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# Graphs with bounded tree-width and large odd-girth are almost bipartite

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## Abstract

We prove that for every  $k$  and every  $\varepsilon > 0$ , there exists  $g$  such that every graph with tree-width at most  $k$  and odd-girth at least  $g$  has circular chromatic number at most  $2 + \varepsilon$ .

## 1 Introduction

It has been a challenging problem to prove the existence of graphs of arbitrarily high girth and chromatic number [2]. On the other hand, graphs with large girth that avoid a fixed minor are known to have low chromatic number (in particular, this applies to graphs embedded on a fixed surface). More precisely, as Thomassen observed [8], a graph that avoids a fixed minor and has large girth is 2-degenerate, and hence 3-colorable. Further, Galluccio, Goddyn and Hell [3] proved the following theorem, which essentially states that graphs with large girth that avoid a fixed minor are almost bipartite.

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**Theorem 1** (Galluccio, Goddyn and Hell, 2001). *For every graph  $H$  and every  $\varepsilon > 0$ , there exists an integer  $g$  such that the circular chromatic number of every  $H$ -minor free graph of girth at least  $g$  is at most  $2 + \varepsilon$ .*

A natural way to weaken the girth-condition is to require the graphs to have high odd-girth (the *odd-girth* is the length of a shortest odd cycle). However, Young [9] constructed 4-chromatic projective graphs with arbitrarily high odd-girth. Thus, the high odd-girth requirement is not sufficient to ensure 3-colorability, even for graphs embedded on a fixed surface. Klostermeyer and Zhang [4], though, proved that the circular chromatic number of every planar graph of sufficiently high odd-girth is arbitrarily close to 2. In particular, the same is true for  $K_4$ -minor free graphs, i.e. graphs with tree-width at most 2. We prove that the conclusion is still true for any class of graphs with bounded tree-width, which answers a question of Pan and Zhu [6, Question 6.2] also appearing as Question 8.12 in the survey by Zhu [10].

**Theorem 2.** *For every  $k$  and every  $\varepsilon > 0$ , there exists  $g$  such that every graph with tree-width at most  $k$  and odd-girth at least  $g$  has circular chromatic number at most  $2 + \varepsilon$ .*

Motivated by tree-width duality, Nešetřil and Zhu [5] proved the following theorem.

**Theorem 3** (Nešetřil and Zhu, 1996). *For every  $k$  and every  $\varepsilon > 0$ , there exists  $g$  such that every graph  $G$  with tree-width at most  $k$  and homomorphic to a graph  $H$  with girth at least  $g$  has circular chromatic number at most  $2 + \varepsilon$ .*

To see that Theorem 2 implies Theorem 3, observe that if  $G$  has an odd cycle of length  $g$ , then  $H$  has an odd cycle of length at most  $g$ .

## 2 Notation

A  $(p, q)$ -coloring of a graph is a coloring  $c$  of the vertices with colors from the set  $\{0, \dots, p-1\}$  such that the colors of any two adjacent vertices  $u$  and  $v$  satisfy  $q \leq |c(u) - c(v)| \leq p - q$ . The *circular chromatic number*  $\chi_c(G)$  of a graph  $G$  is the infimum (and it can be shown to be the minimum) of the ratios  $p/q$  such that  $G$  has a  $(p, q)$ -coloring. For every finite graph  $G$ , it holds that  $\chi(G) = \lceil \chi_c(G) \rceil$  and there is a  $(p, q)$ -coloring of  $G$  for every  $p$  and  $q$  with  $p/q \geq \chi_c(G)$ . In particular, the circular chromatic number of  $G$  is at most  $2 + 1/k$  if and only if  $G$  is homomorphic to a cycle of length  $2k + 1$ .

The reader is referred to the surveys by Zhu [10, 11] for more information about circular colorings.

A *p-precoloring* is a coloring  $\varphi$  of a subset  $A$  of vertices of a graph  $G$  with colors from  $\{0, \dots, p-1\}$ , and its *extension* is a coloring of the whole graph  $G$  that coincides with  $\varphi$  on  $A$ . The following lemma can be seen as a corollary of a theorem of Albertson and West [1, Theorem 1], and it is the only tool we use from this area.

