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# CARLESON MEASURES FOR LOCAL DIRICHLET SPACES

EMMANUEL FRICAIN<sup>1</sup> AND MAHMOOD SHABANKHAH<sup>2</sup>

ABSTRACT. In this paper, using a generalization of Richter-Sundberg representation theorem, we give a complete characterization of Carleson measures for the local Dirichlet space  $\mathcal{D}(\mu)$  when  $\mu$  is finitely atomic. A reproducing kernel thesis result is also established in the case when  $\mu$  is a point mass.

## 1. INTRODUCTION

The classical Dirichlet space  $\mathcal{D}$  consists of holomorphic functions  $f$  in the unit disk  $\mathbb{D}$ , with finite Dirichlet integral, denoted by  $\mathcal{D}(f)$ , and given by

$$\mathcal{D}(f) := \int_{\mathbb{D}} |f'(z)|^2 dA(z),$$

where  $dA$  is the normalized area measure. It is easily verified that  $\mathcal{D}$  is a subspace of the Hardy space  $H^2(\mathbb{D})$ , i.e., the space of holomorphic functions  $f(z) = \sum_n a_n z^n$  in  $\mathbb{D}$  for which

$$\|f\|_2^2 := \sum_{n \geq 0} |a_n|^2 < \infty.$$

In particular,  $\mathcal{D}$  turns into a Hilbert space under the following natural norm

$$\|f\|_{\mathcal{D}}^2 := \|f\|_2^2 + \mathcal{D}(f).$$

It is an immediate consequence of the above norm definitions that  $M_z$ , the operator of multiplication by the independent variable  $z$ , is an isometry on  $H^2$  but not on  $\mathcal{D}$ . In fact, one can verify that

$$\|z^2 f\|_{\mathcal{D}}^2 - \|zf\|_{\mathcal{D}}^2 = \|zf\|_{\mathcal{D}}^2 - \|f\|_{\mathcal{D}}^2, \quad (f \in \mathcal{D}).$$

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A bounded operator  $T$  on a complex separable Hilbert space  $H$  is called a two-isometry if, for every  $x \in H$ ,

$$\|T^2x\|^2 - \|Tx\|^2 = \|Tx\|^2 - \|x\|^2.$$

If  $\bigcap_{n \geq 0} T^n(H) = \{0\}$ ,  $T$  is said to be analytic. In particular,  $M_z$  is an analytic two-isometry on  $\mathcal{D}$ . It is a remarkable result of Richter [8] that every analytic two-isometry satisfying  $\dim \text{Ker} T^* = 1$  is unitarily equivalent to  $M_z$  on some Dirichlet-type space  $\mathcal{D}(\mu)$ . These spaces are also called local Dirichlet spaces.

Let  $\mu$  be a positive finite Borel measure on the unit circle  $\mathbb{T}$ , and let  $P_\mu$  be its harmonic extension to  $\mathbb{D}$ , i.e.,

$$P_\mu(z) = \int_{\mathbb{T}} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta), \quad (z \in \mathbb{D}).$$

The corresponding  $\mathcal{D}(\mu)$  space is defined to be the set of holomorphic functions  $f$  for which

$$\mathcal{D}_\mu(f) := \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) dA(z) < \infty.$$

In particular, if  $\mu$  is taken to be the normalized Lebesgue measure on  $\mathbb{T}$ , then  $P_\mu(z) = 1$ ,  $z \in \mathbb{D}$ , and therefore  $\mathcal{D}(\mu) = \mathcal{D}$ , the classical Dirichlet space. The quantity  $\mathcal{D}_\mu(f)$  is called the Dirichlet integral of  $f$  with respect to  $\mu$ . If  $\mu = \delta_\lambda$ , i.e., the unit point mass at  $\lambda \in \mathbb{T}$ , one writes  $\mathcal{D}_\lambda(f)$  instead of  $\mathcal{D}_\mu(f)$ , and calls it the local Dirichlet integral of  $f$  at  $\lambda$ . By Fubini's Theorem, we have

$$(1.1) \quad \mathcal{D}_\mu(f) = \int_{\mathbb{T}} \mathcal{D}_\lambda(f) d\mu(\lambda).$$

