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SUMMARY

A geometrical, structural tool for treating a curved thin layer with variable thickness is presented in the first two sections. Then developed tools is applied in deriving particular balance laws of thermomechanics. It is done for the case of interfacial layer with nonvanishing thickness and for true, not excess, quantities.

1. INTRODUCTION

The surface phenomena play an essential role in the borderland between the chemistry, physics and mechanics of fluids and solids. Mechanical phenomena associated with fluid interface regions in equilibrium are well-described in terms a surface tension. Since days of Young [29] the interface between two fluids, says, a liquid and its vapour, has been considered from the mechanical point of view as if it were a uniformly stretched massless membrane of zero thickness.

When a system in equilibrium is composed of two or more phases, the interface region between any two phases has a small but perceptible contribution to the mechanical and thermodynamic behaviour of the system. An extensive description of thermostatic behaviour of multicomponent interfaces was established by Gibbs [9]; he used the method of dividing surfaces. However, non-equilibrium situations are somewhat complex. Any comprehensive theory must accommodate the possibility of transport phenomena both within and across the interface; bulk motions may be induced by inhomogeneities of fields and of the matter in the interfacial region

(cf.[17]) and moreover physical adsorption and evaporation can occur there.

At this point one should underline the difference appearing between two models based on the concept of the interface 3D layer: the first referring to the excess quantities and the second referring to the true ones. In the first model one introduces a dividing surface located somewhere in the transition (interface) zone and then the bulk quantities are extrapolated up to this surface by stipulating (cf.[1,7]) that they must satisfy the typical 3D balance equations and the bulk constitutive relations (whatever these may be). The main problem of this model consists in introducing surface excess densities (quantities) to compensate the error introduced by replacing the exact (true) quantities by the extrapolated quantities in the transition zone.

In the second model no extrapolation is made, instead two dividing surfaces are introduced, which make the boundary between the single phase bulk media and the interface zone; in the latter multi-phase behaviour is observed, in which the confined matter possesses constitutive properties different from the surrounding bulk phases.

In both models an averaging procedure is applied in which integration along the thickness is performed to get mean quantities defined as surface fields. In the first model one relates those quantities to the deviations between exact and extrapolated quantities in the layer and in the second the mean quantities are defined as line integrals of the exact fields on some reference (e.g. mean) surface located in between the previous two. Here no physical meaning is ascribed to that surface: however, for convenience one can call it the dividing surface (as in the first model).

In the present paper (as in the second approach) the interface is modeled as a shell-like Eulerean region composed in principle of different material points at different time instants[1].

Having, without any extrapolations, the exact integral relations for the true surface fields in terms of the bulk quantities from the layer one tries to make constitutive description, that takes into account the interface and its interaction with the bulk phases as a whole, without retaining the macroscopically-irrelevant details of its structure. The more

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[1] This is evident in the the case of phase transition. However, the case of adsorption may be also modeled by a material region with extra mass supply sources, cf.[21].

detailed description[2] can be given by developing a theory in which higher order moments of the true fields appear (cf. [4,7]).

In the phenomenological approach we are presenting the interface is modeled as a finite slab and more detailed information about the structure of the dividing surface is introduced by relating the interfacial quantities to their 3D counterparts.

It is one of our aims to draw attention to the fact that in localizing the surface phenomena to their carrier, namely to moving surfaces, we are loosing some information necessary to this constitutive modelling. To get this back we can explore the results of our exact derivation and the formulae in which the interfacial quantities appearing in the interfacial balance laws are defined in terms of the corresponding 3D quantities.

It is our hope that the present method could be applied to analyzing shock wave structure in fluids and in solids.

2. MOVING SHELL-LIKE EULEREAN REGION IN THE CONTINUUM

Let us now assume that the effect of the interface in a continuous material system B occupying at time t in a motion χ a simply-connected region B_t in 3D space E^3 may be attributed to a 3D *moving layer* Z_t of finite thickness.

The assumption about the constant thickness of the layer is rather reasonable when very thin layer of the interfacial medium is modeled. However, constant in time (and in surface coordinates) interfacial layer restricts the class of physical problems successfully treated by either model. Here we shall omit this assumption.

