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Applying Relation Algebra and RelView to Measures in a Social Network ⋆ ⋆⋆

Rudolf Berghammer[1], Agnieszka Rusinowska⋆⋆⋆[2], and Harrie de Swart[3]

[1] Institut für Informatik, Christian-Albrechts-Universität Kiel
Olshausenstraße 40, 24098 Kiel, Germany, rub@informatik.uni-kiel.de
[2] GATE, Université Lumière Lyon 2 - CNRS
93 Chemin des Mouilles - B.P. 167, 69131 Ecully Cedex, France, rusinowska@gate.cnrs.fr
[3] Department of Philosophy, Tilburg University
P.O. Box 90153, 5000 LE Tilburg, The Netherlands, H.C.M.deSwart@uvt.nl

Abstract. We present an application of relation algebra to measure agents' 'strength' in a social network with influence between agents. In particular, we deal with power, success, and influence of an agent as measured by the generalized Hoede-Bakker index and its modifications, and by the influence indices. We also apply relation algebra to determine followers of a coalition and the kernel of an influence function. This leads to specifications, which can be executed with the help of the BDD-based tool RelView after a simple translation into the tool's programming language. As an example we consider the present Dutch parliament.

Keywords: RelView, relation algebra, social network, Hoede-Bakker index, influence index

1 Introduction

In order to measure agents' (or players') 'strength' in a voting situation, a lot of power indices have been proposed and analyzed in the course of more than fifty years (e.g. Banzhaf, 1965; Coleman, 1971, 1986; Felsenthal and Machover, 1998; König and Bräuninger, 1998; Rae, 1969; Shapley and Shubik, 1954). In the voting power literature, one may find theoretical analysis (which includes both the axiomatic and probabilistic approaches to power indices) as well as applications of power indices. They have been especially applied to political science, i.e., to decision-making in the European Union and the national parliaments (e.g. Hosli, 1997; Laruelle and Widgren, 1998; Nurmi and Meskanen, 1999). Since power indices can be applied to all kinds of organizations: to political bodies, international economic organizations, as well as to business settings, searching efficient methods for computing power indices, in particular, in large voting games, is of great importance (Leech, 2003).

Coming from a different direction is an approach proposed in (Hoede and Bakker, 1982), where a social network with players who are to make a 'yes'-'no' decision is considered. In this framework, the Hoede-Bakker index has been introduced. The essential feature of this framework is the distinction between the inclination of a player (to say 'yes' or 'no') and

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⋆⋆⋆ Corresponding author: A. Rusinowska, Tel.: 0033 472866080, Fax: 0033 472866090, rusinowska@gate.cnrs.fr

the final decision of the player, which can be different from his initial inclination, due to influences of others in the network. Such an influence is formally represented by an influence function. The Hoede-Bakker index has been recently studied in (Rusinowska and de Swart 2006, 2007). In the first paper a generalization and some modifications of the Hoede-Bakker index are introduced that coincide with some standard power indices. Although the Hoede-Bakker index has been defined in the framework of influence, in fact it does not measure the influence between players. Influence indices, influence functions, and some other concepts related to influence have been investigated in (Grabisch and Rusinowska, 2009).

Since more than two decades, relation algebra is used successfully for formal problem specification, prototyping, and algorithm development (Brink et al., 1997; Schmidt and Ströhlein, 1993; de Swart et al., 2003, 2006). Relations are well suited for modeling and reasoning about many discrete structures (like graphs, games, Petri nets, orders and lattices) and, due to the easy and/or efficient mechanization using, for instance, Boolean matrices, successor lists or binary decision diagrams (BDDs), also for computations on them. RelView (Behnke et al., 1998; Berghammer et al., 1996, 2003) is a BDD-based tool for the visualization and manipulation of relations and for prototyping and relational programming.

In (Berghammer et al., 2007, 2009; Rusinowska et al., 2006) we have successfully applied relation algebra and RelView to compute the set of all feasible stable governments in a coalition formation model introduced in (Rusinowska et al., 2005). In the present paper we like to apply the same approach to compute measures of agents' 'strength' in a social network. Determining such measures can become quite complex and requires a lot of computations. Hence, using a computer program to compute the measures is extremely useful for real life applications of the concepts in question. To be more precise, *the aim of this paper* is to apply relation algebra and RelView to compute power, success, and influence of an agent as measured by the generalized Hoede-Bakker index, its modifications, and the influence indices, and to determine the followers of a coalition and the kernel of an influence function in a social network with influence between agents.

2 Measures of Players' 'Strength' in a Social Network

The framework studied in the paper is the following. We consider a social network with the set of all agents (players, voters, actors) denoted by $P := \{1, ..., n\}$. The agents make a certain acceptance-rejection decision. Each agent has an inclination either to say 'yes' (denoted by 1) or 'no' (denoted by 0). A Boolean *inclination vector*, denoted by $i = (i_1, ..., i_n)$, indicates the inclinations of all agents. All inclination vectors are assumed to be equally probable. Let $I := \{0, 1\}^n$ be the set of all inclination vectors. It is assumed that agents may influence each other, and due to the influences in the network, the final decision of an agent may be different from his original inclination. In other words, each inclination vector $i \in I$ is transformed into a *decision vector* Bi, where $B : I \to I$ is the influence function, and

the decision vector $Bi = ((Bi)_1, ..., (Bi)_n)$ indicates the final decisions made by all agents. Let $B(I)$ be the set of all decision vectors under B. We assume a *group decision function* $gd : B(I) \to \{0, 1\}$, having the Boolean value 1 if the group decision is 'yes', and the Boolean value 0 if the group decision is 'no'. The set of all influence functions, and the set of all group decision functions will be denoted by $\mathcal{B}$ and $\mathcal{G}$, respectively.

2.1 The Generalized Hoede-Bakker index and its modifications

In this section we recapitulate the generalized Hoede-Bakker index and its modifications as given in (Rusinowska and de Swart, 2006). It generalizes the original Hoede-Bakker index introduced in (Hoede and Bakker, 1982) in order to eliminate certain shortcomings of the latter. First, we introduce some notations. Given an influence function $B \in \mathcal{B}$ and a group decision function $gd \in \mathcal{G}$, we define the two subsets $I^+(B, gd)$ and $I^-(B, gd)$ of the set I of all inclination vectors as follows:

$$I^+(B, gd) := \{i \in I \mid gd(Bi) = 1\}, \qquad I^-(B, gd) := \{i \in I \mid gd(Bi) = 0\}$$

Depending on the functions B and gd, we now introduce for each agent (player) $k \in P$ four decisive sets by the following definitions:

$$I_k^{++}(B, gd) := \{i \in I \mid i_k = 1 \ \wedge \ gd(Bi) = 1\}$$
$$I_k^{+-}(B, gd) := \{i \in I \mid i_k = 1 \ \wedge \ gd(Bi) = 0\}$$
$$I_k^{-+}(B, gd) := \{i \in I \mid i_k = 0 \ \wedge \ gd(Bi) = 1\}$$
$$I_k^{--}(B, gd) := \{i \in I \mid i_k = 0 \ \wedge \ gd(Bi) = 0\}$$

When clear from the context, we will skip '(B, gd)' in the expressions above; so, for instance, we may write I_k^{+-} instead of $I_k^{+-}(B, gd)$. In order to measure the voting strength of the players in a network, where the inclination of an agent may be different from its final decision due to influences from other agents, the subsequent definitions have been introduced in (Rusinowska and de Swart, 2006) (note, that n is the number of players):

Definition 2.1.1 *Given $B \in \mathcal{B}$ and $gd \in \mathcal{G}$, the generalized Hoede-Bakker index of a player $k \in P$ is defined as follows:*

$$\mathrm{GHB}_k(B, gd) := \frac{|I_k^{++}| - |I_k^{+-}| + |I_k^{--}| - |I_k^{-+}|}{2^n} \tag{1}$$

For each $k \in P$ we define modifications of the generalized Hoede-Bakker index as follows:

$$\mathrm{M_1GHB}_k(B, gd) := \frac{|I_k^{++}| - |I_k^{-+}|}{|I^+|} \qquad \mathrm{M_2GHB}_k(B, gd) := \frac{|I_k^{--}| - |I_k^{+-}|}{|I^-|} \tag{2}$$

$$\mathrm{M_3GHB}_k(B, gd) := \frac{|I_k^{++}| + |I_k^{--}|}{2^n} \qquad \mathrm{M_4GHB}_k(B, gd) := \frac{|I_k^{++}|}{|I^+|} \tag{3}$$

Furthermore, we define independently of k: $\mathrm{MGHB}(B, gd) := \frac{|I^+|}{2^n}$

The value of $\mathrm{GHB}_k(B, gd)$ measures a kind of 'net' Success, i.e., Success − Failure, where by a successful player, given $i \in I$, $B \in \mathcal{B}$ and $gd \in \mathcal{G}$, we mean a player $k \in P$ whose inclination i_k coincides with the group decision $gd(Bi)$. In (Rusinowska and de Swart, 2006) it is shown that if all inclination vectors are equally probable, then the generalized Hoede-Bakker index coincides with the absolute Banzhaf index, i.e., it measures 'Decisiveness'. A decisive player is a player who is successful and changing his inclination causes a change of the group decision. In (Rusinowska and de Swart, 2006) it is also proved that the modifications $\mathrm{M_1GHB}$, $\mathrm{M_2GHB}$, $\mathrm{M_3GHB}$ and $\mathrm{M_4GHB}$, coincide with the Coleman's index 'to prevent action', Coleman's index 'to initiate action', the Rae index, and the König-Bräuninger index, respectively. MGHB coincides with Coleman's 'power of a collectivity to act'. Note that the modification $\mathrm{M_3GHB}$ measures Success of a player in such a network.