In [9], Richter and Sundberg proved that  $f \in \mathcal{D}(\delta_\lambda)$  if and only if  $f = c + (z - \lambda)g$ , where  $c$  is a constant and  $g \in H^2$ . In this case,  $c$  is the non-tangential (even oricyclic) limit of  $f$  at  $\lambda$ , denoted by  $f(\lambda)$ , and  $\|g\|_2^2 = \mathcal{D}_\lambda(f)$ . In other words,

$$\mathcal{D}_\lambda(f) = \left\| \frac{f - f(\lambda)}{z - \lambda} \right\|_2^2.$$

A different proof of these facts can be found in [11]. Since  $\mathcal{D}(\delta_\lambda) \subset H^2$ , it follows from (1.1) that  $\mathcal{D}(\mu) \subset H^2$ . The norm, with respect to which  $\mathcal{D}(\mu)$  is a Hilbert space, is given by

$$(1.2) \quad \|f\|_\mu^2 := \|f\|_2^2 + \mathcal{D}_\mu(f).$$

Moreover, polynomials form a dense subset of  $\mathcal{D}(\mu)$  [8].

The Hardy space  $H^2$  is an example of a reproducing kernel Hilbert space, i.e., the point evaluation  $f \mapsto f(z)$  is a bounded functional on  $H^2$ , for every  $z \in \mathbb{D}$ . In particular,

$$f(z) = \langle f, k_z \rangle,$$

for some  $k_z \in H^2$ . One calls  $k_z$  the reproducing kernel at  $z$ . According to (1.2), the point evaluation  $f \mapsto f(z)$  is also a bounded functional on  $\mathcal{D}(\mu)$ , for every  $z \in \mathbb{D}$ . Now, if  $\lambda \in \mathbb{T}$ , using the uniform boundedness principle, we easily see that the functional  $f \mapsto f(\lambda)$  is bounded on  $\mathcal{D}(\delta_\lambda)$ .

An important notion in the theory of complex function spaces is that of Carleson measures. Given a Banach space  $X$  of holomorphic functions in  $\mathbb{D}$ , one says that  $\nu$ , a positive Borel measure on  $\mathbb{D}$ , is a Carleson measure for  $X$  if the identity map  $I : X \rightarrow L^2(\nu)$  is a bounded operator, i.e., there is a constant  $C > 0$  such that

$$(1.3) \quad \int_{\mathbb{D}} |f|^2 d\nu \leq C \|f\|_X^2,$$

for every function  $f$  in  $X$ . If the embedding happens to be compact, then  $\nu$  is called a compact (or vanishing) Carleson measure.

Carleson measures have proved to be objects of fundamental importance in the development of modern function theory. In particular, they have appeared in areas ranging from the celebrated Corona problem and its solution by Carleson [2], to the development of bounded mean oscillation (BMO) functions by C. Fefferman and E. Stein [4], P. Jones [7] and many others.

In the setting of the Hardy space  $H^2$ , a geometric characterization of the corresponding measures was obtained by Carleson, and is as follows. In fact,  $\nu$  is a Carleson measure for  $H^2$  if and only if

$$\nu(S(\zeta, h)) = O(h), \quad (h \rightarrow 0^+),$$

where, for  $\zeta \in \mathbb{T}$  and  $0 < h < 1$ , the set  $S(\zeta, h)$  is given by

$$(1.4) \quad S(\zeta, h) := \left\{ z \in \mathbb{D} : 1 - h < |z| < 1 \text{ and } \left| \frac{z}{|z|} - \zeta \right| < \frac{h}{2} \right\}.$$

See for example [5] for a proof of this result. As one might expect, the characterization of compact Carleson measures for  $H^2$  is obtained by replacing “O” in the above condition by “o”.

