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# On the maximal offspring in a critical branching process with infinite variance

Jean Bertoin\*

## Abstract

We investigate the maximal number  $M_k$  of offsprings amongst all individuals in a critical Galton-Watson process started with  $k$  ancestors. We show that when the reproduction law has a regularly varying tail with index  $-\alpha$  for  $1 < \alpha < 2$ , then  $k^{-1}M_k$  converges in distribution to a Frechet law with shape parameter 1 and scale parameter depending only on  $\alpha$ .

**Key words:** Branching process, maximal offspring, Frechet distribution, stable Lévy process, extreme value theory.

## 1 Introduction and main results

There exists in the literature a variety of results about extremes in branching processes; see the surveys by Yanev [11, 12] and references therein. In particular, Rahinov and Yanev [9] have characterized the asymptotic behavior as  $n \rightarrow \infty$  of the maximal offspring at the  $n$ -th generation conditionally on the event that the extinction has not yet occurred. In a different direction, Pakes [8] considered the order statistic of scores in branching processes, and more precisely asymptotics of the largest score up to and including the  $n$ -th generation, again conditionally on the event that extinction does not occur before the  $n$ -th generation. In this vein, we also mention the recent contributions of Lebedev [6, 7] for eternal branching process. The purpose of this work is to point at simple result in this area which does not seem to have been observed before.

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Let  $F$  denote the distribution function of some probability measure on  $\mathbb{Z}_+$  and write  $\bar{F} = 1 - F$  for its tail. We assume throughout this paper that

$$\sum_{i=0}^{\infty} \bar{F}(i) = 1 \quad \text{and} \quad \bar{F}(n) = n^{-\alpha} \ell(n) \tag{1}$$

where  $1 < \alpha < 2$  and  $\ell$  is a slowly varying function. For each  $k \geq 1$ , we consider a critical Bienaymé-Galton-Watson process  $Z = (Z_n, n \geq 0)$  started from  $Z_0 = k$  ancestors and with reproduction law given by  $F$ . We are interested in the maximal offspring until extinction, viz.

$$M_k = \max\{X_i : 1 \leq i \leq T_k\}$$

where  $T_k = \sum_{n=0}^{\infty} Z_n$  denotes the size of the total population, and  $X_i$  the number of children of the  $i$ -th individual (the choice of the enumeration of individuals is obviously irrelevant). We state our main result.

**Theorem 1** *Under the assumption (1),  $k^{-1}M_k$  converges in distribution as  $k \rightarrow \infty$  to a Frechet law with shape parameter 1. More precisely, for every  $x > 0$ , we have*

$$\lim_{k \rightarrow \infty} \mathbb{P}(M_k \leq kx) = \exp(-\beta/x)$$

where  $\beta > 0$  is the unique solution to the equation

$$\sum_{n=0}^{\infty} \frac{(-1)^n \beta^n}{(n - \alpha)n!} = 0. \tag{2}$$

The graph of the map  $\alpha \mapsto \beta$  is displayed in Figure 1 below.

Of course,  $M_k$  can be viewed as the maximum of  $k$  i.i.d. copies of  $M_1$ , and one should expect a weak limit theorem for  $M_k$  involving an extreme value distribution. Nonetheless it is quite remarkable that the slowly varying function  $\ell$  plays no role in this limit theorem, and that the index  $\alpha$  only appears in the scale parameter of the limiting Frechet distribution, but not in the shape parameter.