**Lemma 4.** *For every  $p$  and  $q$  with  $2 < p/q$ , there exists  $d$  such that any  $p$ -precoloring of vertices with mutual distances at least  $d$  of a bipartite graph  $H$  extends to a  $(p, q)$ -coloring of  $H$ .*

A *k-tree* is a graph obtained from a complete graph of order  $k+1$  by successively adding vertices joined to exactly  $k$  pairwise adjacent vertices. The *tree-width* of a graph  $G$  is the smallest  $k$  such that  $G$  is a subgraph of a  $k$ -tree. Graphs with tree-width at most  $k$  are also called *partial k-trees*.

A *rooted partial k-tree* is a partial  $k$ -tree  $G$  with  $k+1$  distinguished vertices  $v_1, \dots, v_{k+1}$  such that there exists a  $k$ -tree  $G'$  that is a supergraph of  $G$  and the vertices  $v_1, \dots, v_{k+1}$  form a clique in  $G'$ . We also say that the partial  $k$ -tree is *rooted* at  $v_1, \dots, v_{k+1}$ . If  $G$  is a partial  $k$ -tree rooted at  $v_1, \dots, v_{k+1}$  and  $G'$  is a partial  $k$ -tree rooted at  $v'_1, \dots, v'_{k+1}$ , then the graph  $G \oplus G'$  obtained by identifying  $v_i$  and  $v'_i$  is again a rooted partial  $k$ -tree (identify the cliques in the corresponding  $k$ -trees).

Fix  $p$  and  $q$ . If  $G$  is a rooted partial  $k$ -tree, then  $\mathcal{F}(G)$  is the set of all  $p$ -precolorings of the  $k+1$  distinguished vertices of  $G$  that can be extended to a  $(p, q)$ -coloring of  $G$ .

The next lemma is a standard application of results in the area of graphs of bounded tree-width [7].

**Lemma 5.** *Let  $k$  and  $N$  be positive integers such that  $N \geq k+1$ . If  $G$  is a partial  $k$ -tree with at least  $3N$  vertices, then there exist partial rooted  $k$ -trees  $G_1$  and  $G_2$  such that  $G$  is isomorphic to  $G_1 \oplus G_2$  and  $G_1$  has at least  $N+1$  and at most  $2N$  vertices.*

If  $G$  is a partial  $k$ -tree rooted at  $v_1, \dots, v_{k+1}$ , then its *type* is a  $(k+1) \times (k+1)$  matrix  $M$  such that  $M_{ij}$  is the length of the shortest path between the vertices  $v_i$  and  $v_j$ . If there is no such path,  $M_{ij}$  is equal to  $\infty$ . Any matrix  $M$  that is a type of a partial rooted  $k$ -tree satisfies the triangle inequality (setting  $\infty+x = \infty$  for any  $x$ ). A symmetric matrix  $M$  whose entries are non-negative integers and  $\infty$  (and zeroes only on the main diagonal) that satisfies the triangle inequality is a *type*. A type is *bipartite* if  $M_{ij} + M_{jk} + M_{ik} \equiv 0 \pmod{2}$  for any three finite entries  $M_{ij}$ ,  $M_{jk}$  and  $M_{ik}$ . Two bipartite types  $M$

and  $M'$  are *compatible* if  $M_{ij}$  and  $M'_{ij}$  have the same parity whenever both of them are finite. We define a binary relation on bipartite types as follows:  $M \preceq M'$  if and only if  $M$  and  $M'$  are compatible and  $M_{ij} \leq M'_{ij}$  for every  $i$  and  $j$ . Note that the relation  $\preceq$  is a partial order.

We finish this section with the following lemma. Its straightforward proof is included to help us in familiarizing with the just introduced notation.

**Lemma 6.** *Let  $G^1$  and  $G^2$  be two bipartite rooted partial  $k$ -trees with types  $M^1$  and  $M^2$  such that there exists a bipartite type  $M^0$  with  $M^0 \preceq M^1$  and  $M^0 \preceq M^2$ . Then the types  $M^1$  and  $M^2$  are compatible,  $G^1 \oplus G^2$  is a bipartite rooted partial  $k$ -tree and its type  $M$  satisfies  $M^0 \preceq M$ .*

*Proof.* The types  $M^1$  and  $M^2$  are compatible: if both  $M^1_{ij}$  and  $M^2_{ij}$  are finite, then  $M^0_{ij}$  is finite and has the same parity as  $M^1_{ij}$  and  $M^2_{ij}$ . Hence, the entries  $M^1_{ij}$  and  $M^2_{ij}$  have the same parity.