2.2 The influence indices and followers

We recapitulate now some concepts to measure influence between players in the presented framework that have been investigated in (Grabisch and Rusinowska, 2009). We introduce for any $S \subseteq P$ such that $|S| \geq 2$ the set I_S of inclination vectors under which all agents of S have the same inclination, i.e.,

$$I_S := \{i \in I \mid \forall\, k, j \in S : i_k = i_j\},$$

and define $I_k := I$ for all $k \in P$. For all inclination vectors $i \in I_S$ we denote by i_S the value i_k for some player $k \in S$. Due to the definition of the set I_S, the Boolean value $i_S \in \{0, 1\}$ does not depend on the choice of k. Based on these notions, let for each subset $S \subseteq P$ of players (that is regarded as a coalition) and each player $j \in P$ the following sets be introduced:

$$I_{S\to j} := \{i \in I_S \mid i_j = \neg i_S\}, \qquad I^*_{S\to j}(B) := \{i \in I_{S\to j} \mid (Bi)_j = i_S\}$$

In words, $I_{S\to j}$ and $I^*_{S\to j}(B)$ denote the set of all inclination vectors of *potential influence of coalition S on player j*, and the set of all inclination vectors of *influence of S on j under the given influence function $B \in \mathcal{B}$*, respectively.

In (Grabisch and Rusinowska, 2009) the so-called *influence indices* have been defined. The general idea is to compute the weighted number of times coalition S makes a player $j \in P$ change his decision. One particular way of weighting leads to the *possibility influence index* $\overline{d}(B, S \to j)$ which measures the degree of influence, coalition S has on player j, taking into account any possibility of influence. We check therefore how many inclination vectors of potential influence of coalition S on player j are indeed vectors of influence of S on j. We do not verify here the inclinations of the players outside $S \cup \{j\}$.

Definition 2.2.1 *Given $B \in \mathcal{B}$, for each coalition $S \subseteq P$ of players and each player $j \in P \setminus S$, the possibility influence index and the certainty influence index of coalition S on player j are defined respectively as follows:*

$$\overline{d}(B, S \to j) := \frac{|I^*_{S\to j}(B)|}{|I_{S\to j}|} \qquad \underline{d}(B, S \to j) := \frac{|\{i \in I^*_{S\to j}(B) \mid \forall\, k \notin S : i_k = \neg i_S\}|}{2} \tag{4}$$

By a follower of a given coalition we mean a voter who always decides according to the inclination of the coalition in question.

Definition 2.2.2 *Let $\emptyset \neq S \subseteq P$ be a coalition of players and $B \in \mathcal{B}$. Then the set of followers of S under the influence function B is defined as follows:*

$$F_B(S) := \{j \in P \mid \forall\, i \in I_S : (Bi)_j = i_S\} \tag{5}$$

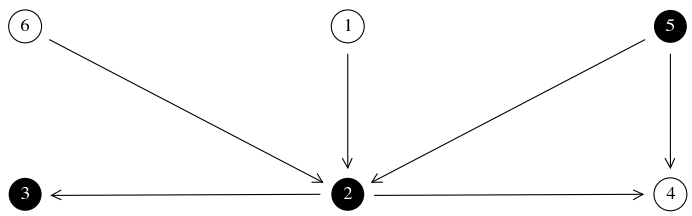
Furthermore, it is worth mentioning the concept of a *kernel* of an influence function B which is the set of the 'true' (minimal) influential coalitions, that is, the collection $\mathcal{K}(B)$ of all subsets S of P such that $F_B(S) \neq \emptyset$ and $F_B(S') = \emptyset$ for all $S' \subset S$.

2.3 Majority and influence by trend-setters

In the preceding two subsections we have defined the different indices and notions dealing with coalitions, influence and followers with respect to an arbitrary influence function $B \in \mathcal{B}$ and an arbitrary group decision function $gd \in \mathcal{G}$. In practice, however, only a very small number of such functions is used. Group decisions almost always are based on majority. This means that for each inclination vector $i \in I$ and each influence function $B \in \mathcal{B}$, the output of $gd : B(I) \to \{0, 1\}$ for the decision vector Bi as input is 1 if the size of the set $\{j \in P \mid (Bi)_j = 1\}$ is at least $[\frac{n}{2}] + 1$, where $[x]$ denotes the largest natural number less than or equal to x. In the remaining cases, $gd(Bi)$ yields 0 as result. Instead of this so-called simple majority, in specific cases also other majority rules are used, e.g., $\frac{2}{3}$-majority.

Influences in a social network essentially are based on dependency relationships, which adequately can be modeled by a dependency graph. The vertices of such a directed loop-free graph are the players. For different players $j, k \in P$ there is an arc from j to k iff j is a so-called trend-setter for k, that is, the vote of k may be influenced by the inclination of j. Then k is called a dependent player. Players without trend-setters (in terms of graph theory: the sources) are said to be independent.

Example 2.3.1. To give a concrete example, the following picture (generated with the help of RelView) shows the dependency graph of a social network with a set P of six players $1, 2, 3, 4, 5$ and 6, where the vertex with label 'k' corresponds to player k, $1 \leq k \leq 6$. Since in Section 3 we will use this social network as running example to illustrate the developed relation-algebraic specifications, in the dependency graph also a coalition S consisting of the three players $2, 3, 5$ is indicated by black vertices.



As one can see from the directed arcs of the graph, the independent players are $1, 5$ and 6 (no ingoing arcs), and the dependent players are $2, 3$ and 4. The vote of player 2 depends on

its three trendsetters (graph-theoretic predecessors) $1, 5$ and 6, the vote of player 3 depends on its unique trend-setter 2, and the vote of player 4 depends on its two trend-setters 2, 5.

Now, assume $i \in I$ to be an inclination vector and we want to define the decision vector Bi in terms of the dependency graph. Of course, for an independent player $k \in P$ we are allowed to define $(Bi)_k := i_k$, i.e., to presume that he does not change his vote. On the other hand, a dependent player $k \in P$ will always follow his sole trend-setter $j \in P$ if there is exactly one. In this case, hence, we put $(Bi)_k := i_j$. It is reasonable to generalize this in such a way that a dependent player always follows his trend-setters if they have the same inclination. However, a problem appears if there are at least two trend-setters player $k \in P$ depends on, and they have different inclinations. Which trend-setter should player k follow? There are several possibilities to define the influence function in such a case. Usually two possibilities are considered:

- *Following only unanimous trend-setters:* Here the vote of player k is equal to the inclination of his trend-setters if they all have the same inclination. Otherwise, player k votes according to his own inclination.
- *Following a majority of trend-setters:* Here k votes as the inclination of the majority of his trend-setters is. Assuming that player k has t trend-setters, this means that if there are at least $[\frac{t}{2}] + 1$ trend-setters of k with the same inclination, k votes according to this inclination. Otherwise, k follows his own inclination.

As in the case of group decisions, also in the second specification of the influence function via trend-setters, simple majority may be replaced by other majority rules. In the remainder of this paper, however, we restrict our analysis to simple majority in the case of the influence rule 'following a majority of trend-setters'.

3 Relation-algebraic Description of Measures in a Social Network

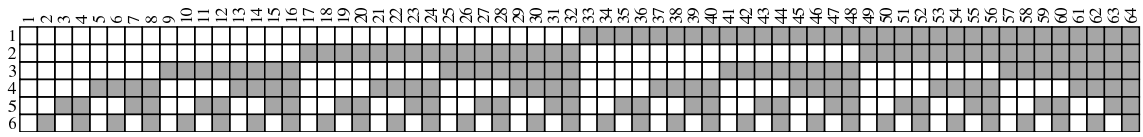
In this section we show how the concepts introduced in Section 2 can be transformed into relation-algebraic specifications that immediately lead to RelView-code. This allows to compute power indices, influence indices, sets of followers and kernels by means of the tool. We demonstrate this by depicting some of the RelView-matrices and -vectors that we have obtained for our running example. For the remainder of the paper, we assume that the reader is familiar with some basic facts of relation algebra, needed to deal with the relation-algebraic specifications and algorithms of the key concepts of Section 2. They can be found, e.g. in (Berghammer et al., 2007, 2009). For more details on relations and relation algebra, see e.g. (Brink et al., 1997; Schmidt and Ströhlein, 1993). Also for a brief description of the RelView tool we refer to (Berghammer et al., 2007, 2009). Details and applications of the RelView tool can be found, e.g. in (Behnke et al., 1998; Berghammer et al., 1996, 2003).

3.1 Modeling inclination vectors and sets of inclination vectors

Relation algebra offers some simple and elegant ways to describe subsets of a given set. For modeling influence vectors, decision vectors, and sets of followers, we will use *column vectors*. Following (Schmidt and Ströhlein, 1993), these are relations v (analogously to linear algebra we use lower-case letters to denote vectors) with $v = v\mathsf{L}$. As for a column vector the range is irrelevant, we consider in the following only vectors $v : X \leftrightarrow \mathbf{1}$ with a specific singleton set $\mathbf{1} := \{\bot\}$ as range. A column vector $v : X \leftrightarrow \mathbf{1}$ can be considered as a Boolean matrix with exactly one column, i.e., as a Boolean column vector, and it *describes* (or: is a description of) the subset $\{x \in X \mid v_{x,\bot}\}$ of its domain X. A non-empty column vector v is a *column point* if $vv^{\mathsf{T}} \subseteq \mathsf{I}$, i.e., it is injective in the relational sense. This means that it represents a singleton subset of its domain or an element from it, if we identify a singleton set $\{x\}$ with the element x. In the Boolean matrix model, hence, a column point $v : X \leftrightarrow \mathbf{1}$ is a Boolean column vector in which exactly one entry is 1. Vectors also allow to formalize the notions of y-columns and x-rows. E.g., for a relation $R : X \leftrightarrow Y$ and $y \in Y$, the column vector $v : X \leftrightarrow \mathbf{1}$ equals the y-column of R if for all $x \in X$ we have $v_{x,\bot}$ iff $R_{x,y}$.

For modeling kernels and subsets of the sets I and $B(I)$, where the influence function B is given by one of the rules 'following only unanimous trend-setters' and 'following a majority of trend-setters' of Subsection 2.3, we will use *row vectors*. These relations are defined as the transposes of column vectors. Again we only will need row vectors v of the specific type $[\mathbf{1} \leftrightarrow Y]$ that correspond to Boolean row vectors. Then v describes the subset $\{y \in Y \mid v_{\bot,y}\}$ of its range Y. The distinction between column vectors and row vectors is not essential. In the context of this paper, however, it is very helpful for the visualization of results of relational computations.

Example 3.1.1. In Example 2.3.1 we have introduced a social network with a set P of six players $1, 2, 3, 4, 5$ and 6. The following picture shows the membership relation[1] $\mathsf{M} : P \leftrightarrow 2^P$ between P and its powerset 2^P as 6×64 Boolean RelView-matrix, where a black square means a 1-entry (i.e., the relationship holds) and a white square means a 0-entry (i.e., the relationship does not hold).