Given a linear operator  $T$  on a reproducing kernel Hilbert space, one may ask whether some basic properties of  $T$  (e.g. boundedness or compactness) can be determined by testing  $T$  on the set of reproducing kernels. In general, there is no reason why this should be true but it turns out, as was proved by Carleson, that this is indeed the case

for the identity map  $I : H^2 \rightarrow L^2(\nu)$ . More explicitly,  $I$  is bounded (compact, respectively) on  $H^2$  if and only if it acts as a bounded (compact, respectively) operator on the set  $\{k_z, z \in \mathbb{D}\}$ . This property is generally referred to as reproducing kernel thesis.

In this paper, we give a complete characterization of Carleson measures for  $\mathcal{D}(\mu)$  when  $\mu$  is finitely atomic. As we will see, in this case, Carleson measures for  $\mathcal{D}(\mu)$  are determined, in a very specific way, from those of  $H^2$ . The key idea is a generalization of Richter-Sundberg representation theorem which will be discussed in the following section. In the last section, a reproducing kernel thesis result is established for the space  $\mathcal{D}(\mu)$ , in the case when  $\mu$  is a point mass.

## 2. A REPRESENTATION THEOREM

The following result extends Richter-Sundberg representation theorem [9] to the case of finitely atomic measures.

**Theorem 2.1.** *Let  $\mu = \sum_{j=1}^n \alpha_j \delta_{\lambda_j}$  be a finitely atomic measure,  $\alpha_j > 0$ ,  $\lambda_j \in \mathbb{T}$ ,  $1 \leq j \leq n$ . Then  $f \in \mathcal{D}(\mu)$  if and only if there exist  $g \in H^2$  and a polynomial  $p$ ,  $\deg p \leq n - 1$ , such that*

$$(2.1) \quad f = p + \prod_{j=1}^n (z - \lambda_j)g.$$

Moreover,  $g$  and  $p$  are unique, and we have

$$(2.2) \quad \|g\|_2 \leq C \|f\|_\mu,$$

for some positive constant  $C = C(n, \mu)$ .

*Proof.* If  $f$  is given by (2.1), then  $f \in \mathcal{D}(\mu)$ . Indeed, we have

$$\prod_{j=1}^n (z - \lambda_j)g \in \bigcap_{j=1}^n \mathcal{D}(\delta_{\lambda_j}) = \mathcal{D}(\mu).$$

Now let  $f \in \mathcal{D}(\mu)$ . We will show that  $f$  has a representation of form (2.1). The case  $n = 1$  being already established in [9], we consider first the case  $n = 2$ . Let  $f \in \mathcal{D}(\mu)$ . Suppose that  $f(\lambda_1) = f(\lambda_2) = 0$ . Since  $f \in \mathcal{D}(\delta_{\lambda_1}) \cap \mathcal{D}(\delta_{\lambda_2})$ , we have

$$f = (z - \lambda_1)g_1 = (z - \lambda_2)g_2,$$

for some  $g_1, g_2 \in H^2$ . A simple calculation shows that  $g_1/(z - \lambda_2) \in H^2$ . In particular,  $f$  can be written as

$$f = (z - \lambda_1)(z - \lambda_2)g,$$

where  $g \in H^2$ . Using induction, we get that if  $f \in \mathcal{D}(\mu)$ , with  $f(\lambda_j) = 0$ ,  $1 \leq j \leq n$ , then

$$f = (z - \lambda_1) \cdots (z - \lambda_n)g,$$

for some  $g \in H^2$ . For the general case, we choose a polynomial  $p$ , of degree less than or equal to  $n - 1$ , such that

$$p(\lambda_j) = f(\lambda_j), \quad (1 \leq j \leq n).$$

Upon applying the preceding argument to  $f - p$ , we obtain (2.1).

