A probabilistic deformation of calculus of variations with constraints

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Abstract. In the framework of a probabilistic deformation of the classical calculus of variations, we consider the simplest problem of constraints, and solve it in two different ways. First by a pathwise argument in the line of Euclidean Quantum Mechanics. Second from an entropic (measure theoretic) perspective.

Contents

1. Introduction

In classical calculus of variations, most concrete problems of optimization are accompanied with various kinds of constraints: isoperimetric, holonomic or not, etc. handled with the method of "Lagrange multipliers" (a misnomer, in fact, since it has been introduced by Euler), see for example [7].

In [3], we presented what can be regarded as a probabilistic deformation of the classical calculus of variations. Its basic principle is to deform in a minimal way the ideas and tools of the classical theory, so that those become compatible with the very irregular paths of diffusion processes. Since then, it has been shown that the same method holds for a much wider class of stochastic processes [11].

By minimal deformation, we mean one involving exclusively the regularizations associated with the existence of the underlying probability measures on path spaces. By construction, our variational framework depends on a positive constant $\hbar$. At the ("classical") limit where this constant vanishes, those paths become smooth and any claim should reduce to a statement of classical calculus of variations.

We are going to consider a problem of constraint in this probabilistic context, from a dual perspective. First, along the original line of [3], namely in the deformed variational perspective. Then, in a measure theoretic perspective, with a strong entropic flavor. It has been known, indeed, for a long time that the diffusions critical points of our regularizartion functionals belong to special time-symmetric class (Bernstein's reciprocal class) which can be naturally described in this way, see for instance [2, 6, 9]. As a matter of fact, the second perspective makes, in a way, clearer the status of Lagrange multipliers in our deformed calculus of variations.

In the present paper, one mostly presents the main ideas without focusing to precise assumptions and detailed proofs. The constraint considered here (prescribed final condition) is the simplest one in a hierarchy of manageable constraints. They will be systematically explored in future publications with a complete mathematical treatment.

Notation

For any measurable set A, $\mathcal{P}(A)$ denotes the set of all probability measures on A. The space $\Omega = C([0,T], \mathbb{R}^d)$ of all continuous paths from the time interval $[0,T]$ to the configuration space $\mathbb{R}^d$ is equipped with its usual σ-field. The canonical process on Ω is $(X_t)_{0 \leq t \leq T}$.
One denotes by $P_t = X_t \# P$ the image law of X_t under $P \in \mathcal{P}(\Omega)$ and $P_{ty} = P(\cdot | X_t = y)$.

2. Stochastic least action principle with final constraint

Let us consider the following action functional on $\mathcal{P}(\Omega)$

$$J(P) = E^P \int_0^T L(X_t, D^P X_t) \, dt + E^P S(X_T), \quad P \in \mathcal{D}_J \qquad (2.1)$$

defined on a domain $\mathcal{D}_J$ of diffusion processes, with measures $P \in \mathcal{P}(\Omega)$, solving stochastic differential equations (SDE) of the form

$$\begin{cases} dX_t &= b(X_t, t) \, dt + \sqrt{\hbar} \, dW_t \\ X_0 &= x \end{cases} \qquad (2.2)$$

where W is the Wiener process, b a Markov measurable "drift" and $\hbar$ a positive constant, our deformation parameter. It should be noted that $b = b^P$ can be regarded as a parametrization of P in $\mathcal{D}_J$. In definition (2.1),

$$L : (q, v) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto L(q, v) \in \mathbb{R}$$

is called the Lagrangian of J, D^P stands for the partial differential operator of parabolic type associated with the diffusion law P, defined on smooth (real valued) functions $\phi(q,t)$ by

$$D^P \phi = \partial_t \phi + b^P(q,t)\partial_q \phi + \frac{\hbar}{2}\partial_q^2 \phi$$