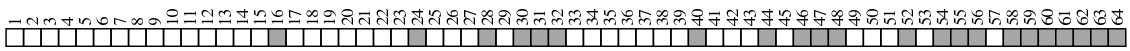
If we consider inclination vectors as relational column vectors, then this membership relation column-wisely enumerates the set I of all inclination vectors, since its 64 columns exactly correspond to the 64 possible inclination vectors of the six players, and these again exactly correspond to the 64 possible subsets of the set of players. For instance, the first column

[1] A membership relation $\mathsf{M} : X \leftrightarrow 2^X$ relates $x \in X$ and $Y \in 2^X$ iff $x \in Y$. It should be emphasized that binary decision diagrams allow a very efficient implementation of M that uses in the worst case $3|X| + 1$ BDD-vertices only. This implementation is part of RelView; see (Berghammer et al., 2002).

corresponds to the inclination vector where each player has the inclination 'no', and the fourth column corresponds to the inclination vector where the players 5 and 6 have the inclination 'yes' and the remaining players have the inclination 'no'.

In the same way, we can obtain a 6×64 Boolean RelView-matrix showing decisions of the players, where the X-column corresponds to the decision vector obtained from the X-column of M representing the inclination vector. Suppose, for instance, that all players are independent, that is, that we deal with the identity function, $Bi = i$ for each $i \in I$.

The next RelView-picture shows a row vector $m : \mathbf{1} \leftrightarrow 2^P$ with 64 columns that describes a subset of the powerset 2^P, i.e., a subset of the set I if we identify $X \in 2^P$ with the inclination vector $i \in I$ where exactly the players of X vote 'yes'.



This row vector describes the set of the inclination vectors where the majority of the players votes 'yes'. This becomes clear if we compare the columns of both RelView-pictures. Doing so, we obtain that for all $X \in 2^P$ the relationship $m_{\perp,X}$ holds iff the number of 1-entries in the X-column of M is strictly larger than the number of 0-entries in the X-column of M.

Besides column vectors, row vectors and membership relations, injective (embedding) mappings are another way of modeling sets. Given a relation $\imath : Z \leftrightarrow X$, that is, an injective mapping in the relational sense of (Schmidt and Ströhlein, 1993), Z may be regarded as a subset of X by identifying it with its image under $\imath$. Then the column vector $\imath^{\mathsf{T}}\mathsf{L} : X \leftrightarrow \mathbf{1}$ describes Z in the above sense. By removing all pairs (x, x) with $x \notin Z$ from the identity relation $\mathsf{I} : X \leftrightarrow X$, the transition in the other direction is also possible, that is, the construction of a relation $inj(v) : Z \leftrightarrow X$ from a given column vector $v : X \leftrightarrow \mathbf{1}$ describing Z in such a way that $inj(v)_{z.x}$ holds iff $z = x$ for all $z \in Z$ and $x \in X$. Such a relation is called the *injective embedding generated by* v and is also used in our applications. Namely, if the row vector $v : \mathbf{1} \leftrightarrow 2^P$ describes a subset $\mathcal{S}$ of 2^P in the sense above, and $\mathsf{M} : P \leftrightarrow 2^P$ is the membership relation, then for all $x \in X$ and $Y \in \mathcal{S}$ we get the equivalence of $(\mathsf{M}\, inj(v^{\mathsf{T}})^{\mathsf{T}})_{x,Y}$ and $x \in Y$. This means that the elements of $\mathcal{S}$ are described precisely by the columns of the relation $\mathsf{M}\, inj(v^{\mathsf{T}})^{\mathsf{T}} : X \leftrightarrow \mathcal{S}$.

3.2 Computing decision vectors and group decisions

We assume a social network with a set P of players. Let $D : P \leftrightarrow P$ be the relation of the dependency graph of the network. The latter property means that there is an arc from a player $j \in P$ to a player $k \in P$ iff $D_{j,k}$ holds. Then the set of the dependent players relation-algebraically is described by the column vector

$$depend(D) := D^{\mathsf{T}}\mathsf{L} \tag{6}$$

of type $[P \leftrightarrow \mathbf{1}]$, where the used L has type $[P \leftrightarrow \mathbf{1}]$, too.

In Subsection 3.1 we have shown that the set I of all inclination vectors immediately can be modeled by the columns of the membership relation $\mathsf{M} : P \leftrightarrow 2^P$. Due to this fact, in

the remainder of this section we regard inclination vectors and the corresponding decision vectors as relational column vectors $i : P \leftrightarrow \mathbf{1}$ and $Bi : P \leftrightarrow \mathbf{1}$, respectively. Our first goal is to develop a column-wise enumeration of the set $B(I)$ of decision vectors with relation-algebraic means, where the influence function B is given by the rule 'following only unambiguous trend-setters'. As a preparatory step, we treat the transformation from i to Bi for a single inclination vector i within relation algebra.

Theorem 3.2.1 *Let $d := depend(D)$. For each inclination vector $i : P \leftrightarrow \mathbf{1}$, the decision vector $Bi : P \leftrightarrow \mathbf{1}$ under the influence rule 'following only unambiguous trend-setters' is given by $Bi = (i \cap (\overline{d} \cup (d \cap D^{\mathsf{T}} i \cap D^{\mathsf{T}} \overline{i}))) \cup (d \cap \overline{D^{\mathsf{T}} \overline{i}})$.*

Proof Let $k \in P$ be an arbitrary player. Using the description (6) of the dependent players and d as abbreviation for $depend(D)$, a formalization of the assumed rule leads to the following specification of $(Bi)_{k,\perp}$:

$$(Bi)_{k,\perp} = \begin{cases} i_{k,\perp} & : \quad \overline{d}_{k,\perp} \vee (d_{k,\perp} \wedge \exists j \in P : D_{j,k} \wedge i_{j,\perp} \wedge \exists j \in P : D_{j,k} \wedge \overline{i}_{j,\perp}) \\ 1 & : \quad d_{k,\perp} \wedge \forall j \in P : D_{j,k} \rightarrow i_{j,\perp} \\ 0 & : \quad d_{k,\perp} \wedge \forall j \in P : D_{j,k} \rightarrow \overline{i}_{j,\perp} \end{cases}$$

If we replace logical constructions by their corresponding relational counter-parts, we obtain the subsequent equivalent specification:

$$(Bi)_{k,\perp} = \begin{cases} i_{k,\perp} & : \quad (\overline{d} \cup (d \cap D^{\mathsf{T}} i \cap D^{\mathsf{T}} \overline{i}))_{k,\perp} \\ \mathsf{L}_{k,\perp} & : \quad (d \cap \overline{D^{\mathsf{T}} \overline{i}})_{k,\perp} \\ \mathsf{O}_{k,\perp} & : \quad (d \cap \overline{D^{\mathsf{T}} i})_{k,\perp} \end{cases}$$

Next, we transform the case distinction in the usual way with the help of disjunctions and conjunctions into a logical formula, viz.:

$$(i_{k,\perp} \wedge (\overline{d} \cup (d \cap D^{\mathsf{T}} i \cap D^{\mathsf{T}} \overline{i}))_{k,\perp}) \ \vee \ (\mathsf{L}_{k,\perp} \wedge (d \cap \overline{D^{\mathsf{T}} \overline{i}})_{k,\perp}) \ \vee \ (\mathsf{O}_{k,\perp} \wedge (d \cap \overline{D^{\mathsf{T}} i})_{k,\perp})$$

Since $\mathsf{L}_{k,\perp}$ is true and $\mathsf{O}_{k,\perp}$ is false, this formula is equivalent to the following one:

$$(i_{k,\perp} \wedge (\overline{d} \cup (d \cap D^{\mathsf{T}} i \cap D^{\mathsf{T}} \overline{i}))_{k,\perp}) \ \vee \ ((d \cap \overline{D^{\mathsf{T}} \overline{i}})_{k,\perp})$$

Now, we again replace in this formula logical constructions by their corresponding relational counter-parts. This yields: $((i \cap (\overline{d} \cup (d \cap D^{\mathsf{T}} i \cap D^{\mathsf{T}} \overline{i}))) \cup (d \cap \overline{D^{\mathsf{T}} \overline{i}}))_{k,\perp}$

If we use this formula as the right-hand side of the original specification, the definition of relational equality shows the claim. □

The relation-algebraic expression $(i \cap (\overline{d} \cup (d \cap D^{\mathsf{T}} i \cap D^{\mathsf{T}} \overline{i}))) \cup (d \cap \overline{D^{\mathsf{T}} \overline{i}})$ is built from i using unions, intersections, complements and left-compositions with constants (i.e., with relation-algebraic expressions free of i) only. Hence (see, for example, (Berghammer, 2006)), if we replace in it the column vector $i : P \leftrightarrow \mathbf{1}$ by the membership relation $\mathsf{M} : P \leftrightarrow 2^P$ that column-wisely enumerates all inclination vectors and adopt simultaneously the type $[P \leftrightarrow \mathbf{1}]$ of d to the type $[P \leftrightarrow 2^P]$ of M by a right-composition with the universal row vector $\mathsf{L} : \mathbf{1} \leftrightarrow 2^P$, we get (with $d := depend(D)$) the relation

$$Dvec(D) := (\mathsf{M} \cap (\overline{d\mathsf{L}} \cup (d\mathsf{L} \cap D^{\mathsf{T}} \mathsf{M} \cap D^{\mathsf{T}} \overline{\mathsf{M}}))) \cup (d\mathsf{L} \cap \overline{D^{\mathsf{T}} \overline{\mathsf{M}}}) \tag{7}$$

of type $[P \leftrightarrow 2^P]$ that column-wisely enumerates the set $B(I)$ of decision vectors. The latter property means: For all $X \in 2^P$, if the X-column of M equals $i : P \leftrightarrow \mathbf{1}$ then, under the assumed rule, the X-column of $Dvec(D)$ equals $Bi : P \leftrightarrow \mathbf{1}$.

