} \mathbf{u} \cdot \tilde{\mathbf{P}}}{(1 - \mathbf{u}^2) E_0} \\ &- \frac{\hbar c^2 \left( \tilde{\mathbf{P}} \times \left( \tilde{\mathbf{P}} \times (\mathbf{u} \times \frac{\partial}{\partial t} \mathbf{u}) \right) \cdot \Sigma \right)}{(1 - \mathbf{u}^2) E_0 (E_0 + \tilde{m})} \end{aligned}$$