As in the reference [15] we admit that a narrow layer Z_t divides the volume phases $B_t^{\pm}$ and that the boundaries between Z_t and both $B_t^{\pm}$ are regular surfaces Σ_t^{+} and Σ_t^{-}, now, however, they are not in general any more equidistant (parallel). As before we can use the parallel surface coordinate system describing an arbitrary point in the layer Z_t. If the position of the reference surface Σ_t is given by

$$y = r(L^1, L^2, t), \tag{2.1}$$

where L^1 and L^2 are Gauss parameters of the surface, then an arbitrary point x in Z_t can be represented as

$$x = r(L^1, L^2, t) + L\, n(r(L^1, L^2, t)) \tag{2.2}$$

where $L \in [\zeta^-, \zeta^+]$ is the third coordinate, measuring the distance of the point x from Σ_t along the unit normal n. Here points of the region Z_t are referred to a fixed rectangular Cartesian coordinate system.

The representation (2.2) means that the zone Z_t is *delineated*. Two scalar fields ζ^- and ζ^+ defined on the hypersurface S give the distance of the boundary surfaces $\Sigma_t^{\pm}$ from Σ_t. The thickness z of the layer

$$z(r,t) := \zeta^+(r,t) - \zeta^-(r,t) \tag{2.3}$$

can then depend on position and time. The interfacial zone Z_t is *delineated* by the surfaces Σ_t^L, distant L from Σ_t and represented by (2.2) with fixed L, and is *delimited* by the surfaces Σ_t^+ and Σ_t^- given by

$$\Sigma_t^{\pm} \equiv \left\{ y \in Z_t \,:\, y = r + \zeta^{\pm}(r,t) n(r),\ r \in \Sigma_t \right\} \tag{2.4}$$

which are not parallel to Σ_t, unless $\zeta^{\pm}(r,t)$ is independent of position r. Let us notice that under the present weaker assumptions the layer can shrink locally to a surface if $\zeta^{\pm} = 0$. Moreover, it is now possible, to describe the situation when the lateral boundary of the whole layer is not a ruled surface. However, to avoid any singularities in the representation of the layer in (L, L^1, L^2) coordinates the maximal thickness of the layer $\zeta := \sup \{\max (\zeta^-(r,t), \zeta^+(r,t)) : (r,t) \in S\}$ should be bounded from above by the maximal curvature of Σ_t.

The geometry of the boundary surfaces $\Sigma_t^{\pm}$ will be related to that of Σ_t as follows. If $a_\alpha^{\pm}$, $\alpha = 1,2$, denote the natural basis vectors of either surface, then due to (2.4), we get

$$a_\alpha^{\pm} = (1_S - \zeta^{\pm} b)\, a_\alpha + \frac{\partial \zeta^{\pm}(r,t)}{\partial L^\alpha}, \tag{2.5}$$

where a_α and b are tangent (basis) vector and curvature tensor of Σ_t, respectively. For the components of the metric tensor we then obtain

$$a_{\alpha\beta}^{\pm} := a_\alpha^{\pm} \cdot a_\beta^{\pm} = a_\alpha \cdot (1_S - \zeta^{\pm} b)^2\, a_\beta + \frac{\partial \zeta^{\pm}}{\partial L^\alpha} \frac{\partial \zeta^{\pm}}{\partial L^\beta}. \tag{2.6}$$

The directed surface element of either surface will be given by

$$n^{\pm}(r) da^{\pm} := a_1^{\pm} \times a_2^{\pm}\, dL^1 dL^2 = (j(\zeta^{\pm}, r) n(r) - A(\zeta^{\pm}) \operatorname{grad}_S \zeta^{\pm}) da \tag{2.7}$$

where on RHS the surface element $da = \sqrt{a}\, dL^1 dL^2$. In what follows we

[2] The other approach can be developed by introducing two different scales: micro- and macro- coordinates, the former responsible for the inner structure of the layer. The tool of nonstandard analysis could be helpful here [28].

denote the ratio $da^{\pm}/da$ by $j^{\pm}$ (a function of r and t). However, the second invariant of the surface tensor $(1_s - L\,b)$, is denoted by $j(L) := \det[\,\delta^{\beta}_{\alpha} - L\,b^{\beta}_{\alpha}] = 1 - 2HL + K\,L^2$, where H and K are invariants of b, i.e. the mean and Gauss curvatures, respectively.