In our "deformation perspective", its proper interpretation is given by the following regularizations (or "deformation") of time derivative of ϕ along the solution of (2.2):

$$D^P \phi(X_t, t) = \lim_{\Delta t \downarrow 0} E^P \left[\frac{\phi(X_{t+\Delta t}, t+\Delta t) - \phi(X_t, t)}{\Delta t} \mid X_t \right] \qquad (2.3)$$

In particular, when $\phi(q,t) = q := \mathrm{pr}_q(q,t)$, with some abuse of notation $D^P \mathrm{pr}_q(X_t, t)$ reduces to

$$D^P X_t = b^P(X_t, t)$$

the second (random) variable of L, regarded as independent of the first one. The existence of such a space of independent variables, in this context, is in itself an interesting geometrical problem which is solved by a deformation of its classical solution, i.e. of classical Contact Geometry. This aspect will not be elaborated here (for a glimpse, see [10, 14]).

Let us consider the following "constrained problem": Find the critical points of the action functional (2.1) in the class of diffusions (2.2) such that, in addition, the following final constraint holds:

$$E^P N(X_T) = 0 \qquad (2.4)$$

for a given measurable function N satisfying some integrability conditions.

Now, we introduce, in complete analogy with the classical case, the new, unconstrained, action functional

$$J^\lambda(P) = E^P \int_0^T L(X_t, D^P X_t)\, dt + E^P(S - \lambda N)(X_T) \qquad (2.5)$$

for $\lambda \in \mathbb{R}$ the Lagrange multiplier.

Then, as expected, the claim is the following

Theorem 2.1. *If, for any $\lambda \in \mathbb{R}$ (or λ in an interval) we can find P^λ in the class (2.2) critical point of the unconstrained action functional (2.5) and if there is a λ_o (in the same interval) for which the constraint (2.4) is satisfied:*

$$E^{P^{\lambda_o}} N(X_T) = 0 \qquad (2.6)$$

then P^{λ_o} is a critical point of the constrained problem and the process associated with $P = P^{\lambda_o}$ solves the stochastic Euler-Lagrange equation:

$$[D^P \partial_v L - \partial_q L](X_t, D^P X_t) = 0,\ 0 < t < T, \quad P\text{-a.s.} \qquad (2.7)$$

with

$$\begin{cases} \partial_v L(X_T, D^P X_T) = -\nabla(S - \lambda_o N)(X_T) \\ X_0 = x \end{cases} \quad P\text{-a.s.} \qquad (2.8)$$

Idea of proof. Let us apply to the unconstrained action J^λ the method of [3] (see also [1, §6.2]) to compute its variation at P in a direction δP, for δP a signed measure such that $P + \delta P$ is a Markov probability measure absolutely continuous with respect to P. By Girsanov's theorem, this implies that there exists a bounded variation random process δX such that the image measure $(X + \delta X)\#P = P + \delta P$. It is enough to differentiate J^λ in the directions δX with sample paths in the Cameron-Martin space. This gives

$$
\begin{aligned}
\langle \nabla J^\lambda(P) | \delta P \rangle \quad &:= \quad \frac{d}{d\epsilon}_{|_{\epsilon=0}} \left\{ E^P \int_0^T L\big((X + \epsilon \delta X)_t, D^P(X + \epsilon \delta X)_t\big) \, dt \right. \\
&\qquad\qquad\qquad \left. + E^P(S - \lambda N)(X_T + \epsilon \delta X_T) \right\} \\
&= \quad E^P \int_0^T \big(\partial_q L(X_t, D^P X_t)\delta X_t + \partial_v L(X_t, D^P X_t)D^P \delta X_t \big) \, dt \\
&\qquad\qquad\qquad + E^P \left[(\nabla S - \lambda \nabla N)(X_T)\delta X_T \right] \qquad (2.9)
\end{aligned}
$$

For the second term under the integral, we appeal to Itô's product formula for δX of bounded variation:

$$
D^P \big[\partial_v L(X_t, D^P X_t)\delta X_t \big] = D^P \big[\partial_v L(X_t, D^P X_t) \big] \delta X_t + \partial_v L(X_t, D^P X_t)D^P \big[\delta X_t \big]
$$

