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► **To cite this version:**

Mickaël dos Santos, Petru Mironescu, Oleksandr Misiats. The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part I: the zero degree case. *Commun. Contemp. Math.*, 2011, 13 (5), pp.885-914. hal-00515893

**HAL Id: hal-00515893**

**<https://hal.science/hal-00515893>**

Submitted on 8 Sep 2010

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# The Ginzburg-Landau functional with a discontinuous and rapidly oscillating pinning term. Part I: the zero degree case

Mickaël DOS SANTOS

Université de Lyon, Université Lyon 1 , Institut Camille Jordan CNRS UMR 5208  
43, boulevard du 11 novembre 1918, F-69622 Villeurbanne, France  
dossantos@math.univ-lyon1.fr

Petru MIRONESCU

Université de Lyon, Université Lyon 1 , Institut Camille Jordan CNRS UMR 5208  
43, boulevard du 11 novembre 1918, F-69622 Villeurbanne, France  
mironescu@math.univ-lyon1.fr

Oleksandr MISIATS

Department of Mathematics, The Pennsylvania State University  
University Park PA 16802, USA  
misiats@math.psu.edu

## Abstract

We consider minimizers of the Ginzburg-Landau energy with pinning term and zero degree Dirichlet boundary condition. Without any assumptions on the pinning term, we prove that these minimizers do not develop vortices in the limit  $\varepsilon \rightarrow 0$ . We next consider the specific case of a periodic discontinuous pinning term taking two values. In this setting, we determine the asymptotic behavior of the minimizers as  $\varepsilon \rightarrow 0$ .

## 1 Introduction

Let  $\Omega \subset \mathbb{R}^2$  be a simply connected domain and let  $a_\varepsilon : \Omega \rightarrow \mathbb{R}$  be a measurable function such that  $0 < b \leq a_\varepsilon \leq 1$ . We associate with  $a_\varepsilon$  a generalized Ginzburg-Landau (GL, in short) type energy

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left\{ |\nabla u(x)|^2 + \frac{1}{2\varepsilon^2} (a_\varepsilon^2(x) - |u(x)|^2)^2 \right\} dx. \quad (1)$$

Here,  $u \in H^1(\Omega, \mathbb{C})$  and  $\varepsilon > 0$  is the GL parameter.

This variant of the standard GL type energy (which corresponds to  $a_\varepsilon \equiv 1$ ) is called GL functional with pinning term  $a_\varepsilon$  or pinned GL functional. We quote here few relevant papers among the vast literature concerning this energy functional.

- In [2], the authors consider the case where  $a_\varepsilon = a \in C^\beta(\Omega)$  is independent of  $\varepsilon$ .

- [18] and [3] treat the case where  $a_\varepsilon = a$  is independent of  $\varepsilon$  and takes the value  $b$  in  $\omega$  and 1 outside  $\omega$ , with  $\omega$  smooth subset of  $\Omega$ . The latter article considers the case of an applied magnetic field.
- In [1],  $a_\varepsilon$  depends on  $\varepsilon$  and is smooth. The oscillation rate of  $a_\varepsilon$  depends on  $\varepsilon$ .

The goal of this article is to study the pinned GL functional with a fast oscillating discontinuous pinning term  $a_\varepsilon$ . This may be viewed as a simplification of more realistic models which describe superconductivity phenomena for composite superconductors. The experimental pictures suggest nearly 2D structure of parallel vortex tubes ([21], Fig I.4). Therefore, the domain  $\Omega$  can be viewed as a cross-section of a multifilamentary wire with a number of thin superconducting filaments. Such multifilamentary wires are widely used in industry, including magnetic energy-storing devices, transformers and power generators [17], [15].

Another important practical issue in modeling superconductivity is to decrease the energy dissipation in superconductors. Here, the dissipation occurs due to currents associated with the motion of vortices ([19], [4]). This dissipation as well the thermomagnetic stability can be improved by *pinning* ("fixing the positions") of vortices. This, in turn, can be done by introducing impurities or inclusions in the superconductor.

Our pinning term is periodic with respect to a  $\delta \times \delta$  grid where  $\delta = \delta(\varepsilon) \rightarrow 0$ . As in [1], due to the fast oscillations, this problem is related to a periodic homogenization problem (depending on the relation between  $\varepsilon$  and  $\delta$ ).

The boundary condition we consider is the Dirichlet one. More specifically, we fix some  $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$ . Our class of test functions is

$$H_g^1 := \{u \in H^1(\Omega, \mathbb{C}) \mid u = g \text{ on } \partial\Omega\}. \quad (2)$$

We consider solutions  $u_\varepsilon$  of the minimization problem

$$\inf_{u \in H_g^1} E_\varepsilon(u). \quad (3)$$

In this article we will consider only the case where the boundary data  $g$  has zero degree. The case where the degree is not zero requires additional techniques and will be investigated in a forthcoming paper.

Recall that the degree (winding number) of  $g$  is defined as

$$\text{deg}_{\partial\Omega}(g) = \frac{1}{2\pi} \int_{\partial\Omega} g \times \partial_\tau g \, d\tau = 0,$$

where:

- For  $z \in \mathbb{C}$ ,  $\Re z$  denotes the real part of  $z$  and  $\Im z$  denotes the imaginary part of  $z$ .
- " $\times$ " stands for the "vectorial product" in  $\mathbb{C}$ ,  $z_1 \times z_2 = \Im(\bar{z}_1 z_2)$ ,  $z_1, z_2 \in \mathbb{C}$ .
- $\tau$  is the unit and direct tangent vector at  $\partial\Omega$ , *i.e.*, denoting  $\nu$  to be the unit outward normal to  $\partial\Omega$ , one has  $\tau = \nu^\perp$ .

- $\partial_\tau$  is the tangential derivative.

This degree is an integer. For a proof of this assertion and for more properties of the topological degree of  $g$ , see *e.g.* [10] or [5].

If  $u_\varepsilon$  is a minimizer of the problem (3), then it satisfies the Euler-Lagrange equation

$$\begin{cases} -\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (a_\varepsilon^2 - |u_\varepsilon|^2) & \text{in } \Omega \\ u_\varepsilon = g & \text{on } \partial\Omega \end{cases}. \quad (4)$$

Following [18], one may prove that in the special case  $g \equiv 1$  there is a unique minimizer  $U_\varepsilon$ . Moreover, this minimizer satisfies  $b \leq U_\varepsilon \leq 1$ . This  $U_\varepsilon$  plays an important role in the study of GL functional with pinning term. Indeed, define, for  $u \in H_g^1$ , a new map  $v = \frac{u}{U_\varepsilon} \in H_g^1$ . Then  $E_\varepsilon$  decouples as follows [18]

$$E_\varepsilon(u) = E_\varepsilon(U_\varepsilon v) = f(\varepsilon) + F_\varepsilon(v), \quad (5)$$

where

$$f(\varepsilon) := E_\varepsilon(U_\varepsilon), \quad F_\varepsilon(v) := \frac{1}{2} \int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{1}{2\varepsilon^2} U_\varepsilon^4 (1 - |v|^2)^2 \right\}. \quad (6)$$

Therefore,  $u$  minimizes  $E_\varepsilon$  in  $H_g^1(\Omega)$  if and only if  $v$  minimizes  $F_\varepsilon$  in  $H_g^1$ . In what follows, we denote by  $v_\varepsilon$  a minimizer of  $F_\varepsilon$  in  $H_g^1$ .

Following again [18], we have  $|v_\varepsilon| \leq 1$  and  $|u_\varepsilon| \leq 1$  in  $\Omega$ .

From (5) and (6) we see that the study of the pinned GL is reduced to the study of the weighted GL functional  $F_\varepsilon$  and to the study of the asymptotics of  $U_\varepsilon$ .

The plan of our work is the following: in Section 2 we prove a "clearing out" result (Theorem 1). More specifically, we prove that  $v_\varepsilon$  is "vortexless" for small  $\varepsilon$ , *i. e.*, that  $|v_\varepsilon| \rightarrow 1$  uniformly in  $\bar{\Omega}$  as  $\varepsilon \rightarrow 0$ . (Recall that  $\deg_{\partial\Omega}(g) = 0$ ; this assumption is essential for our conclusion.) This result is true for any weighted GL functionals. Such general functionals are defined by formula (7) and do not require any assumption except uniform bounds on the weights. In particular, clearing out does not rely on any periodicity assumption. We believe that this result has its own interest.

The clearing out result reduces the study of the behavior of  $v_\varepsilon$  to the one of  $\mathbb{S}^1$ -valued maps. In other words, we will reduce the problem of minimizing  $F_\varepsilon$  in the class of all test functions to the one of minimizing  $F_\varepsilon$  in the class of  $\mathbb{S}^1$ -valued maps. The latter problem will be studied in detail in Section 3. There, the asymptotic analysis of minimizers of the  $F_\varepsilon$  among  $\mathbb{S}^1$ -valued maps, combined with an asymptotic analysis of  $U_\varepsilon$  (analysis performed at the beginning of Section 3), will allow us to conclude Section 3 by describing the behavior of  $u_\varepsilon$  as  $\varepsilon \rightarrow 0$ .

## Acknowledgments

The authors thank Leonid V. Berlyand and Georges Griso for useful and stimulating discussions. Part of this work was done while the first author was visiting the Pennsylvania State University. He thanks the Mathematics Department there for its hospitality. The work of M. Dos Santos and O. Misiats was partially supported by NSF grant DMS-0708324.

## 2 Clearing out for general weighted Ginzburg-Landau type functionals

Let  $b \in (0, 1)$  and let  $\alpha_\varepsilon \in W^{1,\infty}(\Omega)$ ,  $\beta_\varepsilon \in L^\infty(\Omega)$  be such that  $b \leq \alpha_\varepsilon, \beta_\varepsilon \leq 1$ . We associate to  $\alpha_\varepsilon$  and  $\beta_\varepsilon$  the weighted GL type functional defined through the formula

$$F_\varepsilon : H^1(\Omega, \mathbb{C}) \rightarrow \mathbb{R}^+ \\ v \mapsto \frac{1}{2} \int_\Omega \left\{ \alpha_\varepsilon |\nabla v|^2 + \frac{\beta_\varepsilon}{2\varepsilon^2} (1 - |v|^2)^2 \right\}. \quad (7)$$

Let  $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$  be such that  $\deg_{\partial\Omega}(g) = 0$ . For  $\varepsilon > 0$ , we denote by  $v_\varepsilon$  a minimizer of  $F_\varepsilon$  in  $H_g^1$ . One may easily prove that  $v_\varepsilon$  satisfies

$$\begin{cases} -\operatorname{div}(\alpha_\varepsilon \nabla v_\varepsilon) = \frac{\beta_\varepsilon}{\varepsilon^2} v_\varepsilon (1 - |v_\varepsilon|^2) & \text{in } \Omega \\ v_\varepsilon = g & \text{on } \partial\Omega \end{cases}. \quad (8)$$

Since  $\deg_{\partial\Omega}(g) = 0$ , we have [7]  $H_g^1(\Omega, \mathbb{S}^1) = \{v \in H_g^1 \mid |v| = 1 \text{ in } \Omega\} \neq \emptyset$ .

If we take any fixed map  $v \in H_g^1(\Omega, \mathbb{S}^1)$  as a test function for  $F_\varepsilon$ , we find that there is  $C_0$  depending only on  $g$  such that

$$\min_{v \in H_g^1(\Omega)} F_\varepsilon(v) = F_\varepsilon(v_\varepsilon) \leq C_0. \quad (9)$$

### 2.1 Uniform convergence of $|v_\varepsilon|$ to 1

This part is devoted to the proof of the following theorem.

**Theorem 1.** *When  $\varepsilon \rightarrow 0$ , we have  $|v_\varepsilon| \rightarrow 1$  uniformly in  $\bar{\Omega}$ .*

For the convenience of the reader, we split the rather long proof of Theorem 1 into two parts.

#### 2.1.1 Theorem 1 holds far away the boundary

We prove that, for sufficiently small  $\varepsilon$ ,  $|v_\varepsilon|$  is arbitrarily close to 1 outside an  $2\sqrt{\varepsilon}$ -neighborhood of  $\partial\Omega$ .

**Proposition 1.** *Let  $\varepsilon_n \downarrow 0$  and  $\{x_n\}_n \subset \Omega$  be such that  $\operatorname{dist}(x_n, \partial\Omega) \geq 2\sqrt{\varepsilon_n}$ . Then  $|v_{\varepsilon_n}(x_n)| \rightarrow 1$ .*

*Proof.* We write  $\varepsilon$  instead of  $\varepsilon_n$ . Let  $n$  be sufficiently large such that  $\sqrt{\varepsilon} > \varepsilon$  and consider the circular annulus  $B_{\sqrt{\varepsilon}}(x_n) \setminus B_\varepsilon(x_n)$ .

From (9), we have, with  $\mathcal{C}_r := \{|x - x_n| = r\}$ ,

$$C_0 \geq \frac{b}{4} \int_{B_{\sqrt{\varepsilon}}(x_n) \setminus B_\varepsilon(x_n)} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \right\} = \frac{b}{4} \int_\varepsilon^{\sqrt{\varepsilon}} \frac{1}{r} \cdot r \int_{\mathcal{C}_r} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \right\}. \quad (10)$$

By mean value argument, there are  $C_1$  (depending only on  $g, \Omega$  and  $b$ ) and  $r \in (\varepsilon, \sqrt{\varepsilon})$  such that

$$r \int_{\mathcal{C}_r} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \right\} \leq \frac{C_1}{|\ln \varepsilon|}. \quad (11)$$

**Lemma 1.** *Let  $\delta > 0$ . Then, for large  $n$  and for  $r$  as in (11), we have*

1.  $\text{Var}(v_\varepsilon, \mathcal{C}_r) \leq \delta$ , where  $\text{Var}(v_\varepsilon, \mathcal{C}_r) := \int_{\mathcal{C}_r} |\partial_\tau v_\varepsilon|$ ;
2.  $|v_\varepsilon| \geq 1 - 2\delta$  on  $\mathcal{C}_r$ .

*Proof.* Assertion 1. is a direct consequence of the bound (11), which yields

$$\left( \int_{\mathcal{C}_r} |\partial_\tau v_\varepsilon| \right)^2 \leq \left( \int_{\mathcal{C}_r} |\nabla v_\varepsilon| \right)^2 \leq \int_{\mathcal{C}_r} 1 \int_{\mathcal{C}_r} |\nabla v_\varepsilon|^2 = 2\pi r \int_{\mathcal{C}_r} |\nabla v_\varepsilon|^2 \leq \frac{2\pi C_1}{|\ln \varepsilon|}.$$

It follows that, for large  $n$ , we have  $|\text{Var}(v_\varepsilon, \mathcal{C}_r)| \leq \delta$ .