For the uniqueness in the decomposition (2.1), we just note that the polynomial  $p$  is necessarily given by

$$(2.3) \quad p(z) = \sum_{j=1}^n \prod_{1 \leq k \leq n, k \neq j} \frac{z - \lambda_k}{\lambda_j - \lambda_k} f(\lambda_j).$$

It remains to prove the norm inequality (2.2). Since the evaluation functionals  $f \mapsto f(\lambda_j)$  are bounded,  $1 \leq j \leq n$ , we see that

$$\begin{aligned} \|p\|_\mu^2 &= \|p\|_2^2 + \int_{\mathbb{D}} |p'(z)|^2 P_\mu(z) dA(z) \\ &\leq C \|f\|_\mu^2, \end{aligned}$$

where  $C$  is a constant depending on  $n$  and  $\mu$ . In particular, by the triangle inequality,

$$\left\| \prod_{j=1}^n (z - \lambda_j)g \right\|_\mu \leq C \|f\|_\mu.$$

To complete the proof of (2.2) we need to show that

$$\|g\|_2 \leq C \left\| \prod_{j=1}^n (z - \lambda_j)g \right\|_\mu.$$

This is obvious if  $n = 1$ . In fact, given  $\mu = \alpha_1 \delta_{\lambda_1}$ , we have by (1.1) and (1.2),

$$\begin{aligned} \|(z - \lambda_1)g\|_\mu^2 &= \|(z - \lambda_1)g\|_2^2 + \mathcal{D}_\mu((z - \lambda_1)g) \\ &= \|(z - \lambda_1)g\|_2^2 + \alpha_1 \mathcal{D}_{\lambda_1}((z - \lambda_1)g) \\ &= \|(z - \lambda_1)g\|_2^2 + \alpha_1 \|g\|_2^2 \geq \alpha_1 \|g\|_2^2. \end{aligned}$$

For  $n = 2$ , we have

$$\begin{aligned} |\lambda_1 - \lambda_2|^2 \|g\|_2^2 &\leq 2 \left( \|(z - \lambda_1)g\|_2^2 + \|(z - \lambda_2)g\|_2^2 \right) \\ &\leq 2 \left\| \prod_{j=1}^2 (z - \lambda_j)g \right\|_\mu^2. \end{aligned}$$

The general case follows by induction.  $\square$

### 3. CARLESON MEASURES

Theorem 2.1 enables us to give a complete description of Carleson measures in Dirichlet spaces induced by finitely atomic measures.

**Theorem 3.1.** *Let  $\mu = \sum_{j=1}^n \alpha_j \delta_{\lambda_j}$  be a finitely atomic measure,  $\alpha_j > 0$ ,  $\lambda_j \in \mathbb{T}$ ,  $1 \leq j \leq n$ . A finite positive Borel measure  $\nu$  on  $\mathbb{D}$  is a Carleson measure for  $\mathcal{D}(\mu)$  if and only if  $\prod_{j=1}^n |z - \lambda_j|^2 d\nu(z)$  is a Carleson measure for  $H^2$ .*

*Proof.* Assume that  $\nu$  is a Carleson measure for  $\mathcal{D}(\mu)$ . Since for every  $g \in H^2$ ,  $f = \prod_{j=1}^n (z - \lambda_j)g \in \mathcal{D}(\mu)$ , we have

$$\int_{\mathbb{D}} \prod_{j=1}^n |z - \lambda_j|^2 |g|^2 d\nu(z) \leq C \|f\|_\mu^2 \leq C' \|g\|_2^2,$$

where the last inequality follows easily from the definition of norm in  $\mathcal{D}(\mu)$ . Consequently,  $\prod_{j=1}^n |z - \lambda_j|^2 d\nu(z)$  is a Carleson measure for  $H^2$ .

Conversely, if  $\prod_{j=1}^n |z - \lambda_j|^2 d\nu(z)$  is a Carleson measure for  $H^2$ , we will show that  $\nu$  is a Carleson measure for  $\mathcal{D}(\mu)$ . Let  $f \in \mathcal{D}(\mu)$ . By Theorem 2.1,

$$f = p + \prod_{j=1}^n (z - \lambda_j)g,$$

where  $g \in H^2$  and  $p$  is given by (2.3). Since the evaluation functionals  $f \mapsto f(\lambda_j)$ ,  $1 \leq j \leq n$ , are bounded on  $\mathcal{D}(\mu)$ , we see that

$$\begin{aligned} \int_{\mathbb{D}} |f|^2 d\nu &\leq C \left( \int_{\mathbb{D}} |p|^2 d\nu + \int_{\mathbb{D}} \prod_{j=1}^n |z - \lambda_j|^2 |g|^2 d\nu \right) \\ &\leq C (\|f\|_\mu^2 + \|g\|_2^2). \end{aligned}$$