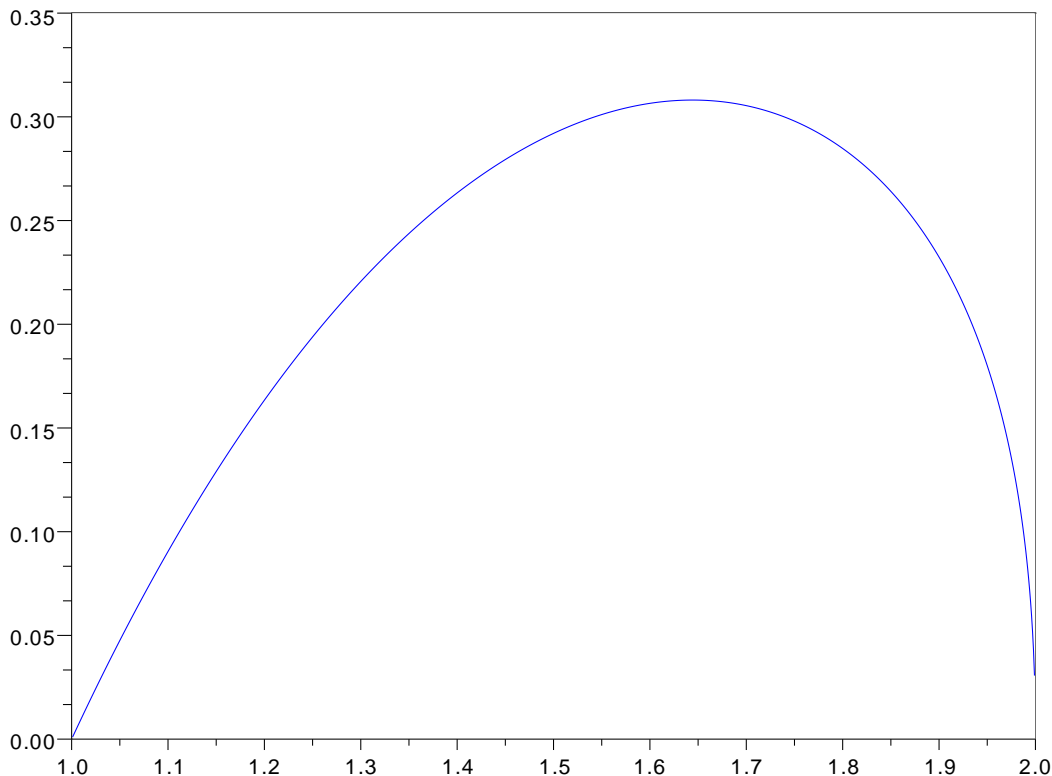
It is also interesting to point at the following consequence of a classical result due to Gnedenko (see, for instance Proposition 1.11 in [10]) combined with Theorem 1:

**Corollary 1** *Under the assumption (1), the tail distribution of  $M_1$  fulfills*

$$\mathbb{P}(M_1 > x) \sim \beta/x \quad \text{as } x \rightarrow \infty,$$

where  $\beta > 0$  is the unique solution to (2).

One should note that  $M_1$  has thus an infinite expectation even though the mean offspring of a typical individual is 1.



**Figure 1 :** *Graph of the map  $\alpha \mapsto \beta$ .*

By Gnedenko's Theorem, the asymptotics stated in Theorem 1 and Corollary 1 are equivalent. Usually, it is more natural to establish first that the tail distribution of some random variable is regularly varying and to deduce then the limiting behavior of the maximum of i.i.d. copies. However this is not the approach that we shall use here. More precisely, the hypothesis (1) is a necessary and sufficient condition for the distribution  $F$  to belong to the domain of attraction of a spectrally positive stable law with index  $\alpha$ . Using the classical connexion between downward skip-free random walks and branching processes and standard arguments of weak convergence on the space of cadlåg paths, this enables us to reduce the proof of Theorem 1 to the determination of the distribution of the maximal size of jumps in a spectrally positive stable Lévy process up to its first hitting time of  $-1$ . The latter is achieved by rather straightforward arguments of fluctuation theory for Lévy processes without negative jumps.

The proof of Theorem 1 is presented in the next section, and miscellaneous comments are

made in Section 3.

## 2 Proof of Theorem 1

Introduce a sequence  $X_1, X_2, \dots$  of i.i.d. variables distributed according to  $F$  and consider the downward-skip-free random walk

$$S_n = X_1 + \dots + X_n - n.$$

It is a well-known fact observed by Harris (see Section 6 in [3]) that the Bienaymé-Galton-Watson branching process with reproduction law  $F$  and started with  $k$  ancestors can be constructed from the random walk stopped at its first hitting time of level  $-k$ ,

$$T_k = \min\{i : S_i = -k\}.$$

In this setting,  $T_k$  corresponds to the total size of the population generated by the branching process (so our notation is coherent) and the maximal offspring until extinction is closely related to the maximal step of the random walk until time  $T_k$ . More precisely, there is the identity

$$M_k = \max\{X_i : 1 \leq i \leq T_k\}.$$

The assumption (1) ensures that the random walk  $S$  belongs to the domain of attraction of a spectrally positive stable Lévy process with index  $\alpha$ , say  $\xi = (\xi_t, t \geq 0)$ . This means that there exists a sequence  $(a_k)_{k \geq 1}$  such that

$$\left(\frac{1}{k}S_{[a_k t]}, t \geq 0\right) \Longrightarrow (\xi_t, t \geq 0), \quad (3)$$

where the notation  $\Longrightarrow$  is for convergence in distribution as  $k \rightarrow \infty$ , in the sense of Skorohod on the space of càdlàg paths. See for instance Theorem 16.14 in [4]. More precisely,  $\xi$  is a process with independent and stationary increments and which fulfills the scaling property:

for any  $c > 0$ ,  $(c^{-1/\alpha}\xi_{ct}, t \geq 0)$  has the same law as  $\xi$ .