Let  $M$  be the type of  $G^1 \oplus G^2$ . Note that it does not hold in general that  $M_{ij} = \min\{M^1_{ij}, M^2_{ij}\}$ . We show that  $M^0 \preceq M$  which will also imply that  $G^1 \oplus G^2$  is bipartite since  $M^0$  is a bipartite type. Consider a shortest path  $P$  between two distinguished vertices  $v_t$  and  $v_{t'}$  and split  $P$  into paths  $P_1, \dots, P_\ell$  delimited by distinguished vertices on  $P$ . Note that  $\ell \leq k$  since  $P$  is a path. Let  $j_0 = t$  and let  $j_i$  be the index of the end-vertex of  $P_i$  for  $i \in \{1, \dots, \ell\}$ . In particular,  $j_\ell = t'$ . Each of the paths  $P_1, \dots, P_\ell$  is fully contained in  $G^1$  or in  $G^2$  (possibly in both if it is a single edge). Since  $M^0 \preceq M^1$  and  $M^0 \preceq M^2$ , the length of  $P_i$  is at least  $M^0_{j_{i-1}j_i}$ , and it has the same parity as  $M^0_{j_{i-1}j_i}$ . Since  $M^0$  is a bipartite type (among others, it satisfies the triangle inequality), the length of  $P$ , which is  $M_{tt'}$ , has the same parity as  $M^0_{j_0j_\ell} = M^0_{tt'}$  and is at least  $M^0_{tt'}$ . This implies that  $M^0 \preceq M$ .  $\square$

### 3 The Main Lemma

In this section, we prove a lemma which forms the core of our argument. To this end, we first prove another lemma that asserts that for every  $k, p$  and  $q$ , the set of types of all bipartite rooted partial  $k$ -trees forbidding a fixed set of  $p$ -precolorings from extending (and maybe some other precolorings, too) has always a maximal element. We state the lemma slightly differently to facilitate its application.

**Lemma 7.** *For every  $k, p$  and  $q$ , there exists a finite number of (bipartite) types  $M^1, \dots, M^m$  such that for any bipartite rooted partial  $k$ -tree  $G$  with type  $M$ , there exists a bipartite rooted partial  $k$ -tree  $G'$  with type  $M^i$  for some  $i \in \{1, \dots, m\}$  such that  $\mathcal{F}(G') \subseteq \mathcal{F}(G)$  and  $M \preceq M^i$ .*

*Proof.* Let  $d \geq 2$  be the constant from Lemma 4 applied for  $p$  and  $q$ . Let  $M^1, \dots, M^m$  be all bipartite types with entries from the set  $\{1, \dots, D^{(k+1)^2}\} \cup \{\infty\}$  where  $D = 4d$ . Thus,  $m$  is finite and does not exceed  $(D^{(k+1)^2} + 1)^{k(k+1)/2}$ .

Let  $G$  be a bipartite rooted partial  $k$ -tree with type  $M$ . If  $M$  is one of the types  $M^1, \dots, M^m$ , then there is nothing to prove (just choose  $i$  such that  $M = M^i$ ). Otherwise, one of its entries is finite and exceeds  $D^{(k+1)^2}$ .

For  $i \in \{1, \dots, (k+1)^2\}$ , let  $J^i$  be the set of all positive integers between  $D^{i-1}$  and  $D^i - 1$  (inclusively). Let  $i_0$  be the smallest integer such that no entry of  $M$  is contained in  $J^{i_0}$ . Since  $M$  has at most  $k(k+1)/2$  different entries, such an index  $i_0$  exists. Note that if  $i_0 = 1$ , then Lemma 4 implies that  $\mathcal{F}(G)$  contains all possible  $p$ -precolorings, and the sought graph  $G'$  is the bipartite rooted partial  $k$ -tree composed of  $k+1$  isolated vertices, with the all- $\infty$  type.

Two vertices  $v_i$  and  $v_j$  at which  $G$  is rooted are *close* if  $M_{ij}$  is less than  $D^{i_0-1}$ . The relation  $\approx$  of being close is an equivalence relation on  $v_1, \dots, v_{k+1}$ . Indeed, it is reflexive and symmetric by the definition, and we now show that it is transitive. Suppose that  $M_{ij}$  and  $M_{jt}$  are both less than  $D^{i_0-1}$ . Then, the distance between  $v_i$  and  $v_t$  is at most  $M_{ij} + M_{jt} \leq 2D^{i_0-1} - 2 \leq D^{i_0} - 1$  since  $D \geq 2$ . Consequently, by the choice of  $i_0$ , the distance between  $v_i$  and  $v_t$  is at most  $D^{i_0-1} - 1$  and thus  $v_i \approx v_t$ .