Having obtained a relation-algebraic specification for the column-wise enumeration of the decision vectors, our next goal is to obtain with the help of (7) a relation-algebraic specification of the group decisions under majority as decision rule via a row vector. To reach the goal, we assume that a row vector $m : \mathbf{1} \leftrightarrow 2^P$ is available such that for all $X \in 2^P$ we have $m_{\perp,X}$ iff $|X| \geq [\frac{|P|}{2}] + 1$. In RelView such a vector can be easily obtained with the help of the base operation *cardfilter*[2] as $m := \overline{cardfilter(\mathsf{L}, w)}^{\mathsf{T}}$, where the first argument $\mathsf{L} : 2^P \leftrightarrow \mathbf{1}$ describes the entire powerset 2^P, and the second argument $w : W \leftrightarrow \mathbf{1}$ determines the threshold for majority by its length, i.e., fulfils $|W| = [\frac{|P|}{2}] + 1$. Based on the specification of m, we can specify the desired row vector as shown now.

Theorem 3.2.2 *Let, based on m and the specification (7), the row vector $gdv(D)$ of type $[\mathbf{1} \leftrightarrow 2^P]$ be defined by $gdv(D) := m\, syq(\mathsf{M}, Dvec(D))$, where $\mathsf{M} : P \leftrightarrow 2^P$ is the membership relation. Then we have for all $X \in 2^P$: If the decision vector $Bi : P \leftrightarrow \mathbf{1}$ equals the X-column of $Dvec(D)$, then $gdv(D)_{\perp,X}$ holds iff the number of 1-entries in Bi is at least $[\frac{|P|}{2}] + 1$.*

Proof We compute as given below, where the assumption that the X-column of $Dvec(D)$ equals Bi is used in the last step, and the inclination vector $i^{(Y)}$ introduced in this step coincides with the Y-column of M.

$$\begin{aligned}
gdv(D)_{\perp,X} &\iff (m\, syq(\mathsf{M}, Dvec(D)))_{\perp,X} \\
&\iff \exists\, Y \in 2^P : m_{\perp,Y} \wedge syq(\mathsf{M}, Dvec(D))_{Y,X} \\
&\iff \exists\, Y \in 2^P : m_{\perp,Y} \wedge (\forall\, k \in P : \mathsf{M}_{k,Y} \leftrightarrow Dvec(D)_{k,X}) \\
&\iff \exists\, Y \in 2^P : |Y| \geq [\tfrac{|P|}{2}] + 1 \;\wedge\; (\forall\, k \in P : \mathsf{M}_{k,Y} \leftrightarrow Dvec(D)_{k,X}) \\
&\iff \exists\, Y \in 2^P : |Y| \geq [\tfrac{|P|}{2}] + 1 \;\wedge\; i^{(Y)} = Bi
\end{aligned}$$

Now the claim follows from the simple fact that the number of 1-entries in the column vector $i^{(Y)}$ equals $|Y|$. □

Summing up, we have for the influence function B defined by the rule 'following only unambiguous trend-setters' and for the group decision function gd defined by simple majority: If the inclination vector $i : P \leftrightarrow \mathbf{1}$ is given by the X-column of the membership relation $\mathsf{M} : P \leftrightarrow 2^P$, then the corresponding decision vector $Bi : P \leftrightarrow \mathbf{1}$ is given by the X-column of the relation $Dvec(D) : P \leftrightarrow 2^P$ and, furthermore, $gd(Bi) = 1$ iff $gdv(D)_{\perp,X}$ holds.

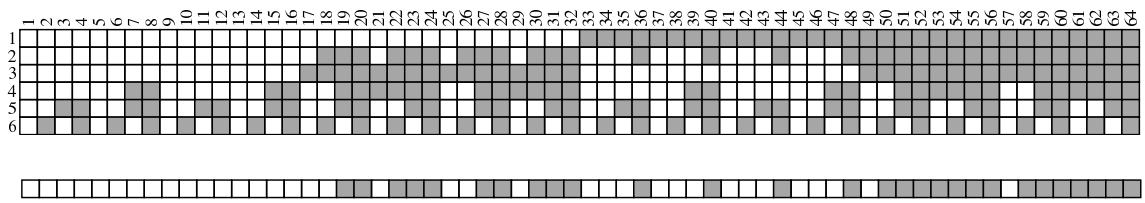
Example 3.2.1. We have transformed the above relation-algebraic specifications into RelView-programs. To give examples how such programs look like, we present in the following the code for both specifications. In the following RelView-programs *Dvec* and *gdv* the calls

[2] If $v : 2^M \leftrightarrow \mathbf{1}$ represents the subset $\mathcal{S}$ of 2^M and the size of the domain of $w : W \leftrightarrow \mathbf{1}$ is at most $|M| + 1$, then for all $X \in 2^M$ we have $cardfilter(v, w)_{X,\perp}$ iff $X \in \mathcal{S}$ and $|X| < |W|$. Hence, the complement of $cardfilter(\mathsf{L}, w)$ represents the subsets of 2^M whose elements have at least size $|W|$.

epsi$(O(D))$ of the pre-defined operation *epsi* compute the membership relation $\mathsf{M} : P \leftrightarrow 2^P$, and the calls *L1n*$(M)$ of the pre-defined operation *L1n* yield the row vector $\mathsf{L} : \mathbf{1} \leftrightarrow 2^P$.

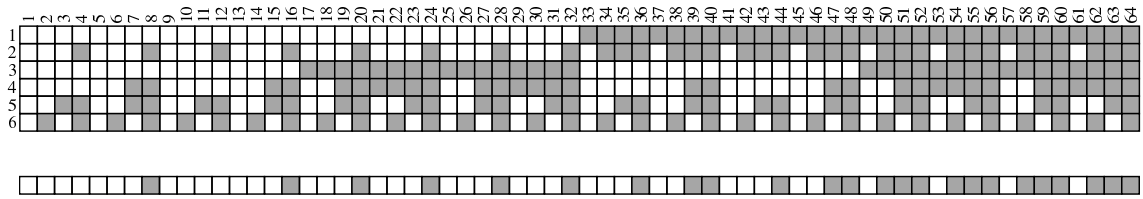
```
Dvec(D)
  DECL M, d
  BEG  M = epsi(O(D)); d = D^*L1n(M)
       RETURN (M & (-d | (d & D^*M & D^*-M))) | (d & -(D^*-M))
  END.

gdv(D,w)
  DECL M, m
  BEG  M = epsi(O(D)); m = -cardfilter(L1n(M)^,w)^
       RETURN m*syq(M,Dvec(D))
  END.
```

Applied to the relation D of our running example and a column vector w of length 4 (the threshold of majority) in the case of the RelView-program for $gdv(D)$, we obtained by their means the following results for $Dvec(D)$ and $gdv(D)$. The 64 columns of the 6×64 RelView-matrix represent the 64 decision vectors obtained from the 64 inclination vectors, and the entries of the 1×64 row vector below this matrix indicate the group decision for each decision vector.



Let us explain these results by the specific inclination vectors treated in Example 3.1.1. For the first column of the membership relation M of Example 3.1.1, where each player votes 'no', we obtain 'no' also as decision of each player as well as of the entire group. The same is the case if the inclination of the players 5 and 6 is 'yes' and that of the remaining players is 'no'; cf. the fourth columns of M, $Dvec(D)$ and $gdv(D)$.

We also have developed a RelView-program that computes the column-wise enumeration of the decision vectors under 'following a majority of the trend-setters' as the influence rule by handling one after another the columns of the membership relation via a loop. If we use this program in the case of our running example, we obtain the following RelView-matrix and row vector for the decision vectors and the group decisions, respectively.



In contrast with the influence rule 'following only unambiguous trend-setters', now the inclinations 'yes' of the agents 5 and 6 and 'no' of the remaining agents yield a decision, where 2 changes his opinion from 'no' to 'yes', because of the 'yes'-vote of the majority of the trend-setters agent 2 depends on. In spite of this change, the group's decision remains 'no'.

An example where the different influence rules yield different group decisions for the same inclination vector is given by the 8th columns of the matrices and row vectors, respectively. If the inclination of the players 4, 5 and 6 is 'yes' and that of the remaining players is 'no', then 'following only unambiguous trend-setters' implies 'inclination equals decision' and the group decision 'no'. Nevertheless, 'following a majority of the trend-setters' implies that also player 2 finally votes 'yes', so that the collective vote becomes 'yes', too.

3.3 Computing power indices

Now, we demonstrate how to compute the indices presented in Subsection 2.1 with relation-algebraic means. The main steps are to determine four row vectors of type $[\mathbf{1} \leftrightarrow 2^P]$ which describe the four sets I_k^{++}, I_k^{+-}, I_k^{-+} and I_k^{--}, respectively. Since RelView yields for each computed relation also the number of its 1-entries (i.e., its set-theoretic size), from the vector descriptions we get the numbers $|I_k^{++}|$, $|I_k^{+-}|$, $|I_k^{-+}|$ and $|I_k^{--}|$, and from these also the various power indices using straightforwardly their specifications of Subsection 2.1. Note, that the set I^+ used in the definition of the indices $\mathrm{M_1GHB}_k$, $\mathrm{M_4GHB}_k$ and MGHB is already described by the row vector $gdv(D)$ of Theorem 3.2.2 or its analogon in the case of the rule 'following a majority of the trend-setters'.

We assume that the player $k \in P$, on which the four sets I_k^{++}, I_k^{+-}, I_k^{-+} and I_k^{--} depend, is described by a column point $p : P \leftrightarrow \mathbf{1}$ in the relational sense. As the definitions of the sets also use the values $gd(Bi)$ for $i \in I$, we assume, furthermore, that the group decision row vector $g := gdv(D)$ is at hand (where the influence rule used for its computation is arbitrary). Then we are able to prove the following result.