Since the differential meaning of (2.3) is $E^P[D^P \phi(X_t, t) \, dt] = E^P[d\phi(X_t, t),]$ the r.h.s. of (2.9) reduces to

$$
\begin{aligned}
E^P \int_0^T [\partial_q L - D^P \partial_v L](X_t, D^P X_t) \, \delta X_t \, dt & \\
+ E^P \big\{ [\partial_v L(X_T, D^P X_T) &+ \nabla(S - \lambda N)(X_T)]\delta X_T \big\} \quad (2.10)
\end{aligned}
$$

A diffusion P^λ in the class (2.2) is extremal (or critical) for the functional J^λ if $\langle \nabla J^\lambda(P) | \delta P \rangle = 0$ for any admissible δP in the Cameron-Martin subspace. By (2.10) this means that, P^λ-a.s.

$$
\begin{aligned}
[\partial_q L - D^{P^\lambda}(\partial_v L)](X_t, D^{P^\lambda} X_t) = 0 \quad &\text{and} \\
\partial_v L(X_T, D^{P^\lambda} X_T) + \nabla(S - \lambda N)(X_T) = 0. &
\end{aligned}
$$

In particular, if $\lambda = \lambda_o$ then the final constraint (2.6) is satisfied, as well as, by definition, for any diffusion satisfying (2.4) whose boundary term N is irrelevant to the action. This means that P^{λ_o} is critical for the above constrained problem and solves (2.7) and (2.8). $\qquad\square$

This argument is along the line of a verification theorem in the sense of stochastic optimal control theory (see [5] for instance). Let us see how it works in practice.

Consider a typical example suggested by "Euclidean Quantum Mechanics" [1], namely the action functional

$$J(P) = E^P \int_s^T \left\{ \frac{1}{2} |D^P X_t|^2 + \frac{1}{2} X_t^2 \right\} dt$$

where instead of a zero initial time, as before, it will be useful to pick any time $s < T$.

We are looking for a critical point of J in the class of diffusions (2.2) such that

$$E_{sx}^P[X_T^2 - m^2] = 0, \quad \text{for } m \text{ a given constant.}$$

According to the theorem, we need first to find, for any $\lambda \in \mathbb{R}$, the critical point of the (unconstrained) functional

$$J^\lambda(P) = E_{sx}^P \int_s^T \left\{ \frac{1}{2} |D^P X_t|^2 + \frac{1}{2} X_t^2 \right\} dt - \lambda E_{sx}^P[X_T^2 - m^2]$$

This is an action of the form (6.25) in [1], for a quadratic potential $V(q) = \frac{1}{2} q^2$ and a final value $S_L(q, T) := -\lambda(q^2 - m^2)$.

A key point of the method advocated in [1] (dating back, in fact, to the mid-eighties, see [13] and references in [3]) is the following change of variables

$$S_L(q, T) = -\hbar \log \eta_T(q)$$

Here, $\eta_T(q) = e^{\frac{\lambda}{\hbar}(q^2 - m^2)}$. Indeed, the local characteristics of the diffusion P^λ, critical for J^λ, will be provided by the (positive) solution of the PDE with final boundary condition

$$\begin{cases} \hbar \partial_t \eta & = & \mathcal{H} \eta, \quad t < T \\ \eta(q, T) & = & \eta_T(q) \end{cases} \tag{2.11}$$

and "Hamiltonian"

$$\mathcal{H} = -\frac{\hbar^2}{2} \partial_q^2 + V(q) \tag{2.12}$$

This solution is given explicitly, in terms of the ("Mehler") integral kernel

$$h(q, T - t, y) = \text{kernel} \left\{ e^{-\frac{T-t}{\hbar} \mathcal{H}} \right\} (q, y)$$

by

$$\eta(q, t) = \int_{\mathbb{R}} h(q, T - t, y) \eta_T(y) \, dy \tag{2.13}$$

The critical process P^λ of J^λ solves a SDE of the form (2.2) for the drift

$$b^\lambda(q, t) = -\nabla S_L(q, t) := \hbar \nabla \log \eta(q, t) = - \left[\frac{\sinh(T - t) + 2\lambda \cosh(T - t)}{\cosh(T - t) + 2\lambda \sinh(T - t)} \right] q \tag{2.14}$$

Now, let us look for a $\lambda_o \in \mathbb{R}$ such that the constraint

$$E_{sx}^{P^\lambda}[(X_T^{\lambda_o})^2] = m^2 \tag{2.15}$$

is satisfied.