Let  $C_1, \dots, C_\ell$  be the equivalence classes of the relation  $\approx$ . Note that  $C_1, \dots, C_\ell$  is a finer partition than that given by the equivalence relation of being connected.

Since  $G$  is bipartite, we can partition its vertices into two color classes, say red and blue. For every  $i \in \{1, \dots, \ell\}$ , contract the closed neighborhood of a vertex  $v$  if  $v$  is a blue vertex and its distance from any vertex of  $C_i$  is at least  $D^{i_0-1}$  and keep doing so as long as such a vertex exists. Observe that the resulting graph is uniquely defined. After discarding the components that do not contain the vertices of  $C_i$ , we obtain a bipartite partial  $k$ -tree  $G_i$  rooted at the vertices of  $C_i$ : it is bipartite as we have always contracted closed neighborhoods of vertices of the same color (blue) to a single (red) vertex, and its tree-width is at most  $k$  since the tree-width is preserved by contractions. Moreover, the distance between any two vertices of  $C_i$  has not decreased since any path between them through any of the newly arising vertices has length at least  $2D^{i_0-1} - 2 \geq D^{i_0-1}$ .

Now, let  $G'$  be the bipartite rooted partial  $k$ -tree obtained by taking the disjoint union of  $G_1, \dots, G_\ell$ . The type  $M'$  of  $G'$  can be obtained from the type of  $G$ : set  $M'_{ij}$  to be  $M_{ij}$  if the vertices  $v_i$  and  $v_j$  are close, and  $\infty$  otherwise. Thus,  $M'$  is one of the types  $M^1, \dots, M^m$  and  $M \preceq M'$ . It remains to show that  $\mathcal{F}(G') \subseteq \mathcal{F}(G)$ .

Let  $c \in \mathcal{F}(G')$  be a  $p$ -precoloring that extends to  $G'$ , and recall that  $D \geq 4$ . For  $i \in \{1, \dots, \ell\}$ , let  $A_i$  be the set of all red vertices at distance at most  $D^{i_0-1}$  and all blue vertices at distance at most  $D^{i_0-1} - 1$  from  $C_i$ , and let  $R_i$  be the set of all red vertices at distance  $D^{i_0-1} - 1$  or  $D^{i_0-1}$  from  $C_i$ . Set  $B_i = A_i \setminus R_i$  ( $B_i$  is the ‘‘interior’’ of  $A_i$  and  $R_i$  its ‘‘boundary’’). The extension of  $c$  to  $G_i$  naturally defines a coloring of all vertices of  $A_i$ :  $G_i$  is the subgraph of  $G$  induced by  $A_i$  with some red vertices of  $R_i$  identified (two vertices of  $R_i$  are identified if and only if they are in the same component of the graph  $G - B_i$ ).

Let  $H$  be the following auxiliary graph obtained from  $G$ : remove the vertices of  $B = B_1 \cup \dots \cup B_\ell$  and, for  $i \in \{1, \dots, \ell\}$ , identify every pair of vertices of  $R_i$  that are in the same component of  $G - B$ . Let  $R$  be the set of vertices of  $H$  corresponding to some vertices of  $R_1 \cup \dots \cup R_\ell$ . Precolor the vertices of  $R$  with the colors given by the colorings of the graphs  $G_i$  (note that two vertices of  $R_i$  in the same component of  $G - B_i$  are also in the same component of  $G - B$ , so this is well-defined). The graph  $H$  is bipartite as only red vertices have been identified. The distance between any two precolored vertices is at least  $d$ : consider two precolored vertices  $r$  and  $r'$  at distance at most  $d - 1$ . Let  $i$  and  $i'$  be such that  $r \in R_i$  and  $r' \in R_{i'}$ . If  $i = i'$ , then  $r$  and  $r'$  are in the same component of  $G - B$  and thus  $r = r'$ . If  $i \neq i'$  then by the definition of  $R_i$  and  $R_{i'}$ , the vertex  $r$  is in  $G$  at distance at most  $D^{i_0-1}$  from some vertex  $v$  of  $C_i$  and  $r'$  is at distance at most  $D^{i_0-1}$  from some vertex  $v'$  of  $C_{i'}$ . So, the distance between  $v$  and  $v'$  is at most  $2D^{i_0-1} + d - 1 \leq D^{i_0} - 1$ . Since  $M$  has no entry from  $J^{i_0}$ , the vertices  $v$  and  $v'$  must be close and thus  $i = i'$ , a contradiction.