Theorem 3.3.1 *Let, depending on the column point $p : P \leftrightarrow \mathbf{1}$ and the row vector $g : \mathbf{1} \leftrightarrow 2^P$, the four vectors $ipp(p,g)$, $ipm(p,g)$, $imp(p,g)$ and $imm(p,g)$ of type $[\mathbf{1} \leftrightarrow 2^P]$ be defined as $ipp(p,g) := p^{\mathsf{T}}\mathsf{M} \cap g$, $ipm(p,g) := p^{\mathsf{T}}\mathsf{M} \cap \overline{g}$, $imp(p,g) := p^{\mathsf{T}}\,\overline{\mathsf{M}} \cap g$ and $imm(p,g) := p^{\mathsf{T}}\,\overline{\mathsf{M}} \cap \overline{g}$, where $\mathsf{M} : P \leftrightarrow 2^P$ is the membership relation. Then for all $X \in 2^P$: If the X-column of M equals the inclination vector $i : P \leftrightarrow \mathbf{1}$, then $ipp(p,g)_{\perp,X}$ holds iff $i \in I_k^{++}$, $ipm(p,g)_{\perp,X}$ holds iff $i \in I_k^{+-}$, $imp(p,g)_{\perp,X}$ holds iff $i \in I_k^{-+}$, and $imm(p,g)_{\perp,X}$ holds iff $i \in I_k^{--}$.*

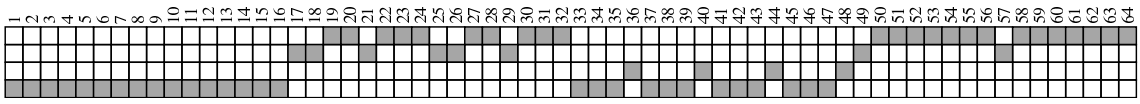
Proof

$$
\begin{aligned}
ipp(p,g)_{\perp,X} &\iff (p^{\mathsf{T}}\mathsf{M} \cap g)_{\perp,X} \\
&\iff \exists\, j \in P : p_{j,\perp} \wedge \mathsf{M}_{j,X} \wedge g_{\perp,X} \\
&\iff \exists\, j \in P : j = k \wedge \mathsf{M}_{j,X} \wedge g_{\perp,X} && p \text{ describes } k \\
&\iff \mathsf{M}_{k,X} \wedge g_{\perp,X} && gd(Bi) = 1 \text{ iff } gdv(D)_{\perp,X} \\
&\iff i_{k,\perp} \wedge gd(Bi) = 1 && \text{assumption}
\end{aligned}
$$

Since the relationship $i_{k,\perp}$ is nothing else than $i_k = 1$ for the k-component of a Boolean vector in the sense of Section 2, the first claim follows from this. In the same way the remaining equivalences can be computed. $\square$

Due to this theorem, the row vector $ipp(p, g)$ precisely designates those columns of the membership relation M which belong to the set I_k^{++}, and the remaining three row vectors of the theorem do the same for the three sets I_k^{+-}, I_k^{-+} and I_k^{--}, respectively. Once more it is very easy to translate the relation-algebraic specifications of Theorem 3.3.1 into the programming language of RelView. Subsequently, we show some results for our running example. We restrict our analysis to the generalized Hoede-Bakker index.

Example 3.3.1. Focussing on player 2 in our example, which is influenced by the three trend-setters 1, 5 and 6, using 'following only unanimous trend-setters' as influence rule, the first row of the following 4×64 RelView-matrix depicts the row vector $ipp(p, g)$, i.e., precisely designates those columns of the membership relation $M : P \leftrightarrow 2^P$ that belong to the set $I_2^{++}(B, gd)$, with gd given by simple majority. The second, third and fourth row of the matrix do the same for $I_2^{+-}(B, gd)$, $I_2^{-+}(B, gd)$ and $I_2^{--}(B, gd)$ respectively.



Counting the 1-entries of the single rows, one can easily obtain the generalized Hoede Bakker index of player 2: $\frac{5}{8}$.

3.4 Computing influence indices, followers and kernels

In the following, we assume a coalition S of players to be described by a column vector $s : P \leftrightarrow \mathbf{1}$, and a single player $j \in P$ to be described by a column point $p : P \leftrightarrow \mathbf{1}$. We want to compute the possibility influence index of S on player j. Since it is defined by means of the sizes of the sets $I_{S\to j}$ and $I^*_{S\to j}(B)$, our task is to describe these sets within relation algebra. A translation of the results into RelView-code then allows to proceed exactly as in the case of the power indices. Both $I_{S\to j}$ and $I^*_{S\to j}(B)$ are subsets of I_S. Therefore, as a preparatory step we describe the latter set of inclination vectors with relation-algebraic means. Doing so, projection relations and the pairing operation come into play.

Theorem 3.4.1 *Assume $s : P \leftrightarrow \mathbf{1}$ as description of the coalition $S \subseteq P$ and the row vector $is(s)$ of type $[\mathbf{1} \leftrightarrow 2^P]$ to be defined as $is(s) := [s^\mathsf{T}, s^\mathsf{T}]\ \overline{\overline{(\overline{\pi\mathsf{M}} \cup \rho\mathsf{M}) \cap (\overline{\rho\mathsf{M}} \cup \pi\mathsf{M})}}$, where $\mathsf{M} : P \leftrightarrow 2^P$ is the membership relation and $\pi : P \times P \leftrightarrow P$ and $\rho : P \times P \leftrightarrow P$ are the projection relations. Then we have for all $X \in 2^P$: If the X-column of M equals the inclination vector $i : P \leftrightarrow \mathbf{1}$, then $is(s)_{\perp,X}$ holds iff $i \in I_S$.*

Proof Since the X-column of M equals i, we have for all pairs $u = (u_1, u_2) \in P \times P$ the following equivalence:

$$\begin{aligned}
i_{u_1,\bot} = i_{u_2,\bot} &\iff \mathsf{M}_{u_1,X} \leftrightarrow \mathsf{M}_{u_2,X} \\
&\iff (\pi\mathsf{M})_{u,X} \leftrightarrow (\rho\mathsf{M})_{u,X} \\
&\iff ((\pi\mathsf{M})_{u,X} \rightarrow (\rho\mathsf{M})_{u,X}) \wedge ((\rho\mathsf{M})_{u,X} \rightarrow (\pi\mathsf{M})_{u,X}) \\
&\iff (\,\overline{\pi\mathsf{M}}\,_{u,X} \vee (\rho\mathsf{M})_{u,X}) \wedge (\,\overline{\rho\mathsf{M}}\,_{u,X} \vee (\pi\mathsf{M})_{u,X}) \\
&\iff ((\,\overline{\pi\mathsf{M}} \cup \rho\mathsf{M}) \cap (\,\overline{\rho\mathsf{M}} \cup \pi\mathsf{M}))_{u,X}
\end{aligned}$$

From this result and since s describes S, we obtain

$$\begin{aligned}
is(s)_{\bot,X} &\iff \overline{[s^\mathsf{T}, s^\mathsf{T}]\ \overline{(\,\overline{\pi\mathsf{M}} \cup \rho\mathsf{M}) \cap (\,\overline{\rho\mathsf{M}} \cup \pi\mathsf{M})}}\ _{\bot,X} \\
&\iff \neg\exists\, u \in P{\times}P : [s^\mathsf{T}, s^\mathsf{T}]_{\bot,u} \wedge \overline{(\,\overline{\pi\mathsf{M}} \cup \rho\mathsf{M}) \cap (\,\overline{\rho\mathsf{M}} \cup \pi\mathsf{M})}\,_{u,X} \\
&\iff \forall\, u \in P{\times}P : [s^\mathsf{T}, s^\mathsf{T}]_{\bot,u} \rightarrow ((\,\overline{\pi\mathsf{M}} \cup \rho\mathsf{M}) \cap (\,\overline{\rho\mathsf{M}} \cup \pi\mathsf{M}))_{u,X} \\
&\iff \forall\, u \in P{\times}P : s_{u_1,\bot} \wedge s_{u_2,\bot} \rightarrow (i_{u_1,\bot} = i_{u_2,\bot}) \\
&\iff \forall\, u \in P{\times}P : u_1 \in S \wedge u_2 \in S \rightarrow (i_{u_1,\bot} = i_{u_2,\bot})
\end{aligned}$$

The latter formula of this calculation exactly says that $i \in I_S$. □

Hence, the row vector $is(s)$ precisely designates those columns of the membership relation M which belong to the set I_S. Next, we attack the relation-algebraic specification of the set $I_{S\rightarrow j}$, where $j \in P$ is described by the column point $p : P \leftrightarrow \mathbf{1}$. In the following theorem we relation-algebraically specify a row vector that precisely designates those columns of M which are inclination vectors of potential influence of S on j.

Theorem 3.4.2 *Let $s : P \leftrightarrow \mathbf{1}$ describe the coalition $S \subseteq P$, the column point $p : P \leftrightarrow \mathbf{1}$ describe the player $j \in P$, the column point $q \subseteq s$ describe some player $k \in S$, and the row vector $potinf(s,p)$ of type $[\mathbf{1} \leftrightarrow 2^P]$ be defined as $potinf(s,p) := ((r \cup r') \cap \overline{r \cap r'}\,)\, inj(is(s)^\mathsf{T})$, where $r := p^\mathsf{T}\mathsf{M}\, inj(is(s)^\mathsf{T})^\mathsf{T}$ and $r' := q^\mathsf{T}\mathsf{M}\, inj(is(s)^\mathsf{T})^\mathsf{T}$ with $\mathsf{M} : P \leftrightarrow 2^P$ as membership relation. Then we have for all $X \in 2^P$: If the X-column of M equals the inclination vector $i : P \leftrightarrow \mathbf{1}$, then $potinf(s,p)_{\bot,X}$ holds iff $i \in I_{S\rightarrow j}$.*

Proof From Theorem 3.4.1 we know that the row vector $is(s)$ describes the subset $\mathcal{S}$ of 2^P that consists of those sets $Y \in 2^P$ for which the Y-column of M is, considered as inclination vector, a member of I_S. Furthermore, $inj(is(s)^\mathsf{T}) : \mathcal{S} \leftrightarrow 2^P$ is the relational description of the identity mapping from $\mathcal{S}$ to 2^P; see Subsection 3.1. Using these facts and the assumption that the X-column of M equals i, we get

$$\begin{aligned}
potinf(s,p)_{\bot,X} &\iff (((r \cup r') \cap \overline{r \cap r'}\,)\, inj(is(s)^\mathsf{T}))_{\bot,X} \\
&\iff \exists\, Y \in \mathcal{S} : ((r \cup r') \cap \overline{r \cap r'}\,)_{\bot,Y} \wedge inj(is(s)^\mathsf{T})_{Y,X} \\
&\iff \exists\, Y \in \mathcal{S} : ((r \cup r') \cap \overline{r \cap r'}\,)_{\bot,Y} \wedge Y = X \\
&\iff X \in \mathcal{S} \wedge ((r \cup r') \cap \overline{r \cap r'}\,)_{\bot,X} \\
&\iff i \in I_S \wedge ((r \cup r') \cap \overline{r \cap r'}\,)_{\bot,X}
\end{aligned}$$

