



HAL
open science

On Poincare-Wirtinger inequalities in spaces of functions of bounded variation

Maïtine Bergounioux

► **To cite this version:**

Maïtine Bergounioux. On Poincare-Wirtinger inequalities in spaces of functions of bounded variation. 2010. hal-00515451v1

HAL Id: hal-00515451

<https://hal.science/hal-00515451v1>

Preprint submitted on 7 Sep 2010 (v1), last revised 5 Oct 2011 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ON POINCARÉ-WIRTINGER INEQUALITIES IN SPACES OF FUNCTIONS OF BOUNDED VARIATION

M. BERGOUNIOUX

ABSTRACT. The goal of this paper is to extend Poincaré-Wirtinger inequalities from Sobolev spaces to spaces of functions of bounded variation of second order. We give an application to an image processing variational problem.

1. INTRODUCTION

A useful tool when dealing with Sobolev spaces and partial differential equations is the Poincaré-Wirtinger inequality that provides norm equivalences under appropriate assumptions. These inequalities usually provide Sobolev embeddings and compactness results (see Adams [1]). The goal of this paper is to extend Poincaré-Wirtinger inequalities from Sobolev spaces to spaces of functions of second order bounded variation. The result is known for the space of functions of first-order bounded variation (see [3]). Indeed, this space is very useful in image processing context and many variational models are developed to deal with denoising of texture extraction. Variational models in image processing can be improved using the so-called BV^2 space that we define in next section ([7, 9, 4, 6]). Generally these models require a priori estimates on functions while first and/or second order derivative estimates are available.

2. THE SPACES OF FUNCTIONS OF BOUNDED VARIATION

We briefly recall the definitions and the main properties of (classical) spaces of functions of bounded variation. One can refer to [2, 3, 8] for a complete study of the BV space and to [7, 4, 9] for the so-called BV^2 space.

Let Ω be an open bounded subset of \mathbb{R}^n , $n \geq 2$ smooth enough (Lipschitz for example). The spaces $BV(\Omega)$ and $BV^2(\Omega)$ of functions of first-order and second-order bounded variation are defined by

$$BV(\Omega) = \{u \in L^1(\Omega) \mid \Phi_1(u) < +\infty\},$$

where

$$(2.1) \quad \Phi_1(u) := \sup \left\{ \int_{\Omega} u(x) \operatorname{div} \xi(x) dx \mid \xi \in \mathcal{C}_c^1(\Omega), \|\xi\|_{\infty} \leq 1 \right\}.$$

and

$$BV^2(\Omega) = \{u \in W^{1,1}(\Omega) \mid \Phi_2(u) < +\infty\},$$

where

Date: September 7, 2010.

2000 Mathematics Subject Classification. Primary 54C40, 14E20; Secondary 46E25, 20C20.

Key words and phrases. Measure theory.

- The Sobolev space $W^{1,1}(\Omega)$ is defined as

$$W^{1,1}(\Omega) = \{ u \in L^1(\Omega) \mid \nabla u \in L^1(\Omega) \}$$

- The second-order total variation is :

$$(2.2) \quad \Phi_2(u) := \sup \left\{ \int_{\Omega} \langle \nabla u, \operatorname{div}(\xi) \rangle_{\mathbb{R}^n} \mid \xi \in \mathcal{C}_c^2(\Omega, \mathbb{R}^{n \times n}), \|\xi\|_{\infty} \leq 1 \right\} < \infty,$$

with $\operatorname{div}(\xi) = (\operatorname{div}(\xi_1), \operatorname{div}(\xi_2), \dots, \operatorname{div}(\xi_n))$, and

$$\forall i, \xi_i = (\xi_i^1, \xi_i^2, \dots, \xi_i^n) \in \mathbb{R}^n \text{ and } \operatorname{div}(\xi_i) = \sum_{k=1}^n \frac{\partial \xi_i^k}{\partial x_k}.$$

The following result makes precise the connection between $BV(\Omega)$ and $BV^2(\Omega)$:

Theorem 2.1. *A function u belongs to $BV^2(\Omega)$ if and only if $u \in W^{1,1}(\Omega)$ and $\frac{\partial u}{\partial x_i} \in BV(\Omega)$ for $i \in \{1, \dots, n\}$. In particular*

$$\Phi_2(u) \leq \sum_{i=1}^n \Phi_1 \left(\frac{\partial u}{\partial x_i} \right) \leq n \Phi_2(u).$$

Proof. See Piffet [9]. □

The space $BV(\Omega)$, endowed with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1} + \Phi_1(u)$, and $BV^2(\Omega)$ endowed with the following norm

$$(2.3) \quad \|f\|_{BV^2(\Omega)} := \|f\|_{W^{1,1}(\Omega)} + \Phi_2(f) = \|f\|_{L^1} + \|\nabla f\|_{L^1} + \Phi_2(f),$$

where Φ_2 is given by (2.2), are Banach spaces.

