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ON SOME GEOMETRY OF PROPAGATION IN DIFFRACTIVE TIME SCALES

CHRISTOPHE CHEVERRY AND THIERRY PAUL

ABSTRACT. In this article, we develop a non linear geometric optics which presents the two main following features. It is valid in *diffractive* times and it extends the classical approaches [7, 17, 18, 24] to the case of fast *variable* coefficients. In this context, we can show that the energy is transported along the rays associated with some non usual *long-time hamiltonian*. Our analysis needs structural assumptions and initial data suitably polarized to be implemented. All the required conditions are met concerning a current model [2, 3, 8, 9, 10, 11, 19, 21] arising in fluid mechanics and which was the original motivation of our work. As a by product, we get results complementary to the litterature concerning the propagation of the Rossby waves which play a part in the description of large oceanic currents, like Gulf stream or Kuroshio.

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1. DETAILED INTRODUCTION

This article is divided in two parts. The first (Section 2) is devoted to the WKB analysis associated with some class of quasilinear systems. The second (Section 3) explains how this approach can be implemented to solve concrete questions coming from quasi-geostrophic models.

1.1. Presentation of the framework. In this paragraph, we introduce the equations, the difficulties, and some significant results.

1.1.1. A class of equations involving a singular parameter ($\varepsilon \in]0, 1]$). Wave propagation phenomena are usually well described by solutions of quasilinear hyperbolic systems of the following type:

$$(1.1) \quad QS(\varepsilon, \varepsilon t, x, u; \partial) u := S_0(\varepsilon, \varepsilon t, x, u) \partial_t u + \sum_{j=1}^d S_j(\varepsilon, \varepsilon t, x, u) \partial_j u + \varepsilon^{-1} \Lambda(\varepsilon, \varepsilon t, x) u - \varepsilon^{-1} F(\varepsilon, \varepsilon t, x, u) = 0.$$

The adimensionalized parameter $\varepsilon \in]0, 1]$ represents a *wave length* (which will tend to zero). We denote by $t \in \mathbb{R}_+$ and $\tau := \varepsilon t \in \mathbb{R}_+$ respectively the *fast* and *slow* time variables. The space variable x is chosen in \mathbb{R}^d with $d \geq 1$. The state variable u describes the physics and is a vector of \mathbb{C}^n with $n \in \mathbb{N}_+^*$. We denote

$$(\varepsilon, \tau, x, u) \in \mathfrak{J} := \mathbb{R}_+ \times [0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{C}^n, \quad \mathcal{T} \in \mathbb{R}_+^*.$$

Let us now describe more precisely the operators appearing in (1.1). For j belonging to $\{0, \dots, d\}$, the operators S_j are smooth vector fields, valued in $\mathcal{S}_n := \{S \in \mathcal{M}_n(\mathbb{C}); S^* := {}^t \bar{S} = S\}$ of hermitian matrices:

$$S_j(\varepsilon, \tau, x, u) = S_j^*(\varepsilon, \tau, x, u) \in \mathcal{C}^\infty(\mathfrak{J}; \mathcal{S}_n), \quad \forall j \in \{0, \dots, d\}.$$

As usual the matrix S_0 is assumed to be positive definite. In other words, there is a constant $c \in \mathbb{R}_+^*$ such that

$$(1.2) \quad c I \leq S_0(\varepsilon, \tau, x, u), \quad \forall (\varepsilon, \tau, x, u) \in \mathfrak{J}.$$

The operator Λ represents **dispersive phenomena** [17]. It takes its values in the set $\mathcal{A}_n := \{A \in \mathcal{M}_n(\mathbb{C}); A^* = -A\}$ of *anti-hermitian* matrices:

$$(1.3) \quad \Lambda(\varepsilon, \tau, x) = -\Lambda^*(\varepsilon, \tau, x) \in \mathcal{C}^\infty(\mathfrak{J}; \mathcal{A}_n).$$

The source term $F \in \mathcal{C}^\infty(\mathfrak{J}; \mathbb{C}^n)$ is supposed to have an expansion of the following type:

$$(1.4) \quad F(\varepsilon, \tau, x, u) = F^0(\tau, x) + \varepsilon F^1(\tau, x) + \varepsilon^2 F^2(\varepsilon, \tau, x, u).$$

Similarly Taylor-expanding around $\varepsilon = 0$ the function Λ , one can incorporate the $O(\varepsilon^2)$ corresponding part into F^2 , hence it is simply assumed that

$$\Lambda(\varepsilon, \tau, x) = \Lambda^0(\tau, x) + \varepsilon \Lambda^1(\tau, x).$$

1.1.2. *About geometrical optics.* The geometrical optics approximation [15] simplifies the description of the evolution of a wave by considering that propagation takes place along rays. The simplest model corresponds to the situation where the high frequency oscillation is along *one phase* only:

$$(1.5) \quad u_\varepsilon(t, x) = U_\varepsilon\left(\varepsilon t, x, \frac{\varphi(\varepsilon t, x)}{\varepsilon}\right), \quad U_\varepsilon(\tau, x, \theta), \quad \theta \in \mathbb{T} := \mathbb{R}/\mathbb{Z}.$$

The amplitude $U_\varepsilon(\tau, x, \theta)$ is assumed to be in $\mathcal{C}^\infty(\mathbb{R}_+^2 \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$ while the phase $\varphi(\tau, x)$ is in $\mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}^d; \mathbb{R})$. Hence, the wave u_ε is roughly constant on the level surfaces of φ (called *wave fronts*) and has strong variations (at a speed of the order of ε^{-1}) in the directions of $\nabla\varphi$.

The monophasic WKB constructions are devised to prove the existence of solutions to (1.1) having the form (1.5). This goes back to the works of Choquet-Bruhat [5] as well as those of Hunter, Majda and Rosales [14]. A rigorous justification of such developments was performed by Guès [12, 13]. Classical results are only valid on a finite time interval, of the form $t \in [0, T]$ with $T \in \mathbb{R}_+^*$. After such a time, two phenomena may prevent from going further in time: the creation of *shocks* and the appearance of *caustics*.

1.1.3. *On shocks.* Discontinuities of order zero on solutions of (1.1) may be caused by the *nonlinearity* of the coefficients of the matrices S_j . This difficulty can be managed [1] in one space dimension ($d = 1$) but it seems out of reach in higher space dimensions. Then, it may be avoided through some linear degeneracy assumption on the coefficients.

In this article, we ensure that no shocks appear in times $t \simeq 1$. For this we require that the hermitian matrices S_j depend little on the state variable. More precisely, we impose that for all $j \in \{0, \dots, d\}$,

$$(H1) \quad S_j(\varepsilon, \tau, x, u) = S_j^0(\tau, x) + \varepsilon S_j^1(\tau, x) + \varepsilon^2 S_j^2(\varepsilon, \tau, x, u)$$

with S_j^0 and S_j^1 in $\mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}^d; \mathcal{S}_n)$, and $S_j^2 \in \mathcal{C}^\infty(\mathbb{J}; \mathcal{S}_n)$.