By standard arguments, the convergence (3) can be extended to processes stopped at their the first-passage times, in the sense if we introduce

$$\tau_1 = \inf\{t : \xi_t = -1\},$$

then

$$\left( \frac{1}{k} S_{[a_k t]}, t \leq T_k/a_k \right) \Longrightarrow (\xi_t, t \leq \tau_1).$$

Because convergence in the sense of Skorohod implies uniform convergence after suitable time-changes, it follows that if we denote by  $\Delta_t^*$  the largest jump made by  $\xi$  on the time-interval  $[0, t]$ , i.e.

$$\Delta_t^* = \max\{\Delta_s = \xi_s - \xi_{s-} : 0 \leq s \leq t\},$$

then

$$\lim_{k \rightarrow \infty} \frac{1}{k} M_k = \Delta_{\tau_1}^* \quad \text{in distribution.}$$

By Gnedenko's Theorem, this entails that  $\Delta_{\tau_1}^*$  has the Frechet distribution with shape parameter 1, and we only need to determine its scale parameter. We state this as a lemma that may be of independent interest.

**Lemma 1** *Let  $\xi$  be a spectrally positive stable Lévy process with index  $\alpha \in (1, 2)$ . The distribution the size of its largest jump until the hitting time of  $-1$  is given by*

$$\mathbb{P}(\Delta_{\tau_1}^* \leq y) = \exp(-\beta/y), \quad y > 0$$

where  $\beta > 0$  is the unique solution to the equation (2).

**Proof:** Thanks to the scaling property, there is no loss of generality in assuming that the Lévy measure  $\Pi$  of  $\xi$  is simply

$$\Pi(dx) = x^{-\alpha-1} dx, \quad x > 0.$$

Recall that the Laplace exponent  $\Psi$  of  $\xi$  is given by Lévy-Khintchin formula

$$\Psi(q) = \int_0^\infty (e^{-qx} - 1 + qx) \Pi(dx) = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)} q^\alpha, \quad q \geq 0$$

and then

$$\mathbb{E}(\exp(-q\xi_t)) = \exp(t\Psi(q)).$$

Fix  $y > 0$  and let  $\delta_y = \inf\{t : \Delta_t > y\}$  denote the first instant when  $\xi$  makes a jump of size greater than  $y$ . Introduce also for every  $t \geq 0$

$$\xi_t^y = \xi_t - \sum_{0 \leq s \leq t} \Delta_s \mathbf{1}_{\{\Delta_s > y\}},$$

i.e.  $\xi^y = (\xi_t^y, t \geq 0)$  is the spectrally positive Lévy process obtained from  $\xi$  by suppressing all

the jumps of size greater than  $y$ . In particular its Laplace exponent is given by

$$\begin{aligned}\Psi^y(q) &= \int_0^y (e^{-qx} - 1 + qx) \Pi(dx) + q \int_y^\infty x \Pi(dx) \\ &= \int_0^y (e^{-qx} - 1 + qx) x^{-1-\alpha} dx + q \frac{y^{1-\alpha}}{\alpha - 1}.\end{aligned}$$

Introduce now the first passage time of  $\xi^y$  at  $-1$ ,

$$\tau_1^y = \inf\{t : \xi_t^y = -1\},$$

and note that the events  $\{\Delta_{\tau_1}^* \leq y\}$  and  $\{\tau_1^y < \delta_y\}$  coincide. Recall further from the Lévy-Itô decomposition of Lévy processes that  $\xi^y$  and *a fortiori*  $\tau_1^y$  are independent of  $\delta_y$ . Since the latter has the exponential distribution with parameter  $\bar{\Pi}(y) = \Pi((y, \infty)) = \alpha^{-1}y^{-\alpha}$ , we deduce that

$$\mathbb{P}(\Delta_{\tau_1}^* \leq y) = \mathbb{P}(\tau_1^y < \delta_y) = \mathbb{E}(\exp(-\alpha^{-1}y^{-\alpha}\tau_1^y)).$$