As in [5] we may define the space $BV_0(\Omega)$ of functions of bounded variation that vanish on the boundary $\partial\Omega$ of Ω . More precisely as Ω is bounded and $\partial\Omega$ is Lipschitz, functions of $BV(\Omega)$ have a trace of class L^1 on $\partial\Omega$, and the trace mapping $T : BV(\Omega) \rightarrow L^1(\partial\Omega)$ is linear and bounded (see [8]). The space $BV_0(\Omega)$ is then defined as the kernel of T . It is a Banach space, endowed with the induced norm.

We next recall standard properties of functions of 1st and 2nd order bounded variation. We first have embedding theorems

Proposition 2.2. [2, 3, 7, 9] *Let Ω be an open subset of \mathbb{R}^n with Lipschitz boundary.*

- (1) $BV(\Omega) \subset L^2(\Omega)$ with continuous embedding, if $n = 2$.
- (2) $BV(\Omega) \subset L^p(\Omega)$ with compact embedding, for every $p \in [1, 2)$, if $n = 2$.
- (3) Assume $n \geq 2$. Then

$$BV^2(\Omega) \hookrightarrow W^{1,q}(\Omega) \text{ with } q \leq \frac{n}{n-1},$$

with continuous embedding. Moreover the embedding is compact if $q < \frac{n}{n-1}$.
In particular

$$BV^2(\Omega) \hookrightarrow L^q(\Omega) \text{ for } q \leq \frac{n}{n-2} \text{ if } n > 2$$

$$BV^2(\Omega) \hookrightarrow L^q(\Omega), \forall q \in [1, \infty[, \text{ if } n = 2.$$

We get lower semi-continuity results as well:

- Theorem 2.3.** (1) *The mapping $u \mapsto \Phi_1(u)$ is lower semi-continuous from $BV(\Omega)$ to \mathbb{R}^+ for the $L^1(\Omega)$ topology.*
 (2) *The mapping $u \mapsto \Phi_2(u)$ is lower semi-continuous from $BV^2(\Omega)$ endowed with the strong topology of $W^{1,1}(\Omega)$ to \mathbb{R} . More precisely, if $\{u_k\}_{k \in \mathbb{N}}$ is a sequence of $BV^2(\Omega)$ that strongly converges to u in $W^{1,1}(\Omega)$ then*

$$\Phi_2(u) \leq \liminf_{k \rightarrow \infty} \Phi_2(u_k).$$

We end this section with a “density ” result in $BV(\Omega)$:

Theorem 2.4 ([3] Theorem 10.1.2. p 375). *The space $C^\infty(\overline{\Omega})$ is dense in $BV(\Omega)$ in the following sense : for every $u \in BV(\Omega)$ there exist a sequence $(u_n)_{n \geq 0} \in C^\infty(\overline{\Omega})$ such that*

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_{L^1} = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \Phi_1(u_n) = \Phi_1(u) .$$

3. POINCARÉ-WIRTINGER INEQUALITIES

3.1. Poincaré-Wirtinger inequality in $BV(\Omega)$. We first recall the classical Poincaré-Wirtinger inequality for the Sobolev-space $W^{1,1}(\Omega)$ (see [3] p. 161–180 for example, or [1]).

Theorem 3.1. *Let Ω be an open subset of \mathbb{R}^n , which is bounded in one direction. Then, there exists a constant C_Ω such that*

$$\forall v \in W_0^{1,1}(\Omega) \quad \|v\|_{L^1(\Omega)} \leq C_\Omega \|\nabla v\|_{L^1(\Omega)} .$$

Moreover, if Ω is an open bounded set of class \mathcal{C}^1 , then there exists a constant C_Ω such that

$$\forall u \in W^{1,1}(\Omega) \quad \|u - m(u)\|_{L^1(\Omega)} \leq C_\Omega \|\nabla u\|_{L^1(\Omega)} ,$$

where where $m(u) := \frac{1}{|\Omega|} \int_\Omega u(x)dx$ is the mean-value of u .