Next, we apply that the column point p describes the player $j \in P$, again we apply the assumption and get in the case $X \in \mathcal{S}$ the equivalence

$$r_{\perp,X} \iff \exists\, l \in P : p_{l,\perp} \wedge \mathsf{M}_{l,X} \iff \exists\, l \in P : j = l \wedge l \in X \iff j \in X \iff i_{j,\perp}.$$

In the same way[3] from the description of $k \in P$ by the column vector q and the assumption we obtain that $r'_{\perp,X}$ is equivalent to $i_{k,\perp}$, i.e., to the k-entry of i to be 1. The latter fact implies the equivalence of $r'_{\perp,X}$ and $i_S = 1$ for the Boolean value i_S used in the specification of $I_{S\to j}$. A consequence of the just shown properties is

$$\begin{aligned}((r \cup r') \cap \overline{r \cap r'}\,)_{\perp,X} &\iff (r_{\perp,X} \vee r'_{\perp,X}) \wedge \neg(r_{\perp,X} \wedge r'_{\perp,X})\\ &\iff (i_{j,\perp} \vee i_S = 1) \wedge \neg(i_{j,\perp} \wedge i_S = 1)\\ &\iff (i_{j,\perp} \leftrightarrow \neg(i_S = 1))\\ &\iff (i_j = \neg i_S)\end{aligned}$$

since again the relationship $i_{j,\perp}$ is nothing else than the validity of $i_j = 1$ in the sense of Section 2. A combination of this fact with the result of the above calculation yields the claim:

$$\mathit{potinf}(s,p)_{\perp,X} \iff i \in I_S \wedge (i_j = \neg i_S) \qquad \square$$

To obtain a row vector $\mathit{inf}(s,p,D)$ of type $[\mathbf{1} \leftrightarrow 2^P]$ that precisely designates those columns of the membership relation $\mathsf{M} : P \leftrightarrow 2^P$ which are inclination vectors of influence of S on j, i.e., members of $I^*_{S\to j}(B)$, we use the equation $I^*_{S\to j}(B) = I_{S\to j} \cap \{i \in I_S \mid (Bi)_j = i_S\}$.

The definition of the set $\{i \in I_S \mid (Bi)_j = i_S\}$ is rather similar to that of the set $I_{S\to j}$; cf. Subsection 2.2. Compared with the latter one, only the expressions Bi and i_S are used instead of i and $\neg i_S$. This immediately leads to the relation-algebraic specification of the set $I^*_{S\to j}(B)$ by the row vector

$$\mathit{inf}(s,p,D) := \mathit{potinf}(s,p) \cap \overline{(r \cup r') \cap \overline{r \cap r'}}\ \mathit{inj}(\mathit{is}(s)^{\mathsf{T}}) \tag{8}$$

with now r and r' given as follows (cf. Theorem 3.4.2):

$$r := p^{\mathsf{T}} \mathit{Dvec}(D)\, \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})^{\mathsf{T}} \qquad r' := q^{\mathsf{T}} \mathsf{M}\, \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})^{\mathsf{T}}$$

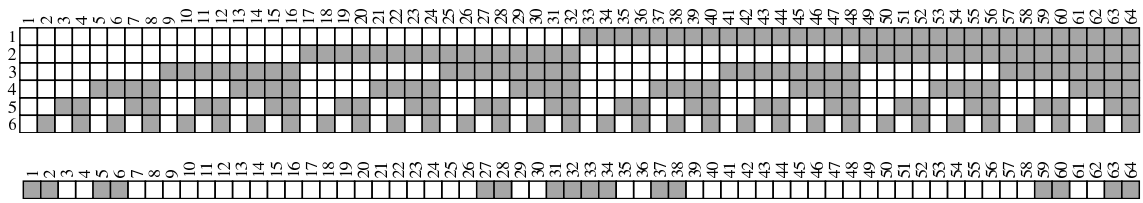
This is due to the fact that the decision vectors column-wisely are enumerated via the relation $\mathit{Dvec}(D) : P \leftrightarrow 2^P$ (where the concrete influence rule is irrelevant) and, for the inclination vector $i : P \leftrightarrow \mathbf{1}$ being the X-column of the membership relation $\mathsf{M} : P \leftrightarrow 2^P$, the relationship $((r \cup r') \cap \overline{r \cap r'}\,)_{\perp,X}$ does not hold iff $i_{j,\perp}$ and $i_S = 1$ are equivalent.

In the following, we demonstrate by means of our running example how results of the RelView-programs (that immediately are obtained from the developed relation-algebraic specifications by writing them in the programming language of the tool) look like.

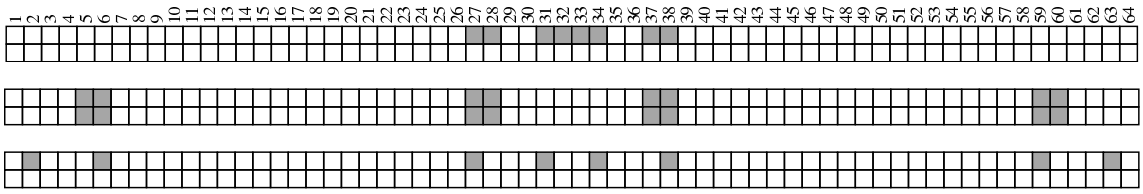
Example 3.4.1. Let us consider the coalition S with players $2, 3$ and 5, which have been indicated by black vertices in the dependency graph of Example 2.3.1. For this coalition, the set I_S contains 16 inclination vectors. This follows from the following two RelView-pictures.

[3] In terms of matrices, r equals the j-row of $\mathsf{M}\, \mathit{inj}(\mathit{is}(s)^{\mathsf{T}})^{\mathsf{T}}$ and r' the k-row of the same relation.

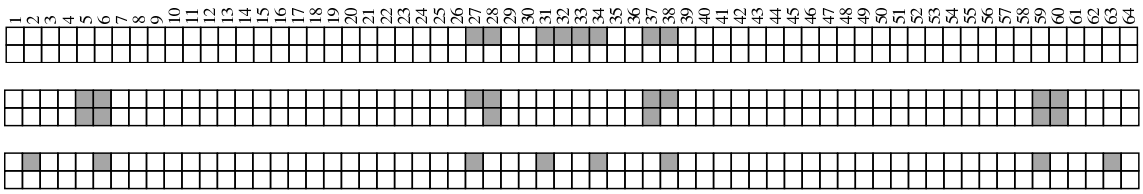
The first one shows again the membership relation $\mathsf{M} : P \leftrightarrow 2^P$ of Example 3.1.1 and the second one the row vector $is(s) : \mathbf{1} \leftrightarrow 2^P$, where the column vector $s : P \leftrightarrow \mathbf{1}$ describes S.



The row vector precisely designates those columns of the matrix where the entries $2, 3$ and 5 have the same colour. Below we show the RelView-representations of the sets $I_{S\to j}$ and $I^*_{S\to j}(B)$ for those players j which are not contained in the coalition S. The first row of the following 2×64 RelView-matrix indicates the columns of the membership relation M which are inclination vectors from the set $I_{S\to 1}$, and the second row indicates the inclination vectors that belong to $I^*_{S\to 1}(B)$, where 'following only unambiguous trend-setters' is the influence rule. The next two RelView-matrices do the same for the sets $I_{S\to 4}$ and $I^*_{S\to 4}(B)$ and the sets $I_{S\to 6}$ and $I^*_{S\to 6}(B)$, respectively. From the three pictures we obtain that, under the assumed rule, the possibility influence indices of S on the players 1 and 6 are 0, and the possibility influence index of S on the player 4 is 1.



The next three RelView-matrices are analogous to the just presented ones, however, now with 'following the majority of the trend-setters' as influence rule.



Comparing these matrices with the above ones, we get that in the case of our running example both influence rules lead to the same sets and, hence, the same indices.[4] In words, the results say: Whatever of the two influence rules is applied, the coalition S is without any influence on the players 1 and 6 and in the case of player 4 there is a possibility of influence of S on 4 and it is even maximal.

[4] It should be mentioned that the equal results for player $k := 4$ in our running example are caused by the fact that it has exactly two trend-setters. In such a case the value $(Bi)_k$ computed from i via 'following only unambiguous trend-setters' is the same as that computed via 'following the majority of the trend-setters'. To give an example where both rules lead to different results, we want to mention that, for $S' := \{3, 5\}$ and player 2, we get $|I_{S'\to 2}| = 16$ and $|I^*_{S'\to 2}(B)| = 4$ if B is given by 'following only unambiguous trend-setters', respectively $|I^*_{S'\to 2}(B)| = 12$ if B is given by 'following the majority of the trend-setters'.

The next theorem shows how sets of followers can be described relation-algebraically by means of column vectors in the sense of Subsection 3.1. In it, the relations R and Q column-wisely enumerate the sets I_S and $B(I_S)$, respectively, and the column point q again is used for specifying for $i \in I_S$ the specific Boolean value i_S. Once more it is arbitrarily which influence rule is used for the definition of the influence function B.