In order to prove 2., we argue by contradiction. Assume that there are  $\delta > 0$ , a subsequence  $\{n_k\}_k$  and points  $x_{n_k} \in \mathcal{C}_r$  such that  $|v_\varepsilon(x_{n_k})| < 1 - 2\delta$  (here  $\varepsilon = \varepsilon_{n_k}$ ).

From the estimate 1. on  $\text{Var}(v_\varepsilon, \mathcal{C}_r)$ , one has, for large  $k$ ,  $|v_\varepsilon| < 1 - \delta$  on  $\mathcal{C}_r$ .

Consequently,  $r \int_{\mathcal{C}_r} (1 - |v_{\varepsilon_{n_k}}|^2)^2 \geq 2\pi r^2 \delta^2$ . Since  $r \geq \varepsilon$ , this inequality contradicts the estimate (11) for small  $\varepsilon$ .  $\square$

So far, we proved the existence of a circle around  $x_n$  such that, on that circle,  $|v_\varepsilon|$  is close to 1 and  $v_\varepsilon$  varies little. More specifically: if  $0 < \gamma < 1$  then, for large  $n$ , there exists  $S_\varepsilon \subset \overline{B_1(0)}$  such that

- $\text{dist}(S_\varepsilon, 0) \geq 1 - \gamma$ ,
- $S_\varepsilon$  is the smallest of the two regions delimited by a chord in the closed unit disc,
- $v_\varepsilon(\mathcal{C}_r) \subset S_\varepsilon$ .

The following lemma implies that, under the above assumptions on  $S_\varepsilon$  and on  $r$ , we have, for large  $n$ ,  $|v_\varepsilon(x_n)| \geq 1 - \gamma$ . This inequality completes the proof of Proposition 1, which is the first step in the proof of Theorem 1.  $\square$

**Lemma 2.** *Let  $C$  be a chord in the closed unit disc,  $C$  different from a diameter. Let  $S$  be the smallest of the two regions enclosed by the chord and the boundary of the disc.*

*Let  $O$  be a Lipschitz bounded open set and let  $g \in H^{1/2}(\partial O, S)$ .*

*Let  $\tilde{\alpha}, \tilde{\beta} \in L^\infty(O, \mathbb{R})$  satisfy  $\text{ess inf } \tilde{\alpha} > 0$ ,  $\text{ess inf } \tilde{\beta} > 0$ .*

*If  $v$  minimizes GL type energy*

$$\tilde{F}(v) = \int_O \left\{ \tilde{\alpha}(x) |\nabla v|^2 + \tilde{\beta}(x) (1 - |v|^2)^2 \right\}$$

*in  $H_g^1(O)$ , then  $v(O) \subset S$ .*

**Remark 1.** *This statement generalizes Lemma 8 in [6] (there  $\tilde{\alpha} = 1, \tilde{\beta} = 1/(2\varepsilon^2)$ ). However, the proof in [6] does not apply directly to our situation.*

*Proof.* Clearly, one may assume that  $O$  is connected.

We start by noting that  $v$  has the following properties:

- $v$  is continuous in  $O$  (this relies on the equation satisfied by  $v$ , on Theorem 2 in [20] and on Sobolev embeddings).

- $|v| \leq 1$ . Indeed, consider the test function  $v' = \begin{cases} v, & \text{if } |v| \leq 1 \\ \frac{v}{|v|}, & \text{if } |v| > 1 \end{cases}$ . Since  $v'$  has more energy than  $v$ ,

we find that  $|v| \leq 1$  a. e. and thus  $|v| \leq 1$ .

Without loss of generality, we may assume that, for some  $\mu \in (0, 1)$ , we have  $C = \{z \in B_1(0) : \Re z = \mu\}$  and  $S = \{z \in \overline{B_1(0)}; \Re z \geq \mu\}$ .

The map  $w := |\Re v| + i\Im v$  equals  $g$  on  $\partial O$  and has the same energy as  $v$ . Thus  $w$  minimizes  $\tilde{F}$ . In particular,  $w$  is continuous. Therefore, if we prove that  $w(O) \subset S$ , we will have  $v(O) \subset S$ . In conclusion, we reduced the problem to the case where  $\Re v \geq 0$ .

Let  $P$  be the orthogonal projection on  $S$ . When  $z \in B_1(0) \cap \{\Re z \geq 0\}$ , we have

$$P(z) = \begin{cases} z, & \text{if } \Re z \geq \mu \\ \mu + i\Im z, & \text{if } |\Im z| \leq \sqrt{1 - \mu^2} \text{ and } \Re z < \mu \\ \mu + i(\text{sign } \Im z)\sqrt{1 - \mu^2}, & \text{if } |\Im z| > \sqrt{1 - \mu^2} \text{ and } \Re z < \mu \end{cases} \quad (12)$$

One may check easily that

$$|z| \leq |P(z)| \leq 1 \text{ for } z \in B_1(0) \cap \{\Re z \geq 0\}. \quad (13)$$

Set  $\psi(z) := P(w(z))$ , which equals  $g$  on  $\partial O$ . Since  $P$  is 1-Lipschitz, we have  $|\nabla \psi| \leq |\nabla w|$ . On the other hand, (13) implies  $|w| \leq |\psi| \leq 1$ . Consequently,  $\tilde{F}(\psi) \leq \tilde{F}(w)$ .

Since  $w$  is a minimizer,  $\psi$  is also a minimizer. Using the previous pointwise estimates and the equality of the energies, one may conclude that  $|\psi(z)| = |w(z)|$  for each  $z$  (by continuity of  $w$  and  $\psi$ ) and  $|\nabla \psi| = |\nabla w|$  a.e.

By solving the equation  $|z| = |P(z)|$ , we see that  $|\psi| = |w|$  implies that  $w$  takes values in  $S \cup V$ , where  $V := \{z' \in \mathbb{S}^1 \mid 0 \leq \Re z' < \mu\}$ .

We have to prove that  $U := w^{-1}(V) = \emptyset$ . We argue by contradiction and assume  $U \neq \emptyset$ . Then  $U$  is open, since  $U = O \setminus w^{-1}(S)$  with  $S$  a closed set.

We first prove that  $w$  is locally constant in  $U$ . Indeed, in  $U$ ,  $w$  satisfies  $\text{div}(\alpha \nabla w) = 0$ . Since  $w \in H^1(U, \mathbb{S}^1)$ , we may write, in  $U$ ,  $w = e^{i\varphi}$ , where  $\varphi \in H^1$  [8]. Let  $\zeta \in C_c^\infty(U)$ . If we multiply the equation  $\text{div}(\alpha \nabla(\cos \varphi)) = 0$  by  $\zeta \cos \varphi$  and the equation  $\text{div}(\alpha \nabla(\sin \varphi)) = 0$  by  $\zeta \sin \varphi$  and add the two results, we obtain  $\int \alpha \zeta |\nabla \varphi|^2 = 0$ , so that  $\varphi$  (and thus  $w$ ) is locally constant in  $U$ .

Let  $W \neq \emptyset$  be a connected component of  $U$ , so that  $w \equiv s \in V$  in  $W$ . Consider the non empty set  $Y := w^{-1}(\{s\})$ . Then  $Y$  is open in  $O$  (since  $w$  is locally constant in  $U$ ), and clearly  $Y$  is closed in  $O$ . Therefore,  $Y = O$ , i. e.,  $w \equiv s$  in  $O$ . This contradicts the facts that  $g : \partial O \rightarrow S$ ,  $\text{tr}_{\partial O} w = g$  and  $s \notin S$ .  $\square$

### 2.1.2 Theorem 1 holds close to the boundary

We prove that, inside an  $o_\varepsilon(1)$ -strip along  $\partial\Omega$  and for sufficiently small  $\varepsilon$ ,  $|v_\varepsilon|$  is arbitrarily close to 1.

The key argument will be provided by the following lemma.

**Lemma 3.** *Let  $(x_\varepsilon)_{\varepsilon>0} \subset \Omega$  be such that  $r_\varepsilon := \text{dist}(x_\varepsilon, \partial\Omega) \rightarrow 0$ . Then we have, for all  $C \geq 2$ ,  $F_\varepsilon(v_\varepsilon, B_{Cr_\varepsilon}(x_\varepsilon)) \rightarrow 0$ .*

*Proof.* Note that it suffices to prove the result for  $C = 2$ . (For larger values of  $C$ , it suffices to replace  $x_\varepsilon$  by the point at distance  $\frac{C+1}{2}r_\varepsilon$  from  $x_\varepsilon$  and at distance  $\frac{C+3}{2}r_\varepsilon$  from  $\partial\Omega$ .)

Let  $\delta > 0$ . We will prove that there is  $\varepsilon_\delta > 0$  such that for  $\varepsilon < \varepsilon_\delta$ , we have  $F_\varepsilon(v_\varepsilon, B_{2r_\varepsilon}(x_\varepsilon)) \leq \delta$ . For the convenience of the reader, the proof is divided into four steps.

#### Step 1: Flattening of $\Omega$ and choice of a good triangle

Without loss of generality, we may assume that  $\partial\Omega$  is flat near  $x_\varepsilon$ . The general case is obtained by flattening the boundary. This will affect the equation satisfied by  $v_\varepsilon$  and the energy associated with it, but not the conclusion of the proof below (which relies only on energy bounds and qualitative conclusions derived from the equation of  $v_\varepsilon$ ). From now on, we assume that  $\Omega \subset \mathbb{R}_+^2$  and  $\partial\Omega \subset \mathbb{R}$  in a neighborhood of fixed size of  $x_\varepsilon$ . We also assume, without loss of generality, that  $x_\varepsilon = (0, r_\varepsilon)$ .

For  $\ell > 0$ , we set

$$T_\ell := \{(s, t) \mid t = s + \ell, s \in [-\ell, 0]\} \cup \{(s, t) \mid t = -s + \ell, s \in (0, \ell)\} \subset \mathbb{R}_+^2$$

(thus  $T_\ell$  is the union of two segments).

Denote by  $\omega_\ell$  the (solid) triangle enclosed by  $T_\ell$  and  $\mathbb{R}$ . Then we have  $B(x_\varepsilon, 2r_\varepsilon) \cap \Omega \subset \omega_{5r_\varepsilon}$ .

Our goal is to construct, for an appropriate small  $\ell$  (depending on  $x_\varepsilon$  and such that  $\ell > 5r_\varepsilon$ ) a test function  $h : \omega_\ell \rightarrow \mathbb{C}$  such that  $\text{tr}_{\partial\omega_\ell} h = \text{tr}_{\partial\omega_\ell} v_\varepsilon$  and  $F_\varepsilon(h, \omega_\ell) \rightarrow 0$ . Since  $v_\varepsilon$  is a global minimizer of  $F_\varepsilon$  in  $H_g^1(\Omega, \mathbb{C})$ , it follows that  $v_\varepsilon$  is also a minimizer of  $F_\varepsilon$  in  $H_{\text{tr}_{\partial\omega_\ell} v}^1(\omega_\ell, \mathbb{C})$ .

Our goal is to prove that  $F_\varepsilon(v_\varepsilon, \omega_\ell) \rightarrow 0$ . Since  $B_{2r_\varepsilon}(x_\varepsilon) \subset \omega_\ell$ , the lemma will follow.

Let  $\varepsilon_1 > 0$  be such that for  $\varepsilon < \varepsilon_1$ ,  $5r < \sqrt{r}$ . Let  $w$  be the harmonic extension of  $g$  to  $\Omega$ . We claim that

1.  $\exists C_1 > 0$  (independent of  $\varepsilon$ ) and  $\exists \ell \in (5r, \sqrt{r})$  such that

$$\ell \int_{T_\ell} \left\{ |\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2} (1 - |v_\varepsilon|^2)^2 + |\nabla w|^2 \right\} \leq \frac{C_1}{|\ln r|}, \quad (14)$$

2.  $|v_\varepsilon(x)| \xrightarrow{x \in T_\ell, x \rightarrow \partial\Omega} 1$ ,

3.  $|v_\varepsilon| \geq 1/2$  on  $T_\ell$  (for sufficiently small  $\varepsilon$ ).

The claim 1. comes directly from (9) and a mean value argument.

Claim 2. is proved in Lemma 4 below, using an argument essentially due to Boutet de Monvel and Gabber [14].



In order to prove Claim 3., we start by noting that

$$\left( \int_{T_\ell} |\partial_\tau |v_\varepsilon|| \right)^2 \leq C\ell \int_{T_\ell} |\partial_\tau |v_\varepsilon||^2 \leq C\ell \int_{T_\ell} |\nabla v_\varepsilon|^2 \leq \frac{C'}{|\ln r|}. \quad (15)$$

Consequently, there exists  $0 < \varepsilon_2 \leq \varepsilon_1$  such that, for  $\varepsilon < \varepsilon_2$ , the variation of  $|v_\varepsilon|$  on  $T_\ell$  is smaller than  $1/2$ . Since, by Lemma 4, we have  $|v_\varepsilon| = 1$  at the endpoints of  $T_\ell$ , we obtain that Claim 3. holds.

**Lemma 4.** *Let  $\alpha \in W^{1,\infty}(\Omega)$ ,  $\beta \in L^\infty(\Omega; \mathbb{R}_+)$  be such that  $\inf \alpha > 0$ . Let  $v$  be a critical point of  $u \mapsto \int \alpha |\nabla u|^2 + \int \beta(1 - |u|^2)^2$  in the class  $H_g^1(\Omega)$ , where  $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$ . Then  $|v| \in C(\overline{\Omega})$ .*

*Proof.* We first note that  $|v| \leq 1$  a. e. (by the maximum principle. This is obtained, e. g., by noting that  $U := 1 - |v|^2$  satisfies  $\begin{cases} -\operatorname{div}(\alpha \nabla U) + 4\beta |v|^2 U = 2\alpha |\nabla v|^2 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases}$ , and consequently  $U \geq 0$  in  $\Omega$ .) We next split  $v = v_1 + v_2$ , where  $v_1$  is the harmonic extension of  $g$ . It follows that  $v_2$  satisfies  $\begin{cases} -\Delta v_2 = \alpha^{-1} \nabla \alpha \cdot \nabla v + 2\alpha^{-1} \beta v(1 - |v|^2) & \text{in } \Omega \\ v_2 = 0 & \text{on } \partial\Omega \end{cases}$ . Since  $|v| \leq 1$  and  $\alpha \in W^{1,\infty}$ , we obtain  $v_2 \in H^2(\Omega) \cap H_0^1 \subset C_0(\overline{\Omega})$ . On the other hand, we have  $v_1 \in C(\Omega)$  and  $|v_1| \in C(\overline{\Omega})$  (the last point is essentially due to Boutet de Monvel and Gabber [14]; see also [11], Theorem A.3.2). Therefore, we have  $|v| \in C(\overline{\Omega})$ .  $\square$

Now that  $\ell$  was properly chosen, we construct our test function  $h$ . This function will coincide with  $v_\varepsilon$  outside  $\omega_\ell$ . Therefore, we will only explain how to construct  $h$  inside  $\omega_\ell$ . In order to obtain a globally  $H^1$ -map, we will set  $h$  equal  $v_\varepsilon$  on  $T_\ell$ . Let  $h$  be of the form  $h = \rho e^{i\psi}$ ; in order to have  $h = v_\varepsilon$  on  $T_\ell$ , we will make sure that  $\rho = |v|$  and  $e^{i\psi} = \frac{v_\varepsilon}{|v_\varepsilon|}$  on  $T_\ell$ . In Step 2, we construct  $\rho$ . In Step 3, we construct  $\psi$ . Finally, in Step 4 we estimate the energy of  $h$  and conclude.