A consequence of the previous results is a Poincaré-Wirtinger inequality in the BV-space

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set of class \mathcal{C}^1 . Then there exists a constant $C > 0$ such that*

$$\forall u \in BV(\Omega) \quad \|u - m(u)\|_{L^1(\Omega)} \leq C \Phi_1(u) .$$

Proof. The result is mentioned in [3], p. 399 (proof of Lemma 10.3.2), but we give the proof for convenience. Let $u \in BV(\Omega)$ and $(u_n)_{n \geq 0} \in C^\infty(\overline{\Omega})$ be a sequence such that

$$\lim_{n \rightarrow +\infty} \|u_n - u\|_{L^1} = 0 \quad \text{and} \quad \lim_{n \rightarrow +\infty} \Phi_1(u_n) = \Phi_1(u) .$$

It is clear that $m(u_n) \rightarrow m(u)$. In addition $u_n \in W^{1,1}(\Omega)$ since Ω is bounded. We use theorem 3.1 to infer

$$\forall n \quad \|u_n - m(u_n)\|_{L^1(\Omega)} \leq C \|\nabla u_n\|_{L^1} = \Phi_1(u_n) .$$

Passing to the limit gives the result. □

We have the same result without any regularity assumption on the set Ω for functions in $BV_0(\Omega)$ (the proof is the same):

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset of \mathbb{R}^n , Then there exists a constant $C > 0$ such that*

$$\forall u \in BV_0(\Omega) \quad \|u\|_{L^1(\Omega)} \leq C\Phi_1(u) .$$

3.2. Poincaré-Wirtinger inequality in $BV^2(\Omega)$. We may extend the previous inequalities to functions in $BV^2(\Omega)$:

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set of class \mathcal{C}^1 . Then there exists a constant $C > 0$ such that*

$$\forall u \in BV^2(\Omega) \quad |\nabla u - M(\nabla u)|_{L^1(\Omega), \mathcal{N}} \leq C_{\mathcal{N}}\Phi_2(u) ,$$

where $M(V) := \{m(V_1), \dots, m(V_n)\}$ is the (vectorial) mean-value of V and

$$|V|_{L^1(\Omega), \mathcal{N}} = \mathcal{N}(\|V_1\|_{L^1(\Omega)}, \dots, \|V_n\|_{L^1(\Omega)}) ,$$

where \mathcal{N} denotes any norm in \mathbb{R}^n (for example the ℓ^1 -norm).

Proof. As $u \in BV^2(\Omega)$ we use Proposition 2.1 to infer that for every $i \in \{1, \dots, n\}$ $\frac{\partial u}{\partial x_i} \in BV(\Omega)$. With Theorem 3.3 we get the existence of C such that

$$\forall i \in \{1, \dots, n\} \quad \left\| \frac{\partial u}{\partial x_i} - m\left(\frac{\partial u}{\partial x_i}\right) \right\|_{L^1(\Omega)} \leq C\Phi_1\left(\frac{\partial u}{\partial x_i}\right) \leq nC\Phi_2(u) .$$

This gives the result. \square

Corollary 3.5. *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set. Then, there exists a constant $C > 0$ such that*

$$\forall u \in BV^2(\Omega) \text{ such that } \frac{\partial u}{\partial x_i} = 0 \text{ on } \partial\Omega \quad \left\| \frac{\partial u}{\partial x_i} \right\|_{L^1(\Omega)} \leq C\Phi_2(u) .$$

3.3. Examples. Let us detail the particular case where $u \in BV^2(\Omega)$ and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$. This is the usual case in the context of image processing. Indeed, one often performs a reflexion of the image along its frame to avoid undesirable boundary effects. We assume also that $\Omega = \prod_{i=1}^n]a_i, b_i[$ (the frame is a square). This is a bounded open subset (not \mathcal{C}^1).

By definition, for every $i \in \{1, \dots, n\}$, $\frac{\partial u}{\partial x_i} \in BV_0(\Omega)$; indeed,

$$\frac{\partial u}{\partial x_i} \in BV(\Omega) \text{ and } \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial n}|_{]a_i, b_i[} = 0 .$$

Therefore we use Corollary 3.5 to get : for all $u \in BV^2(\Omega)$ such that $\frac{\partial u}{\partial n} = 0$

$$(3.1) \quad \Phi_1(u) \simeq \|\nabla u\|_{L^1(\Omega)} \leq C\Phi_2(u) ,$$

where C denotes a generic constant.