As observed in the introduction, the structure of the diffusions P^λ critical for action functionals such as J^λ is very special. Their probability density, for any fixed $\mathcal{H}$, is of the Bernstein reciprocal form [3, 1]

$$P(X_t \in A) = \int_A \eta^*(q,t)\eta(q,t)\,dq \qquad (2.16)$$

with t in the existence interval of P, A a Borel set and η^* a positive solution of the equation adjoint to (2.11) with respect to the time parameter t :

$$\begin{cases} -\hbar\partial_t \eta^* = \mathcal{H}\eta^*, & s \le t \le T \\ \eta^*(q,s) = \eta_s^*(q) \end{cases}$$

The product form (2.16) is, in fact, the key one for the relation with the entropic approach of §3 (cf. Conclusion).

Our constraint expectation (2.15) is a conditional one, knowing that $X_s = x$. This means that $\eta_s^*(q)\,dq = \delta_x(dq)$. The relevant normalized probability density of P_t^λ is therefore of the special product form

$$\mathcal{N}^{-1} h(x, t-s, q)\eta(q,t)\,dq$$

with η given by (2.13) and a normalizing factor $\mathcal{N} = \eta(x,s)$.

For $\eta_T(q)$ as prescribed before, the l.h.s. of (2.15) means

$$E_{sx}^{P^{\lambda_o}}[(X_T)^2] = \frac{1}{2\alpha}\left(1 + \frac{\beta^2}{2\alpha}\right)$$

where $\alpha = \hbar^{-1}(\frac{1}{2}\coth(T-s) - \lambda_o)$, $\beta = (\hbar\sinh(T-s))^{-1}x$.

The constraint (2.15) becomes equivalent to a quadratic polynomial in λ_o, whose coefficient depend on m, T, x and s. So P^{λ_o} with drift b^{λ_o} of the form (2.14) is a solution of the constrained problem and $P = P^{\lambda_o}$ (or, better, its associated diffusion) solves

$$\begin{cases} DD^P X_t = X_t, & \forall s \le t \le T; \\ X_s = x; & \quad P\text{-a.s.} \qquad (2.17) \\ D^P X_T = b^{\lambda_o}(X_T, T) \end{cases}$$

which is a stochastic Euler-Lagrange equation with boundary conditions, for the given Lagrangian. As expected on a classical basis, an endpoint constraint such as (2.15) does not affect the dynamics of the extremal process of our stochastic action functional (given by (2.18)), but only the boundary conditions it should satisfy.

Let us observe that, in the perspective of the above calculus of variations, it is natural to regard X_t as a (random) variable subjected to a variational procedure.

In the next §, it will be more appropriate to consider the process as a fixed canonical one and the probability laws as the true variables of optimization.

3. Random dynamical particles performing an unexpected final event

In 1932, Erwin Schrödinger addressed (in [12]) a problem of Brownian particles performing an unexpected event. His answer founded what we call, since the mid eighties, Euclidean Quantum Mechanics (cf [3], [13], for instance).

The problem of classical statistical physics is the following one. Suppose that you observe at time $t = 0$, a large number n of *independent* Brownian particles with some configuration distribution on $\mathbb{R}^d$ close to the probability measure μ and that at some later time T you observe that their distribution ν differs significantly from the expected distribution $\mu * \gamma_T$: the convolution of the initial state μ with the centered Gaussian measure with variance T. What is the most probable evolution of the whole system between these two states?