**Step 2 :** Choice of the modulus  $\rho$  of the test function  $h$

Let  $\rho : \overline{\omega}_\ell \rightarrow [0, 1]$  be defined by

$$\rho(s, t) = \begin{cases} \frac{t}{s + \ell} (|v_\varepsilon(s, s + \ell)| - 1) + 1, & \text{if } s < 0 \\ \frac{t}{-s + \ell} (|v_\varepsilon(s, -s + \ell)| - 1) + 1, & \text{if } s > 0 \end{cases}.$$

Clearly,  $\rho \in H^1(\omega_\ell, [0, 1])$ ,  $\rho = |v_\varepsilon|$  on  $T_\ell$  and  $\rho = 1$  on  $\partial\omega_\ell \cap \partial\Omega$ .

For further use, we estimate  $\int_{\omega_\ell} \left\{ |\nabla \rho|^2 + \frac{1}{\varepsilon^2} (1 - \rho^2)^2 \right\}$ . We denote  $\omega_\ell^- = \{x = (s, t) \in \omega_\ell \mid s < 0\}$

(this is the left half of the triangle  $\omega_\ell$ ). We will estimate the quantity  $\int_{\omega_\ell^-} \left\{ |\nabla \rho|^2 + \frac{1}{\varepsilon^2} (1 - \rho^2)^2 \right\}$ .

By symmetry, a similar estimate will hold in  $\omega_\ell^+ := \omega_\ell \setminus \overline{\omega_\ell^-}$ , and thus in  $\omega_\ell$ .

We have

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_{\omega_\ell^-} (1 - \rho^2)^2 &\leq \frac{4}{\varepsilon^2} \int_{\omega_\ell^-} (1 - \rho)^2 \leq \frac{C}{\varepsilon^2} \int_{-\ell}^0 ds \int_0^{\ell+s} \frac{t^2}{(s+\ell)^2} (|v_\varepsilon(s, s+\ell)| - 1)^2 dt \\ &\leq \frac{C\ell}{\varepsilon^2} \int_{-\ell}^0 (|v_\varepsilon(s, s+\ell)| - 1)^2 ds \leq \frac{C\ell}{\varepsilon^2} \int_{T_\ell} (|v_\varepsilon| - 1)^2 ds \leq \frac{C}{|\ln r|}. \end{aligned}$$

(The last inequality comes from Claim 1.)

In order to estimate  $\int_{\omega_\ell^-} |\nabla \rho|^2$ , we start from the identity

$$\int_{\omega_\ell^-} |\nabla \rho|^2 = \int_{-\ell}^0 ds \int_0^{\ell+s} dt \{ |\partial_s \rho|^2 + |\partial_t \rho|^2 \}.$$

On the one hand,

$$\begin{aligned} \int_{-\ell}^0 ds \int_0^{\ell+s} dt |\partial_t \rho|^2 &= \int_{-\ell}^0 \frac{(|v_\varepsilon(s, s+\ell)| - 1)^2}{s+\ell} ds = \int_{-\ell}^0 \frac{ds}{s+\ell} \left( \int_{-\ell}^s \frac{d}{dk} [|v_\varepsilon|(k, k+\ell)] \right)^2 \\ &\leq \sqrt{2}\ell \int_{T_\ell} |\nabla v_\varepsilon|^2 \leq \frac{C}{|\ln r|}. \end{aligned}$$

On the other hand, we have

$$|\partial_s \rho|^2 \leq 2 \left( \frac{t^2}{(s+\ell)^4} (|v_\varepsilon(s, s+\ell)| - 1)^2 + \frac{t^2}{(s+\ell)^2} (\nabla |v_\varepsilon|(s, s+\ell) \cdot (1, 1))^2 \right) = 2(A_1 + A_2).$$

Since

$$\int_{-\ell}^0 \int_0^{\ell+s} A_1 \leq \int_{-\ell}^0 \int_0^{\ell+s} \frac{1}{(s+\ell)^2} (|v_\varepsilon(s, s+\ell)| - 1)^2 = \int_{-\ell}^0 \frac{1}{s+\ell} (|v_\varepsilon(s, s+\ell)| - 1)^2 \leq \frac{C}{|\ln r|}$$

and

$$\int_{-\ell}^0 \int_0^{\ell+s} A_2 \leq 2\ell \int_{T_\ell} |\nabla v_\varepsilon|^2 \leq \frac{C}{|\ln r|},$$

we find that  $\int_{\omega_\ell} |\nabla \rho|^2 \leq \frac{C}{|\ln r|}$ . In conclusion, the following estimate holds:

$$\int_{\omega_\ell} \left\{ |\nabla \rho|^2 + \frac{1}{\varepsilon^2} (1 - \rho^2)^2 \right\} \leq \frac{C}{|\ln r|}. \quad (16)$$

### Step 3 : Construction of an auxiliary phase $\psi$

Recall that  $|w(z)| \rightarrow 1$  uniformly as  $z \rightarrow \partial\Omega$  [11]. Thus, there is some  $0 < \varepsilon_3 \leq \varepsilon_2$  such that for  $\varepsilon < \varepsilon_3$  we have  $|w| \geq 1/2$  in  $\omega_\ell$ . For  $\varepsilon < \varepsilon_3$ , we may write, in  $\omega_\ell$ ,  $w = |w|e^{i\varphi}$  with  $\varphi \in H^1(\omega_\ell, \mathbb{R})$ . Note that, by choice of  $\ell$ , we have  $|v_\varepsilon| \geq 1/2$  on  $T_\ell$  and  $v_\varepsilon \in H^1(T_\ell)$ . Therefore, we may write  $v_\varepsilon = |v_\varepsilon|e^{i\phi}$  on  $T_\ell$ , with  $1/2 \leq |v_\varepsilon| \leq 1$  and  $\phi \in H^1(T_\ell)$ .

Since  $v_\varepsilon - w \in C(\overline{\Omega})$  (cf the proof of Lemma 4) and  $v_\varepsilon = w$  on  $\partial\Omega$ , it follows that  $\lim_{z \rightarrow \partial\Omega} (v_\varepsilon - w)(z) = 0$ . Therefore, we have

$$\lim_{\substack{z \rightarrow \partial\Omega \\ z \in T_\ell \cap \partial\omega_\ell^-}} e^{i(\phi(z) - \varphi(z))} = \lim_{\substack{z \rightarrow \partial\Omega \\ z \in T_\ell \cap \partial\omega_\ell^+}} e^{i(\phi(z) - \varphi(z))} = 1.$$

Consequently, there are  $k_+, k_- \in \mathbb{Z}$  such that

$$\lim_{\substack{z \rightarrow \partial\Omega \\ z \in T_\ell \cap \partial\omega_\ell^-}} \frac{\phi(z) - \varphi(z)}{2\pi} = k_- \quad \text{and} \quad \lim_{\substack{z \rightarrow \partial\Omega \\ z \in T_\ell \cap \partial\omega_\ell^+}} \frac{\phi(z) - \varphi(z)}{2\pi} = k_+.$$

By (14) and the fact that  $|v_\varepsilon|, |w| \geq \frac{1}{2}$  on  $T_\ell$ , we obtain  $\ell \int_{T_\ell} \{|\nabla\phi|^2 + |\nabla\varphi|^2\} \leq \frac{C}{|\ln r|}$ .

Thus, for small  $\varepsilon$ , the variations of  $\phi$  and  $\varphi$  are small on  $\partial\omega_\ell \setminus \partial\Omega$  and consequently, there is  $0 < \varepsilon_4 < \varepsilon_3$  such that for  $\varepsilon < \varepsilon_4$ , we have  $k_- = k_+$ . Without loss of generality, we may assume  $k_- = k_+ = 0$ .

Let  $\psi : \overline{\omega}_\ell \rightarrow \mathbb{R}$  be defined by

1.  $\text{tr}_{\partial\omega_\ell} \psi = \text{tr}_{\partial\omega_\ell} (\phi - \varphi)$ ,
2.  $\psi(s, t) = \begin{cases} \frac{t}{\ell + s} [\phi(s, s + \ell) - \varphi(s, s + \ell)], & \text{if } s < 0 \\ \frac{t}{\ell - s} [\phi(s, -s + \ell) - \varphi(s, -s + \ell)], & \text{if } s > 0 \end{cases}$ .

For further use, we estimate the Dirichlet energy of  $\psi$ . It suffices to estimate the energy in  $\omega_\ell^-$ ; a similar estimate holds in  $\omega_\ell$ .

We have

$$\int_{\omega_\ell^-} |\nabla\psi|^2 = \int_{-\ell}^0 ds \int_0^{\ell+s} dt \{|\partial_s\psi|^2 + |\partial_t\psi|^2\} = B_1 + B_2.$$

First, we obtain, denoting  $\xi = \phi - \varphi$ ,

$$B_1 = \int_{-\ell}^0 \int_0^{\ell+s} |\partial_s\psi|^2 \leq 2 \int_{-\ell}^0 \int_0^{\ell+s} \left\{ \left| \frac{\xi(s, s + \ell)}{\ell + s} \right|^2 + \left| \frac{d}{ds} \xi(s, s + \ell) \right|^2 \right\} = 2(B_{11} + B_{12}).$$

Now

$$\begin{aligned} B_{11} &= \int_{-\ell}^0 \frac{1}{\ell + s} |\xi(s, s + \ell)|^2 \leq \int_{-\ell}^0 \frac{1}{\ell + s} \left| \int_{-\ell}^s \frac{d}{d\alpha} \xi(\alpha, \alpha + \ell) d\alpha \right|^2 \\ &\leq C \int_{-\ell}^0 \int_{T_\ell} |d\xi|^2 \leq \ell \int_{T_\ell} |d\xi|^2 \leq \frac{C}{|\ln r|}. \end{aligned}$$

Next, we have

$$B_{12} = \int_{-\ell}^0 \int_0^{\ell+s} |d\xi|^2(s, s + \ell) \leq \ell \int_{T_\ell} |d\xi|^2 \leq \frac{C}{|\ln r|}.$$

Similarly, we have  $B_2 \leq \frac{C}{|\ln r|}$ .

Finally, we find that

$$\int_{\omega_\ell} |\nabla\psi|^2 \leq \frac{C}{|\ln r|}. \tag{17}$$

**Step 4:** Conclusion (proof of Lemma 3 completed)

Consider the following test function

$$h := \begin{cases} v & \text{in } \Omega \setminus \omega_\ell \\ \rho e^{2(\varphi+\psi)} & \text{in } \omega_\ell \end{cases}.$$

Clearly  $h \in H_g^1$  and

$$F_\varepsilon(v_\varepsilon, B_{2r_\varepsilon}(x_\varepsilon)) \leq F_\varepsilon(v_\varepsilon, \omega_\ell) \leq F_\varepsilon(h, \omega_\ell) \leq \frac{C}{|\ln r|} + 4 \int_{\omega_\ell} |\nabla w|^2. \quad (18)$$

The last estimate follows by combining (16) with (17) and the fact that  $|\nabla h|^2 = |\nabla \rho|^2 + \rho^2 |\nabla(\varphi+\psi)|^2$ .

Since  $\int_{\omega_\ell} |\nabla w|^2 \rightarrow 0$  as  $\varepsilon \rightarrow 0$ , we find that  $F_\varepsilon(v, B_{2r}(x)) < \delta$  for small  $\varepsilon$ .  $\square$

The next result completes the proof of Theorem 1.

**Proposition 2.** *Let  $\varepsilon_n \downarrow 0$  and  $\{x_n\}_n \subset \Omega$  be such that  $\text{dist}(x_n, \partial\Omega) \rightarrow 0$ . Then  $|v_{\varepsilon_n}(x_n)| \rightarrow 1$ .*

*Proof of Proposition 2.* Let  $\delta \in (0, 1)$ . Denote  $d_n := \text{dist}(x_n, \partial\Omega)$  and  $v_n := v_{\varepsilon_n}$ . Since there is  $C_0 > 0$  such that  $F_{\varepsilon_n}(v_n) \leq C_0$ , we may choose  $C_1 > 1$  and  $r_n \in (d_n/C_1, d_n)$  such that

$$\frac{2\pi C_0}{\ln C_1} < \frac{\delta}{10^4} \quad (19)$$

and

$$r_n \int_{\mathcal{C}_n} \left\{ |\nabla v_n|^2 + \frac{1}{\varepsilon_n^2} (1 - |v_n|^2)^2 \right\} \leq \frac{C_0}{\ln C_1}, \text{ with } \mathcal{C}_n = \{x \in \Omega \mid |x - x_n| = r_n\}. \quad (20)$$

As in the proof of 1. in Lemma 1, we have

$$[\text{Var}(v_n, \mathcal{C}_n)]^2 \leq \frac{2\pi C_0}{\ln C_1}. \quad (21)$$

Using (21) and the bound (19), we find that one of the two cases occurs:

1.  $|v_n| \geq 1 - \frac{\delta}{10}$  on  $\mathcal{C}_n$ ,
2.  $|v_n| < 1 - \frac{\delta}{10^3}$  on  $\mathcal{C}_n$ .

In the first case, using (21) and Lemma 2, we obtain  $|v_n(x_n)| \geq 1 - \delta$ .

Assume that for infinitely many  $n$  the second case occurs. Up to subsequence, we may assume that it is true for each  $n$ .

For large  $n$ , let  $y_n := \Pi_{\partial\Omega}(x_n)$  be the orthogonal projection of  $x_n$  on  $\partial\Omega$  and let  $x'_n$  be the intersection point of the segment  $[x_n, y_n]$  with  $\mathcal{C}_n$ . For large  $n$  and for all  $z \in T_n := \left\{ z \in \mathcal{C}_n \mid |x'_n - z| \leq \frac{r}{2} \right\}$  we have

$$|z - w_z| \leq 3d_n. \quad (22)$$

Here,  $w_z$  is the first intersection point with  $\partial\Omega$  of the ray starting from  $x$  and passing through  $z$ .