4. AN APPLICATION IN IMAGE PROCESSING

In [4] we have investigated a second order model for image processing:

$$F(v) = \frac{1}{2} \|u_d - v\|_{L^2(\Omega)}^2 + \lambda \Phi_2(v) + \delta \|v\|_{W^{1,1}(\Omega)},$$

where Ω is a squared bounded open set, $\lambda, \delta > 0$, $u_d \in L^2(\Omega)$ and $v \in BV^2(\Omega)$ such that $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$. We are looking for a solution to the optimization problem

$$(\mathcal{P}) \quad \inf\{ F(v) \mid v \in \widetilde{BV^2}(\Omega) \},$$

where

$$\widetilde{BV^2}(\Omega) = \{ v \in BV^2(\Omega) \mid \frac{\partial v}{\partial n}|_{\partial\Omega} = 0 \}.$$

We chose $\delta > 0$ because we were not able to prove existence of solution without this assumption. Now, with Poincaré-Wirtinger tool we can get rid of the δ -term so that the model is improved. More precisely

Theorem 4.1. *Assume $\lambda > 0$ and $\delta = 0$. Then problem (\mathcal{P}) has at least a solution.*

Proof. Let $v_n \in \widetilde{BV^2}(\Omega)$ be a minimizing sequence, i.e.

$$\lim_{n \rightarrow +\infty} F(v_n) = \inf\{ F(v) \mid v \in \widetilde{BV^2}(\Omega) \} < +\infty.$$

The sequence $(v_n)_{n \in \mathbb{N}}$ is bounded in $BV^2(\Omega)$. Indeed $\Phi_2(v_n)$ is bounded and $\Phi_1(v_n)$ as well with relation (3.1). As v_n is L^2 -bounded it is also bounded in $W^{1,1}(\Omega)$. With the compactness result of Proposition 2.2 this yields that $(v_n)_{n \in \mathbb{N}}$ strongly converges (up to a subsequence) in $W^{1,1}(\Omega)$ to $v^* \in BV^2(\Omega)$. With the continuity of the trace operator we get $v^* \in \widetilde{BV^2}(\Omega)$. Theorem 2.3 gives the following:

$$\Phi_2(v^*) \leq \liminf_{n \rightarrow +\infty} \Phi_2(v_n).$$

So

$$F(v^*) \leq \liminf_{n \rightarrow +\infty} F(v_n) = \min_{v \in \widetilde{BV^2}(\Omega)} F(v),$$

and v^* is a solution to (\mathcal{P}) . □

REFERENCES

1. Adams, R.A. : *Sobolev spaces*. Academic Press, Springer Verlag (1978).
2. L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems*. Oxford mathematical monographs, Oxford University Press (2000).
3. H. Attouch, G. Buttazzo and G. Michaille, *Variational analysis in Sobolev and BV spaces : applications to PDEs and optimization*. MPS-SIAM series on optimization, (2006)
4. M. Bergounioux and L. Piffet, *A second-order model for image denoising*, Set-Valued Analysis and Variational Analysis, to appear
5. M. Bergounioux and E. Trélat, *A variational method using fractional order Hilbert spaces for tomographic reconstruction of blurred and noised binary images*, to appear J. Functional Analysis, (2010); doi:10.1016/j.jfa.2010.05.016
6. K. Bredies, K. Kunisch and T. Pock, *Total Generalized Variation*, preprint (2009)
7. F. Demengel, *Fonctions à hessien borné*, Annales de l'institut Fourier, tome **34**, no 2, pp. 155-190 (1984)
8. L.C. Evans, R. Gariepy, *Measure theory and fine properties of functions*, CRC Press, (1992)
9. L. Piffet, *Modèles variationnels pour l'extraction de textures 2D*, PhD Thesis, Orléans, (2010)

UNIVERSITÉ D'ORLÉANS, UFR SCIENCES, MAPMO, UMR 6628, ROUTE DE CHARTRES, BP
6759, 45067 ORLÉANS CEDEX 2

Current address: Université d'Orléans, UFR Sciences, MAPMO, UMR 6628, Route de Chartres,
BP 6759, 45067 Orléans cedex 2

E-mail address: `maitine.bergounioux@univ-orleans.fr`