1.1.4. *On caustics.* On the domain $(t, x) \in [0, T] \times \mathbb{R}^d$ with $T \in \mathbb{R}_+^*$, the geometry of propagation is given by the structure of the principal symbol associated with (1.1), namely $P^0(0, x; \xi)$ where

$$P^0(\tau, x; \xi) := i \sum_{j=1}^d \xi_j S_j^0(\tau, x) + \Lambda^0(\tau, x), \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d.$$

To obtain solutions $u_\varepsilon(t, x)$ of (1.1) of the form prescribed by (1.5), valid in times $t \simeq 1$, one first selects an eigenvalue $i\lambda(0, x; \xi)$ of the matrix $S_0^0(0, x)^{-1} P^0(0, x; \xi)$. The profile $\mathcal{U}_0(x, \theta) := U_0(0, x, \theta)$ has to be polarized along the eigenspace associated with the eigenvalue $i\lambda(0, x; \nabla\varphi(t, x))$ of the matrix $S_0^0(0, x)^{-1} P^0(0, x; \nabla\varphi(t, x))$, where the phase $\varphi(t, x)$ is obtained by solving the eikonal (Hamilton-Jacobi type) equation

$$(1.6) \quad \partial_t \varphi + \lambda(0, x; \nabla\varphi) = 0, \quad \varphi(0, x) = \varphi_0(x).$$

The rays $t \mapsto X(t, x; \nabla \varphi_0(x))$ are obtained by projecting onto \mathbb{R}^d the hamiltonian flow associated with the following system of ODEs:

$$(1.7) \quad \begin{cases} dX/dt = \nabla_{\xi} \lambda(0, X; \Xi), & X(0, x; \xi) = x, \\ d\Xi/dt = -\nabla_x \lambda(0, X; \Xi), & \Xi(0, x; \xi) = \xi. \end{cases}$$

Those curves may *focalize*, or *even cross* [15, 16]. This mechanism prevents from solving (1.6) in the class of \mathcal{C}^1 functions. It does not occur when

- i)* The matrices S_j^0 and Λ^0 do not depend on the space variable x ;
- ii)* The initial data φ_0 is linear in x , meaning that there exists a direction $\eta \in \mathbb{R}^d$ such that $\varphi_0(x) = \eta \cdot x$.

Condition *i)* yields parallel rays. It implies that $\lambda(0, x; \xi) \equiv \mu(\xi)$ so that $\Xi(t, x; \xi) = \xi$ and $X(t, x; \xi) = x + \nabla_{\xi} \mu(\xi) t$. From *ii)* one gets $\nabla \varphi_0 \equiv \eta$ and one recovers plane phases $\varphi(t, x) = -\mu(\eta) t + \eta \cdot x$.

The two restrictions *i)* and *ii)* appeared in the pioneering work by Donnat, Joly, Métivier and Rauch [6] where they were used to propagate the WKB analysis all the way to times $t \simeq \varepsilon^{-1}$ or $\tau \simeq 1$. They have since been considered as prerequisites in contributions dealing with diffractive nonlinear geometrical optics [17, 18, 24]. Let us also mention [20] where the long time semiclassical evolution involving non classical phenomena is studied for the linear quantum dynamics.

1.1.5. *The analysis in diffractive times.* In this article, we will consider diffractive times $\tau \simeq 1$ without assuming conditions *i)* and *ii)*. We will allow *variable* coefficients S_j^0 and Λ^0 , along with *nonlinear* phases. Some attempts in this direction have been performed in [7, 14] but (after rescaling) it was in the context of *almost planar* phases, meaning in particular that φ is in the form $\varphi(t, \varepsilon x)$ instead of $\varphi(t, x)$.

In order to get to times $\tau \simeq 1$, one still needs a degeneracy assumption on the curvature of the characteristic variety. The stronger version of that property consists in requiring (after an adequate change of variables in ε , t , x and u) the existence of a spectral value such that

$$(H2) \quad \lambda(0, x; \xi) = 0, \quad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

At first sight, condition (H2) seems to be of no interest. Indeed, as long as $t \simeq 1$, nothing happens. The phase and the principal profile U_0 remain both unchanged. One finds $\varphi(t, \cdot) = \varphi_0(\cdot)$ and $U_0(t, \cdot) = U_0(0, \cdot)$. On the other hand, for $t \simeq \varepsilon^{-1}$ or $\tau \simeq 1$, one expects that the dispersive effects (and the production of Schrödinger equations) which motivate the articles [6, 17, 18, 24] do not appear. Indeed, the hypothesis (H2) implies that the Hessian matrix $D_{\xi}^2 \lambda(0, x, \cdot)$ is zero.

However, precisely because we do not assume *i)* and *ii)*, other phenomena can occur. Without *i)* and *ii)*, the discussion concerning oscillating solutions of (1.1), like in (1.5), is in fact rather complex as soon as diffractive times $\tau \simeq 1$ are reached. The corresponding study is new. It is motivated both by mathematical and physical issues.

1.2. The main results. In this Section 1.2, we state the main results of our paper, postponing to the next Section 1.3 the statement of the bunch of hypothesis ($H\star$ for *linear* and $HN\star$ for *nonlinear*).

1.2.1. *Approximate solutions.* Our first statement guarantees the existence in diffractive times $t \simeq \varepsilon^{-1}$ or $\tau \simeq 1$ of approximate solutions to (1.1).

Theorem 1. [*Approximate solutions*] *In the linear case (S_j independent of u), we assume that conditions (H1) to (H8) hold. In the nonlinear (more general) case, we have to complete these prerequisites with conditions (HN1) to (HN5). These are structural assumptions on the expressions S_j , Λ and F appearing in the system (1.1), which are made precise further in the text. Consider a phase $\varphi_0 \in C^\infty(\mathbb{R}^d; \mathbb{R})$, which is non stationary in the sense that*

$$(1.8) \quad \exists (c, C) \in (\mathbb{R}_+^*)^2; \quad c \leq |\nabla \varphi_0(x)| \leq C, \quad \forall x \in \mathbb{R}^d.$$

Select a profile $\mathcal{U}_\varepsilon \in H^\infty([0, 1] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$ with an asymptotic expansion in powers of ε involving a leading term \mathcal{U}_0 such that $\partial_\theta \mathcal{U}_0(x, \theta)$ is polarized in the kernel of $P^0(0, x; \nabla \varphi_0(x)) := i \sum_{j=1}^d \partial_j \varphi_0 S_j^0(0, x) + \Lambda^0(0, x)$. Look at an oscillatory initial datum of the form

$$(1.9) \quad u_\varepsilon(0, x) = \mathcal{U}_\varepsilon\left(x, \frac{\varphi_0(x)}{\varepsilon}\right), \quad \varepsilon \in]0, 1].$$

Then, for all $N \in \mathbb{N}$, there is a family $\{u_\varepsilon^a\}_{\varepsilon \in]0, 1]}$, involving monophasic oscillations of the form

$$(1.10) \quad u_\varepsilon^a(t, x) = \sum_{k=0}^{N+1} \varepsilon^k U^k\left(\varepsilon t, x, \frac{\varphi(\varepsilon t, x)}{\varepsilon}\right), \quad U^k \in H^\infty$$

which is an approximate solutions to the system (1.1) in diffractive times. More precisely, the functions $u_\varepsilon^a(t, x)$ are defined on a time interval $[0, T/\varepsilon]$ for all $\varepsilon \in]0, 1]$ with $T \in \mathbb{R}_+^$. They satisfy (1.9) and*

$$(1.11) \quad QS(\varepsilon, \varepsilon t, x, u_\varepsilon^a; \partial) u_\varepsilon^a = O(\varepsilon^N), \quad \text{in } L^\infty([0, T/\varepsilon]; H_\varepsilon^s(\mathbb{R}^d)).$$

In addition, the presence of variable coefficients can induce a modification of φ_0 in times $t \simeq 1/\varepsilon$ or $\tau \simeq 1$, via some (non trivial) eikonal equation

$$(1.12) \quad \partial_\tau \varphi(\tau, x) = h(\tau, x; \nabla \varphi(\tau, x)), \quad \varphi(0, x) = \varphi_0(x).$$

To our knowledge, Theorem 1 cannot be derived, after a change of scalings (in ε , t and x) or a change of variables (in u) from well-established results.

Section 2.1 introduces our notations and our strategy. The hierarchy of equations is presented in Section 2.2 and initiated in Section 2.3. As already explained, the main effect (for small times $t \simeq 1$) of the penalization term is to polarize $\partial_\theta U^0$ in the kernel of P^0 (having dimension $p \in \mathbb{N}^*$). Then, comes the question of the propagation in diffractive times.