An appeal to (2.2) yields the desired conclusion.  $\square$

*Example 3.2.* It is clear that every Carleson measure for  $H^2$  is a Carleson measure for  $\mathcal{D}(\mu)$  as well. The converse, however, is not true. Let  $\alpha \in (0, 1)$ , and let  $d\nu(z) = (1 - |z|)^{-\alpha} dm(z)$ , where  $dm$  is the Lebesgue measure on  $[0, 1]$ . Since  $dm$  is a Carleson measure for  $H^2$ , the preceding result implies that  $d\nu$  is indeed a Carleson measure for  $\mathcal{D}(\delta_1)$ . We claim that  $d\nu$  is not a Carleson measure for  $H^2$ . For  $\zeta \in \mathbb{T}$  and  $0 < h < 1$ , consider  $S(\zeta, h)$  as given by (1.4). If  $S(\zeta, h) \cap [0, 1] \neq \emptyset$ , then

$$\nu\left(S(\zeta, h)\right) = \int_{1-h}^1 \frac{dr}{(1-r)^\alpha} = \frac{h^{1-\alpha}}{1-\alpha},$$

which proves the claim because

$$\lim_{h \rightarrow 0^+} \frac{\nu\left(S(\zeta, h)\right)}{h} = \infty.$$

In fact, given  $0 < \beta < 1$ , if we choose  $\alpha$  such that  $1 - \beta < \alpha < 1$ , then the same example gives a Carleson measure  $\nu$  for  $\mathcal{D}(\delta_1)$  which satisfies  $\nu(S(\zeta, h))/h^\beta \rightarrow +\infty$ , as  $h \rightarrow 0^+$ . That means that we cannot translate directly our characterization of Carleson measures for  $\mathcal{D}(\mu)$  in terms of the asymptotic of  $\nu(S(\zeta, h))$ .

As the following result shows, compact Carleson measures for  $\mathcal{D}(\mu)$  are characterized in a similar manner.

**Theorem 3.3.** *Let  $\mu = \sum_{j=1}^n \alpha_j \delta_{\lambda_j}$  be a finitely atomic measure. A finite positive Borel measure  $\nu$  is a compact Carleson measure for  $\mathcal{D}(\mu)$  if and only if  $\prod_{j=1}^n |z - \lambda_j|^2 d\nu(z)$  is a compact Carleson measure for  $H^2$ .*

*Proof.* Suppose that  $\nu$  is a compact Carleson measure for  $\mathcal{D}(\mu)$ . Let  $(g_k)_{k \geq 1} \subset H^2$  such that  $(g_k)_k$  converges weakly to zero, as  $k \rightarrow \infty$ . Put  $f_k = \prod_{j=1}^n (z - \lambda_j) g_k$ ,  $k \geq 1$ . Since the operator  $T : H^2 \rightarrow \mathcal{D}(\mu)$  of multiplication by  $\prod_{j=1}^n (z - \lambda_j)$  is bounded, it follows that  $f_k \rightarrow 0$  weakly in  $\mathcal{D}(\mu)$ . So, the assumption that  $I : \mathcal{D}(\mu) \rightarrow L^2(\nu)$  is compact implies that

$$\int_{\mathbb{D}} |f_k|^2 d\nu(z) = \int_{\mathbb{D}} |g_k|^2 \prod_{j=1}^n |z - \lambda_j|^2 d\nu(z) \rightarrow 0, \quad (k \rightarrow \infty).$$

Therefore,  $\prod_{j=1}^n |z - \lambda_j|^2 d\nu(z)$  is a compact Carleson measure for  $H^2$ .