Theorem 3.4.3 *Assume $s : P \leftrightarrow \mathbf{1}$ to describe the coalition $S \subseteq P$, and the column point $q \subseteq s$ to describe some player $k \in S$. Furthermore, let $\mathsf{M} : P \leftrightarrow 2^P$ be the membership relation. If the column vector $follow(D, s)$ of type $[P \leftrightarrow \mathbf{1}]$ is defined as $follow(D, s) := syq(Q^\mathsf{T}, R^\mathsf{T} q)$, with relations $R := \mathsf{M}\, inj(is(s)^\mathsf{T})^\mathsf{T}$ and $Q := Dvec(D)\, inj(is(s)^\mathsf{T})^\mathsf{T}$, then for all $j \in P$ we have $follow(D, s)_{j,\perp}$ iff $j \in F_B(S)$.*

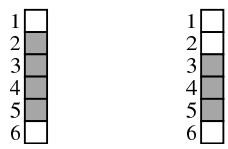
Proof As in the proof of Theorem 3.4.2, we denote the subset of 2^P that is described by the row vector $is(s) : \mathbf{1} \leftrightarrow 2^P$ with $\mathcal{S}$. Then both R and Q have the type $[P \leftrightarrow \mathcal{S}]$. Furthermore, we are able to compute as given below:

$$\begin{aligned} follow(D, s)_{j,\perp} &\iff syq(Q^\mathsf{T}, R^\mathsf{T} q)_{j,\perp} \\ &\iff \forall\, X \in \mathcal{S} : Q^\mathsf{T}_{X,j} \leftrightarrow (R^\mathsf{T} q)_{X,\perp} \\ &\iff \forall\, X \in \mathcal{S} : Q_{j,X} \leftrightarrow (q^\mathsf{T} R)_{\perp,X} \\ &\iff \forall\, i \in I_S : (Bi)_{j,\perp} \leftrightarrow i_S = 1 \\ &\iff j \in F_S(B) \end{aligned}$$

The fourth step of this calculation uses that there is a one-to-one corrrespondence between the sets 2^P and I, that $X \in 2^P$ belongs to $\mathcal{S}$ iff the corresponding inclination vector $i \in I$ belongs to I_S, that $Dvec(D)$ column-wisely enumerates the decision vectors Bi and that $(q^\mathsf{T} R)_{\perp,X}$ iff $i_S = 1$ (see the proof of Theorem 3.4.2). □

Let us again demonstrate what the RelView-program obtained from this theorem yields in the case of our running example with the coalition S consisting of the players $2, 3$ and 5.

Example 3.4.2. In the following two RelView-pictures two column vectors are depicted which describe two subsets of the set P. The left column vector describes the set of followers of S under the influence rule 'following only unambiguous trend-setters' and the right column vector does the same with 'following the majority of the trend-setters'.



So, the followers of S under the first rule are $2, 3, 4$ and 5 and those under the second rule are $3, 4$ and 5.

Having a RelView-program at hand for computing sets of followers, it is an easy task to implement another one that computes the kernel of an influence function by applying the

former program to all subsets of P. Applied to our running example, the second program proved that there is no difference whether the influence function B is defined via the rule 'following only unambiguous trend-setters' or the rule 'following the majority of the trend-setters'. Both rules yield the same result, viz. $\{\{6\}, \{5\}, \{2\}, \{1\}\}$. Of course, this is a special case. Experiments with RelView showed that, in general, the kernels of both rules we have introduced in this paper turn out to be different.

4 The Dutch Parliament Example

In the last section, we have used an artificial running example to illustrate our relation-algebraic approach to measure players' 'strength' in a social network. In the following we present another application of RelView. It stems from the real world and is based on the structure of the Second Chamber (*Tweede Kamer*) of the present Dutch Parliament. Although we think that our assumptions are not far from reality, one should be careful to draw conclusions about Dutch politics from this example. We mainly want to show how our model and software works and may be applied to the real world.

4.1 The present Dutch parliament

There are presently ten parties in the Dutch parliament, viz. (in alphabetic order) the parties CDA - Christen-Democratisch Appel (Christian Democrats), CU - Christen Unie (Christian Union), D66 - Democraten66 (Democrats 66), GL - GroenLinks (Green Left), PvdA - Partij van de Arbeid (Labor Party), PvdD - Partij voor de Dieren (Animal Party), PVV - Partij voor de Vrijheid (Party for Freedom), SGP - Staatkundig Gereformeerde Partij (Political Reformed Party), SP - Socialistische Partij (Socialist Party), and VVD - Volkspartij voor Vrijheid en Democratie (People's Party for Freedom and Democracy). Hence, we have

$$P := \{\mathrm{CDA}, \mathrm{CU}, \mathrm{D66}, \mathrm{GL}, \mathrm{PvdA}, \mathrm{PvdD}, \mathrm{PVV}, \mathrm{SGP}, \mathrm{SP}, \mathrm{VVD}\}.$$

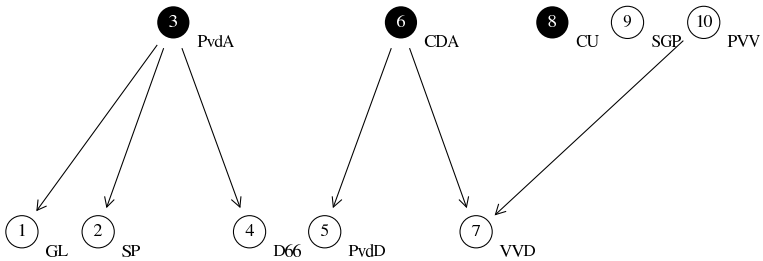
In the following table the Dutch parties of the present parliament are shown ($k \in P$), placed in a specific order, together with the numbers of seats (w_k), where the total number of seats is equal to 150. The main data source for these matters can be found in the *Dutch Parliamentary Election Studies* [17], containing among others data about left-right self rating scales. All experts will agree with the one-dimensional socio-economic left-right scale, though they will disagree in the end on the number of relevant dimensions. Probably one should take other dimensions into account too, for instance, an immaterial dimension, but although data are being collected, among others in national election research, there is much less agreement among experts on these other dimensions. In principle, our software can be adapted to the multidimensional case, but would become more complex. The specific placement of the parties below from GL to PVV is based on a socio-economic left-right scale for the postwar period in the Netherlands, developed in (Morgan, 1976).

$k \in P$	GL	SP	PvdA	D66	PvdD	CDA	VVD	CU	SGP	PVV
w_k	7	25	33	3	2	41	22	6	2	9

A winning coalition in this Dutch example is a coalition with at least 76 seats in parliament. The three parties CDA, CU and PvdA are presently forming the Dutch cabinet. We may assume that PvdA is a trend-setter for the two parties D66 and GL: the latter parties usually follow the former one. Furthermore, we assume that PVV is a trend-setter for VVD. This may be only partially true, but we assume it as hypothesis for our computations. Apart from that, we assume more influence relationships, which display both office seeking and policy seeking motivations of the Dutch parties. Let us assume that for some of the parties, the stronger (direct) neighbor on the left-right scale is a trend-setter of a party if this neighbor has more seats than the party in question. So, apart from being trend-setter for D66, the PvdA is assumed to be also trend-setter for SP and CDA is assumed to be trend-setter for VVD and PvdD (hence, VVD is assumed to have two trend-setters, PVV and CDA).

4.2 Results of computations

Below is the RelView-representation of the dependency relation and the present coalition $S = \{\mathrm{CDA}, \mathrm{CU}, \mathrm{PvdA}\}$.



We have used the RelView-versions of the relation-algebraic specifications of Section 3 to determine by means of the tool for this quasi-realistic example all the concepts mentioned in Section 2. Since each party has at most two trend-setters, both influence rules of Subsection 2.3 lead to the same description of the influence function B. We have for each inclination vector i that $(Bi)_{\mathrm{D66}} = (Bi)_{\mathrm{SP}} = (Bi)_{\mathrm{GL}} = i_{\mathrm{PvdA}}$, $(Bi)_{\mathrm{PvdD}} = i_{\mathrm{CDA}}$, $(Bi)_{\mathrm{VVD}}$ depends on the inclinations i_{CDA} and i_{PVV} and $(Bi)_j = i_j$ for j being PvdA, CDA, CU, SGP, PVV.

There are situations where parties decide without taking their numbers of seats into account, for instance, in parliamentary committees with one representative per party. In such situations the group decision gd is given by simple majority of the number of parties. For each Dutch party $k \in P$, we determined the generalized Hoede-Bakker index $\mathrm{GHB}_k(B, gd)$ and its modifications. Moreover, for each party outside the cabinet, that is, for all $j \in P \setminus S$, we have computed the possibility influence indices of the cabinet on j, i.e., $\overline{d}(B, S \to j)$, as well as the set of followers of the cabinet $F_B(S)$ under B, and the kernel $\mathcal{K}(B)$ of B. Here are some results. Because of their sizes we are not able to present the corresponding RelView-matrices. We only present the decisive numbers that, as already mentioned, either

are directly delivered by RelView as numbers of 1-entries of computed results or can be easily computed from these numbers.

Let us start with the power indices. In the following table we listen for the Dutch parliament example the sizes of the sets underlying their definitions as computed by RelView using the relation-algebraic specifications of the Subsections 3.2 and 3.3.

$k \in P$	GL	SP	PvdA	D66	PvdD	CDA	VVD	CU	SGP	PVV
$\lvert I_k^{++}\rvert$	216	216	400	216	216	288	224	256	256	272
$\lvert I_k^{+-}\rvert$	296	296	112	296	296	224	288	256	256	240
$\lvert I_k^{-+}\rvert$	216	216	32	216	216	144	208	176	176	160
$\lvert I_k^{--}\rvert$	296	296	480	296	296	368	304	336	336	352

From these numbers and the fact that $|I^+| = 432$ and $|I^-| = 1024 - 432 = 592$, we immediately are able to compute all power indices introduced in Subsection 2.1. In the following table we show the values for the power index GHB_k only.

$k \in P$	GL	SP	PvdA	D66	PvdD	CDA	VVD	CU	SGP	PVV
GHB_k	0	0	0.72	0	0	0.28	0.03	0.16	0.16	0.22

Since the specification of the row vector $gdv(D) : \mathbf{1} \leftrightarrow 2^P$ of Theorem 3.2.2 bases on 'simple majority of number of parties' as group decision function gd, in our concrete example meaning that $gd(Bi) = 1$ iff the size of the set $\{j \in P \mid (Bi)_j = 1\}$ is at least 6, in the above results all parties are treated as if they have exactly one seat. However, in plenary meetings of the Dutch parliament the number of seats is decisive. Here a proposal is accepted by the parliament iff more than 75 seats vote 'yes'.[5] As a consequence, the majority-of-parties-based definition of gd we have used so far may lead to wrong results, if, e.g., 5 parties with few seats vote 'no' while the proposal is accepted because more than 75 seats vote 'yes'. An example for this is: $i_{\mathrm{CDA}} = i_{\mathrm{SGP}} = i_{\mathrm{PVV}} = 0$, and $i_j = 1$ for $j \notin \{\mathrm{CDA}, \mathrm{SGP}, \mathrm{PVV}\}$. The influences given by the above graph changes the inclinations 'yes' of the two parties PvdD and VVD to the decision 'no', the remaining parties vote according to their inclinations. With five 'no' and five 'yes' the majority-of-parties-based definition of gd leads to group decision 'no'. But the parliament votes 'yes' due to the $2 + 41 + 22 + 2 + 9 = 76$ seats of the five parties PvdD, CDA, VVD, SGP and PVV together.