Because the coefficients $S_j^0(\tau, x)$ and $\Lambda^0(\tau, x)$ may depend on the variable x , the discussion must be organized differently from what is usually done. In fact, it needs a refinement of the analysis inside the kernel of $P^0(\tau, x, \xi)$.

A crucial step (Lemma 2.3) is to get the Hamiltonian $h(\tau, x; \xi)$ of (1.12). It corresponds to the eigenvalue of some hermitian matrix $H(\tau, x, \xi)$ exhibited in Section 2.4, see (2.31). One notices that there can be (in diffractive times) as much as p *different geometries* (or p different types of rays).

The transport equation on U^0 is solved in Section 2.5. The induction giving access to the other profiles U^k with $k \geq 1$ is presented in Section 2.6. This concludes the formal WKB analysis.

1.2.2. *Stability issues.* Due to (H3) and (H5), the system (1.1) is compatible with energy estimates in the space L^2 . Therefore, in the *linear* case, we can infer from the preceding construction the existence of exact solutions u_ε close (in the sense of L^2) to the approximate solutions u_ε^a . This is what says our next Theorem, proven in Section 2.7.

Theorem 2. *(The linear case) Let us assume conditions (H \star) and suppose that QS is independent of u . Consider a family $\{u_\varepsilon^a\}_{\varepsilon \in]0,1]}$ of approximate solutions of order N to (1.1), given by Theorem 1. Then, for all $\varepsilon \in]0,1]$, the exact solution u_ε of the Cauchy problem*

$$(1.13) \quad QS(\varepsilon, \varepsilon t, x, u_\varepsilon; \partial) u_\varepsilon = 0, \quad u_\varepsilon(0, \cdot) \equiv u_\varepsilon^a(\cdot)$$

is defined on the domain $[0, \mathcal{T}/\varepsilon] \times \mathbb{R}^d$ and it is such that

$$(1.14) \quad \sup_{t \in [0, \mathcal{T}/\varepsilon]} \| (u_\varepsilon - u_\varepsilon^a)(t, \cdot) \|_{L^2(\mathbb{R}^d; \mathbb{C}^n)} = O(\varepsilon^{N-1}), \quad \forall \varepsilon \in]0,1].$$

The nonlinear situation is more delicate to deal with. It requires uniform estimates in L^∞ on the family $\{u_\varepsilon\}_\varepsilon$. Such an information can be obtained only through Sobolev estimates.

Theorem 3. *Select any approximate solution u_ε^a given by Theorem 1, with $2 + d < N$. Suppose that*

$$(1.15) \quad \nabla_x S_j^0 \equiv 0, \quad \forall j \in \{0, 1, \dots, d\}.$$

Then, the exact solution u_ε of the Cauchy problem (1.13) is defined on the domain $[0, \mathcal{T}/\varepsilon] \times \mathbb{R}^d$. It remains close to the approximate solutions u_ε^a in the sense that, for all s and N with $1 + d/2 < s < N/2$, one has

$$(1.16) \quad \sup_{t \in [0, \mathcal{T}/\varepsilon]} \| (u_\varepsilon - u_\varepsilon^a)(t, \cdot) \|_{H^s \cap L^\infty} = O(\varepsilon^{N-2-d}).$$

The condition (1.15) is rather restrictive but it seems necessary. Still, it allows variable coefficients at the level of the matrix $\Lambda^0(\tau, x)$. The proof of Theorem 3 relies on energy estimates performed in the weighted space $H_{\varepsilon^2}^s$, where for $\iota \in \mathbb{N}$:

$$(1.17) \quad \| u \|_{H_{\varepsilon^\iota}^s} := \sum_{|\alpha| \leq s} \| (\varepsilon^\iota \partial_x)^\alpha u \|_{L^2}, \quad (\varepsilon^\iota \partial_x)^\alpha \equiv \varepsilon^{2|\alpha|} \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}.$$

The control (1.16) hides a lost of ε^{-2} by spatial derivative ∂_j . As we will see in the next Section 1.4, this information $H_{\varepsilon^2}^s$ is not always sure to be optimal. Nevertheless, it is sufficient to justify our WKB analysis.

Although all hypothesis $(H\star)$ and $(HN\star)$ will be introduced in context in the following Sections, for the comfort of the reader, we state them below.

1.3. Statement of the hypothesis. Note that our statements and the following assumptions could be localized in a conic region (in ξ) containing the set $\{(x, \nabla\varphi_0(x)); x \in \mathbb{R}^d\}$. For the simplicity of exposition, we will not take into account such a refinement. Let us first state the assumptions necessary for the *linear* results.

(H1): For all $j \in \{0, \dots, d\}$, we impose

$$S_j(\varepsilon, \tau, x, u) = S_j^0(\tau, x) + \varepsilon S_j^1(\tau, x) + \varepsilon^2 S_j^2(\varepsilon, \tau, x, u)$$

with S_j^0 and S_j^1 in $\mathcal{C}^\infty(\mathbb{R}_+ \times \mathbb{R}^d; \mathcal{S}_n)$, and $S_j^2 \in \mathcal{C}^\infty(\mathfrak{I}; \mathcal{S}_n)$.

(H2): Let $P^0(\tau, x; \xi) := i \sum_{j=1}^d \xi_j S_j^0(\tau, x) + \Lambda^0(\tau, x)$. We suppose that the matrix $S_0^0(0, x)^{-1} P^0(0, x; \xi)$ has an eigenvalue $i \lambda(0, x; \xi)$ satisfying

$$\lambda(0, x; \xi) = 0, \quad \forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

(H3): The matrix S^0 is divergence free, in the sense that

$$\operatorname{div} S^0 := \sum_{j=1}^d (\partial_j S_j^0)(\tau, x) = 0, \quad \forall (\tau, x) \in [0, T] \times \mathbb{R}^d.$$

(H4): There is a positive integer $p \in \mathbb{N}^*$ such that

$$\dim(\ker P^0(\tau, x; \xi)) = p, \quad \forall (\tau, x, \xi) \in [0, T] \times \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}).$$

(H5): $\left[\begin{array}{l} \text{There is a compact set } \mathcal{K} \subset \mathbb{R}^d \text{ such that the fields } S_j, \Lambda \text{ and} \\ F \text{ evaluated at } (\varepsilon, \tau, x, u) \text{ are constant if } \tau \in [0, T] \text{ and } x \notin \mathcal{K}. \end{array} \right.$

(H6): The source term F^0 must be well prepared

$$F^0(\tau, x) \perp \ker \Lambda^0(\tau, x), \quad \forall (\tau, x) \in [0, T] \times \mathbb{R}^d.$$

Let $\Pi(\tau, x; \xi)$ be the unitary projector onto the kernel of $P^0(\tau, x; \xi)$. Introduce the two following matrices

$$G(\tau, x; \xi) := i \sum_{j=1}^d \Pi S_j^0 (\partial_j \Pi) \Pi, \quad P^1(\tau, x; \xi) := i \sum_{j=1}^d \xi_j S_j^1 + \Lambda^1$$

and the Hermitian square root of $\Pi S_0^0 \Pi$, that is the matrix M satisfying

$$M \equiv \Pi M \Pi \equiv M^*, \quad M^* \circ M = \Pi S_0^0 \Pi.$$

Let us also define

$$H := (\Pi M \Pi)^{-1} \Pi (G + iP^1) \Pi (\Pi M \Pi)^{-1} = H^* \equiv \Pi H \Pi.$$

(H7): We assume that

$$\left[\begin{array}{l} \text{There is an eigenvalue } h \text{ of } H \text{ whose multiplicity} \\ \mu(\tau, x, \xi) \equiv \mu \in \mathbb{N}^* \text{ does not depend on } (\tau, x, \xi) \in [0, T] \times T_{\setminus 0}^*. \end{array} \right.$$

Given $m \in \mathbb{Z}$ and some application $f(\tau, x; \xi)$ defined on $[0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{R}^d$, we use the shorthand notation

$$(1.18) \quad |f\rangle_m(\tau, x) := f(\tau, x; m \nabla \varphi(\tau, x)), \quad m \in \mathbb{Z}$$

which simply amounts to replacing everywhere the vector ξ by $m \nabla \varphi$. For instance $|f\rangle_0(\tau, x) = f(\tau, x; 0)$. Let us denote by $\Pi_0^h(\tau, x)$ the unitary projector onto the kernel of $H(\tau, x; 0)$. Introduce the projector $Q := I - \Pi$ and the matrix $G^0 := |(QP^0Q)^{-1}\rangle_0 F^0$.