Note that

$$z \in T_n \Leftrightarrow z = x_n + (x'_n - x_n)e^{i\theta} \text{ with } \theta \in [-\pi/6, \pi/6]. \quad (23)$$

For  $\theta \in [-\pi/6, \pi/6]$  we denote  $I_\theta := [z, w_z]$ , where  $z = z(\theta)$  is given by (23). Since  $|v_n(z)| < 1 - \frac{\delta}{10^3}$  and  $|v_n(w_z)| = 1$  we have

$$\frac{\delta^2}{10^6} \leq \left( \int_{I_\theta} \partial_\tau |v_n| \right)^2 \leq 3d_n \int_{I_\theta} |\partial_\tau v_n|^2. \quad (24)$$

Denote  $A := \bigcup_{\theta \in [-\pi/6, \pi/6]} I_\theta$  and write each  $x \in A$  as  $x = x_n + se^{i\theta}$  ( $s \geq r_n$ ). By (22), (23) and (24) we

have

$$\int_A |\nabla v_n|^2 \geq \int_{-\pi/6}^{\pi/6} d\theta \int_{I_\theta} |\partial_\tau v_n|^2 s \, ds \geq \frac{\pi}{3C_1} \inf_{z \in T_n} d_n \int_{I_\theta} |\partial_\tau v_n|^2 \geq \frac{\pi\delta^2}{9 \cdot 10^6 \cdot C_1}.$$

Since  $C_1$  is independent of  $n$  and  $A \subset B_{3d_n}(x_n)$ , the above estimate contradicts Lemma 3.

Hence, for sufficiently large  $n$ , we have  $|v_n| \geq 1 - \frac{\delta}{10}$  on  $\mathcal{C}_n$ . This estimate together with Lemma 2 implies  $|v_n(x_n)| \geq 1 - \delta$ .  $\square$

## 2.2 A corollary of Theorem 1

From Theorem 1 one may easily prove that the contribution of the modulus is negligible. Indeed we have

**Corollary 1.** *The following hold.*

1. We have  $\int_\Omega \left\{ |\nabla |v_\varepsilon||^2 + \frac{1}{\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \right\} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ .

*In particular, we have  $|v_\varepsilon| \rightarrow 1$  in  $H^1(\Omega)$ .*

2. Assume that (possibly along some subsequence) we have  $\alpha_\varepsilon \rightarrow \kappa$  in  $L^2(\Omega)$ . Write  $g = e^{i\varphi_0}$  (see [7]), where  $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$ . Write, for small  $\varepsilon$ ,  $v_\varepsilon = |v_\varepsilon|e^{i\varphi_\varepsilon}$ , where  $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega, \mathbb{R})$ . Then

$$\varphi_\varepsilon \rightarrow \varphi^* \text{ in } H^1(\Omega), \text{ where } \varphi^* \text{ is the solution of } \begin{cases} -\operatorname{div}(\kappa \nabla \varphi^*) = 0 & \text{in } \Omega \\ \varphi^* = \varphi_0 & \text{on } \partial\Omega \end{cases}.$$

The above statement implicitly uses two results on lifting, for which we refer to [8, 9]. The first one is that each zero degree map  $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$  may be lifted as  $g = e^{i\varphi_0}$  for some  $\varphi_0 \in H^{1/2}(\partial\Omega; \mathbb{R})$ . The second is that each map in  $u \in H^1_g(\Omega; \mathbb{S}^1)$  may be written as  $u = e^{i\varphi}$ , with  $\varphi \in H^1_{\varphi_0}(\Omega; \mathbb{R})$ . Consequently, each map  $u \in H^1_g(\Omega; \mathbb{R}^2)$  such that  $0 < \operatorname{ess\,inf} |u| \leq \operatorname{ess\,sup} |u| < \infty$  may be written as  $u = \rho e^{i\varphi}$ , where  $\rho = |u| \in H^1_1(\Omega; \mathbb{R}_+)$  and  $\varphi \in H^1_{\varphi_0}(\Omega; \mathbb{R})$ .

*Proof.* We start by noting that  $b \leq \kappa \leq 1$ .

Let  $v_\varepsilon$  be a minimizer of  $F_\varepsilon$  in  $H^1_g$ . By Theorem 1, we may write, for small  $\varepsilon$ ,  $v_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ , with  $1/2 \leq \rho_\varepsilon := |v_\varepsilon| \leq 1$  and  $\varphi_\varepsilon \in H^1_{\varphi_0}(\Omega, \mathbb{R})$ .

Recall that  $F_\varepsilon(v_\varepsilon) \leq C_0$  (with  $C_0$  depending only on  $g, \Omega$  and  $b$ ). Thus, for small  $\varepsilon$ , we have  $\int_\Omega |\nabla \varphi_\varepsilon|^2 \leq \frac{8C_0}{b}$ . If we set  $w_\varepsilon := e^{i\varphi_\varepsilon} \in H_g^1$ , then we have

$$F_\varepsilon(v_\varepsilon) = \frac{1}{2} \int_\Omega \left\{ \alpha_\varepsilon (\rho_\varepsilon^2 |\nabla \varphi_\varepsilon|^2 + |\nabla \rho_\varepsilon|^2) + \frac{\beta_\varepsilon}{2\varepsilon^2} (1 - \rho_\varepsilon^2)^2 \right\} \leq F_\varepsilon(w_\varepsilon) = \frac{1}{2} \int_\Omega \alpha_\varepsilon |\nabla \varphi_\varepsilon|^2.$$

Consequently,

$$\int_\Omega \left\{ |\nabla \rho_\varepsilon|^2 + \frac{1}{\varepsilon^2} (1 - \rho_\varepsilon^2)^2 \right\} \leq \frac{2}{b} \int_\Omega (1 - \rho_\varepsilon^2) |\nabla \varphi_\varepsilon|^2 \leq \frac{16C_0}{b^2} \|1 - \rho_\varepsilon^2\|_{L^\infty(\Omega)} \xrightarrow{\varepsilon \rightarrow 0} 0.$$

We now prove 2. We start by noting that  $\varphi_\varepsilon - \varphi^*$  satisfies

$$\begin{cases} -\operatorname{div}[\alpha_\varepsilon \rho_\varepsilon^2 \nabla(\varphi_\varepsilon - \varphi^*)] = \operatorname{div}[(\alpha_\varepsilon \rho_\varepsilon^2 - \kappa) \nabla \varphi^*] & \text{in } \Omega \\ \varphi_\varepsilon - \varphi^* = 0 & \text{on } \partial\Omega \end{cases}.$$

By the Lax-Milgram theorem, we find that

$$\|\nabla(\varphi_\varepsilon - \varphi^*)\|_{L^2} \leq C \|(\alpha_\varepsilon \rho_\varepsilon^2 - \kappa) \nabla \varphi^*\|_{L^2}. \quad (25)$$

We will next use the following simple fact: if  $|f_n| \leq C$  and  $f_n \rightarrow f$  in  $L^2$  and if  $g_n \rightarrow g$  in  $L^2$ , then  $f_n g_n \rightarrow fg$  in  $L^2$ . This implies that  $\alpha_\varepsilon \rho_\varepsilon^2 - \kappa \rightarrow 0$  in  $L^2$  as  $\varepsilon \rightarrow 0$ . Finally, (25) implies that  $\varphi_\varepsilon \rightarrow \varphi^*$  in  $H^1$ .  $\square$

### 2.3 More on the convergence of $v_\varepsilon$

This part provides a more quantitative version of Theorem 1. Specifically, under some additional hypotheses on the boundary data  $g$  or on the behavior of the weight  $\alpha_\varepsilon$ , we derive estimates on the rate of convergence of  $|v_\varepsilon|$  to 1 or derive better convergence of the phase  $\varphi_\varepsilon$  of  $v_\varepsilon$  respectively.

In what follows, we assume that  $g \in W^{1-1/q, q}(\partial\Omega, \mathbb{S}^1)$  for some  $q > 2$ . Let  $\varphi_0 \in W^{1-1/q, q}(\partial\Omega, \mathbb{R})$  be such that  $e^{i\varphi_0} = g$  (for the existence of  $\varphi_0$ , see, e. g., [8]). For a fixed measurable function  $\kappa : \Omega \rightarrow [b, 1]$ , let  $\varphi^* \in W^{1, q}(\Omega, \mathbb{R})$  be the solution of  $\begin{cases} -\operatorname{div}(\kappa \nabla \varphi^*) = 0 & \text{in } \Omega \\ \varphi^* = \varphi_0 & \text{on } \partial\Omega \end{cases}$ .

**Proposition 3.** *There is  $p \in (2, q]$ ,  $\alpha \in (0, 1)$ ,  $C > 0$  (depending only on  $q, b, \Omega$  and  $g$ ) such that, for  $0 < \varepsilon < 1$  and  $v_\varepsilon$  a minimizer of  $F_\varepsilon$  in  $H_g^1$ , we have*

1.  $\{v_\varepsilon\}$  is bounded in  $W^{1, p}$  by a constant  $C$  which depends only on  $g, b$  and  $\Omega$ .
2.  $\{v_\varepsilon\}$  is relatively compact in  $C^{0, \alpha}(\overline{\Omega})$ .
3.  $1 - |v_\varepsilon| \leq C\varepsilon^\gamma$  and  $\int_\Omega \left\{ |\nabla |v_\varepsilon||^2 + \frac{1}{\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \right\} \leq C\varepsilon^\gamma$  with  $\gamma = \frac{2\alpha}{2 + \alpha}$ .

4. Furthermore, if (possibly after passing to a subsequence) we have  $\alpha_\varepsilon \rightarrow \kappa$  in  $L^2$ , then we have  $\varphi_\varepsilon \rightarrow \varphi^*$  in  $W^{1,p}$ .

Here, we write, for small  $\varepsilon$  and in virtue of Theorem 1,  $v_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$ , with  $\varphi_\varepsilon \in H^1_{\varphi_0}$ ,  $\rho_\varepsilon := |v_\varepsilon| \in [1/2, 1]$ .

*Proof.* Let  $\varphi$  be any fixed  $W^{1,q}$ -extension of  $\varphi_0$ . Then  $\varphi_\varepsilon - \varphi$  satisfies

$$\begin{cases} -\operatorname{div} [\alpha_\varepsilon \rho_\varepsilon^2 \nabla(\varphi_\varepsilon - \varphi)] = \operatorname{div}(\alpha_\varepsilon \rho_\varepsilon^2 \nabla \varphi) & \text{in } \Omega \\ \varphi_\varepsilon - \varphi = 0 & \text{on } \partial\Omega \end{cases}. \quad (26)$$

Since

$$\|\alpha_\varepsilon \rho_\varepsilon^2 \nabla \varphi\|_{L^q(\Omega)} \leq C,$$

it follows from Theorem 1 in [20] that there are  $p_1 \in (2, q]$  and  $C > 0$  (depending only on  $b$  and  $\Omega$ ) such that  $\|\nabla(\varphi_\varepsilon - \varphi)\|_{L^{p_1}(\Omega)} \leq C$ . Thus  $\{\varphi_\varepsilon\}$  is bounded in  $W^{1,p_1}(\Omega)$ .

We next prove that  $\|1 - \rho_\varepsilon\|_{L^{p_1/2}} \leq C\varepsilon^2$ . For this purpose, we start with the equation satisfied by  $\rho_\varepsilon$ :

$$\begin{cases} \operatorname{div}(\alpha_\varepsilon \nabla \rho_\varepsilon) + \frac{\beta_\varepsilon}{\varepsilon^2} \rho_\varepsilon (1 - \rho_\varepsilon^2) = \alpha_\varepsilon \rho_\varepsilon |\nabla \varphi_\varepsilon|^2 & \text{in } \Omega \\ 1 - \rho_\varepsilon = 0 & \text{on } \partial\Omega \end{cases}. \quad (27)$$

Let  $\eta_\varepsilon := 1 - \rho_\varepsilon$  and  $p_1 > 2$  be as in the conclusion of Theorem 1 in [20]. Set  $r := p_1/2$  and consider a sequence  $\{\phi_k\} \subset C^\infty([0, 1], [0, 1])$  such that

$$\phi_k \text{ is nondecreasing, } \phi_k(0) = 0 \text{ and } \phi_k(s) \rightarrow |s|^{r-1} \text{ as } k \rightarrow \infty, \forall s \in [0, 1].$$

Let  $A_\varepsilon := \beta_\varepsilon \rho_\varepsilon (1 + \rho_\varepsilon)$ , which satisfies, for small  $\varepsilon$ ,  $3b/4 \leq A_\varepsilon \leq 2$ . Set  $B_\varepsilon := \alpha_\varepsilon \rho_\varepsilon |\nabla \varphi_\varepsilon|^2$ , which is bounded in  $L^{p_1/2}$ . If we multiply (27) by  $\phi_k(\eta_\varepsilon)$ , we find that

$$\int_\Omega \alpha_\varepsilon |\nabla \eta_\varepsilon|^2 \phi'_k(\eta_\varepsilon) + \frac{1}{\varepsilon^2} \int_\Omega A_\varepsilon \eta_\varepsilon \phi_k(\eta_\varepsilon) = \int_\Omega B_\varepsilon \phi_k(\eta_\varepsilon).$$

Consequently, we have

$$\int_\Omega \eta_\varepsilon \phi_k(\eta_\varepsilon) \leq C\varepsilon^2 \int_\Omega B_\varepsilon \phi_k(\eta_\varepsilon). \quad (28)$$

Note that, in (28), the constant  $C$  depends only on  $b$ . By letting  $k \rightarrow \infty$ , we obtain, with  $s$  being the conjugate exponent of  $r$ , that

$$\int_\Omega \eta_\varepsilon^r \leq C\varepsilon^2 \int_\Omega B_\varepsilon \eta_\varepsilon^{r-1} \leq C\varepsilon^2 \left( \int_\Omega \eta_\varepsilon^r \right)^{\frac{1}{s}} \|B_\varepsilon\|_{L^r}.$$

This implies that  $\|1 - \rho_\varepsilon\|_{L^{p_1/2}} \leq C\varepsilon^2$  which we wanted to prove.

Going back to (27), we observe that  $\eta_\varepsilon$  satisfies  $\operatorname{div}(\alpha_\varepsilon \nabla \eta_\varepsilon) = h_\varepsilon$ , where  $h_\varepsilon$  is bounded in  $L^{p_1/2}(\Omega)$ . Using again [20], we find that there is some  $p_2 > 2$  such that  $\nabla \eta_\varepsilon$  is bounded in  $L^{p_2}(\Omega)$ .

It follows that  $v_\varepsilon$  is bounded in  $W^{1,p}(\Omega)$ , with  $p := \min(p_1, p_2) > 2$ .