Since the distance between any two precolored vertices is at least  $d$ , the precoloring extends to  $H$  by Lemma 4 and in a natural way it defines a coloring of  $G$ . We conclude that every  $p$ -precoloring that extends to  $G'$  also extends to  $G$  and thus  $\mathcal{F}(G') \subseteq \mathcal{F}(G)$ .  $\square$

We now prove our main lemma, which basically states that there is only a finite number of bipartite rooted partial  $k$ -trees that can appear in a minimal non- $(p, q)$ -colorable graph with tree-width  $k$  and a given odd girth.

**Lemma 8.** *For every  $k, p$  and  $q$ , there exist a finite number  $m$  and bipartite rooted partial  $k$ -trees  $G^1, \dots, G^m$  with types  $M^1, \dots, M^m$  such that for any bipartite rooted partial  $k$ -tree  $G$  with type  $M$  there exists  $i$  such that  $\mathcal{F}(G^i) \subseteq \mathcal{F}(G)$  and  $M \preceq M^i$ .*

*Proof.* Let  $M^1, \dots, M^m$  be the types from Lemma 7. We define the graph  $G^i$  as follows: for every  $p$ -precoloring  $c$  that does not extend to a bipartite partial rooted  $k$ -tree with type  $M^i$ , fix a partial rooted  $k$ -tree  $G_c^i$  with type

$M^i$  such that  $c$  does not extend to  $G_c^i$ . Set  $G^i = \bigoplus_c G_c^i$ , where  $c$  runs over all such  $p$ -precolorings. If the above sum of partial  $k$ -trees is non-empty, then the type  $M$  of  $G^i$  is  $M^i$ . Indeed,  $M \preceq M^i$  by the definition of  $G^i$ , and Lemma 6 implies that  $M^i \preceq M$ . If all the  $p$ -precolorings of the  $k+1$  vertices in the root extend to each partial  $k$ -tree of type  $M^i$ , then let  $G^i$  be the graph consisting of  $k+1$  isolated vertices. This happens in particular for the all- $\infty$  type.

Let us verify the statement of the lemma. Let  $G$  be a bipartite rooted partial  $k$ -tree and let  $M$  be the type of  $G$ . If  $\mathcal{F}(G)$  is composed of all  $p$ -precolorings, the sought graph  $G^i$  is the one composed of  $k+1$  isolated vertices. Hence, we assume that  $\mathcal{F}(G)$  does not contain all  $p$ -precolorings, i.e., there are  $p$ -precolorings that do not extend to  $G$ . By Lemma 7, there exists a bipartite rooted partial  $k$ -tree  $G'$  with type  $M'$  such that  $M \preceq M' = M^i$  for some  $i$  and  $\mathcal{F}(G') \subseteq \mathcal{F}(G)$ . For every  $p$ -precoloring  $c$  that does not extend to  $G'$  (and there exists at least one such  $p$ -precoloring  $c$ ), some graph  $G_c^i$  has been glued into  $G^i$ . Hence,  $\mathcal{F}(G^i) \subseteq \mathcal{F}(G') \subseteq \mathcal{F}(G)$ . Since the type of  $G^i$  is  $M^i$ , the conclusion of the lemma follows.  $\square$

## 4 Proof of Theorem 2

We are now ready to prove Theorem 2, which is recalled below.

**Theorem 2.** *For every  $k$  and every  $\varepsilon > 0$ , there exists  $g$  such that every graph with tree-width at most  $k$  and odd-girth at least  $g$  has circular chromatic number at most  $2 + \varepsilon$ .*

*Proof.* Fix  $p$  and  $q$  such that  $2 < p/q \leq 2 + \varepsilon$ . Let  $G^1, \dots, G^m$  be the bipartite partial  $k$ -trees from Lemma 8 applied for  $k, p$  and  $q$ . Set  $N$  to be the largest order of the graphs  $G^i$  and set  $g$  to be  $3N$ . We assert that each partial  $k$ -tree with odd-girth  $g$  has circular chromatic number at most  $p/q$ . Assume that this is not the case and let  $G$  be a counterexample with the fewest vertices.