(H8): The source term F^1 must be well prepared in the sense that

$$\Pi_0^h (|\Pi M \Pi\rangle_0)^{-1} \left\{ F^1 - |P^1\rangle_0 G^0 - \sum_{j=1}^d S_j^0 |Q\rangle_0 \partial_j G^0 \right\} \equiv 0.$$

We can now state the assumptions necessary for the *nonlinear* results. The conditions **(HN*)** are mainly technical assumptions in order to control what happens at the level of harmonics. In contrast with the linear hypothesis, they require to know what is φ on $[0, \mathcal{T}] \times \mathbb{R}^d$. For instance, the definition of the set \mathcal{HA} does depend on \mathcal{T} . Also, the conditions **(HN3)** and **(HN4)** must be tested in the whole domain $[0, \mathcal{T}] \times \mathbb{R}^d$. Therefore, the procedure is first to solve (1.6) on $[0, \mathcal{T}] \times \mathbb{R}^d$ and then to look at **(HN*)**.

(HN1): There is an integer $p_0 \in \mathbb{N}$ satisfying $p_0 \geq p$ and $\dim(\ker P^0(\tau, x; 0)) = p_0$, $\forall (\tau, x) \in [0, \mathcal{T}] \times \mathbb{R}^d$.

(HN2): $|\Pi\rangle_0 S_j^0 |\Pi\rangle_0 \equiv 0$, $\forall j \in \{1, \dots, d\}$.

Consider a solution φ of the eikonal equation (1.6). Associated with φ , define the set

$$\mathcal{HA} := \{0\} \cup \{m \in \mathbb{Z}^*; m \partial_\tau \varphi = |h\rangle_m \text{ on } [0, \mathcal{T}] \times \mathbb{R}^d\}.$$

(HN3): There is a constant $c \in \mathbb{R}_+^*$ such that for every $m \in \mathcal{HA}$:

$$c < \inf_{(\tau, x) \in [0, \mathcal{T}] \times \mathbb{R}^d} \min_{h \neq \mu \in \text{spec } H} |(\mu - h)(\tau, x; m \nabla \varphi(\tau, x))|.$$

(HN4): There is a constant $c \in \mathbb{R}_+^*$ such that for every $m \notin \mathcal{HA}$:

$$c < \inf_{(\tau, x) \in [0, \mathcal{T}] \times \mathbb{R}^d} \min_{\mu \in \text{spec } H} |m \partial_\tau \varphi - \mu(\tau, x; m \nabla \varphi(\tau, x))|.$$

(HN5): $\sup_{m \in \mathbb{Z}} \|(QP^0Q)^{-1}\rangle_m\|_{H^s([0, \mathcal{T}] \times \mathbb{R}^d)} < \infty$, $\forall s \in \mathbb{R}$.

These assumptions **(H*)** and **(HN*)** are not so restrictive. They are verified in various contexts including the propagation of Rossby waves (Section 3) and of electromagnetic waves. They do not imply *i*) and *ii*). The point *i*) can be lifted since the matrices $S_j^0(\tau, x)$ or $\Lambda^0(\tau, x)$ may well depend on x while the function $\lambda(0, x; \xi)$, in view of **(H2)**, does not. The restriction *ii*) can also be lifted as one can start with an arbitrary phase φ_0 and no caustics will appear (at least as long as εt or τ remain small enough).

1.4. A model arising in fluid mechanics. The present work is motivated by physical considerations. Indeed, our WKB construction allows to account for some wave-like features of oceanic circulation, called *Rossby waves*, which are produced by the variations of the Coriolis force with latitude.

The link between this problem and our discussion is presented in detail in [2, 3, 8, 9, 10]. Basically, we have to deal with a two-dimensional system of compressible Euler type. The space variable is $x = (x_1, x_2) \in \mathbb{R}^2$. The state variable is $u = {}^t(p, v_1, v_2) \in \mathbb{R}^3$ and it must satisfy

$$(1.19) \quad \begin{cases} d_\tau p + \varepsilon^{-1} f(\bar{p} + \varepsilon p^s + \varepsilon^2 p) (\partial_1 v_1 + \partial_2 v_2) = \varepsilon^{-2} F_0^r, \\ d_\tau v_1 + \varepsilon^{-1} f(\bar{p} + \varepsilon p^s + \varepsilon^2 p) \partial_1 p - \varepsilon^{-2} b(\varepsilon, x) v_2 = \varepsilon^{-2} F_1^r, \\ d_\tau v_2 + \varepsilon^{-1} f(\bar{p} + \varepsilon p^s + \varepsilon^2 p) \partial_2 p + \varepsilon^{-2} b(\varepsilon, x) v_1 = \varepsilon^{-2} F_2^r. \end{cases}$$

The data $u^s = {}^t(p^s, v_1^s, v_2^s)$ represents a state of rest. It is a smooth function of $(\varepsilon, x) \in \mathbb{R}_+ \times \mathbb{R}^2$. The introduction of $b(\varepsilon, x)$ is due to the Coriolis force. In basic models, we have to deal with the choice $b(\varepsilon, x) = \sin x_2$. The source term $F^r := {}^t(F_0^r, F_1^r, F_2^r)$ allows to take into account other influences (like wind, \dots). It is a smooth application which can depend on the variables $(\varepsilon, \tau, x, u) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^3$. The notation d_τ is for the particular derivative $d_\tau := \partial_\tau + v_1^s \partial_1 + v_2^s \partial_2 + \varepsilon v_1 \partial_1 + \varepsilon v_2 \partial_2$.

In contrast to (1.1), the system (1.19) involves directly the diffractive time variable τ , explaining the singular power ε^{-2} in front of b . The modeling work leading to (1.19) is done in Paragraph 3.1. In the context of (1.19), a version of Theorem 1 is the following :

Theorem 4. *Consider a family of oscillatory initial data, like in (1.9). Suppose that, for all $\theta \in \mathbb{T}$, the profile $\mathcal{U}_\varepsilon(\cdot, \theta)$ is supported in a domain $\mathcal{D} \subset \mathbb{R}^2$ adjusted such that*

$$(1.20) \quad \exists c \in \mathbb{R}_+^*; \quad \mathcal{D} \subset \{x \in \mathbb{R}^2; |b^0(x)| \geq c > 0\}, \quad b^0 := b(0, \cdot).$$

*Assume moreover that the phase φ_0 is nonstationary in the sense of (1.8) and that it satisfies assumptions **(Hi)** or **(Hii)** given below :*

$$\begin{aligned} \text{(Hi)} \quad & \left[v^s \cdot \nabla b^0 \equiv 0 \text{ and } \varphi_0 = \chi(b^0) \text{ with } \chi \text{ belonging to } \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}); \right. \\ \text{(Hii)} \quad & \left[0 < \inf_{x \in \mathbb{R}^2} |(\partial_1 \varphi_0 \partial_2 b^0 - \partial_2 \varphi_0 \partial_1 b^0)(x)|. \right. \end{aligned}$$

Then, there is a family $\{u_\varepsilon^a(\tau, x)\}_{\varepsilon \in]0, 1]}$, as in (1.10), made of approximate solutions to the oscillatory Cauchy problem (1.9)-(1.19). For all $\varepsilon \in]0, \varepsilon_0]$ with $\varepsilon_0 \in \mathbb{R}_+^$, it is defined in diffractive times $\tau \in [0, \mathcal{T}]$ with $\mathcal{T} \in \mathbb{R}_+^*$. More precisely, those approximate solutions oscillate with a phase solving the eikonal equation (1.12) where the function h must be replaced by the Rossby Hamiltonian h^r given by*

$$(1.21) \quad h^r(\tau, x; \xi) := -v^s \cdot \xi + \frac{\xi_1 \partial_2 b^0(x) - \xi_2 \partial_1 b^0(x)}{b^0(x)^2 + \xi_1^2 + \xi_2^2}, \quad (x, \xi) \in \mathbb{R}^4.$$

The energy of Rossby waves is transported along the rays which are associated with h^r , up to some explicit damping and source terms. Moreover, there are exact solutions $\{u_\varepsilon(\tau, x)\}_{\varepsilon \in]0,1]}$ of (1.19) corresponding to the approximate solutions u_ε^a in the sense of (1.16).