We next prove that  $|1 - \rho_\varepsilon| \leq C\varepsilon^\gamma$  and  $\int_\Omega \left\{ |\nabla |v_\varepsilon||^2 + \frac{1}{\varepsilon^2} (1 - |v_\varepsilon|^2)^2 \right\} \leq C\varepsilon^\gamma$ , where  $\gamma := \frac{p-2}{p-1}$ .

Indeed, let  $\alpha := 1 - \frac{2}{p}$ , so that  $v_\varepsilon$  is bounded in  $C^\alpha(\bar{\Omega})$  and  $\int_{\Omega} (1 - \rho_\varepsilon) \leq C\varepsilon^2$ . Let  $x_0 = x_0(\varepsilon)$  be a minimum point of  $\rho_\varepsilon$  in  $\bar{\Omega}$ . Since  $\Omega$  is smooth, for  $r > 0$  sufficiently small we have  $|B_r(x_0) \cap \Omega| \geq Cr^2$ . It follows that

$$C\varepsilon^2 \geq \int_{B_r(x_0)} (1 - \rho_\varepsilon) \geq C(1 - \rho_\varepsilon(x_0) - Cr^\alpha)r^2.$$

With  $r := \varepsilon^{\frac{2}{\alpha+2}}$ , we find that  $1 - \rho_\varepsilon(x_0) = \sup_{\bar{\Omega}} \{1 - \rho_\varepsilon\} \leq C\varepsilon^\gamma$ .

The above estimate together with the inequality  $F_\varepsilon(v_\varepsilon) \leq F_\varepsilon(e^{i\varphi_\varepsilon})$  yield the bound on  $\nabla\rho_\varepsilon$ :

$$\int_{\Omega} \left\{ \alpha_\varepsilon |\nabla\rho_\varepsilon|^2 + \frac{\beta_\varepsilon}{2\varepsilon^2} (1 - \rho_\varepsilon^2)^2 \right\} \leq \int_{\Omega} \alpha_\varepsilon (1 - \rho_\varepsilon^2) |\nabla\varphi_\varepsilon|^2 \leq C\varepsilon^\gamma.$$

Finally, 4. follows from the equation

$$\begin{cases} -\operatorname{div} [\alpha_\varepsilon \rho_\varepsilon^2 \nabla(\varphi_\varepsilon - \varphi^*)] = \operatorname{div} [(\alpha_\varepsilon \rho_\varepsilon^2 - \kappa) \nabla\varphi^*] & \text{in } \Omega \\ \varphi_\varepsilon - \varphi^* = 0 & \text{on } \partial\Omega \end{cases}.$$

Indeed, since  $(\alpha_\varepsilon \rho_\varepsilon^2 - \kappa) \nabla\varphi^* \rightarrow 0$  in  $L^{p_3}(\Omega)$  for a suitable  $p_3$  such that  $\nabla\varphi^* \in L^{p_3}$ , we obtain, using again [20], that  $\varphi_\varepsilon \rightarrow \varphi^*$  in  $W^{1,p_4}$ , for a suitable  $p_4 > 2$ . We conclude by choosing  $p := \min\{p_1, \dots, p_4\}$ .  $\square$

### 3 The Ginzburg-Landau functional with a periodic pinning term

In this part, we apply the results obtained in the previous section to the study of a GL energy with a discontinuous periodic pinning term. Inside unit square  $Y = [0, 1]^2$ , consider a smooth subset  $\omega \prec Y$ , which will play a role of inclusion (or impurity). The relative size of this inclusion (with respect to the size of the square) will be controlled by some parameter  $\lambda > 0$  in the following way: for  $x_0 \in \omega$ , we set  $\omega_\lambda = \lambda\omega + (1 - \lambda)x_0$ . We now define the pinning term  $a = a(x, \lambda)$  so that it takes different constant values inside and outside of the inclusion:

$$a(x, \lambda) = \begin{cases} b, & \text{if } x \in \omega_\lambda \\ 1, & \text{if } x \in Y \setminus \omega_\lambda \end{cases}, \quad (29)$$

where  $b \in (0, 1)$  is a fixed (material) parameter. We extend  $a$  to a periodic function in  $\mathbb{R}^2$ .

The analysis we develop here could apply to the more complicated situation where  $x_0$  is allowed to depend on  $\lambda$ ; however, we will not pursue in this direction here.

Let  $\Omega \subset \mathbb{C}$  be a smooth, bounded, simply connected domain. For  $1 > \delta > 0$ , denote  $\{C_n^\delta, n \geq 1\}$  a partition of  $\mathbb{R}^2$  into squares with side  $\delta$ ; for simplicity, we suppose that the origin is an edge of one of the squares. We may assume, with no loss of generality, that the squares that lie inside  $\Omega$  are labelled

$$C_n^\delta \text{ with } 1 \leq n \leq N_\delta. \text{ Denote } \Omega_\delta := \bigcup_{n=1}^{N_\delta} C_n^\delta.$$



We define the pinning term in  $\Omega$  as

$$a_\varepsilon(x) = \begin{cases} a(x/\delta, \lambda), & \text{if } x \in \bar{\Omega}_\delta \\ 1, & \text{if } x \in \Omega \setminus \bar{\Omega}_\delta \end{cases};$$

the notation  $a_\varepsilon$  is justified by the fact that we will later let  $\delta$  depend on the GL parameter  $\varepsilon$ . The following energy will be associated with this pinning term:

$$E_\varepsilon(u) = \frac{1}{2} \int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2} (a_\varepsilon^2 - |u|^2)^2 \right\}.$$

Under the Dirichlet condition  $\text{tr}_{\partial\Omega} u = 1$ , one has the existence of the unique minimizer  $U_\varepsilon$  of  $E_\varepsilon$  [18]. The following lemma is straightforward.

**Lemma 5.** *There exists a constant  $C$  (independent of  $\varepsilon \in (0, 1)$ ) such that*

$$E_\varepsilon(U_\varepsilon) \leq C\lambda \min\left(\frac{1}{\varepsilon\delta}, \frac{\lambda}{\varepsilon^2}\right)$$

and

$$|\nabla U_\varepsilon| \leq \frac{C}{\varepsilon}.$$

When  $\varepsilon < \lambda\delta$ , the above lemma is obtained by considering as a test function an  $\varepsilon$ -regularization of  $a_\varepsilon$ . When  $\varepsilon \geq \lambda\delta$ , it suffices to estimate the energy of the test function 1.

As explained in [18], if  $u$  is of modulus 1 on  $\partial\Omega$  and we set  $v := u/U_\varepsilon$ , then the energy  $E_\varepsilon$  decouples as follows:

$$E_\varepsilon(u) = E_\varepsilon(U_\varepsilon) + F_\varepsilon(v),$$

where

$$F_\varepsilon(v) := \frac{1}{2} \int_\Omega \left\{ U_\varepsilon^2 |\nabla v|^2 + \frac{U_\varepsilon^4}{2\varepsilon^2} (1 - |v|^2)^2 \right\}.$$

We next note that, by the maximum principle, we have  $b \leq U_\varepsilon \leq 1$ . Thus  $F_\varepsilon$  satisfies the assumptions of Theorem 1, Corollary 1 and Proposition 3. Therefore, if we let  $u_\varepsilon$  minimize  $E_\varepsilon$  in  $H_g^1$ , where  $g : \partial\Omega \rightarrow \mathbb{S}^1$  is of zero degree, if  $U_\varepsilon$  minimizes  $E_\varepsilon$  in  $H_1^1$  and if we decompose  $u_\varepsilon = U_\varepsilon v_\varepsilon$ , then the conclusions of these results apply to  $v_\varepsilon$ .

To be more specific, we fix  $g \in H^{1/2}(\partial\Omega, \mathbb{S}^1)$  such that  $\text{deg}_{\partial\Omega}(g) = 0$ . Then:

1. there is some  $\varphi_0 \in H^{1/2}(\Omega, \mathbb{R})$  is such that  $g = e^{i\varphi_0}$
2. we decompose a minimizer  $u_\varepsilon$  of  $E_\varepsilon$  in  $H_g^1$  as  $u_\varepsilon = U_\varepsilon v_\varepsilon$ , where  $U_\varepsilon$  minimizes  $E_\varepsilon$  in  $H_1^1$  and  $v_\varepsilon$  minimizes  $F_\varepsilon$  in  $H_g^1$
3. using Theorem 1 we have, for small  $\varepsilon$ ,  $|u_\varepsilon| \geq b/2$ . Thus we may decompose, for small  $\varepsilon$ ,  $u_\varepsilon = |u_\varepsilon| e^{i\varphi_\varepsilon}$  with  $\varphi_\varepsilon \in H_{\varphi_0}^1(\Omega, \mathbb{R})$
4. consequently, for small  $\varepsilon$  we have  $v_\varepsilon = |v_\varepsilon| e^{i\varphi_\varepsilon}$  with  $|u_\varepsilon| = U_\varepsilon |v_\varepsilon|$ .

From Corollary 1, we know that  $|v_\varepsilon| \rightarrow 1$  uniformly and in  $H^1$ . Consequently, we will obtain the asymptotics of  $u_\varepsilon$  from the one of  $U_\varepsilon$  and of  $\varphi_\varepsilon$ .

The remaining part of this section is devoted to the asymptotic analysis of  $U_\varepsilon$  and  $v_\varepsilon$ ; as a byproduct, this will give the asymptotics of  $u_\varepsilon$ . It turns out that the analysis is governed by the relation between  $\varepsilon$  and  $\delta$ , as well as by the size of  $\lambda$ . Possibly after passing to subsequences and rescaling, we may assume, with no loss of generality, that we are in one of the four following cases:

Section 3.1:  $\lambda \rightarrow 0$ , the dilute case,

Section 3.2:  $\lambda = 1, \delta = \varepsilon$ , the critical case,

Section 3.3:  $\lambda = 1, \varepsilon \ll \delta$ , the physical case,

Section 3.4:  $\lambda = 1, \delta \ll \varepsilon$ , the non-physical case.

### 3.1 The dilute limit $\lambda \rightarrow 0$

#### 3.1.1 Behavior of $U_\varepsilon$

In this case, the energy bound given by Lemma 5 immediately implies

**Proposition 4.** *We have*

$$U_\varepsilon \rightarrow 1 \text{ in } L^2(\Omega). \quad (30)$$

#### 3.1.2 Limit of $\varphi_\varepsilon$

**Proposition 5.** *Let  $\varphi_*$  be the harmonic extension of  $\varphi_0$  in  $\Omega$ . Then, as  $\varepsilon \rightarrow 0$ ,*

1.  $\varphi_\varepsilon \rightarrow \varphi_*$  in  $H^1$
2. *if, in addition, there is some  $q > 2$  such that  $g \in W^{1-1/q, q}(\partial\Omega)$ , then we have  $\varphi_\varepsilon \rightarrow \varphi_*$  in  $W^{1, p}$  for some suitable  $p \in (2, q]$ .*

*Proof.* The first part is a direct consequence of Corollary 1 and of Proposition 4. The second part is a direct consequence of Propositions 3 and 4.  $\square$

### 3.2 The case $\lambda = 1, \delta = \varepsilon$

#### 3.2.1 Limit of $U_\varepsilon$

Recall that  $Y := [0, 1]^2$ . Let

$$H_{\text{per}}^1(Y, \mathbb{R}) = \{u \in H^1(Y, \mathbb{R}) \mid \text{the extension by } Y\text{-periodicity of } u \text{ in } \mathbb{R}^2 \text{ is in } H_{\text{loc}}^1(\mathbb{R}^2)\}.$$

We define similarly  $H_{\text{per}}^1(Y, \mathbb{C})$ . For simplicity, we ignore the reference to  $\mathbb{R}$  or  $\mathbb{C}$  when irrelevant. Note that  $u \in H^1(Y)$  extends to a  $Y$ -periodic  $H_{\text{loc}}^1$ -map if and only if

$$\begin{aligned} \text{tr}_{\{y_1=0\}} u(0, \cdot) &= \text{tr}_{\{y_1=1\}} u(1, \cdot) \text{ and } \text{tr}_{\{y_2=0\}} u(\cdot, 0) = \text{tr}_{\{y_2=1\}} u(\cdot, 1) \\ \Leftrightarrow y_1(1 - y_1) [u(y_1, y_2) - u(y_1, 1 - y_2)] &+ y_2(1 - y_2) [u(y_1, y_2) - u(1 - y_1, y_2)] \in H_0^1(Y). \end{aligned}$$

Using these characterizations of  $H_{\text{per}}^1(Y)$ , we find that  $H_{\text{per}}^1(Y)$  is weakly  $H^1$ -closed. (For more properties of  $H_{\text{per}}^1(Y)$ , see, e. g., [13], part 3.4.)

It follows that there exists  $\hat{u}$  which is a minimizer of

$$\mathcal{E}(u) = \frac{1}{2} \int_Y \left\{ |\nabla u|^2 + \frac{1}{2}(u^2 - a^2)^2 \right\} \text{ in the class } H_{\text{per}}^1(Y, \mathbb{R}).$$

**Theorem 2.** *The following hold:*

1. *The functional  $\mathcal{E}$  has a unique (modulo multiplication by  $\pm 1$ ) minimizer  $\hat{u}$  in  $H_{\text{per}}^1(Y, \mathbb{R})$ . Among the (exactly) two minimizers, one is positive, the other one negative*
2. *If  $\hat{u}$  is the positive minimizer of  $\mathcal{E}$  in  $H_{\text{per}}^1(Y, \mathbb{R})$ , then we have  $U_\varepsilon \rightharpoonup \int_Y \hat{u}$  in  $L^2(\Omega)$  as  $\varepsilon \rightarrow 0$ .*

*Proof.* We first investigate property 1. This is done via the following two lemmas.

**Lemma 6.** *The energy functional  $\mathcal{E}$  admits a positive global minimizer in  $H_{\text{per}}^1(Y, \mathbb{R})$ . Furthermore, all global minimizers have constant sign and satisfy*

$$-\Delta \hat{u} = \hat{u}(a^2 - \hat{u}^2) \text{ in } Y, \quad (31)$$

$$b \leq |\hat{u}| \leq 1, \quad (32)$$

$$\partial_\nu \hat{u}(0, y_2) = -\partial_\nu \hat{u}(1, y_2) \text{ and } \partial_\nu \hat{u}(y_1, 0) = -\partial_\nu \hat{u}(y_1, 1). \quad (33)$$

*Proof.* (31) is clear. In order to prove (32), let  $u \in H_{\text{per}}^1(Y, \mathbb{R})$  minimize  $\mathcal{E}$ . Let

$$v := \begin{cases} |u|, & \text{if } b \leq |u| \leq 1 \\ 1, & \text{if } |u| > 1 \\ b, & \text{if } |u| < b \end{cases}.$$

It is clear that  $v \in H_{\text{per}}^1(Y, \mathbb{R})$ . On the other hand, we have

$$\mathcal{E}(v) = \frac{1}{2} \int_{\{b \leq |u| \leq 1\}} \left\{ |\nabla u|^2 + \frac{1}{2}(a^2 - u^2)^2 \right\} + \frac{1}{4} \int_{\{|u| > 1\}} (a^2 - 1)^2 + \frac{1}{4} \int_{\{|u| < b\}} (a^2 - b^2)^2.$$

By the minimality of  $\mathcal{E}(u)$ , we find that  $b \leq |u| \leq 1$  a. e. Noting that, if  $u$  is a minimizer, then  $u$  is continuous, we find that either  $u$  is either positive, or negative. In addition, either  $b \leq u \leq 1$  or  $-1 \leq u \leq -b$ .