A modern answer to this problem is in terms of large deviations when n tends to infinity of the empirical measure

$$L_n^Y := \frac{1}{n} \sum_{i=1}^n \delta_{Y^i}$$

for the n random paths $Y^1, \ldots, Y^n$ (δ_Y is the Dirac measure at $Y : t \in [0, T] \mapsto Y_t \in \mathbb{R}^d$), see [6]. Note that L_n^Y is a random element of $\mathcal{P}(\Omega)$. In this section, inspired by the above example, we address a similar problem where the n particles are no longer Brownian but are independent and identically distributed with the law

$$R^V := z_V^{-1} \exp\left(-\frac{1}{\hbar} \int_0^T V(X_t) \, dt\right) R$$

where R is the Wiener measure with variance $\hbar$ and initial law $X_0 \# R = \mu = \delta_x$, i.e. the law of (2.2) with $b = 0$, and V some real function such that

$$z_V := E^R \exp\left(-\frac{1}{\hbar} \int_0^T V(X_t) \, dt\right) < \infty.$$

The initial configuration $L_n^Y(0) = \frac{1}{n} \sum_{i=1}^n \delta_{Y_0^i}$ is almost surely δ_x and, according to (2.4), one observes the mean value of a real function N on the final configuration:

$$\int_{\mathbb{R}^d} N(q) \, L_n^Y(T)(dq) = \frac{1}{n} \sum_{i=1}^n N(Y_T^i) \approx c$$

which may deviate significantly from the value predicted by the law of large numbers.

Sanov's theorem tells us that the large deviations of L_n^Y where the Y_i's are identically distributed with the reference law R^V is governed by the relative entropy

$$H(P|R^V) = \begin{cases} E^P\left(\log \frac{dP}{dR^V}\right) & \text{if } P \prec R^V \\ +\infty & \text{otherwise} \end{cases}$$

This approximately means that $\mathbb{P}(L_n^Y \in \mathcal{A}) \underset{n\to\infty}{\asymp} \exp[-n \inf_{P\in\mathcal{A}} H(P|R^V)]$ for $\mathcal{A}$ a subset of $\mathcal{P}(\Omega)$. A precise statement is in terms of a large deviation principle, see [4]. This means that for any closed set $\mathcal{A}$, $\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_n^Y \in \mathcal{A}) \leq -\inf_{P\in\mathcal{A}} H(P|R^V)$ and for any open set $\mathcal{A}$, $\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(L_n^Y \in \mathcal{A}) \geq -\inf_{P\in\mathcal{A}} H(P|R^V)$ where $\mathcal{P}(\Omega)$ is endowed with the usual weak topology. With regular enough subsets $\mathcal{A}$ (for instance open convex subsets) and $\mathcal{C}^\delta = \{P \in \mathcal{P}(\Omega); E^P[N(X_T)] \in [c-\delta, c+\delta]\}$ (one introduces $\delta > 0$ to make sure that $\mathbb{P}(L_n^Y \in \mathcal{C}^\delta) > 0$), one can expect the following *conditional law of large numbers*

$$\lim_{\delta\downarrow 0} \lim_{n\to\infty} \mathbb{P}(L_n^Y \in \mathcal{A} \mid L_n^Y \in \mathcal{C}^\delta) = \left\{ \begin{array}{ll} 1, & \text{if } \mathcal{A} \ni P^o \\ 0, & \text{otherwise} \end{array} \right.$$

where P^o is a minimizer of $P \mapsto H(P|R^V)$ on $\mathcal{C} := \lim_{\delta\downarrow 0} \mathcal{C}^\delta = \{P \in \mathcal{P}(\Omega); E^P[N(X_T)] = c\}$. To see this, remark that (formally)

$$\mathbb{P}(L_n^Y \in \mathcal{A} \mid L_n^Y \in \mathcal{C}^\delta) \underset{n\to\infty}{\asymp} \exp[-n(\inf_{P\in\mathcal{A}\cap\mathcal{C}^\delta} H(P|R^V) - \inf_{P\in\mathcal{C}^\delta} H(P|R^V))].$$