This statement should be compared with the results announced in [3] and proved in [2]. In those papers, the discussion is essentially based on the study of a linearization of the system (1.19) around a particular stationary solution, and the methods come from *semi-classical analysis and dynamical systems*. They consist indeed in diagonalizing the linearized system using ε -pseudodifferential symmetrizers and in obtaining dynamical information, in terms of the wave front set of the initial data.

Theorem 4 concerns more restrictive initial data, but the preparation of the data (the polarization on *Rossby waves*) has the advantage of giving rise to a more precise description. It allows to catch quantitative informations and to point out nonlinear mechanisms influencing the propagation.

In Paragraph 3.2, we check that the structure of (1.19) is compatible with Assumptions (H1), \dots , (H7) and (H8) required by Theorem 1.

The paragraph 3.3 is devoted to the description of rays transporting *Rossby waves*. The hamiltonien h^r exhibited in (1.21) is a generalization of what is produced in [3]. Moreover, its domain of validity is proved to be the whole cotangent space $T^*(\mathbb{R}^2) \setminus \{0\}$. On the other hand, the detailed analysis of the corresponding bicharacteristics, and in particular the reasons why it is possible to find *trapped trajectories*, is performed in [2].

The Part 3.4 aims to make sure that the requirements (HN1), \dots , (HN4) and (HN5) are satisfied. It means, in the context of (1.19), a precise study of harmonics. In comparison with what is obtained in [2], the present analysis allows to catch more nonlinearity. The size of the oscillating parts can be larger by a factor $\varepsilon^{-1-\eta}$, with $\eta > 0$.

Another specificity of the current text is that it includes a discussion about *quasilinear transparencies*. In the Paragraph 3.5, we show (see Lemma 3.1) that the obstructions to take arbitrary large times \mathcal{T} in Theorem 4 are not coming from the nonlinearities but only from the restriction (1.20) or from the possible formation of caustics when solving the eikonal equation (1.12).

1.5. About the propagation of electromagnetic waves. Our approach can bring useful information in other physical contexts. For instance, it can be used to explore questions linked with *light propagation* in inhomogeneous media and with *lasers* in a plasma [17, 24, 23]. In the Paragraph 4, as an illustration, we explain the case of *ferromagnetism*.

2. THE WKB ANALYSIS.

This Section 2 is devoted to the proof of Theorems 1 and 3. Parts 2.1 up to 2.6 explain the construction of the approximate solutions u_ε^a . Part 2.7 deals with nonlinear stability issues.

2.1. Assumptions and notations. We are interested in the system (1.1) for $t \simeq \varepsilon^{-1}$. For the sake of simplicity, we will manipulate matrices S_j and Λ which do not depend on the variables t and εx . The influence of t and εx could be incorporated in the analysis but it would induce technicalities which are not central in what follows. For this reason, we deal only with $x \in \mathbb{R}^d$ and only with the slow variable $\tau \in \mathbb{R}_+$.

Reasoning directly with $\tau \in \mathbb{R}_+$, we are faced with a system (containing singularities both in ε^{-1} and ε^{-2}):

$$(2.1) \quad \begin{aligned} QS(\varepsilon, \tau, x, u; \partial) u &= S_0(\varepsilon, \tau, x, u) \partial_\tau u + \varepsilon^{-1} \sum_{j=1}^d S_j(\varepsilon, \tau, x, u) \partial_j u \\ &+ \varepsilon^{-2} \Lambda(\varepsilon, \tau, x) u - \varepsilon^{-2} F(\varepsilon, \tau, x, u) = 0. \end{aligned}$$

We denote by T^* the cotangent space in x , and its elements by (x, ξ) . The null section of T^* is denoted by T_0^* , and its complement is $T_{\setminus 0}^*$. Thus

$$T_0^* := \{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d; \xi = 0\}, \quad T_{\setminus 0}^* := \{(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d; \xi \neq 0\}.$$

To shorten the notation, we define, for $k \in \{0, 1\}$, the following differential operators in x , using the notation introduced in (H1):

$$\begin{aligned} S^k(\tau, x; \partial_x) &:= \sum_{j=1}^d S_j^k(\tau, x) \partial_j, & \partial_j &:= \frac{\partial}{\partial x_j}, \quad j \in \{1, \dots, d\}, \\ P^k(\tau, x; \partial_x) &:= S^k(\tau, x; \partial_x) + \Lambda^k(\tau, x), & \partial_0 &:= \partial_\tau \equiv \frac{\partial}{\partial \tau} \end{aligned}$$

as well as their symbols

$$\begin{aligned} \mathcal{A}_n \ni S^k(\tau, x; \xi) &:= \sum_{j=1}^d i \xi_j S_j^k(\tau, x), & (\tau, x, \xi) &\in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d, \\ \mathcal{A}_n \ni P^k(\tau, x; \xi) &:= \sum_{j=1}^d i \xi_j S_j^k(\tau, x) + \Lambda^k(\tau, x). \end{aligned}$$

With that notation, and expanding (2.1) in powers of ε , we find

$$\begin{aligned} QS(\varepsilon, \tau, x, u; \partial) u &\equiv \frac{1}{\varepsilon^2} \left\{ \Lambda^0(\tau, x) u - F^0(\tau, x) \right\} \\ &+ \frac{1}{\varepsilon} \left\{ S^0(\tau, x; \partial_x) u + \Lambda^1(\tau, x) u - F^1(\tau, x) \right\} \\ &+ \varepsilon^0 \left\{ S_0^0(\tau, x) \partial_\tau u + S^1(\tau, x; \partial_x) u + \Lambda^2(\varepsilon, \tau, x) u - F^2(\varepsilon, \tau, x, u) \right\} \\ &+ \varepsilon \left\{ S_0^1(\tau, x) \partial_\tau u + \varepsilon S_0^2(\varepsilon, \tau, x, u) \partial_\tau u + \sum_{j=1}^d S_j^2(\varepsilon, \tau, x, u) \partial_j u \right\} = 0. \end{aligned}$$

The three main constraints to keep in mind are the following:

- The matrix S^0 is divergence free, in the sense that

$$(H3) \quad \operatorname{div} S^0 := \sum_{j=1}^d (\partial_j S_j^0)(\tau, x) = 0, \quad \forall (\tau, x) \in [0, T] \times \mathbb{R}^d.$$

This assumption ensures that the differential operator $S^0(\tau, x; \partial_x)$ is anti-selfadjoint, which guarantees conservation laws.