We next prove that minimizers  $\hat{u}$  satisfy (33). Indeed, for all  $\phi \in H_{\text{per}}^1(Y) \cap C(\bar{\Omega})$  we have

$$0 = \int_Y \nabla \hat{u} \cdot \nabla \phi - \hat{u} \phi (a^2 - \hat{u}^2) = - \int_{\partial Y} \phi \partial_\nu \hat{u}. \quad (34)$$

We next note that

$$\begin{aligned} 0 = \int_{\partial Y} \phi \partial_\nu \hat{u} &= \int_0^1 (\partial_\nu \hat{u}(0, t) + \partial_\nu \hat{u}(1, t)) \phi(0, t) + \int_0^1 (\partial_\nu \hat{u}(t, 0) + \partial_\nu \hat{u}(t, 1)) \phi(t, 0) \\ &= T_1(\phi_1(t)) + T_2(\phi_2(t)), \end{aligned}$$

with  $\phi_1(t) = \phi(0, t)$  and  $\phi_2(t) = \phi(t, 0)$ .

Since for each  $\psi \in C_0^\infty((0, 1), \mathbb{R})$  there is some  $\phi \in H_{\text{per}}^1(Y, \mathbb{R})$  such that  $\phi_1(t) = \psi(t)$  and  $\phi_2 \equiv 0$ , (34) implies that the map

$$\begin{aligned} T_1 : C_0^\infty((0, 1), \mathbb{R}) &\rightarrow \mathbb{R} \\ \psi &\mapsto \int_0^1 (\partial_\nu \hat{u}(0, t) + \partial_\nu \hat{u}(1, t)) \psi(t) \end{aligned}$$

is identically zero. It follows that  $\partial_\nu \hat{u}(0, t) + \partial_\nu \hat{u}(1, t) = 0$ . A similar argument leads to  $\partial_\nu \hat{u}(t, 0) + \partial_\nu \hat{u}(t, 1) = 0$ .  $\square$

**Lemma 7.** *The energy  $\mathcal{E}$  has a unique positive minimizer in  $H_{\text{per}}^1(Y, \mathbb{R})$ .*

*Proof.* Let  $u, v$  be two positive minimizers and let  $w := v/u \in H_{\text{per}}^1$ . By the energy decoupling formula [18] (which adapts to the periodic case), we have

$$E_\varepsilon(u) = E_\varepsilon(v) = E_\varepsilon(u) + \frac{1}{2} \int \left\{ u^2 |\nabla w|^2 + \frac{1}{2} u^4 (1 - w^2)^2 \right\}.$$

Thus  $w \equiv 1$ , which implies  $u = v$ .  $\square$

As a next (and rather long) step in the proof of Theorem 2, we examine the asymptotic behavior of the energy carried by  $U_\varepsilon$ .

**Proposition 6.** *We have  $\lim_{\varepsilon \rightarrow 0} \varepsilon^2 E_\varepsilon(U_\varepsilon) = |\Omega| \mathcal{E}(\hat{u})$ .*

*Proof.* We use the *unfolding operator* (see [12], definition 2.1). More specifically, we define, for  $p \in (1, \infty)$ ,

$$\begin{aligned} \mathcal{T}_\varepsilon : L^p(\Omega) &\rightarrow L^p(\Omega \times Y) \\ \phi &\mapsto \mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi \left( \varepsilon \left[ \frac{x}{\varepsilon} \right] + \varepsilon y \right), & \text{if } (x, y) \in \hat{\Omega}_\varepsilon \times Y, \\ 0 & \text{if } (x, y) \in \Lambda_\varepsilon \times Y \end{cases}, \\ \hat{\Omega}_\varepsilon &:= \bigcup_{\substack{Y_\varepsilon^K \subset \Omega \\ Y_\varepsilon^K = \varepsilon(K+Y), K \in \mathbb{Z}^2}} \overline{Y_\varepsilon^K}, \Lambda_\varepsilon := \Omega \setminus \hat{\Omega}_\varepsilon \text{ and } \left[ \frac{x}{\varepsilon} \right] := \left( \left[ \frac{x_1}{\varepsilon} \right], \left[ \frac{x_2}{\varepsilon} \right] \right). \end{aligned}$$

Here, for  $s \in \mathbb{R}$ ,  $[s]$  is the integer part of  $s$ .

We will use the following results:

- i)  $\mathcal{T}_\varepsilon$  is linear and continuous, of norm at most 1 ([12], prop. 2.5);

ii)  $\mathcal{T}_\varepsilon(uv) = \mathcal{T}_\varepsilon(u)\mathcal{T}_\varepsilon(v)$  and  $\mathcal{T}_\varepsilon\left(\frac{u}{v}\right) = \frac{\mathcal{T}_\varepsilon(u)}{\mathcal{T}_\varepsilon(v)}\mathbb{1}_{\hat{\Omega}_\varepsilon \times Y}$  ([12], equation (2.2));

iii) "Unfolding criterion for integrals" (u. c. i., [12], prop. 2.6) : If  $\phi_\varepsilon \in L^1(\Omega)$  is such that  $\int_{\Lambda_\varepsilon} |\phi_\varepsilon| \rightarrow 0$ , then we have

$$\int_{\Omega} \phi_\varepsilon - \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\phi) \rightarrow 0;$$

iv)  $\varepsilon\mathcal{T}_\varepsilon(\nabla\phi)(x, y) = \nabla_y\mathcal{T}_\varepsilon(\phi)(x, y)$  for  $\phi \in W^{1,p}(\Omega)$  ([12], equation (3.1)).

As a first step in the proof of Proposition 6, we prove that  $\limsup_{\varepsilon} \varepsilon^2 E_\varepsilon(U_\varepsilon) \leq |\Omega|\mathcal{E}(\hat{u})$ . Indeed, we consider the test function  $H_\varepsilon \in H_1^1$  defined by

$$H_\varepsilon(x) := \rho_\varepsilon(x)\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) + 1 - \rho_\varepsilon(x),$$

with

$$\rho_\varepsilon(x) := \min\left(1, \frac{\text{dist}(x, \partial\Omega)}{\varepsilon}\right) \text{ and } \left\{\frac{x}{\varepsilon}\right\} = \frac{x}{\varepsilon} - \left[\frac{x}{\varepsilon}\right] \in Y.$$

Then we have

$$\mathcal{T}_\varepsilon(H_\varepsilon) \rightarrow \hat{u}(y) \text{ in } L^4(\Omega \times Y) \text{ and } \mathcal{T}_\varepsilon(\varepsilon\nabla H_\varepsilon)(x, y) \rightarrow \nabla_y\hat{u}(y) \text{ in } L^2(\Omega \times Y). \quad (35)$$

Indeed, the first convergence in (35) is a consequence of the fact that  $\mathcal{T}_\varepsilon(H_\varepsilon) - \hat{u}(y)$  is bounded in  $L^\infty(\Omega \times Y)$  and that its support is contained inside  $\{x \in \Omega \mid \text{dist}(x, \partial\Omega) < 3\varepsilon\} \times Y$ . This implies at once that  $\mathcal{T}_\varepsilon(H_\varepsilon) \rightarrow \hat{u}(y)$  in  $L^4(\Omega \times Y)$ .

In order to establish the second convergence in (35), we start from the identity

$$\begin{aligned} \mathcal{T}_\varepsilon(\varepsilon\nabla H_\varepsilon) &= \mathcal{T}_\varepsilon(\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\varepsilon\nabla\left(\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right)\right)\right] + \mathcal{T}_\varepsilon(\varepsilon\nabla\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) - 1\right] \\ &= \nabla_y\hat{u}(y)\mathbb{1}_{\hat{\Omega}_\varepsilon}(x) + (\mathcal{T}_\varepsilon(\rho_\varepsilon) - 1)\nabla_y\hat{u}(y)\mathbb{1}_{\hat{\Omega}_\varepsilon}(x) + \nabla_y\mathcal{T}_\varepsilon(\rho_\varepsilon)\mathcal{T}_\varepsilon\left[\hat{u}\left(\left\{\frac{x}{\varepsilon}\right\}\right) - 1\right] \\ &\equiv \nabla_y\hat{u}(y)\mathbb{1}_{\hat{\Omega}_\varepsilon}(x) + R_\varepsilon. \end{aligned}$$

Since  $\rho_\varepsilon \equiv 1$  in  $\{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}$  and since  $\varepsilon|\nabla\rho_\varepsilon|$  is bounded in  $L^\infty(\Omega)$ , it is clear that the support of  $R_\varepsilon$  is included in  $\{x \in \Omega \mid \text{dist}(x, \partial\Omega) < 3\varepsilon\} \times Y$  and that  $R_\varepsilon$  is bounded in  $L^\infty(\Omega \times Y)$ . Thus  $R_\varepsilon \rightarrow 0$  in  $L^2(\Omega \times Y)$ . It then suffices to note that  $\nabla_y\hat{u}(y)\mathbb{1}_{\hat{\Omega}_\varepsilon}(x) \rightarrow \nabla_y\hat{u}(y)$  in  $L^4(\Omega \times Y)$  in order to obtain the desired convergence result.

Similarly, we have  $\mathcal{T}_\varepsilon(a_\varepsilon)(x, y) \rightarrow a(y)$  in  $L^4(\Omega \times Y)$ .

Finally,

$$\begin{aligned}
\limsup_{\varepsilon} \varepsilon^2 E_{\varepsilon}(U_{\varepsilon}) &\leq \lim_{\varepsilon} \varepsilon^2 E_{\varepsilon}(H_{\varepsilon}) = \lim_{\varepsilon} \frac{1}{2} \int_{\Omega} \left\{ |\varepsilon \nabla H_{\varepsilon}|^2 + \frac{1}{2} (H_{\varepsilon}^2 - a_{\varepsilon}^2)^2 \right\} \\
&= \left[ \text{with } \phi = |\varepsilon \nabla H_{\varepsilon}|^2 + \frac{1}{2} (H_{\varepsilon}^2 - a_{\varepsilon}^2)^2 \right] \\
&= \lim_{\varepsilon} \frac{1}{2} \int_{\Omega \times Y} \mathcal{T}_{\varepsilon}(\phi) = [\text{here, we use u. c. i.}] \\
&= \lim_{\varepsilon} \frac{1}{2} \int_{\hat{\Omega}_{\varepsilon} \times Y} \left\{ |\nabla \hat{u}|^2 + \frac{1}{2} (\hat{u}^2 - \mathcal{T}_{\varepsilon}(a_{\varepsilon}))^2 \right\} \\
&= \frac{1}{2} \int_{\Omega \times Y} \left\{ |\nabla \hat{u}(y)|^2 + \frac{1}{2} (\hat{u}(y)^2 - a(y)^2)^2 \right\} = |\Omega| \mathcal{E}(\hat{u}).
\end{aligned}$$

In order to complete the proof of Proposition 6, it suffices to establish the inequality

$$\liminf_{\varepsilon} \varepsilon^2 E_{\varepsilon}(U_{\varepsilon}) \geq |\Omega| \mathcal{E}(\hat{u}).$$

In order to obtain this estimate, we perform the following change of functions: for  $u \in A := \{u \in H_1^1(\Omega) \text{ such that } b \leq u \leq 1\}$ , we let  $v := u^2$ . We clearly have  $v \in B := \{v \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\}$ . Both  $A$  and  $B$  are convex and closed in  $H_1^1$ . We have the following equivalences

$$\begin{aligned}
u \text{ minimizes } E_{\varepsilon} \text{ in } H_1^1(\Omega) &\Leftrightarrow u \text{ minimizes } E_{\varepsilon} \text{ in } \{u \in H_1^1(\Omega) \text{ such that } b \leq u \leq 1\} \\
&\Leftrightarrow u = \sqrt{v} \text{ minimizes } E_{\varepsilon} \text{ in } \{\sqrt{v} \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\} \\
&\Leftrightarrow v = u^2 \text{ minimizes } G_{\varepsilon} \text{ in } \{v \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\}
\end{aligned}$$

with

$$G_{\varepsilon}(v) := \frac{1}{4} \int_{\Omega} \left\{ \frac{|\nabla v|^2}{2v} + \frac{1}{\varepsilon^2} (a_{\varepsilon}^2 - v)^2 \right\}.$$

Let  $U_{\varepsilon}$  be the minimizer of  $E_{\varepsilon}$  in  $H_1^1$ . Then  $V_{\varepsilon} := U_{\varepsilon}^2$  is the global minimizer of  $G_{\varepsilon}$  in  $\{v \in H_1^1(\Omega) \text{ such that } b^2 \leq v \leq 1\}$ . Let, for  $v \in C := \{v \in H_{\text{per}}^1(Y, \mathbb{R}) \text{ such that } v \geq b^2\}$ ,

$$\mathcal{G}(v) := \frac{1}{4} \int_Y \left\{ \frac{|\nabla v|^2}{2v} + (a^2 - v)^2 \right\}.$$

It is clear that  $\mathcal{G}$  has a unique minimizer in  $C$ , namely  $\hat{v} := \hat{u}^2$ .

With these notations, we have

$$\liminf_{\varepsilon} \varepsilon^2 E_{\varepsilon}(U_{\varepsilon}) = \liminf_{\varepsilon} \varepsilon^2 G_{\varepsilon}(V_{\varepsilon}) = \liminf_{\varepsilon} \frac{1}{4} \int_{\Omega} \left\{ \frac{|\varepsilon \nabla V_{\varepsilon}|^2}{2V_{\varepsilon}} + (a_{\varepsilon}^2 - V_{\varepsilon})^2 \right\} = \liminf_{\varepsilon} \frac{1}{4} \int_{\Omega} \tilde{\phi}_{\varepsilon}(V_{\varepsilon}),$$

where  $\tilde{\phi}_{\varepsilon}(V_{\varepsilon}) := \frac{|\varepsilon \nabla V_{\varepsilon}|^2}{2V_{\varepsilon}} + (a_{\varepsilon}^2 - V_{\varepsilon})^2$ . Using the bound  $|\nabla U_{\varepsilon}| \leq \frac{C}{\varepsilon}$  [7], we see that  $\int_{\Lambda_{\varepsilon}} \tilde{\phi}_{\varepsilon}(V_{\varepsilon}) \rightarrow 0$ .