Since $H(\cdot|R^V)$ is strictly convex and $\mathcal{C}$ is a convex set, P^o is unique and this roughly means that conditionally on $L_n^Y \in \mathcal{C}$, as n tends to infinity L_n^Y tends to the solution P^o of the minimization problem

$$\text{minimize } H(P|R^V) \text{ subject to } E^P[N(X_T)] = c \qquad (3.1)$$

Clearly

$$\begin{aligned} H(P|R^V) &= E^P \log \frac{dP}{dR^V} = E^P \log \frac{dP}{dR} - E^P \log \frac{dR^V}{dR} \\ &= H(P|R) + E^P \frac{1}{\hbar} \int_0^T V(X_t)\,dt + \log z_V \end{aligned}$$

and with Girsanov's theorem one proves that

$$H(P|R) = E^P \int_0^T \frac{1}{2\hbar} |D^P X_t|^2 \, dt. \qquad (3.2)$$

Finally, P^o is the critical point of a special functional of type (2.1):

$$J(P) = \frac{1}{\hbar} E^P \int_0^T \left(\frac{1}{2} |D^P X_t|^2 + V(X_t) \right) dt = H(P|R^V) - \log z_V$$

under the constraint $E^P[N(X_T)] = c$, and applying Theorem 2.1 one sees that (P^o, λ_o) satisfies

$$\left\{ \begin{array}{rcll} DD^{P^o} X_t &=& \nabla V(X_t), & 0 < t < T; \\ X_0 &=& x \\ D^{P^o} X_T &=& \lambda_o \nabla N(X_T) \\ E^{P^o} N(X_T) &=& c \end{array} \right. \qquad P^o\text{-a.s.} \qquad (3.3)$$

On the other hand, one can characterize P^o as the solution of (3.1), using general results about entropy minimization. For each real λ, let

$$\Lambda(\lambda) := \log \int_{\mathbb{R}^d} e^{\frac{\lambda}{\hbar} N(y)} R_T^V(dy) \in (-\infty, +\infty].$$

The convex conjugate of Λ is defined for all real a by $\Lambda^*(a) := \sup_{\lambda \in \mathbb{R}} \{a\lambda - \Lambda(\lambda)\} \in (-\infty, +\infty]$.

Theorem 3.1. *It is assumed that N satisfies $\Lambda(\lambda) < \infty$ for all $\lambda \in \mathbb{R}$ and that c in (3.1) stands in the interior of the convex hull of the support of R_T^V. Then:*

1. *There exists a unique solution P^o to (3.1) and it is of the form*

$$P^o = \exp\left(\frac{\lambda_o}{\hbar} N(X_T) - \Lambda(\lambda_o)\right) R^V \tag{3.4}$$

   where $\lambda = \lambda_o$ is the unique solution to $\Lambda'(\lambda) = c, \lambda \in \mathbb{R}$.
2. *$H(P^o|R^V) = \Lambda^*(c) < \infty$.*
3. *$P^o(d\omega) = \int_{\mathbb{R}^d} R_{T,y}^V(d\omega)\,\rho^o(dy)$ where $\rho^o = P_T^o$ uniquely solves the entropy minimization problem:*

$$\text{minimize } H(\rho|R_T^V) \text{ subject to } \int_{\mathbb{R}^d} N(y)\,\rho(dy) = c, \ \rho \in \mathcal{P}(\mathbb{R}^d).$$

4. *P^o is the Markov diffusion associated with (2.2) for the drift*

$$b_o(q,t) = \hbar \nabla \log \eta(q,t)$$

   where η is the solution to

$$\begin{cases} \hbar \partial_t \eta &= \mathcal{H}\eta, \quad t < T \\ \eta(y,T) &= \exp\left(\frac{\lambda_o}{\hbar} N(y) - \Lambda(\lambda_o)\right) \end{cases} \tag{3.5}$$

   and $\mathcal{H} = -\frac{\hbar^2}{2}\Delta + V(q)$ is the Hamiltonian defined in (2.12).
5. *P^o and λ_o satisfy (3.3).*

Idea of proof. Except for the proof of (4), where regularity conditions are left vague, the proof is rigorous.