For the converse, assume that  $(f_k)_k$  is a sequence in  $\mathcal{D}(\mu)$  which converges weakly to zero. We will show that  $\|f_k\|_{L^2(\nu)} \rightarrow 0$ , as  $k \rightarrow \infty$ . Indeed, by Theorem (2.1),  $f_k = p_k + \prod_{j=1}^n (z - \lambda_j) g_k$ , where  $g_k \in H^2$



and

$$p_k(z) = \sum_{j=1}^n \prod_{m \neq j} \frac{z - \lambda_m}{\lambda_j - \lambda_m} f_k(\lambda_j), \quad (k \geq 1).$$

By (2.2), we see that  $(g_k)_k$  is a bounded sequence in  $H^2$ . Moreover, since the evaluation functionals  $f \mapsto f(\lambda_j)$  are bounded on  $\mathcal{D}(\mu)$ , for  $1 \leq j \leq n$ , and  $f_k$  converges to zero uniformly on compact subsets of  $\mathbb{D}$ , we see that  $g_k \rightarrow 0$  uniformly on compacts. In other words,  $(g_k)_k$  is a sequence converging weakly to zero in  $H^2$ . Hence,

$$\begin{aligned} \int_{\mathbb{D}} |f_k|^2 d\nu &\leq C \left( \int_{\mathbb{D}} |p_k(z)|^2 d\nu(z) + \int_{\mathbb{D}} |g_k|^2 \prod_{j=1}^n |z - \lambda_j|^2 d\nu(z) \right) \\ &\leq C \left( \sum_{j=1}^n |f_k(\lambda_j)|^2 + \int_{\mathbb{D}} |g_k|^2 \prod_{j=1}^n |z - \lambda_j|^2 d\nu(z) \right) \rightarrow 0, \end{aligned}$$

as  $k \rightarrow \infty$ . The proof is now complete.  $\square$

*Remark 3.4.* In [3], R. Chartrand also characterizes Carleson measures for local Dirichlet spaces but his definition of Carleson measures is different from ours.

#### 4. REPRODUCING KERNEL THESIS

Local Dirichlet spaces  $\mathcal{D}(\mu)$  are reproducing kernel Hilbert spaces. Here, we will prove a reproducing kernel thesis result in the case when  $\mu$  is a point mass (Theorem 4.1). We have been unable to verify this for spaces induced by more general atomic measures. Indeed, one of the difficulties here is to obtain an explicit expression for the reproducing kernels. As far as we know, except for the case of  $\mathcal{D}(\delta_\lambda)$ , such an explicit expression is not known.

Improving an earlier result of Richter and Sundberg [9], Sarason [11] showed that if  $\mu = \delta_\lambda$ , then  $\mathcal{D}(\mu)$  can be identified with a de Branges–Rovnyak space. More precisely, Sarason proved that  $\mathcal{D}(\delta_\lambda) = \mathcal{H}(b_\lambda)$ , with the equality of norms, where

$$(4.1) \quad b_\lambda(z) = \frac{(1 - a_0)\bar{\lambda}z}{1 - a_0\bar{\lambda}z}, \quad (z \in \mathbb{D}),$$

where  $a_0$  is the smallest root of  $(a_0 - 1)^2 = a_0$ . Recall here that, given  $b$  in the closed unit ball of  $H^\infty$ , the de Branges–Rovnyak space  $\mathcal{H}(b)$  is defined as the range of the operator  $(I - T_b T_b^*)^{1/2}$ , equipped with the range norm, where  $T_b$  denotes the Toeplitz operator on  $H^2$  (which, in this case, corresponds to the operator of multiplication by  $b$  on  $H^2$ ).

Then we know [10] that  $\mathcal{H}(b)$  is a reproducing kernel Hilbert space, whose reproducing kernel functions are given by

$$k_w^b(z) = \frac{1 - \overline{b(w)}b(z)}{1 - \overline{w}z}, \quad (w, z \in \mathbb{D}).$$

Now if  $\mu = \alpha\delta_\lambda$ , with  $\alpha > 0$ , then one still has  $\mathcal{D}(\mu) = \mathcal{H}(b_\lambda)$ , except that  $a_0$  in (4.1) has to be replaced by the smallest root of the equation  $(a_0 - 1)^2 = \alpha a_0$  instead. This is in fact proved in [6] where it is also shown that this is the only case where  $\mathcal{D}(\mu)$  arises as a de Branges–Rovnyak space.