This property, together with the properties i)-iv) of the unfolding operator, imply

$$\liminf_{\varepsilon} \int_{\Omega} \tilde{\phi}_{\varepsilon}(V_{\varepsilon}) = \liminf_{\varepsilon} \int_{\Omega \times Y} \mathcal{T}_{\varepsilon}(\tilde{\phi}(V_{\varepsilon})), \quad (36)$$

where

$$\mathcal{T}_\varepsilon(\tilde{\phi}_\varepsilon(V_\varepsilon)) = \begin{cases} \frac{|\nabla_y \mathcal{T}_\varepsilon(V_\varepsilon)|^2}{2\mathcal{T}_\varepsilon(V_\varepsilon)} + (\mathcal{T}_\varepsilon(V_\varepsilon) - \mathcal{T}_\varepsilon(a_\varepsilon))^2 & \text{in } \hat{\Omega}_\varepsilon \times Y \\ 0 & \text{in } \Lambda_\varepsilon \times Y \end{cases} := \phi_\varepsilon^y(\mathcal{T}_\varepsilon(V_\varepsilon)).$$

For  $W \in L^2(\Omega, H^1(Y))$  such that  $W \geq b^2$  a.e. in  $\hat{\Omega}_\varepsilon \times Y$ , define

$$\phi_\varepsilon^y(W) := \left( \frac{|\nabla_y W|^2}{2W} + (W - \mathcal{T}_\varepsilon(a_\varepsilon))^2 \right) \mathbb{1}_{\hat{\Omega}_\varepsilon \times Y}.$$

Similarly, for  $W \in L^2(\Omega, H^1(Y, \mathbb{R}))$  satisfying  $W \geq b^2$  a.e. in  $\Omega \times Y$ , we denote

$$\phi^y(W) = \frac{|\nabla_y W(x, y)|^2}{2W(x, y)} + (W(x, y) - a(y))^2.$$

One may prove that  $\phi^y$  is a convex function of its argument  $W$ .

Using the strong convergence in  $L^4(\Omega \times Y)$ , as  $\varepsilon \rightarrow 0$ , of the family of  $\mathcal{T}_\varepsilon(a_\varepsilon)$  to the map  $(x, y) \mapsto a(y)$ , it is not difficult to prove that the assumptions  $W_\varepsilon \in L^2(\Omega, H^1(Y, \mathbb{R}))$ ,  $W_\varepsilon \geq b^2$  a.e. in  $\Omega \times Y$  and  $|W_\varepsilon|, |\nabla_y W_\varepsilon| \leq C$  in  $\Omega \times Y$  imply

$$\int_{\Omega \times Y} \{\phi_\varepsilon^y(W_\varepsilon) - \phi^y(W_\varepsilon)\} \rightarrow 0. \quad (37)$$

Since  $\varepsilon \nabla V_\varepsilon$  is bounded in  $L^\infty(\Omega)$  [7] and since  $V_\varepsilon$  is bounded in  $L^2$ , Corollary 3.2 in [12] implies that there exists some  $\hat{V} \in L^2(\Omega, H_{\text{per}}^1(Y))$  such that, up to a subsequence, we have

$$\mathcal{T}_\varepsilon(V_\varepsilon) \rightharpoonup \hat{V} \text{ in } L^2(\Omega \times Y) \text{ and } \nabla_y(\mathcal{T}_\varepsilon(V_\varepsilon)) \rightharpoonup \nabla_y \hat{V} \text{ in } L^2(\Omega \times Y). \quad (38)$$

Let  $W_\varepsilon := \mathcal{T}_\varepsilon(V_\varepsilon) + \mathbb{1}_{\Lambda_\varepsilon \times Y}$ , which satisfies the assumptions leading to (37) and, in addition, satisfies

$$W_\varepsilon - \mathcal{T}_\varepsilon(V_\varepsilon) \rightarrow 0 \text{ in } L^2(\Omega \times Y) \text{ and } \nabla_y W_\varepsilon = \nabla_y \mathcal{T}_\varepsilon(V_\varepsilon).$$

To resume, the definition of  $W_\varepsilon$  combined with (38) yields

$$W_\varepsilon \rightharpoonup \hat{V} \text{ in } L^2(\Omega \times Y), \nabla_y W_\varepsilon \rightharpoonup \nabla_y \hat{V} \text{ in } L^2(\Omega \times Y), |W_\varepsilon|, |\nabla_y W_\varepsilon| \leq C \text{ and } W_\varepsilon \geq b^2 \quad (39)$$

(here, weak convergence is obtained after possibly passing to a subsequence.)

We are now in position to prove that  $\liminf_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) \geq |\Omega| \mathcal{E}(\hat{u})$ . Indeed, using the fact that  $\mathcal{T}_\varepsilon(a_\varepsilon) \rightarrow a$  in  $L^4(\Omega \times Y)$  and the convexity of  $\phi^y$ , we obtain

$$\liminf_\varepsilon \varepsilon^2 E_\varepsilon(U_\varepsilon) = [\text{from (36)}] = \liminf_\varepsilon \frac{1}{4} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\phi_\varepsilon(V_\varepsilon)) \quad (40)$$

$$= [\text{since } W_\varepsilon = \mathcal{T}_\varepsilon(V_\varepsilon) \text{ in } \hat{\Omega}_\varepsilon \times Y] = \liminf_\varepsilon \frac{1}{4} \int_{\Omega \times Y} \phi_\varepsilon^y(W_\varepsilon) \quad (41)$$

$$= [\text{using (37), (39)}] = \liminf_\varepsilon \frac{1}{4} \int_{\Omega \times Y} \phi^y(W_\varepsilon) \quad (42)$$

$$\geq [\text{using (39) and the convexity of } \phi^y] \geq \frac{1}{4} \int_{\Omega \times Y} \phi^y(\hat{V}) \quad (43)$$

$$= \int_\Omega \mathcal{G}(\hat{V}(x, \cdot)) dx \geq \int_\Omega \mathcal{G}(\hat{v}) dx = |\Omega| \mathcal{E}(\hat{u}). \quad (44)$$

It follows that

$$\lim_{\varepsilon} \varepsilon^2 E_{\varepsilon}(U_{\varepsilon}) = |\Omega| \mathcal{E}(\hat{u}).$$

The proof of Proposition 6 is complete.  $\square$

We are now in position to complete the proof of Theorem 2, point 2., by identifying the weak limit of  $U_{\varepsilon}$ . From (40), it follows that, for a. e.  $x \in \Omega$ ,  $\hat{V}(x, \cdot)$  is a positive global minimizer of  $\mathcal{G}$ . For such  $x$ , we have  $\hat{V}(x, \cdot) = \hat{v}(\cdot)$ .

By combining the following facts:

$$\lim_{\varepsilon} \frac{1}{4} \int_{\Omega \times Y} \mathcal{T}_{\varepsilon}(\tilde{\phi}_{\varepsilon}(V_{\varepsilon})) = |\Omega| \mathcal{E}(\hat{u}) = |\Omega| \mathcal{G}(\hat{v}), \quad \mathcal{T}_{\varepsilon}(V_{\varepsilon}) \rightharpoonup \hat{v}, \quad \nabla_y \mathcal{T}_{\varepsilon}(V_{\varepsilon}) \rightharpoonup \nabla_y \hat{v} \text{ in } L^2(\Omega \times Y),$$

we obtain

$$\lim_{\varepsilon} \int_{\Omega \times Y} (\mathcal{T}_{\varepsilon}(V_{\varepsilon}) - \mathcal{T}_{\varepsilon}(a_{\varepsilon}))^2 = \lim_{\varepsilon} \int_{\Omega \times Y} (\hat{v} - a^2)^2.$$

The above equality implies

$$\lim_{\varepsilon} \int_{\Omega \times Y} \mathcal{T}_{\varepsilon}(V_{\varepsilon})^2 = \lim_{\varepsilon} \int_{\Omega \times Y} \hat{v}^2,$$

which in turn implies  $\mathcal{T}_{\varepsilon}(V_{\varepsilon}) \rightarrow \hat{v}$  in  $L^2(\Omega \times Y)$ . Since  $\hat{v} = \hat{u}^2$  and  $V_{\varepsilon} = U_{\varepsilon}^2$ , we obtain

$$\int_{\Omega \times Y} (\mathcal{T}_{\varepsilon}(U_{\varepsilon}) - \hat{u})^2 \leq \frac{1}{4b^2} \int_{\Omega \times Y} (\mathcal{T}_{\varepsilon}(U_{\varepsilon})^2 - \hat{u}^2)^2 \rightarrow 0,$$

that is, we find that  $\mathcal{T}_{\varepsilon}(U_{\varepsilon}) \rightarrow \hat{u}$  in  $L^2(\Omega \times Y)$ . This fact combined with Proposition 2.9 iii) in [12] implies  $U_{\varepsilon} \rightharpoonup \mathcal{M}_Y(\hat{u}) \equiv \int_Y \hat{u}(y) dy$ , which is the desired conclusion.  $\square$

### 3.2.2 Limit of $v_{\varepsilon}$ in $H^1$

Recall that we are in the critical case  $\lambda = 1$ ,  $\delta = \varepsilon$ .

In order to state the main result of this section we recall the following standard existence result (see, e. g., Theorem 4.27 in [13])

**Proposition 7.** *Let  $f \in (H_{\text{per}}^1(Y))'$  have zero average. Then there exists a unique solution  $h \in H_{\text{per}}^1(Y)$  of*

$$\operatorname{div}(\hat{u}^2 \nabla h) = f \text{ and } \mathcal{M}_Y(h) = 0.$$

In view of this proposition, let  $\chi_j \in H_{\text{per}}^1(Y)$  be the unique solution of

$$\operatorname{div}(\hat{u}^2 \nabla \chi_j) = \partial_j(\hat{u}^2) \text{ and } \mathcal{M}_Y(\chi_j) = 0. \quad (45)$$

Recall that the homogenized matrix  $\mathcal{A}$  of  $\hat{u}^2 \left(\frac{x}{\varepsilon}\right) \operatorname{Id}_{\mathbb{R}^2}$  is given by

$$\mathcal{A} = \int_Y \hat{u}^2 \begin{pmatrix} 1 - \partial_1 \chi_1 & -\partial_1 \chi_2 \\ -\partial_2 \chi_1 & 1 - \partial_2 \chi_2 \end{pmatrix} \quad (46)$$

(see, e. g., [16] chapter 1 or [13] chapter 6).



**Proposition 8.** *Let  $\varphi_*$  be the unique solution of*

$$\begin{cases} \operatorname{div}(\mathcal{A}\nabla\varphi_*) = 0 & \text{in } \Omega \\ \varphi_* = \varphi_0 & \text{on } \partial\Omega \end{cases}. \quad (47)$$

Let  $g = e^{i\varphi_0}$ . Also, for small  $\varepsilon$ , represent a minimizer  $u_\varepsilon$  of  $E_\varepsilon$  in  $H_g^1$  as  $u_\varepsilon = U_\varepsilon \rho_\varepsilon e^{i\varphi_\varepsilon}$ , where  $\varphi_\varepsilon \in H_{\varphi_0}^1(\Omega)$ .

Then  $\varphi_\varepsilon \rightharpoonup \varphi_*$  in  $H^1(\Omega)$  as  $\varepsilon \rightarrow 0$ .

*Proof.* This argument is an adaptation of the proof of Theorem 4 in [22].

First note that  $\mathcal{T}_\varepsilon(U_\varepsilon^2)(x, y) \rightarrow \hat{u}^2(y)$  in  $L^2(\Omega \times Y)$  and  $|v_\varepsilon|^2 = \rho_\varepsilon^2 \rightarrow 1$  in  $L^2(\Omega)$  imply that

$$\mathcal{T}_\varepsilon(\rho_\varepsilon^2 U_\varepsilon^2)(x, y) \rightarrow \hat{u}^2(y) \text{ in } L^2(\Omega \times Y).$$

Recalling that  $\varphi_\varepsilon$  is the solution of

$$\begin{cases} -\operatorname{div}(\rho_\varepsilon^2 U_\varepsilon^2 \nabla \varphi_\varepsilon) = 0 & \text{in } \Omega \\ \varphi_\varepsilon = \varphi_0 & \text{on } \partial\Omega \end{cases},$$

we find, using Proposition 6. iv) and the fact that  $\rho_\varepsilon^2 U_\varepsilon^2 \nabla \varphi_\varepsilon \in H_{\text{loc}}^1(\Omega)$ , that

$$0 = \varepsilon \mathcal{T}_\varepsilon(-\operatorname{div}(\rho_\varepsilon^2 U_\varepsilon^2 \nabla \varphi_\varepsilon))(x, y) = -\operatorname{div}_y(\mathcal{T}_\varepsilon(\rho_\varepsilon^2 U_\varepsilon^2)(x, y) \mathcal{T}_\varepsilon(\nabla \varphi_\varepsilon)(x, y)). \quad (48)$$

In order to prove that  $\varphi_\varepsilon \rightharpoonup \varphi_*$  it suffices to prove that if, possibly up to a subsequence, we have  $\varphi_\varepsilon \rightharpoonup \varphi^*$ , then  $\varphi^*$  solves (47).