It is well-known (see for instance Theorem VII.1 in [1] or Theorem 3.12 in [5]) that the right-hand side can be expressed as

$$\mathbb{E}(\exp(-\alpha^{-1}y^{-\alpha}\tau_1^y)) = \exp(-b(y)),$$

where  $b(y) > 0$  is the unique solution to  $\Psi^y(b(y)) = \alpha^{-1}y^{-\alpha}$ . Thus we are led to solving the equation

$$\int_0^y (e^{-b(y)x} - 1 + b(y)x) x^{-1-\alpha} dx + b(y) \frac{y^{1-\alpha}}{\alpha - 1} = \alpha^{-1}y^{-\alpha}.$$

The change of variables  $x = yu$  yields

$$\int_0^1 (e^{-b(y)yu} - 1 + b(y)yu) u^{-1-\alpha} du + \frac{b(y)y}{\alpha - 1} = \frac{1}{\alpha},$$

and putting  $\beta = yb(y)$ , we finally arrive at

$$\int_0^1 (e^{-\beta u} - 1 + \beta u) u^{-1-\alpha} du + \frac{\beta}{\alpha - 1} = \frac{1}{\alpha}. \quad (4)$$

Using the series expansion of  $e^{-\beta u}$ , one sees that Equations (4) and (2) are equivalent.

Finally, note that the function

$$q \rightarrow \int_0^1 (e^{-qu} - 1 + qu) u^{-1-\alpha} du + \frac{q}{\alpha - 1}$$

is convex and strictly increasing on  $[0, \infty)$ , and thus the equation (4) has a unique solution.  $\square$

### 3 Some comments

1. Amateurs of special functions will recognize that Equation (2) can be re-expressed in terms of the incomplete gamma function. Specifically, recall that for  $\Re a > 0$

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{a+n}}{(a+n)n!}$$

is the incomplete gamma function; see 8.354.1 in [2]. The modified function

$$\gamma^*(a, x) = \frac{x^{-a}}{\Gamma(a)} \gamma(a, x)$$

is analytic with respect to  $a$  and  $x$  (cf. 8.353.1 in [2]), and therefore the solution  $\beta$  to (2) is the unique positive root to the equation  $\gamma^*(-\alpha, \beta) = 0$ .

2. The distribution function  $G(x) = \mathbb{P}(M_1 \leq x)$  of the maximal offspring for a single ancestor solves

$$G(x) = \sum_{i \leq x} G^i(x) f(i), \quad x \geq 0,$$

where  $f(i) = F(i) - F(i-1)$  is the mass function of the reproduction law; and it is easily checked that this equation determines  $G(x)$ . However we have not been able to derive Corollary 1 directly from this characterization.

3. In connexion with Lemma 1, we recall that the distribution of the maximal jump size up to time  $t$ ,  $\Delta_t^*$ , is given by

$$\mathbb{P}(\Delta_t^* \leq x) = \exp(-t\bar{\Pi}(x)) = \exp(-tcx^{-\alpha}),$$

where  $\bar{\Pi}$  stands for the tail of the Lévy measure and  $c > 0$  is some constant. Let us assume for simplicity that  $\xi$  is normalized as in the proof of Lemma 1, so that  $c = 1/\alpha$ . In this situation, the Laplace transform of  $\tau_1$  is