Using this identification, we prove the following reproducing kernel thesis in the special case of a point mass.

**Theorem 4.1.** *Let  $\nu$  be a finite positive Borel measure on  $\mathbb{D}$ , and let  $\mu = \alpha\delta_\lambda$ , where  $\lambda \in \mathbb{T}$  and  $\alpha > 0$ . The following statements are equivalent.*

1) *There is a constant  $C$  such that*

$$\int_{\mathbb{D}} |f|^2 d\nu \leq C \|f\|_\mu^2,$$

*for every  $f \in \mathcal{D}(\mu)$ ;*

2) *There is a constant  $C'$  such that*

$$(4.2) \quad \int_{\mathbb{D}} |k_w^\mu|^2 d\nu \leq C' \|k_w^\mu\|_\mu^2, \quad (w \in \mathbb{D}),$$

*where  $k_w^\mu$  is the reproducing kernel for  $\mathcal{D}(\mu)$  at  $w$ .*

*Proof.* Suppose that (4.2) holds. By the paragraph preceding the theorem, we have

$$(4.3) \quad \int_{\mathbb{D}} \left| \frac{1 - \overline{b_\lambda(w)}b_\lambda(z)}{1 - \overline{w}z} \right|^2 d\nu(z) \leq C' \frac{1 - |b_\lambda(w)|^2}{1 - |w|^2}, \quad (w \in \mathbb{D}),$$

where  $b_\lambda$  is given by (4.1) and  $a_0 \in (0, 1)$  satisfies  $(a_0 - 1)^2 = \alpha a_0$ . We will show that (4.3) implies

$$(4.4) \quad \int_{\mathbb{D}} \frac{1}{|1 - \overline{w}z|^2} |z - \lambda|^2 d\nu(z) \leq C \frac{1}{1 - |w|^2}, \quad (w \in \mathbb{D}),$$

for some positive constant  $C$ . Inequality (4.4), in conjunction with Carleson's reproducing kernel thesis for  $H^2$ , implies that  $|z - \lambda|^2 d\nu(z)$  is a Carleson measure for  $H^2$ . The desired conclusion then follows from Theorem 3.1.

To prove (4.4), it is sufficient to show that

$$\frac{1 - |w|^2}{|1 - \overline{w}z|^2} |z - \lambda|^2 \leq C \left| \frac{1 - \overline{b_\lambda(w)}b_\lambda(z)}{1 - \overline{w}z} \right|^2 \frac{1 - |w|^2}{1 - |b_\lambda(w)|^2}, \quad (z, w \in \mathbb{D}),$$

or equivalently

$$(4.5) \quad \inf_{z, w \in \mathbb{D}} \frac{|1 - \overline{b_\lambda(w)} b_\lambda(z)|^2}{(1 - |b_\lambda(w)|^2)|z - \lambda|^2} > 0.$$

Using the inequality

$$\frac{(1 - |\gamma|^2)(1 - |\beta|^2)}{|1 - \overline{\gamma}\beta|^2} = 1 - \left| \frac{\gamma - \beta}{1 - \overline{\gamma}\beta} \right|^2 \leq 1, \quad (\gamma, \beta \in \mathbb{D}),$$

we see that (4.5) is proved as soon as we show that

$$(4.6) \quad \inf_{z \in \mathbb{D}} \frac{1 - |b_\lambda(z)|^2}{|z - \lambda|^2} > 0.$$

Note that condition (4.6) follows immediately from

$$(4.7) \quad 1 - |b_\lambda(z)|^2 \geq \frac{a_0|z - \lambda|^2}{|\lambda - a_0z|^2}, \quad z \in \mathbb{D}.$$

To prove (4.7), we can of course assume that  $\lambda = 1$ , and then it reduces to

$$(4.8) \quad 1 - |b_1(z)|^2 \geq \frac{a_0|z - 1|^2}{|1 - a_0z|^2}, \quad z \in \mathbb{D}.$$