- There is a positive integer $p \in \mathbb{N}^*$ such that

$$(H4) \quad \dim (\ker P^0(\tau, x; \xi)) = p, \quad \forall (\tau, x, \xi) \in [0, T] \times T_{\setminus 0}^*.$$

For any point $(\tau, x, \xi) \in [0, T] \times T^*$, we denote by $\Pi(\tau, x; \xi)$ the unitary projector onto the kernel of $P^0(\tau, x; \xi)$. We also denote by $|\Pi\rangle_0$ the operator $|\Pi\rangle_0 := \Pi(\tau, x; 0)$, see (1.18) for the notation. Then $\Pi \circ \Pi \equiv \Pi$ and one has

$$(2.2) \quad (P^0 \Pi)(\tau, x; \xi) = (\Pi P^0)(\tau, x; \xi) = 0, \quad \forall (\tau, x, \xi) \in [0, T] \times T^*.$$

One notices also that

$$(2.3) \quad \Pi(\tau, x; \xi) \equiv {}^t \bar{\Pi}(\tau, x; \xi) \in \mathcal{S}_n, \quad \forall (\tau, x, \xi) \in [0, T] \times T^*.$$

Assumption (H4) implies that Π is smooth on $T_{\setminus 0}^*$, which will be useful in the discussion, when it comes to differentiating in x and ξ .

In the special case $p = 1$, assumption (H4) may be stated equivalently in the following way. There is a C^∞ vector field $X(\tau, x; \xi)$ on $[0, T] \times T_{\setminus 0}^*$ with values in $\mathbb{C}^n \setminus \{0\}$, such that for any $(\tau, x, \xi) \in [0, T] \times T_{\setminus 0}^*$, one has

$$P^0(\tau, x; \xi) X(\tau, x; \xi) = 0, \quad \ker P^0(\tau, x; \xi) \equiv \text{Vect} \langle X(\tau, x; \xi) \rangle.$$

Then, one deduces the explicit formula for the projector Π :

$$(2.4) \quad \Pi(\tau, x; \xi) U := \frac{{}^t \bar{X}(\tau, x; \xi) \cdot U}{|X(\tau, x; \xi)|^2} X(\tau, x; \xi), \quad |X|^2 := {}^t \bar{X} \cdot X.$$

Assumption (H4) specifies (H2), except at points (τ, x, ξ) with $\xi = 0$. Since the map $(\tau, x, \xi) \mapsto \dim (\ker P^0(\tau, x; \xi))$ is upper semi-continuous, it is natural to supplement (H4) with:

- There is an integer $p_0 \in \mathbb{N}$ satisfying $p_0 \geq p$ and

$$(HN1) \quad \dim (\ker P^0(\tau, x; 0)) = p_0, \quad \forall (\tau, x) \in [0, T] \times \mathbb{R}^d.$$

The reason why the assumptions are different for points of $T_{\setminus 0}^*$ and of T_0^* is due to physical applications, in which it often happens that $p_0 > p$. As usual, one requires above constant multiplicity. That is a serious constraint but it is inevitable if one seeks WKB expansions at any order in ε .

If $p = p_0$, the Assumption (HN1) is nothing but the continuation of (H4) to points (τ, x, ξ) of $[0, T] \times T_0^*$. This allows to extend the regularity of the projector $\Pi(\tau, x; \xi)$ to the whole cotangent space $[0, T] \times T^*$. For $\xi = 0$, one has $P^0(\tau, x; 0) = \Lambda^0(\tau, x)$ and the constraint (2.2) becomes

$$(2.5) \quad (\Lambda^0 |\Pi\rangle_0)(\tau, x) = (|\Pi\rangle_0 \Lambda^0)(\tau, x) = 0, \quad \forall (\tau, x) \in [0, T] \times \mathbb{R}^d.$$

For monophasic, *linear* geometrical optics, the phase φ is *non stationary* in x and the oscillations are *pure* (carried by harmonics of the type $m\varphi$ with $m \neq 0$ fixed), so one does not need to consider the set T_0^* ; hence (HN1) is not relevant. However, in a nonlinear situation that is no longer the case. That accounts for the denomination (HN1) for that assumption.

- Looking at the eigenvalue $\lambda(\tau, x, \xi) \equiv 0$, we can observe that the solutions of (1.7) are simply $X(t, x, \xi) = x$ for all $t \in \mathbb{R}_+$. Thus, the constraint (H4) can be viewed as the strong form of a *capture condition* on the rays. The speed of propagation associated with (2.1) is of the order $O(\varepsilon^{-1})$. But waves which are approximately polarized in the kernel of P^0 (like those on which we will focus) remain located at a fixed distance of X , as long as $\tau \simeq 1$, and this uniformly with respect to the parameter $\varepsilon \in]0, 1]$.

In this article, we focus on such waves. Thus, we will be able to localize the discussion on a set $\{(\tau, x) ; |x| + c\tau \leq C\}$ for constants c and C independent of the parameter $\varepsilon \in \mathbb{R}_+$. The following assumption is therefore natural and does not reduce generality:

(H5) $\left[\begin{array}{l} \text{There is a compact set } \mathcal{K} \subset \mathbb{R}^d \text{ such that the fields } S_j, \Lambda \text{ and} \\ F \text{ evaluated at } (\varepsilon, \tau, x, u) \text{ are constant if } \tau \in [0, T] \text{ and } x \notin \mathcal{K}. \end{array} \right.$

In the following, we shall sometimes denote $\partial_0 := \partial_\tau$. Profiles $U \in L^2(\mathbb{T})$ are decomposed into Fourier series

$$U(\theta) = \sum_{m \in \mathbb{Z}} U_m e^{im\theta}, \quad U_m \equiv \mathcal{F}_m(U) := \int_{\mathbb{T}} U(\theta) e^{-im\theta} d\theta.$$

Given $m \in \mathbb{Z}$ and some application $f(\tau, x; \xi)$ defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, we use the shorthand notation

$$|f\rangle_m(\tau, x) := f(\tau, x; m \nabla \varphi(\tau, x)), \quad m \in \mathbb{Z},$$

which simply amounts to replacing everywhere the vector ξ by $m \nabla \varphi$. For example, $|\Pi\rangle_0(\tau, x)$ is the unitary projector on the kernel of $\Lambda^0(\tau, x)$ and replacing ξ by $m \nabla \varphi$ in condition (2.2) yields

$$(|P^0\rangle_m |\Pi\rangle_m)(\tau, x) = (|\Pi\rangle_m |P^0\rangle_m)(\tau, x) = 0, \quad \forall (\tau, x) \in [0, T] \times \mathbb{R}^d.$$

The symbol $|\cdot\rangle$ (without the index m) is to designate the action on $L^2(\mathbb{T})$ associated with the Fourier multipliers $|\cdot\rangle_m$ with $m \in \mathbb{Z}$. For instance

$$(|\Pi\rangle U)(\tau, x, \theta) := \sum_{m \in \mathbb{Z}} |\Pi\rangle_m(\tau, x) (\mathcal{F}_m U)(\tau, x) e^{im\theta}.$$

Note that the norm of $|\Pi\rangle_m$ is bounded by 1 for all m , so the operator $|\Pi\rangle$ is defined and continuous on $H^s(\mathbb{T}; \mathbb{C}^n)$ for all $s \in \mathbb{R}$. We define

$$Q(\tau, x; \xi) := I - \Pi(\tau, x; \xi), \quad {}^t \bar{Q} \equiv Q, \quad Q \circ Q \equiv Q.$$

The linear map $P^0(\tau, x; \xi)$ is one-to-one on the vector space $Q(\tau, x; \xi)(\mathbb{C}^n)$. Therefore, it has a partial inverse (right and left) denoted by $(Q P^0 Q)^{-1}$ and characterized by the relations

$$(Q P^0 Q)^{-1} P^0 \equiv P^0 (Q P^0 Q)^{-1} \equiv Q.$$

2.2. The hierarchy of equations. To simplify, we will work in the whole space \mathbb{R}^d and postpone the discussion about the localization of the solutions. We look for approximate solutions $u_\varepsilon^a(\tau, x)$ to (2.1) as monophasic oscillations like in (1.10), where the phase $\varphi(\tau, x)$ is smooth with bounded derivatives. More precisely, we impose

$$\varphi \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}), \quad \nabla \varphi := (\partial_1 \varphi, \dots, \partial_d \varphi) \in \mathcal{C}_b^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}).$$