The boundary surfaces $\Sigma_t^{\pm}$ move with velocities which are related to the velocity c of the reference surface Σ_t and the rate of change of ζ. Performing the time differentiation in (2.4) we obtain the following velocities $c^{\pm}$ of displacement of $\Sigma_t^{\pm}$:

$$c^{\pm} = c - \zeta^{\pm}(\mathrm{grad}_s c_n + b\,c) + \frac{\partial \zeta^{\pm}}{\partial t}\,n \qquad , c_n = c\cdot n. \tag{2.8}$$

To finish the geometrical preparation to the next section concerning the general balance law, write the product of $j^{\pm}$ and the normal speed of displacement of either boundary surface $\Sigma_t^{\pm}$

$$j^{\pm}c^{\pm}\cdot n^{\pm}(r) = j(\zeta^{\pm},r)\Big(c_n + \frac{\delta\,\zeta^{\pm}}{\delta\,t}\Big) + \zeta^{\pm}\mathrm{grad}_s\zeta^{\pm}A_s(\zeta^{\pm})\cdot\mathrm{grad}_s c_n, \tag{2.9}$$

where $A_s(L) := 1 + L\,\tilde{b}$ and $\tilde{b} := b - 2H1$.

3. BALANCE LAWS FOR A MOVING NON-MATERIAL SHELL-LIKE REGION

In a previous paper [15] it was assumed that the lateral boundary of the whole interface (transition) region, i.e. $Z_t \setminus \Sigma_t^{+} \cup \Sigma_t^{-}$ is a ruled surface. It turns out that this assumption can be dropped out on the global level (since liquid drops do not possess ruled lateral boundaries), keeping, however, this assumption, on the local level, i.e. during the passage from global to local form of balance laws. It can be done by assuming the integral form of the laws is valid for any sub-layer which is a proper subset of the whole layer bounded by subsurfaces of $\Sigma_t^{\pm}$ and a lateral boundary which is a ruled surface. In such case the natural boundary conditions given on the lateral boundary of Z_t need to be recalculated in an appropriate way.

According to our denotation the normal unit vector of $\Sigma_t^{\pm}$ will point from the region "-" to "+". The surface field ρ^S defined by

$$\rho^S(r(L^1,L^2,t),t) = \int_{\zeta^-}^{\zeta^+} \rho(r(L^{\alpha},t)+Ln(r),t)\,j(L,r(L^{\alpha},t))\,dL \tag{3.1}$$

with $j(L,r(L^{\alpha},t))$, $\alpha=1,2$, given by (2.8), will be called the *surface mass density*. Let us notice that the definition of the surface mass does not take into account the type and the form of the 3D motion governed by the particle velocity field v in the layer (especially its tangential components), since it is not know in advance. As a consequence we shall get an extra term in the mass in the 2D continuity equation (4.2). However, constraining the velocity field to a particular form we could define the surface mass density in a different way appropriate to this form. Particular examples of 3D motions in a material zone are discussed in [16].

After choosing the moving reference surface Σ_t as a geometrical object the remaining material structure of Z_t is preserved by defining ρ^S and next surface densities and fluxes of physical quantities as suitable integrals of the corresponding volume ones along the thickness of Z_t, e.g. we can define for f representing a density of a bulk quantity (i.e. a 3D density field) in the layer Z_t, the corresponding surface field f^S as

$$f^S = \int_{\zeta^-}^{\zeta^+} j(L,r)f(r+Ln(r),t)\,dL =: \langle j\,f\rangle. \tag{3.2}$$

The *geometrical* surface speed field c_n describes how Σ_t, and consequently the zone Z_t moves in E^3; no physical meaning, however, can be ascribed to a tangential component of c; in the literature, however, such a component is searched for, which is "reasonable" from a physical point of view, thus getting a "complete" velocity field to be used in the balance of linear momentum (cf. [8, 10, 18]).

In phase transition problems, for example, the material particles constituting the interfacial matter at instant t differ from those at another instant t'. Hence together with the field $c_n n$, we introduce an average velocity v^S of particles belonging to the layer. In terms of the 3D material (particle) velocity field v, the densities ρ and ρ^S, the *'surface' material point velocity* v^S is given by the relation for the *surface momentum density*

$$\rho^S v^S = \int_{\zeta^-}^{\zeta^+} \rho v\,j(L,r)\,dL, \tag{3.3}$$

together with (3.1). A continuous 2D system modeled by Σ_t will be called

non-material if[3] $c_n \equiv \mathbf{c}\cdot\mathbf{n} \neq V^S\cdot\mathbf{n}$, i.e. in the mean the material points occupying the interface layer will not all the time stay in it. The difference

$$d_n := (\mathbf{c} - V^S)\cdot\mathbf{n} \tag{3.4}$$

is Galilean invariant and is relevant to phase transition and adsorption processes if it does not vanish. It can be regarded as a quantity which needs to be determined by a constitutive equation [13].