But using the identity  $(1 - a_0)^2 = \alpha a_0$ , we have

$$\begin{aligned} 1 - |b_1(z)|^2 &= 1 - \frac{\alpha a_0|z|^2}{|1 - a_0z|^2} = \frac{|1 - a_0z|^2 - \alpha a_0|z|^2}{|1 - a_0z|^2} \\ &= \frac{1 + a_0^2|z|^2 - 2a_0\Re(z) - \alpha a_0|z|^2}{|1 - a_0z|^2}. \end{aligned}$$

Therefore (4.8) is equivalent to

$$1 + a_0^2|z|^2 - 2a_0\Re(z) - \alpha a_0|z|^2 \geq a_0(1 + |z|^2 - 2\Re(z))$$

or

$$1 - a_0 + a_0(a_0 - \alpha - 1)|z|^2 \geq 0.$$

Using now the relation  $a_0(a_0 - \alpha - 1) = a_0 - 1$ , we obtain immediately the last inequality for every  $z \in \mathbb{D}$ . That proves (4.8) and the result.  $\square$

*Remark 4.2.* In [1], a reproducing kernel thesis is proved for the embedding of  $\mathcal{H}(b)$  into  $L^2(\nu)$  in the case where the function  $b$  satisfied some additional hypothesis. More precisely, assume that  $b$  satisfies the so-called connected level set condition (which means that for some  $\varepsilon \in (0, 1)$ , the level set  $\Omega(b, \varepsilon) := \{z \in \mathbb{D} : |b(z)| < \varepsilon\}$  is connected) and assume further that the spectrum of  $b$ ,  $\sigma(b) = \{\zeta \in \mathbb{T} : \liminf_{z \rightarrow \zeta} |b(z)| < 1\}$ , is contained in the closure of  $\Omega(b, \varepsilon)$ . Then the

operator  $f \mapsto f$  is bounded from  $\mathcal{H}(b)$  into  $L^2(\nu)$  if and only if it is bounded on reproducing kernels.

In our case (corresponding to a local Dirichlet space), the hypothesis  $\sigma(b_\lambda) \subset \text{Clos } \Omega(b_\lambda, \varepsilon)$  is not satisfied, whence we cannot apply [1, Theorem 6.8] to get Theorem 4.1. Indeed, we easily check that  $\sigma(b_\lambda) = \mathbb{T} \setminus \{\lambda\}$ ; now if  $\sigma(b_\lambda) \subset \text{Clos } \Omega(b_\lambda, \varepsilon)$ , then we would have  $\lambda \in \text{Clos } \Omega(b_\lambda, \varepsilon)$ , but this is absurd by continuity of  $b_\lambda$  on the closed unit disc.

As one might expect, an analogous result can be obtained for compact Carleson measures.

**Theorem 4.3.** *Let  $\nu$  be a finite positive Borel measure on  $\mathbb{D}$ , and let  $\mu = \alpha\delta_\lambda$ , where  $\lambda \in \mathbb{T}$  and  $\alpha > 0$ . The following statements are equivalent.*

- 1)  $\nu$  is a compact Carleson measure for  $\mathcal{D}(\mu)$ ;
- 2)

$$(4.9) \quad \int_{\mathbb{D}} |k_w^\mu|^2 d\nu = o(\|k_w^\mu\|_\mu^2), \quad (|w| \rightarrow 1^-).$$

*Proof.* That 1) implies 2) is easily established by noting that normalized reproducing kernels  $(k_w^\mu)_w$  in  $\mathcal{D}(\mu)$  converge weakly to zero, as  $|w| \rightarrow 1^-$ .

For the converse, we apply the same method as in the proof of Theorem 4.1 to find that

$$(4.10) \quad \int_{\mathbb{D}} \frac{1}{|1 - \bar{w}z|^2} |z - \lambda|^2 d\nu(z) = o\left(\frac{1}{1 - |w|^2}\right), \quad (|w| \rightarrow 1^-).$$

This implies that  $|z - \lambda|^2 d\nu$  is a compact Carleson measure for  $H^2$ . The result then follows by Theorem 3.3.  $\square$

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