We fix $\varphi(0, \cdot)$ by prescribing the initial data $\varphi_0 \equiv \varphi(0, \cdot)$ with φ_0 as in (1.8). We are interested in *non stationary* phases in the sense that

$$(2.6) \quad \exists (c, C) \in (\mathbb{R}_+^*)^2; \quad c \leq |\nabla \varphi(\tau, x)| \leq C, \quad \forall (\tau, x) \in [0, \mathcal{T}] \times \mathbb{R}^d.$$

We will denote by \mathbb{T} the torus \mathbb{R}/\mathbb{Z} , and elements of \mathbb{T} are denoted θ . The solutions u_ε of (2.1) are therefore sought under the form

$$(2.7) \quad u_\varepsilon(\tau, x) = U_\varepsilon\left(\tau, x, \frac{\varphi(\tau, x)}{\varepsilon}\right), \quad U_\varepsilon \in \mathcal{C}^\infty([0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n),$$

where the function $U_\varepsilon(\tau, x, \theta)$ is smooth on $\mathbb{R}_+ \times [0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{T}$. In particular, it may be Taylor-expanded in ε near $\varepsilon = 0$, under the form

$$U_\varepsilon(\tau, x, \theta) = \sum_{k=0}^N \varepsilon^k U^k(\tau, x, \theta) + O(\varepsilon^N), \quad N \gg 1.$$

Plugging the expansion (2.7) into (2.1) and re-ordering in terms of powers of ε yields a hierarchy of equations which starts at order ε^{-2} . One gets

$$\sum_{j=-2}^{+\infty} \varepsilon^j \Gamma^j(\tau, x; U^0, \dots, U^{j+2}) = 0.$$

Introduce the following differential operators

$$\begin{aligned} D^{-2}(\tau, x; \partial_\theta) &:= \sum_{j=1}^d \partial_j \varphi S_j^0 \partial_\theta + \Lambda^0, \\ D^{-1}(\tau, x; \partial_x, \partial_\theta) &:= \partial_\tau \varphi S_0^0 \partial_\theta + \sum_{j=1}^d \partial_j \varphi S_j^1 \partial_\theta + \Lambda^1 + \sum_{j=1}^d S_j^0 \partial_j, \\ D^0(\tau, x; \partial_\tau, \partial_x, \partial_\theta) &:= \partial_\tau \varphi S_0^1 \partial_\theta + S_0^0 \partial_\tau + \sum_{j=1}^d S_j^1 \partial_j, \end{aligned}$$

as well as the nonlinear expression

$$(2.8) \quad \begin{aligned} \mathfrak{NL}(\tau, x, U) &:= \sum_{j=1}^d \partial_j \varphi S_j^2(0, \tau, x, U) \partial_\theta U \\ &\quad + \Lambda^2(0, \tau, x, U) U - F^2(0, \tau, x, U). \end{aligned}$$

Easy computations allow to find $\Gamma^{-2} = D^{-2} U^0 - F^0$ and

$$\begin{aligned} \Gamma^{-1} &= D^{-2} U^1 + D^{-1} U^0 - F^1, \\ \Gamma^0 &= D^{-2} U^2 + D^{-1} U^1 + D^0 U^0 + \mathfrak{NL}(\tau, x, U^0). \end{aligned}$$

For $k \geq 1$, one obtains Γ^k by linearizing the nonlinear terms in Γ^0 . The equations are therefore of the same type as in the case of Γ^0 , up to a source term, denoted \mathfrak{B}^k , which only depends on the profiles U^j for j going from 0 to $k-1$. Some computations allow to deduce that

$$\begin{aligned} \Gamma^k &= D^{-2} U^{k+2} + D^{-1} U^{k+1} + D^0 U^k \\ &\quad + [(U^k \cdot \nabla_u) \mathfrak{NL}](\tau, x, U^0) + \mathfrak{B}^k(\tau, x, U^0, \dots, U^{k-1}). \end{aligned}$$

Let us now describe briefly the strategy. To obtain an approximate solution

$$U_\varepsilon^a(\tau, x, \theta) = \sum_{k=0}^{N+1} \varepsilon^k U^k(\tau, x, \theta), \quad N \gg 1$$

at order ε^N , in the sense that

$$QS(\varepsilon, \tau, x, u_\varepsilon^a; \partial) u_\varepsilon^a = O(\varepsilon^N), \quad u_\varepsilon^a(\tau, x) = U_\varepsilon^a\left(\tau, x, \frac{\varphi(\tau, x)}{\varepsilon}\right),$$

it is enough to solve the system

$$(2.9) \quad \Gamma^j(\tau, x; U^0, \dots, U^{j+2}) \equiv 0, \quad -2 \leq j \leq N-1.$$

We shall deal with the constraints $\Gamma^j \equiv 0$ one after the other (for j going from -2 to $N-1$). The cases $j = -2$ to $j = 0$ are dealt with in detail in Parts 2.3 to 2.5 respectively. This gives an algorithm providing successively pieces of the U^k , as presented in the induction property (\mathcal{P}_k) in Part 2.6. In the end one recovers all the U^k for $k \leq N+1$.

2.3. The preliminary polarization condition. The equation $\Gamma^{-2} \equiv 0$ is the same as

$$(2.10) \quad |P^0\rangle_m U_m^0 \equiv 0, \quad \forall m \in \mathbb{Z}^*$$

combined with (for $m = 0$):

$$(2.11) \quad |P^0\rangle_0 U_0^0 - F^0 \equiv \Lambda^0 U_0^0 - F^0 \equiv 0.$$

Composing (2.11) on the left with $|\Pi\rangle_0$, one obtains the necessary condition

$$(H6) \quad |\Pi\rangle_0 F^0 \equiv 0.$$

The polarization condition (H6) is sufficient to solve (2.11). To sum up:

Proposition 2.1. *Under the assumptions which are given in Theorem 1, the equation $\Gamma^{-2} \equiv 0$ reduces to the following constraint:*

$$(2.12) \quad U^0 = |\Pi\rangle U^0 + G^0 \quad \text{with} \quad G^0 := |(QP^0Q)^{-1}\rangle_0 F^0.$$

At this stage, the part $|Q\rangle U^0 \equiv G^0$ is entirely determined, while $|\Pi\rangle U^0$ may yet be chosen arbitrary. For $m \in \mathbb{Z}_*$, assumption (H4) and (2.6) yield

$$\dim(\ker |P^0\rangle_m(\tau, x)) = p, \quad \forall (\tau, x) \in [0, \mathcal{T}] \times \mathbb{R}^d.$$

One has p degrees of freedom on U_m^0 . In particular, one can demand that the oscillation u_ε be non trivial, by choosing a coefficient U_1^0 in such a way that

$$(2.13) \quad U_1^0 \equiv \mathcal{F}_1(U^0) \equiv |\Pi\rangle_1 U_1^0 \neq 0.$$

In the case when $p = 1$, using (2.4) one finds that for $m \in \mathbb{Z}^*$,

$$(2.14) \quad U_m^0(\tau, x) = u_m^0(\tau, x) |X\rangle_m(\tau, x), \quad u_m^0 \in \mathcal{C}^\infty([0, \mathcal{T}] \times \mathbb{R}^d; \mathbb{C})$$

while for $m = 0$, one must have

$$(2.15) \quad U_0^0 = u_0^0 |X\rangle_0 + G^0, \quad u_0^0 \in \mathcal{C}^\infty([0, \mathcal{T}] \times \mathbb{R}^d; \mathbb{C}).$$

The restriction (2.13) is guaranteed as soon as $u_1^0(\tau, x) \neq 0$.