Using Theorem 3.5 in [12], we have the existence of  $\hat{\varphi} \in L^2(\Omega, H_{\text{per}}^1(Y))$  such that

$$\mathcal{T}_\varepsilon(\nabla \varphi_\varepsilon) \rightharpoonup \nabla \varphi^* + \nabla_y \hat{\varphi} \text{ in } L^2(\Omega \times Y) \text{ and } \mathcal{M}_Y(\hat{\varphi}) = 0. \quad (49)$$

By inserting (49) into (48) and passing to the weak limits in  $L^2(\Omega, H^{-1}(Y))$ , we obtain

$$-\operatorname{div}_y[\hat{u}^2(y)(\nabla \varphi^*(x) + \nabla_y \hat{\varphi}(x, y))] = 0$$

which is equivalent to

$$-\operatorname{div}_y[\hat{u}^2(y)\nabla_y \hat{\varphi}(x, y)] = \nabla_y \hat{u}^2(y) \cdot \nabla \varphi^*(x).$$

This equality combined with (45) implies that

$$\hat{\varphi}(x, y) = -\chi_1 \partial_{x_1} \varphi^* - \chi_2 \partial_{x_2} \varphi^*.$$

Consequently, we have

$$\nabla \varphi^* + \nabla_y \hat{\varphi} = \begin{pmatrix} 1 - \partial_1 \chi_1 & -\partial_1 \chi_2 \\ -\partial_2 \chi_1 & 1 - \partial_2 \chi_2 \end{pmatrix} \nabla \varphi^*.$$

On the other hand, let  $\xi \in \mathcal{D}(\Omega)$ . Then, for sufficiently small  $\varepsilon$  we have (cf Proposition 2.5. (i) in [12])

$$\begin{aligned} 0 &= \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \rho_\varepsilon^2 U_\varepsilon^2 \nabla \varphi_\varepsilon \cdot \nabla \xi = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(\rho_\varepsilon^2 U_\varepsilon^2) \mathcal{T}_\varepsilon(\nabla \varphi_\varepsilon) \cdot \mathcal{T}_\varepsilon(\nabla \xi) \\ &= \int_{\Omega} \left\{ \int_Y \hat{u}^2(y) (\nabla \varphi^* + \nabla_y \hat{\varphi}) \right\} \cdot \nabla \xi = \int_{\Omega} \operatorname{div}_x \left\{ \int_Y \hat{u}^2(y) (\nabla \varphi^* + \nabla_y \hat{\varphi}) \right\} \xi. \end{aligned}$$

Therefore one has

$$\operatorname{div}_x \left[ \int_Y \hat{u}^2(y) (\nabla \varphi^* + \nabla_y \hat{\varphi}) \right] = \operatorname{div}_x \left[ \int_Y \hat{u}^2(y) \begin{pmatrix} 1 - \partial_1 \chi_1 & -\partial_1 \chi_2 \\ -\partial_2 \chi_1 & 1 - \partial_2 \chi_2 \end{pmatrix} \nabla \varphi^* \right] = \operatorname{div}_x (\mathcal{A} \nabla \varphi^*) = 0$$

and consequently  $\varphi^*$  solves (47).  $\square$

### 3.3 The case $\lambda = 1, \varepsilon \ll \delta$

**Theorem 3.** *Assume that  $\lambda = 1, \delta \rightarrow 0$  and  $\varepsilon/\delta \rightarrow 0$ . Then, as  $\varepsilon \rightarrow 0$ , we have*

1.  $\rho_\varepsilon = |u_\varepsilon| \rightharpoonup \mathcal{M}_Y(a)$  in  $L^2(\Omega)$ ,

2.  $\varphi_\varepsilon \rightharpoonup \varphi_*$  in  $H^1(\Omega)$ ,

3.  $\rho_\varepsilon^2 \nabla \varphi_\varepsilon \rightharpoonup \mathcal{A} \nabla \varphi_*$  in  $L^2(\Omega)$ ,

where  $\varphi_*$  solves the homogenized problem

$$\begin{cases} \operatorname{div}(\mathcal{A} \nabla \varphi_*) = 0 & \text{in } \Omega \\ \varphi_* = \varphi_0 & \text{on } \partial\Omega \end{cases} \quad (50)$$

Here,  $\mathcal{A}$  is the homogenized matrix of  $a^2 \left( \frac{x}{\delta} \right) \operatorname{Id}_{\mathbb{R}^2}$ .

*Proof.* Theorem 1 combined with Lemma 5 yields  $\rho_\varepsilon - a_\varepsilon \rightarrow 0$  in  $L^2(\Omega)$ . On the other hand, we have  $a_\varepsilon \rightarrow \mathcal{M}_Y(a)$  weakly in  $L^2(\Omega)$  (see, e. g., [13] Theorem 2.6), so that 1. follows.

In order to prove 2. and 3., we start from the equation

$$\begin{cases} \operatorname{div}(\rho_\varepsilon^2 \nabla \varphi_\varepsilon) = 0 & \text{in } \Omega \\ \varphi_\varepsilon = \varphi_0 & \text{on } \partial\Omega \end{cases} \quad (51)$$

satisfied by  $\varphi_\varepsilon$ . In view of the fact that  $\rho_\varepsilon - a_\varepsilon \rightarrow 0$  in  $L^2(\Omega)$ , it is natural to compare  $\varphi_\varepsilon$  to the solution  $\hat{\varphi}_\varepsilon$  of

$$\begin{cases} \operatorname{div}(a_\varepsilon^2 \nabla \hat{\varphi}_\varepsilon) = 0 & \text{in } \Omega \\ \hat{\varphi}_\varepsilon = \varphi_0 & \text{on } \partial\Omega \end{cases} \quad (52)$$

The difference  $\psi_\varepsilon := \hat{\varphi}_\varepsilon - \varphi_\varepsilon$  is solution of

$$\begin{cases} \operatorname{div}(a_\varepsilon^2 \nabla \psi_\varepsilon) = \operatorname{div}[(\rho_\varepsilon^2 - a_\varepsilon^2) \nabla \varphi_\varepsilon] & \text{in } \Omega \\ \psi_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \quad (53)$$

We claim  $\psi_\varepsilon \rightarrow 0$  in  $H^1(\Omega)$ . Indeed, we first note that, by (51),  $\varphi_\varepsilon$  is bounded in  $H^1$ . Using the fact that  $b^2 \leq a_\varepsilon^2 \leq 1$  and (53) we obtain, via the Lax-Milgram theorem, that, with  $C, C' > 0$  and  $p < 2$  independent of  $\varepsilon$ , we have

$$\|\nabla \psi_\varepsilon\|_{L^2} \leq C \|(\rho_\varepsilon^2 - a_\varepsilon^2) \nabla \varphi_\varepsilon\|_{L^2} \leq C' < \infty$$

and (with  $r := 2/(2-p)$ )

$$\|\nabla\psi_\varepsilon\|_{L^p} \leq C\|(\rho_\varepsilon^2 - a_\varepsilon^2)\nabla\varphi_\varepsilon\|_{L^p} \leq C\|\rho_\varepsilon^2 - a_\varepsilon^2\|_{L^{rp}}\|\nabla\varphi_\varepsilon\|_{L^2}.$$

Consequently,  $\psi_\varepsilon$  is bounded in  $H_0^1$  and converges strongly to 0 in  $W^{1,p}(\Omega)$ . It follows that  $\psi_\varepsilon \rightarrow 0$  in  $H^1(\Omega)$ .

Now, using the classic periodic homogenization result (see, e. g., [16] chapter 1 or [13] chapter 6), we know that  $\hat{\varphi}_\varepsilon \rightarrow \varphi_*$  in  $H^1(\Omega)$  and  $a_\varepsilon^2\nabla\hat{\varphi}_\varepsilon \rightarrow \mathcal{A}\nabla\varphi_*$  in  $L^2(\Omega)$ . These facts combined with the weak convergences  $\psi_\varepsilon \rightarrow 0$  in  $H^1(\Omega)$  and  $(a_\varepsilon^2\nabla\hat{\varphi}_\varepsilon - \rho_\varepsilon^2\nabla\varphi_\varepsilon) \rightarrow 0$  in  $L^2(\Omega)$  complete the proof of the theorem.  $\square$

### 3.4 The case $\lambda = 1, \delta \ll \varepsilon$

In this case,  $\varepsilon$  need not tend to 0. Up to subsequences, we may assume that either  $\varepsilon = 1$  or  $\varepsilon \rightarrow 0$ .

**Theorem 4.** *The following hold.*

1. *Assume that  $\varepsilon = 1$  and that  $\delta \rightarrow 0$ , and denote the energy by  $E_\delta$  rather than  $E_\varepsilon$ . If  $u_\delta$  is a minimizer of  $E_\delta$ , then  $u_\delta \rightarrow \hat{u}$  in  $H^1(\Omega)$ , where  $\hat{u}$  solves*

$$\begin{cases} -\Delta\hat{u} = \hat{u}(\mathcal{M}_Y(a^2) - \hat{u}^2) & \text{in } \Omega \\ \hat{u} = g & \text{on } \partial\Omega \end{cases}. \quad (54)$$

2. *Assume that  $\varepsilon \rightarrow 0$  and that  $\delta/\varepsilon \rightarrow 0$ . If  $u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon}$  is a minimizer of  $E_\varepsilon$ , then we have*

- (i)  $\rho_\varepsilon \rightarrow \sqrt{\mathcal{M}_Y(a^2)}$  strongly in  $L^2(\Omega)$ ,
- (ii)  $\varphi_\varepsilon \rightarrow \varphi_*$  in  $H^1(\Omega)$ .

Here,  $\varphi_*$  denotes the harmonic extension of  $\varphi_0$ .

*Proof.* In case 1., we start by noting that  $\|u_\delta\|_{H^1(\Omega)}$  is uniformly bounded with respect to  $\delta$ . Let  $\hat{u}$  be such that, possibly after passing to a subsequence,  $u_\delta$  weakly converges to  $\hat{u}$  in  $H^1$ . In order to identify  $\hat{u}$ , we let  $\delta \rightarrow 0$  in the weak form of the GL equation satisfied by  $u_\delta$ , namely:

$$\int_{\Omega} \nabla u_\delta \cdot \nabla \psi \, dx = \int_{\Omega} u_\delta (a_\delta^2 - u_\delta^2) \psi \, dx, \quad \forall \psi \in C_0^\infty(\Omega)$$

and find that (54) holds.

In order to prove 2., we consider a partition of  $\mathbb{R}^2$  by a family  $\{C_k^\varepsilon\}$  of  $\delta \times \delta$  squares. We may assume that

$$\{C_k^\varepsilon \mid C_k^\varepsilon \subset \{x \in \Omega \mid \text{dist}(x, \partial\Omega) > \varepsilon\}\} = \{C_k^\varepsilon \mid k \in \{1, \dots, N_\varepsilon\}\}.$$

Clearly, we have  $N_\varepsilon = |\Omega|\delta^{-2} + \mathcal{O}(\varepsilon\delta^{-2})$ . Denote  $\Omega'_\varepsilon := \bigcup_{k=1}^{N_\varepsilon} C_k^\varepsilon$ .

For  $C_0 > 0$  (independent of  $\varepsilon$ ) consider

$$\mathcal{H}_\varepsilon^{C_0} = \{w \in H_g^1 \mid |\nabla w| \leq \frac{C_0}{\varepsilon} \text{ in } \Omega'_\varepsilon \text{ and } |w| \leq 1 \text{ in } \Omega\}.$$

Recall [7] that, for  $\varepsilon < 1$  and a suitable  $C_0$ , each minimizer  $u_\varepsilon$  of  $E_\varepsilon$  in  $H_g^1$  belongs to  $\mathcal{H}_\varepsilon^{C_0}$ .

For  $w \in \mathcal{H}_\varepsilon^{C_0}$ , we have

$$\int_\Omega (|w|^2 - a_\varepsilon^2)^2 = \int_\Omega (|w|^2 - \mathcal{M}_Y(a^2))^2 + |\Omega| [\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2] + H_\varepsilon(w). \quad (55)$$

Here, the reminder  $H_\varepsilon$  satisfies  $|H_\varepsilon(w)| \leq o_\varepsilon(1)$ , with  $o_\varepsilon(1)$  independent of  $w$ . Indeed, we have

$$\begin{aligned} \int_\Omega (|w|^2 - a_\varepsilon^2)^2 - \int_\Omega (|w|^2 - \mathcal{M}_Y(a^2))^2 &= \int_\Omega [a_\varepsilon^2 - \mathcal{M}_Y(a^2)] a_\varepsilon^2 \\ &\quad + \mathcal{M}_Y(a^2) \int_\Omega (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) \\ &\quad - 2 \int_\Omega [a_\varepsilon^2 - \mathcal{M}_Y(a^2)] |w|^2. \end{aligned}$$

We next note the three following facts. First, we have

$$\int_\Omega [a_\varepsilon^2 - \mathcal{M}_Y(a^2)] a_\varepsilon^2 = \sum_k \left\{ \int_{C_\varepsilon^k} [a_\varepsilon^2 - \mathcal{M}_Y(a^2)] a_\varepsilon^2 \right\} + \mathcal{O}(\varepsilon) = |\Omega| [\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2] + \mathcal{O}(\varepsilon).$$

Next, it holds that

$$\int_\Omega (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) = \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} (a_\varepsilon^2 - \mathcal{M}_Y(a^2)) = \mathcal{O}(\varepsilon).$$

Finally, we have

$$\begin{aligned} \left| \int_\Omega [a_\varepsilon^2 - \mathcal{M}_Y(a^2)] |w|^2 \right| &\leq \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} |a_\varepsilon^2 - \mathcal{M}_Y(a^2)| |w|^2 \\ &\leq \mathcal{O}(\varepsilon) + \sum_k \int_{C_\varepsilon^k} |a_\varepsilon^2 - \mathcal{M}_Y(a^2)| = o_\varepsilon(1). \end{aligned}$$

Thus (55) holds. Consequently, for  $u \in \mathcal{H}_\varepsilon^{C_0}$ , one has

$$E_\varepsilon(u) = \frac{|\Omega|}{4\varepsilon^2} (\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2) + G_\varepsilon(u) + o\left(\frac{1}{\varepsilon^2}\right), \quad (56)$$

where

$$G_\varepsilon(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (\mathcal{M}_Y(a^2) - |u|^2)^2.$$

We next claim that  $\int_{\Omega} (|u_{\varepsilon}|^2 - \mathcal{M}_Y(a^2))^2 dx \rightarrow 0$ . Indeed, we consider a test function in the spirit of [18], more specifically we let  $w_{\varepsilon} = |w_{\varepsilon}|e^{i\varphi_*}$ , where  $\varphi_*$  is the harmonic extension of  $\varphi_0$  and

$$|w_{\varepsilon}|(x) = \begin{cases} 1 - \frac{1 - \sqrt{\mathcal{M}_Y(a^2)}}{\varepsilon} \text{dist}(x, \partial\Omega), & \text{if } \text{dist}(x, \partial\Omega) < \varepsilon \\ \sqrt{\mathcal{M}_Y(a^2)}, & \text{otherwise} \end{cases}.$$

Note that, for a suitable  $C_0$ , we have  $w_{\varepsilon} \in \mathcal{H}_{\varepsilon}^{C_0}$ . A straightforward computation yields  $G_{\varepsilon}[w_{\varepsilon}] \leq \frac{C}{\varepsilon}$ . Consequently, we obtain

$$E_{\varepsilon}(u_{\varepsilon}) \leq E_{\varepsilon}(w_{\varepsilon}) \leq \frac{|\Omega|}{4\varepsilon^2} (\mathcal{M}_Y(a^4) - \mathcal{M}_Y(a^2)^2) + o(\varepsilon^{-2}).$$

This estimate combined with (56) implies that  $|u_{\varepsilon}| \rightarrow \sqrt{\mathcal{M}_Y(a^2)}$  strongly in  $L^2(\Omega)$ .

Using the second part of Corollary 1, we obtain that  $\varphi_{\varepsilon} \rightarrow \varphi_*$  in  $H^1(\Omega)$  where  $\varphi_*$  is the harmonic extension of  $\varphi_0$ .

The proof of Theorem 4 is complete. □

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