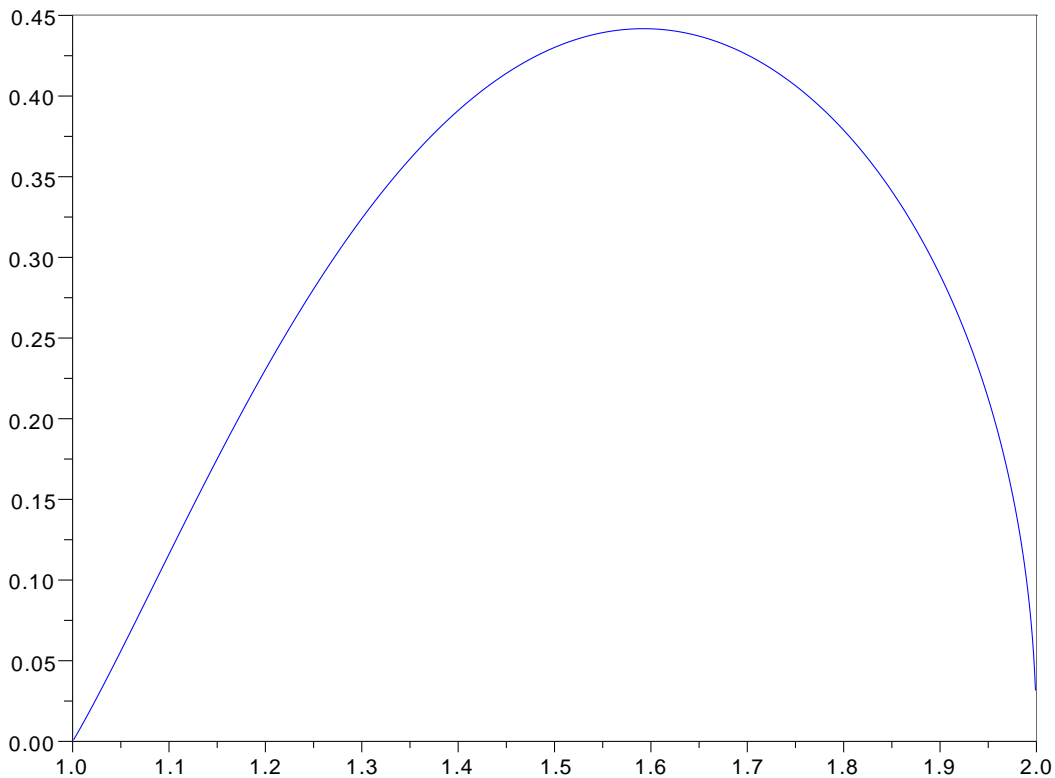
$$\mathbb{E}(\exp(-q\tau_1)) = \left( \frac{\alpha(\alpha-1)}{\Gamma(2-\alpha)} q \right)^{1/\alpha},$$



and we deduce that if  $T$  is a random time distributed as  $\tau_1$  and independent of  $\xi$ , then

$$\mathbb{P}(\Delta_T^* \leq x) = \mathbb{E} \left( \exp \left( -\frac{1}{\alpha x^\alpha} \tau_1 \right) \right) = \exp \left( - \left( \frac{\alpha - 1}{\Gamma(2 - \alpha)} \right)^{1/\alpha} x^{-1} \right).$$

We stress that, thanks to the scaling property, the distribution of  $\Delta_T^*$  does not depend on the normalization that has been chosen for  $\xi$ . We see that  $\Delta_T^*$  has again the Fréchet law with shape parameter 1, but the scale parameter differs from the solution to (2); see Figure 2 below and compare with Figure 1.



**Figure 2 :** Graph of the map  $\alpha \mapsto \left( \frac{\alpha-1}{\Gamma(2-\alpha)} \right)^{1/\alpha}$ .

4. It is natural to rephrase Lemma 1 in terms of the excursion measure of the reflected stable Lévy process (we refer to Chapters IV and VII in [1] for background). More precisely, if we write  $\underline{\xi}_t = \min_{0 \leq s \leq t} \xi_s$  for the continuous minimum process of  $\xi$ , then the reflected process  $\xi - \underline{\xi}$  is Markovian and  $-\underline{\xi}$  serves as local time at the regular boundary point 0. Denote by  $\underline{n}$  the Itô excursion measure of the reflected process. We have

**Corollary 2** *The distribution of the maximal jump  $\Delta^*$  under the excursion measure  $\underline{n}$  is given by*

$$\underline{n}(\Delta^* > x) = \beta/x, \quad x > 0$$

where  $\beta$  is the unique solution of (2).

**Proof:** Denote by  $\tau_t = \inf\{s : \xi_s < -t\}$  the right-continuous inverse of the local time  $-\underline{\xi}$  and recall that the path-valued process

$$e_t = (\xi_{t+\tau_{s-}} - s, 0 \leq s \leq \tau_s - \tau_{s-}), \quad t \geq 0$$

is a homogeneous Poisson point process with intensity  $\underline{n}$ . Observe also that the jumps of  $\xi$  and of  $\xi - \underline{\xi}$  are the same. It follows that

$$\mathbb{P}(\Delta_{\tau_1}^* \leq x) = \exp(-\underline{n}(\Delta^* > x))$$

and we conclude from Lemma 1. □

It may be interesting to compare Corollary 2 with the distribution of the height of the excursion,  $h = \max_{s \geq 0} e_s$  which is given by

$$\underline{n}(h > x) = (\alpha - 1)/x, \quad x > 0,$$

see for instance Theorem VII.8 in [1] and its proof.

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