The classical balance law for a quantity ψ with its Galilean-invariant flux (current) $\mathbf{w}$ and the source (supply + production) term p in the non-material, in general, region Z_t moving with the velocity $\mathbf{c}$, is

$$\frac{d_c}{dt}\int_{Z_t}\psi\,dv = -\int_{\partial Z_t}(\psi\,(\mathbf{v}-\mathbf{c}) + \mathbf{w})\cdot\mathbf{N}\,da + \int_{Z_t} p\,dv. \tag{3.5}$$

where $\mathbf{N}$ is the outward unit normal to ∂Z_t and the derivative $\frac{d_c}{dt}$ follows the region Z_t. After partition of both volumetric and surface measures into product measures we can get the final form of the integral balance law for layer Z_t

$$\int_{\Sigma_t}(\frac{d_c}{dt}\psi^S + \psi^S \mathrm{div}_S \mathbf{c}_\tau)da + \int_{\partial\Sigma_t}(\mathbf{W}^{S'}\{\psi\} - \psi^S\mathbf{c}_\tau)\tilde{\mathbf{n}}ds = -\int_{\Sigma_t}((\tilde{\mathbf{w}}j\mathbf{n})^- + (\tilde{\mathbf{w}}j\mathbf{n})^+)da + \int_{\Sigma_t} p^S da.$$

Here we have used $\mathbf{c}_\tau$ to denote the tangential part of $\mathbf{c}$ and put $\tilde{\mathbf{n}}$ for the unit normal to the curve ∂Z_t that is both tangent and outwardly directed with respect to Σ_t. The 'weighted' limiting bulk-field values $(\tilde{\mathbf{w}}j\mathbf{n})^\pm$ are

$$(\tilde{\mathbf{w}}j\mathbf{n})^\pm := \tilde{\mathbf{w}}(\mathbf{r} + \zeta^\pm\mathbf{n}(\mathbf{r}), t)(j\mathbf{n})^\pm, \text{ where } \tilde{\mathbf{w}} := \psi\,(\mathbf{v}-\mathbf{c}) + \mathbf{w} \tag{3.6}$$

is the new flux appearing under the surface integral over ∂Z_t, and we have used the fact that $\mathbf{N}da = (j\mathbf{n})^\pm da$ on $\Sigma_t^\pm$. Since the definition of a general surface density has been given by (3.2) we ought to define the surface flux $\mathbf{W}^{S'}\{\psi\}$ corresponding to the flux $\tilde{\mathbf{w}}$ (note the prime over s):

$$\mathbf{W}^{S'}\{\psi\} \equiv \langle\psi \circledcirc (\mathbf{v} + L\,\mathrm{grad}_S c_n)\,A_S(L)\rangle + \langle \mathbf{w}\,A_S(L)\rangle. \tag{3.7}$$

The above definition together with (3.2) give at the same time the only possible relationships between the surface quantities and their bulk

[3] The surface is p -material if equality holds [12].

counterparts (better to say - their primitives), in order for the interfacial balance law localized on the surface Σ_t to be compatible with, and derivable from, the 3D law. The latter is postulated for ψ in the integral form (3.5). Let us notice that the surface flux $\mathbf{W}^{S'}\{\psi\}$ is Galilean invariant (due to the Cayley-Hamilton identity). In order to obtain the local, differential form of the law we have to perform a localization procedure by applying the integral law to an arbitrary *subzone* Z'_t of Z_t. Here by *a subzone* we mean an arbitrary (shell-like) subregion Z'_t of $Z_t \subset_P t$ bounded by subsurfaces $\Sigma'^{\pm}_t$ of $\Sigma^{\pm}_t$ with a subsurface Σ'_t of Σ_t and with a nonvanishing lateral boundary which is a ruled surface, to which the Stokes and Green-Gauss theorems can be applied. After calculating the time derivative of the first integral and applying such integral law obtained to such an arbitrary Σ_t 'which is a support of an arbitrary subzone Z'_t of the layer Z_t, we get, modulo the continuity of the integrand[4],

$$\frac{d_c}{dt}\psi^S + \psi^S\mathrm{div}_S\mathbf{c} + \mathrm{div}_S(\mathbf{W}^{S'}\{\psi\} - \psi^S\mathbf{c}_\tau) = \tag{3.8}$$
$$= -\left\{(\{\psi(\mathbf{v}-\mathbf{c}) + \mathbf{w}\}j\mathbf{n})^- + (\{\psi(\mathbf{v}-\mathbf{c}) + \mathbf{w}\}j\mathbf{n})^+\right\} + p^S.$$