The interior of the convex hull of the support of R_T^V is equal to the interior of $\{a, \Lambda^*(a) < \infty\}$: $\text{int dom}\,\Lambda^*$. Since P^o solves (3.1), it is known (see [9] for instance) that under the assumption $\Lambda(\lambda) < \infty$ for all $\lambda \in \mathbb{R}$ and c in (3.1) is in $\text{int dom}\,\Lambda^*$, P^o has the form (3.4). On the other hand, a direct computation shows that $\Lambda'(\lambda_o) = \int_{\mathbb{R}^d} N\,dP_T^o$. This proves (1) and (2).

The proof of (3) is a consequence of the general tensorization formula for the relative entropy:

$$H(P|R^V) = H(P_T|R_T^V) + \int_{\mathbb{R}^d} H(P_{T,y}|R_{T,y}^V)\,P_T(dy)$$

Since the given constraint only concerns P_T and $H(P_{T,y}|R_{T,y}^V) = 0$ if and only if $P_{T,y} = R_{T,y}^V$, we have $P_{T,y}^o = R_{T,y}^V$ for P_T^o-a.e. y and $P_T^o = \rho^o$.

Let us give an idea of the proof of (4). One writes (3.4): $P^o = \eta(X_T, T) R^V$ with $\eta(y, T) = \exp\left(\frac{\lambda_o}{\hbar} N(y) - \Lambda(\lambda_o)\right)$. If the solution $\eta = e^{-\frac{1}{\hbar} S_L}$ of (3.5) doesn't vanish we have $\frac{1}{\hbar} V(q) = \eta(q, t)^{-1}(\partial_t + \frac{\hbar}{2}\Delta)\eta(q, t)$ and

$$
\begin{aligned}
T_t \;&:=\; \eta(X_0, 0)^{-1} \exp\left(-\frac{1}{\hbar} \int_0^t V(X_s)\, ds\right) \eta(X_t, t) \\
&=\; \eta(X_0, 0)^{-1} \exp\left(-\int_0^t \frac{(\partial_s + \frac{\hbar}{2}\Delta)\eta(X_s, s)}{\eta(X_s, s)}\, ds\right) \eta(X_t, t) \\
&=\; \exp -\frac{1}{\hbar}\left(S_L(X_t, t) - S_L(X_0, 0) - \int_0^t (\partial_s + \frac{\hbar}{2}\Delta)S_L(X_s, s) - \frac{1}{2}\int_0^t |\nabla S_L(X_s, s)|^2\, ds\right) \\
&=\; \exp -\frac{1}{\hbar}\left(\int_0^t \nabla S_L(X_s, s) \cdot dX_s - \frac{1}{2}\int_0^t |\nabla S_L(X_s, s)|^2\, ds\right)
\end{aligned}
$$

is a positive local R-martingale. If T is not only a local but also a genuine R-martingale, then $P^o = z_V^{-1}\eta(x, 0)T_T R$. Now, Girsanov's theorem tells us that the drift of P^o is $b^o = -\nabla S_L$, the desired result.

In particular, at time T, $D^{P^o}X_T = b^o(X_T, T) = \hbar\nabla \log \eta(X_T, T) = \lambda_o\nabla N(X_T)$, P^o-almost surely. Comparing with (3.3), one sees that λ_o in Theorems 2.1 and 3.1 is the same. This proves (5) and completes the proof of the theorem. □

Theorem 3.1 seems to have very little to do with our pathwise line of thought (§2). However, consider the function S_L defining the R-martingale T_t and therefore the Markovian diffusion of (4) in Theorem 3.1.