The graph  $G$  has at least  $3N$  vertices (otherwise, it has no odd cycles and thus is bipartite). By Lemma 5,  $G$  is isomorphic to  $G_1 \oplus G_2$ , where  $G_1$  and  $G_2$  are rooted partial  $k$ -trees and the number of vertices of  $G_1$  is between  $N+1$  and  $2N$ . By the choice of  $g$ , the graph  $G_1$  has no odd cycle and thus it is a bipartite rooted partial  $k$ -tree. By Lemma 8, there exists  $i$  such that  $\mathcal{F}(G^i) \subseteq \mathcal{F}(G_1)$  and  $M_1 \preceq M^i$  where  $M_1$  is the type of  $G_1$  and  $M^i$  is the type of  $G^i$ . Let  $G'$  be the partial  $k$ -tree  $G^i \oplus G_2$ .

First,  $G'$  has fewer vertices than  $G$  since the number of vertices of  $G^i$  is at most  $N$  and the number of vertices of  $G_1$  is at least  $N+1$ . Second,  $G'$



has no  $(p, q)$ -coloring: if it had a  $(p, q)$ -coloring, then the corresponding  $p$ -precoloring of the  $k+1$  vertices shared by  $G^i$  and  $G_2$  would extend to  $G_1$  since  $\mathcal{F}(G^i) \subseteq \mathcal{F}(G_1)$  and thus  $G$  would have a  $(p, q)$ -coloring, too. Finally,  $G'$  has no odd cycle of length less than  $g$ : if it had such a cycle, replace any path between vertices  $v_j$  and  $v_{j'}$  of the root of  $G^i$  with a path of at most the same length between them in  $G_1$  (recall that  $M_1 \preceq M^i$ ). If such paths for different pairs of  $v_j$  and  $v_{j'}$  on the considered odd cycle intersect, take their symmetric difference. In this way, we obtain an Eulerian subgraph of  $G = G_1 \oplus G_2$  with an odd number of edges such that the number of its edges is less than  $g$ . Consequently, this Eulerian subgraph has an odd cycle of length less than  $g$ , which violates the assumption on the odd-girth of  $G$ . We conclude that  $G'$  is a counterexample with fewer vertices than  $G$ , a contradiction.  $\square$

We end by pointing out that the approach used yields an upper bound of  $3(k+1) \cdot 2^{2^{p^{k+1}}((4d)^{(k+1)^2}+1)^{k^2}}$  for the smallest  $g$  such that all graphs with tree-width at most  $k$  and odd-girth at least  $g$  have circular chromatic number at most  $p/q$ , whenever  $p/q > 2$ . More precisely, the value of  $N$  cannot exceed  $(k+1) \cdot 2^{2^{p^{k+1}}((4d)^{(k+1)^2}+1)^{k^2}}$ . To see this, we consider all pairs  $P = (C, M)$  where  $C$  is a set of  $p$ -precolorings of the root and  $M$  is a type such that there is a bipartite rooted partial  $k$ -tree of type  $M$  to which no coloring of  $C$  extends. Let  $n_P$  be the size of a smallest such partial  $k$ -tree. We obtain a sequence of at most  $2^{p^{k+1}} \times \left( (4d)^{(k+1)^2} + 1 \right)^{k^2}$  integers. The announced bound follows from the following fact: if the sequence is sorted in increasing order, then each term is at most twice the previous one.

Indeed, consider the tree-decomposition of the partial  $k$ -tree  $G_P$  chosen for the pair  $P$ . If the bag containing the root has a single child, then we delete a vertex of the root, and set a vertex in the single child to be part of the root. We obtain a partial  $k$ -tree to which some  $p$ -precolorings of  $C$  do not extend. Thus,  $n_P \leq 1 + n_{P'}$  for some pair  $P'$  and  $n_{P'} < n_P$ . If the bag containing the root has more than one child, then  $G_P$  can be obtained by identifying the roots of two smaller partial  $k$ -trees  $G$  and  $G'$ . By the minimality of  $G_P$ , the orders of  $G$  and  $G'$  are  $n_{P_1}$  and  $n_{P_2}$  for two pairs  $P_1$  and  $P_2$  such that  $n_{P_i} < n_P$  for  $i \in \{1, 2\}$ . This yields the stated fact, which in turn implies the given bound, since the smallest element of the sequence is  $k+1$ .

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