Using the Thomas displacement derivative $\delta_n/\delta t$ (cf.[26,27]), together with the relationship

$$\frac{d_c}{dt}\psi^S + \psi^S\mathrm{div}_S\mathbf{c} - \mathrm{div}_S(\psi^S\mathbf{c}_\tau) = \frac{\delta_n}{\delta t}\psi^S - 2Hc_n\psi^S.$$

we arrive at

$$\frac{\delta_n}{\delta t}\psi^S - 2Hc_n\psi^S + \mathrm{div}_S\,\mathbf{W}^{S'}\{\psi\} = \left\{(\{\psi(\mathbf{v}-\mathbf{c}) + \mathbf{w}\}j\mathbf{n})^- \right. \tag{3.9}$$
$$\left. (\{\psi(\mathbf{v}-\mathbf{c}) + \mathbf{w}\}j\mathbf{n})^+\right\} + p^S.$$

The constant thickness case can be simplified to

$$\frac{\delta_n}{\delta t}\psi^S - 2Hc_n\psi^S + \mathrm{div}_S\mathbf{W}^{S'}\{\psi\} = [\![\,\psi(\mathbf{v}-\mathbf{c}) + \mathbf{w}\,]\!]\cdot\mathbf{n} + \tilde{p}^S. \tag{3.10}$$

where for an arbitrary field g defined on B_t, we put

[4] A more general case leads to additional equations responsible for contact line effects.

(3.11) $\quad [\![g]\!] := g(r + \zeta^- n(r), t) - g(r + \zeta^+ n(r), t).$

The last equation is very well known in thermodynamics with surface singularities (called also : thermodynamics with singular surfaces, cf. [6,10-12,18,22]). The *surface supply* term $\tilde{p}^S$ is given by

(3.12) $\quad \tilde{p}^S = p^S + [\![h\{\psi(v-c)-w\}]\!] \cdot n,$

where $h(\zeta^\pm, r) := (K(r)\zeta^\pm - 2H(r))\zeta^\pm$, and as previously $r \in \Sigma_t$.

The variable thickness case ends with a relation similar to (3.10) in which, however, the surface supply term $\tilde{p}^S$ is different and if we denote it by $\tilde{p}^S_v$, then due to (2.31) and (2.32) it is equal to

$$\tilde{p}^S_v = p^S + [\![h\{\psi(v-c)+w\}]\!] \cdot n - [\![\psi \otimes (v + \zeta^\pm \operatorname{grad}_S c_n) A_S(\zeta^\pm) \operatorname{grad}_S \zeta^\pm]\!]$$

(3.13) $$- [\![w A_S(\zeta^\pm) \operatorname{grad}_S \zeta^\pm]\!] - [\![\frac{\delta_n}{\delta t} \zeta^\pm \psi j]\!].$$

A brief inspection of this term in comparison with the previous one shows the contribution of a new tangential part. This can be particularly important even in the case of the equilibrium equation for the interfacial stress tensor.

4. PARTICULAR BALANCE LAWS

Let us consider the particular quantities to be balanced by (3.10).

a) *Mass balance equation*: ψ is equal to ρ, and if the mass is conserved in the bulk medium, the flux of mass w and supply p of ρ (compare denotation in (3.11)) are zero. The surface flux $W^{S'}\{\rho\}$ is given by

(4.1) $\quad W^{S'}\{\rho\} = \langle \rho\ (v + L \operatorname{grad}_S c_n)\ A_S(L) \rangle = : \langle m(L) \rangle,$

which can be split into two parts

$$W^{S'}\{\rho\} = \rho^S v^S_\tau + W_\rho = \rho^S v^S_\tau + \langle \rho (A_S(L) - j(L) 1_S) v + L\rho \operatorname{grad}_S c_n A_S(L) \rangle$$

In the obvious way this equation leads to the definition of the extra mass flux W_ρ. Hence the local balance equation for the mass is