As a function, since $S_L = -\hbar \log \eta$ for η a positive solution of Eq. (3.5), S_L solves the following Hamilton-Jacobi-Bellman (HJB) equation:

$$
\frac{\partial S_L}{\partial t} - \frac{1}{2}|\nabla S_L|^2 + \frac{\hbar}{2}\Delta S_L + V = 0 \qquad t < T.
$$

Assuming that the solution S_L is smooth enough (This is not a trivial assumption since, generically they are not: this is for this equation that the notion of viscosity solution was initially designed: cf [5]), the gradient of HJB provides, since $b_o = -\nabla S_L$,

$$
DD^{P^o}X_t = \nabla V(X_t).
$$

This is the a.s Euler-Lagrange equation (3.3), solved by the process critical for the associated stochastic action functional $J(P)$. So the key elements of our pathwise analysis are also present, although implicitly, in the entropic approach.

It has been shown that the study of the symmetries of HJB equation is very rewarding for computations with such critical diffusions (cf [10]).

Let us make a final remark about the product form (2.16). It was the main motivation of Schrödinger's investigation in [12], since it looks like Born's interpretation of the wave function: $|\psi_t(q)|^2\, dq = \psi_t(q)\bar\psi_t(q)\, dq$ i.e. the probability of presence of a quantum particle when ψ solves Schrödinger's equation for the same Hamiltonian $\mathcal{H}$.

Formula (3.4): $P^o = \eta_T(X_T)\,R^V$ which is a direct consequence of the minimization of the relative entropy $H(\cdot|R^V)$, can also be given (approximately) a product form

$$P = \eta_0^*(X_0)\exp\left(-\frac{1}{\hbar}\int_0^T V(X_r)\,dr\right)\eta_T(X_T)\,R_\alpha \qquad (3.6)$$

Here, $R_\alpha(d\omega) := \int_{\mathbb{R}^d} R_q(d\omega)\,\alpha(dq)$ where R_q is the Wiener measure issued from q and α denotes Lebesgue measure. It is an unbounded "reversible" measure of the Wiener process, in the traditional sense of symmetric diffusions (cf [8]). In fact, since it is assumed that $R_0^V = \delta_x$, P^o is only approximately of the form (3.6). Under this P, the law P_t of X_t is given by

$$
\begin{aligned}
P_t(dq) &= E^{R_\alpha}\left[\eta_0^*(X_0)\exp\left(-\frac{1}{\hbar}\int_0^T V(X_r)\,dr\right)\eta_T(X_T)\Big|X_t = q\right]dq \\
&= \eta^*(q,t)\eta(q,t)\,dq
\end{aligned}
$$

where we used the Markov property of R_α at time t, $0 < t < T$, and we have introduced

$$
\begin{aligned}
\eta^*(q,t) &= E^{R_\alpha}\left[\eta_0^*(X_0)\exp\left(-\frac{1}{\hbar}\int_0^t V(X_r)\,dr\right)\Big|X_t = q\right] \\
\eta(q,t) &= E^{R_\alpha}\left[\exp\left(-\frac{1}{\hbar}\int_t^T V(X_r)\,dr\right)\eta_T(X_T)\Big|X_t = q\right].
\end{aligned}
$$

Using forward and backward exponential martingales, one shows that

$$
\begin{cases} \hbar\partial_t\eta(q,t) = \mathcal{H}\eta(q,t) \\ \eta(\cdot,0) = \eta_0 \end{cases}
\quad\text{and}\quad
\begin{cases} -\hbar\partial_t\eta^*(q,t) = \mathcal{H}\eta^*(q,t) \\ \eta^*(\cdot,0) = \eta_0^* \end{cases}.
$$

Given the fact that η and η^* can be regarded as time reversed of each other, the product form of $P_t(dq)$ is, therefore, the expression of a time dependent version of reversibility generalizing considerably the above-mentioned notion of symmetric diffusions. Explored since the mid-eighties (cf [13], [3], etc ...), this idea is the basic one in the program of stochastic deformation whose simplest variational problem with constraint was considered here.

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