If we assume that each party votes as a block, then the following alternative definition of the group decision function precisely describes how the Dutch parliament makes a decision:

$$gd(Bi) = 1 \iff \textstyle\sum_{k \in P_i^+} w_k > 75, \tag{9}$$

where $P_i^+ := \{j \in P \mid (Bi)_j = 1\}$ is the set of parties which vote 'yes' under i and w_k denotes the number of seats of $k \in P$ according to the table of Subsection 4.1.

[5] Of course, this only holds if each member attends the House.

To obtain a relation-algebraic specification of $gdv(D) : \mathbf{1} \leftrightarrow 2^P$ also for the group decision function introduced by (9), we assume N to be the set of the 150 Dutch parliament seats and the distribution of the seats over the parties to be given by a relation $W : N \leftrightarrow P$ such that $W_{n,k}$ iff n is owned by k, for all $n \in N$ and $k \in P$. The relation W is a mapping in the relational sense and for each $k \in P$ the k-column of W consists of exactly w_k 1-entries. Now, let $i : P \leftrightarrow \mathbf{1}$ be an inclination vector and $Bi : P \leftrightarrow \mathbf{1}$ the corresponding decision vector (as relation-algebraically specified in Theorem 3.2.1). Then we have for all $n \in N$ the equivalence

$$(WBi)_{n,\perp} \iff \exists\, k \in P : W_{n,k} \wedge (Bi)_{k,\perp} \iff \exists\, k \in P_i^+ : W_{n,k}$$

such that the column vector $WBi : N \leftrightarrow \mathbf{1}$ describes the set of seats $N_i^+ \in 2^N$ which are owned by a party that votes 'yes' under inclination vector i. Since the relation-algebraic expression that specifies the column vector Bi is built from i using unions, intersections, complements and left-compositions with constants only, the same holds for the expression WBi. Hence, a replacement of Bi in the latter by the column-wise enumeration of all decision vectors, i.e., by the relation $Dvec(D) : P \leftrightarrow 2^P$ of (7), leads to the column-wise enumeration of all sets P_i^+. With respect to the row vector $gdv(D)$ we are looking for, this means that for the relation $WDvec(D) : N \leftrightarrow 2^P$ and for all sets $X \in 2^P$ the following property has to hold: If the inclination vector $i : P \leftrightarrow \mathbf{1}$ equals the X-column of the membership relation $\mathsf{M} : P \leftrightarrow 2^P$, then $gdv(D)_{\perp,X}$ holds iff the X-column of $WDvec(D)$ contains at least 76 1-entries. From this, a relation-algebraic specification of $gdv(D)$ can be obtained exactly as in the case of Theorem 3.2.2 for the majority-of-parties-based group decision function, i.e., by using a threshold vector, the operation *cardfilter* and a symmetric quotient construction. If we use this version of *gdv* instead of the version of Subsection 3.2, then we obtain the following sizes of the sets underlying the definitions of the various power indices.

$k \in P$	GL	SP	PvdA	D66	PvdD	CDA	VVD	CU	SGP	PVV
$\lvert I_k^{++}\rvert$	256	256	416	256	256	352	256	288	288	352
$\lvert I_k^{+-}\rvert$	256	256	96	256	256	160	256	224	224	160
$\lvert I_k^{-+}\rvert$	256	256	96	256	256	160	256	224	224	160
$\lvert I_k^{--}\rvert$	256	256	416	256	256	352	256	288	288	352

In this new case $|I^+| = |I^-| = 512$ and thus from the values of the table above also all other power indices of Subsection 2.1 can be obtained. In the following table we again show the values for the power index GHB_k only.

$k \in P$	GL	SP	PvdA	D66	PvdD	CDA	VVD	CU	SGP	PVV
GHB_k	0	0	0.62	0	0	0.37	0	0.12	0,12	0.37

Comparing this table with the table presenting GHB_k for the previous case, we see that in case of the two largest parties PvdA (33 seats) and CDA (41 seats), the net Success of PvdA (measured by $\mathrm{GHB}_{\mathrm{PvdA}}$) increases but the net Success of CDA decreases if decisions are transfered from the plenum to parliamentary commissions.

We have used RelView also to find all inclination vectors for which majority-of-seats-based decision differs from majority-of-parties-based decision and obtained the following result: If the majority-of-parties-based decision is 'yes', then the same holds for the majority-of-seats-based decision. The other direction does not hold. There are exactly 80 inclination vectors which lead to 'yes' if the decision bases on the majority of seats and to 'no' if it bases on the majority of parties. In each of these cases the inclination of PVV is 'yes'.

Having discussed power indices in great detail, let us now present the RelView results concerning the influence indices, sets of followers and kernels. Note that for computing these values the group decision function gd does not play a role. The next table shows for all Dutch parties not in the coalition $S = \{\mathrm{CDA}, \mathrm{CU}, \mathrm{PvdA}\}$ (the present Dutch cabinet) the sizes of the sets defining the possibility influence indices $\overline{d}(B, S \to j)$ as computed by RelView.

$j \notin S$	GL	SP	D66	PvdD	VVD	SGP	PVV
$\lvert I_{S\to j}\rvert$	128	128	128	128	128	128	128
$\lvert I^*_{S\to j}\rvert$	128	128	128	128	64	0	0

Hence, the possibility influence index of S on GL, SP, D66 and PvdD is 1 (denoting maximal influence), the possibility influence index of S on VVD is 0.5 and the possibility influence index of S on SGP and PVV is 0 (denoting no influence).

Let us, finally, present the results concerning the set of followers and kernels. We obtained with the help of the RelView tool that

$$\{\mathrm{GL,SP,PvdA,D66,PvdD,CDA,CU}\}$$

is the set of the seven followers of the present cabinet S, and that the set of singleton sets

$$\{\{\mathrm{PvdA}\}, \{\mathrm{CDA}\}, \{\mathrm{CU}\}, \{\mathrm{SGP}\}, \{\mathrm{PVV}\}\}$$

is the kernel of the influence function induced by the above dependency graph and both influence rules we have considered in Subsection 2.3.

5 Conclusions and Discussion

The paper disseminates our results on applications of relation algebra and the RelView tool to Game Theory and Social Choice Theory. In (Berghammer et al., 2007, 2009) we have already presented such applications to coalition formation, where with the help of relation algebra and RelView the set of all feasible stable governments has been determined. In the present paper, we apply relation algebra and RelView to network formation, i.e., to compute some measures of agents' strength, like power, success, and influence, in a network.

Algorithms to compute several power indices different from ours are given in (Alonso-Meijide et al., 2008). Crama and Leruth (2007) have proposed an algorithmic approach for the computation of power indices associated with corporate networks. Algaba et al. (2007) use a combinatorial method based on generated functions to compute the normalized Banzhaf

index in pseudo-polynomial time. Since computing the exact value of the power indices is a hard algorithmic problem, it is interesting to see that RelView can deal with far from simple cases, due to its very efficient BDD-implementation.

The algorithms used are expressed by the extremely short RelView programs, see, for instance, the RelView program in Example 3.2.1. Another advantage of RelView is that although all problems are hard since they deal with sets of exponential size, the BDD based implementation of RelView is of immense help, compared with naive implementations of relations. And a third advantage is the extremely formal correctness proofs of the algorithms.

What we particularly like in our approach is its usefulness with respect to applying the tools to organizations and trend-setter structures with a larger number of players. One of the straightforward ideas is to apply the measures computed by RelView to parliaments. This is what we presented in the previous section. For networks with more than, say 5 or 6 players, where some of the players have trend-setters that they follow, calculating the measures and concepts of influence is far too complicated to be done by hand due to the sizes of the set of inclinations. In case of networks with, say, 25 players, even a naive Boolean vector approach within a conventional programming language like C leads to serious difficulties. It is hardly to imagine to generate all the $2^{25} = 33.554.432$ Boolean column vectors of length 25 one after another and to transform them into the corresponding decision vectors within reasonable time. Due to the very efficient BDD-implementation of relations, the RelView tool allows to do this in a very efficient and elegant way in many cases, viz. if programs essentially are described by relation-algebraic expressions and do not use loops that range over huge sets. To give an impression of the amazing power of the BDD-implementation of relations, we want to mention that RelView needs on a Sun Fire-280R workstation (750 MHz, 4 GByte main memory, running Solaris) only 0.04 seconds to compute the group decision vector in the case of the Dutch parliament. Note that the symmetric quotient $syq(\mathsf{M}, WDvec(D))$ used here has type $[2^N \leftrightarrow 2^N]$. Regarded as a Boolean matrix, this means that it has 2^{150} rows and columns.

Of course, we also are not able to compute within reasonable time with RelView the kernel of an influence function in the case of, say again, 25 players, if we apply the program for computing sets of followers to all subsets of P, i.e., all possible column vectors $\mathsf{M}p : P \leftrightarrow \mathbf{1}$ with $p : \mathbf{1} \leftrightarrow 2^P$ being a row point, via a simple loop. A strategy that may help in such a situation is to estimate the sizes of the sets of the kernel. If we are able to show by an analysis of the dependency relationships, for instance, that each set of the kernel consists of at most 3 players, then we can use the base-operation *cardfilter* to obtain a row vector $v : \mathbf{1} \leftrightarrow 2^P$ that represents the set $\{X \in 2^P \mid |X| \leq 3\}$. In the case of 25 players, instead of all 33.554.432 column vectors $\mathsf{M}p : P \leftrightarrow \mathbf{1}$ we then only have to loop through the $\sum_{m=0}^{3} \binom{25}{m} = 14.426$ possible row vectors $\mathsf{M}p$ with the row point p contained in v.

A further advantage of the approach of the present paper is that, because of the concise form of the RelView-programs and the expressiveness of the relation-algebraic language,

the combination of both tools additionally allows to experiment with given concepts and concepts being still under development without having a large overhead and, furthermore, to animate computations and to visualize their results. There are many more possibilities to combine relation algebra and RelView to investigate and solve problems from Game Theory and Social Choice Theory. One of them could be an application of the tools in question to bargaining theory, in particular, to determining solutions of bargaining games.

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