(4.2) $\quad \frac{\delta_n}{\delta t} \rho^S - 2Hc_n \rho^S + \operatorname{div}_S(\rho^S v^S_\tau) + \operatorname{div}_S W_\rho = [\![j\, \rho(v - c)]\!] \cdot n +$

$$[\![\rho \otimes (v + \zeta^\pm \operatorname{grad}_S c_n) A_S(\zeta^\pm) \operatorname{grad}_S \zeta^\pm]\!] - [\![\frac{\delta_n}{\delta t} \zeta^\pm \rho j]\!],$$

where from (4.1) yields the explicit form of the surface extra mass flux W_ρ

(4.3) $\quad W_\rho := \langle L\rho v \rangle b - K \langle L^2 \rho P v \rangle + \{\langle L\rho \rangle 1_S + \langle L^2 \rho \rangle \tilde{b}\} \operatorname{grad}_S c_n.$

The last two terms on the RHS of (4.3) disappear in the *constant thickness case*. However, the first two moments of mass (i.e. $\langle L\rho \rangle$ and $\langle L^2 \rho \rangle$) and of momentum (i.e. $\langle L\rho v \rangle$ and $\langle L^2 \rho v \rangle$) lead to nonvanishing, in general, extra flux of the mass, and moreoverthe flux of the surface mass is different, in general, from the density of linear momentum density [24].

b) *Linear momentum balance equation*: $\psi = \rho v$, the Cauchy stress T with minus sign serves as the flux of linear momentum, and the body force ρb is the supply term. For the surface flux $W^{S'}\{\rho v\}$ we have

(4.4) $\quad W^{S'}\{\rho v\} \equiv \langle \rho v \otimes (v + L \operatorname{grad}_S c_n)\ A_S(L) \rangle - \langle T\ A_S(L) \rangle,$

which can be split into two parts

$$W^{S'}\{\rho v\} = V^S \otimes W^{S'}\{\rho\} + T_S .$$

The Galilean invariant *interfacial surface stress tensor* T_S can be written as the sum of two invariant parts

$$S(r,t) := \langle T A_S(L) \rangle - \langle \rho (v - V^S) \otimes (v - V^S) A_S(L) \rangle$$

$$W_{\rho v}(r,t) := \langle \rho (v - V^S) \otimes ((A_S(L) - j(L) 1_S) V^S + L\rho \operatorname{grad}_S c_n\ A_S(L) \rangle,$$

or two other components contributing to T_S as S^1 and S^2, where we put

(4.5) $\quad S^1(r,t) := - \langle T A_S(L) \rangle,\ S^2(r,t) := \langle (v - V^S) \otimes m(L) \rangle, \quad T_S = S^1 + S^2.$

Due to (4.5) the local balance equation the linear momentum will be

(4.6) $\quad \frac{\delta_n}{\delta t} \left[\rho^S V^S \right] - 2Hc_n \rho^S V^S + \operatorname{div}_S(V^S \otimes (\rho^S v^S_\tau + W_\rho) + T_S) = \rho^S b^S +$

$$[\![(\rho v \otimes (v \cdot n - c_n) - T n) j]\!] - [\![\rho v \otimes (v + \zeta \operatorname{grad}_S c_n) A_S(\zeta) \operatorname{grad}_S \zeta]\!]$$

$$+ [\![T A_S(\zeta) \operatorname{grad}_S \zeta]\!] - [\![\frac{\delta_n}{\delta t} \zeta \rho v j]\!].$$

The last three terms on the RHS of (4.6) disappear in the *constant thickness case*. The above expressions for the interfacial stress tensor T_S show that even in the equilibrium case, when the diffusion terms S^2 and $W_{\rho v}$ are put equal to zero the symmetry of the tangential components of S cannot

hold automatically. Moreover, the normal component nT_s of the surface stress tensor T_s contains a contribution from the diffusion terms unless the tangential component v_τ of the velocity field v is constant along each segment of the layer. In that case $\langle fv\rangle_\tau = \langle f\rangle v_\tau$ for an arbitrary field f, and $v_\tau = v_\tau^{\pm} = V_\tau^S$. Such condition has been admitted by Dell'Isola & Romano (1987). Note that in the case of a spherical interface with non-vanishing thickness (e.g. a soup bubble) under equilibrium conditions $v - V^S = 0$, the surface stress will be

$$T_s = \langle p(1 + L/r)\rangle 1_s,$$

where r is a radius of the bubble. The term $\langle p\rangle$ we can interpret as the classical surface tension, here the additional part $\langle pL/r\rangle$ appears, which is normally very small, unless the thickness of the bubble is comparable with the radius r. This will be the case of very fine bubbles.

c) *Angular momentum balance equation*: $\psi = \mathbf{x} \times \rho\mathbf{v}$. We restrict ourself only to nonpolar continua. The master angular momentum balance law is well known in the 3D theory; its interfacial counterpart requires, if the previous balance law is satisfied,

$$\langle j(1 \times T)\rangle = 0, \tag{4.7}$$

which is automatically satisfied if T is symmetric.

Before closing the discussion the equation of motions we shall write the explicit relation for the normal and antisymmetric parts of the surface stress tensor T_s in the natural basis $\{a_\alpha, n\}$. They are

$$T_s^{n\alpha} = -\langle jT^{n\alpha}\rangle + \langle j\rho(v\cdot n - \underset{\sim}{c_n})(v^\alpha - V^{S\alpha})\rangle + \langle (v - V^S)\cdot n\, w_\rho^\alpha\rangle$$

$$- \langle j\, Lb_\delta^\alpha T^{n\delta}\rangle, \quad T_s^{12} - T_s^{21} = -\langle jL(b_2^1(T^{22} - T^{11}) + (b_1^1 - b_2^2)T^{12}\rangle,$$

where $T_s^{n\alpha} := n\cdot T_s a^\alpha$, $T^{12} := a^1\cdot T\, a^2$, etc.

d) *Energy balance equation*: $\psi = \rho(e + 0.5v\cdot v) =: \rho E$, where e represents the specific internal energy, the sum $-vT + q$ serves as the flux of the total energy, where q is the heat flux vector, the sum $\rho(b\cdot v + r)$ is the supply term, where r represents the body heat supply density. For the surface flux $W^{S'}\{\rho E\}$ we have

$$W^{S'}\{\rho E\} \equiv \langle E\, m(L)\rangle - \langle (vT - q)A_s(L)\rangle.$$

If we define

$$\rho^S e^S := \langle j\tilde{\rho e}\rangle, \quad \tilde{e} := e + 0.5(v - V^S)^2, \quad \rho^S \tilde{r}^S := \langle j\rho(r + b\cdot(v - V^S))\rangle,$$

$$q^S := \langle (\tilde{\rho e}(v - V^S) + q - (v - V^S)T)A_s(L)\rangle,$$

$$W_E := \langle \tilde{\rho e}(A_s(L) - j(L)1_s)V^S + L\tilde{\rho e}\, \mathrm{grad}_s c_n\, A_s(L)\rangle,$$

then the local energy balance equation will be

$$\frac{\delta n}{\delta t}\left[\rho^S(e^S + 0.5\, V^S\cdot V^S)\right] - 2Hc_n\rho^S(e^S + 0.5\, V^S\cdot V^S) \tag{4.8}$$

$$+ \mathrm{div}_s(\rho^S(e^S + 0.5V^S\cdot V^S)V_\tau^S + q^S + W_E + V^S T_s + 0.5V^S\cdot V^S\, W_\rho)$$

$$= -\left[\frac{\delta n}{\delta t}\zeta\, \rho(e + 0.5v\cdot v)j\right] + \rho^S b^S\cdot V^S + \left[(\rho(e + 0.5v\cdot v)(v - c) + q - vT)j\right]\cdot n$$

$$- \left[\rho(e + 0.5v\cdot v)\otimes(v + \zeta\, \mathrm{grad}_s c_n)A_s(\zeta)\mathrm{grad}_s\zeta\right] + \left[(vT - q)A_s(\zeta)\mathrm{grad}_s\zeta\right] + \rho^S\tilde{r}^S.$$

The last three terms on the RHS of (4.8) disappear in the *constant thickness case*. The above expressions for the interfacial heat flux Q^S and the supply terms $\rho^S\tilde{r}^S$ lead to the following relations

$$Q^S = q^S + W_E, \qquad \rho^S r^S \neq \rho^S\tilde{r}^S,$$

which mean that even the case of a nonconductor of heat at the 3D level leads to the nonvanishing interfacial heat flux, and the vanishing heat supply term ρr at the 3D level leads to the interfacial heat supply $\rho^S\tilde{r}^S$ equal to $\langle j\rho\, b\cdot(v - V^S)\rangle$, which does not need to vanish if b is different from zero.

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