2.4. The eikonal equation. In this Section 2.4, we will determine the phase φ , a part of $|\Pi\rangle U^0$, and a part of $|Q\rangle U^1$.

2.4.1. *Introduction.* This Paragraph 2.4.1 is devoted to the study of equation $\Gamma^{-1} \equiv 0$. After proving some preparatory algebraic results, we begin the analysis by deducing from the equation $\Gamma^{-1} \equiv 0$ an equation on the phase. The result is summarized below.

Proposition 2.2. *Under the assumptions which are given in Theorem 1, the equation $\Gamma^{-1} \equiv 0$ implies that the phase satisfies the eikonal equation (1.12) where the hamiltonian $h(\tau, x, \xi)$ is defined on T^* and is an eigenvalue of the matrix $H(\tau, x, \xi)$ defined in (2.31) in terms of Λ^0 , Λ^1 , S_j^0 with $j \geq 0$ and S_j^1 with $j \geq 1$. Up to shrinking $\mathcal{T} \in \mathbb{R}_+^*$, we can obtain (2.6).*

Finally additional information on U^0 and U^1 is also deduced. The statement requires some additional notation so we postpone it to Paragraph 2.4.7.

2.4.2. *A preliminary algebraic computation.* The following lemma, which is classical in this context, will be very useful in the following.

Lemma 2.1. *One has*

$$(2.16) \quad |\Pi\rangle_m S_j^0 |\Pi\rangle_m \equiv 0, \quad \forall (j, m) \in \{1, \dots, d\} \times \mathbb{Z}^*.$$

Proof. Differentiating the relation (2.2) with respect to the direction ξ_j gives, for any (τ, x, ξ) in $[0, \mathcal{T}] \times T_{\setminus 0}^*$:

$$(2.17) \quad i S_j^0(\tau, x) \Pi(\tau, x; \xi) + P^0(\tau, x; \xi) (\partial_{\xi_j} \Pi)(\tau, x; \xi) = 0.$$

One then applies the operator Π to that equation and one uses again (2.2), to find

$$(2.18) \quad \Pi(\tau, x; \xi) S_j^0(\tau, x) \Pi(\tau, x; \xi) = 0, \quad \forall (\tau, x, \xi) \in [0, \mathcal{T}] \times T_{\setminus 0}^*.$$

Finally noticing that $m \nabla \varphi(\tau, x)$ is nonzero due to (1.8) and to the fact that $m \neq 0$, we get directly (2.16). \square

This result 2.1 is due to the fact that the group velocity $(\nabla_{\xi} \lambda)(\tau, x; \xi)$ which is associated with a trivial eigenvalue $\lambda \equiv 0$ is simply zero.

Note that $|\Pi\rangle_0$ is defined using only Λ^0 . Since the matrices S_j^0 and Λ^0 have been chosen independently, there is no reason for (2.16) to be satisfied when $m = 0$. This leads to the following supplementary assumption :

$$(HN2) \quad |\Pi\rangle_0 S_j^0 |\Pi\rangle_0 \equiv 0, \quad \forall j \in \{1, \dots, d\}.$$

The mode $m = 0$ is not sollicited in a linear situation. Thus, the condition (HN2) is useful only in a nonlinear framewok.

Remark 2.1. *The condition $p = p_0$ clearly relates the behaviours of P^0 on T_0^* and $T_{\setminus 0}^*$. In particular, the map $\Pi : T^* \longrightarrow \mathcal{S}_n$ becomes continuous. Taking the limit $\xi \rightarrow 0$ in (2.18) gives*

$$\lim_{|\xi| \rightarrow 0} \Pi(\tau, x; \xi) S_j^0(\tau, x) \Pi(\tau, x; \xi) = (|\Pi\rangle_0 S_j^0 |\Pi\rangle_0)(\tau, x) = 0.$$

This is precisely (HN2).

2.4.3. Some polarization constraints. Since the constraint $\Gamma^{-1} \equiv 0$ is linear, it may be decomposed into conditions on the Fourier coefficients Γ_m^{-1} . For every $m \in \mathbb{Z}^*$, one gets

$$(2.19) \quad |P^0\rangle_m U_m^1 + i m \partial_\tau \varphi S_0^0 U_m^0 + |P^1\rangle_m U_m^0 + \sum_{j=1}^d S_j^0 \partial_j U_m^0 = 0$$

while for $m = 0$ one has

$$(2.20) \quad |P^0\rangle_0 U_0^1 + |P^1\rangle_0 U_0^0 + \sum_{j=1}^d S_j^0 \partial_j U_0^0 = F^1 \equiv \mathcal{F}_0 F^1.$$

Let us introduce the following matrix, for each point $(\tau, x, \xi) \in [0, T] \times T^*$:

$$(2.21) \quad G(\tau, x; \xi) := i \sum_{j=1}^d [\Pi S_j^0 (\partial_j \Pi) \Pi](\tau, x; \xi).$$

The application $G(\tau, x; \xi)$ inherits properties which are stated below and which are proved in the Appendix 5, paragraph 5.2.

Lemma 2.2. *For all $(\tau, x, \xi) \in [0, T] \times T^*$, the operator $G(\tau, x; \xi)$ can be identified with the action of some hermitian matrix which is of size $p_0 \times p_0$ when $\xi \in T_0^*$ and of size $p \times p$ when $\xi \in T_{\setminus 0}^*$. The application G is of class C^∞ on $[0, T] \times T_{\setminus 0}^*$ with values in \mathcal{S}_n .*

Suppose moreover that

$$(2.22) \quad \textit{The fields of matrices } S_j \textit{ and } \Lambda \textit{ are real-valued.}$$

Then, the function $G \in C^\infty([0, T] \times T_{\setminus 0}^; \mathcal{S}_n)$ satisfies*

$$(2.23) \quad G(\tau, x; -\xi) = -\bar{G}(\tau, x; \xi), \quad \forall (\tau, x, \xi) \in [0, T] \times T^*.$$

Let us also define the matrix

$$(2.24) \quad \mathfrak{G}_m := \sum_{j=1}^d |\Pi\rangle_m S_j^0 (\partial_j |\Pi\rangle_m) |\Pi\rangle_m, \quad m \in \mathbb{Z}.$$

Now, we can come back to the study of (2.19) and (2.20).

• **The case $m = 0$.** We recall (2.12) which yields

$$\partial_j U_0^0 = |\Pi\rangle_0 \partial_j U_0^0 + (\partial_j |\Pi\rangle_0) |\Pi\rangle_0 U_0^0 + |Q\rangle_0 \partial_j G^0.$$

Compose (2.20) on the left by $|\Pi\rangle_0$. Then, use (HN2) in order to get the polarization constraint

$$(2.25) \quad [|\Pi P^1 \Pi\rangle_0 + \mathfrak{G}_0] U_0^0 = |\Pi\rangle_0 F^1 - |\Pi P^1\rangle_0 G^0 - \sum_{j=1}^d |\Pi\rangle_0 S_j^0 |Q\rangle_0 \partial_j G^0. \quad \bullet$$

• **The case $m \in \mathbb{Z}^*$.** Using (2.12), one has

$$\partial_j U_m^0 = \partial_j (|\Pi\rangle_m U_m^0) = |\Pi\rangle_m \partial_j U_m^0 + (\partial_j |\Pi\rangle_m) |\Pi\rangle_m U_m^0.$$

To get rid of U_m^1 in (2.19) we can apply (right and left) the operator $|\Pi\rangle_m$. Using Lemma 2.1, we find the polarization constraint

$$(2.26) \quad [i m \partial_\tau \varphi |\Pi S_0^0 \Pi\rangle_m + |\Pi P^1 \Pi\rangle_m + \mathfrak{G}_m] U_m^0 = 0. \quad \bullet$$

Now, let us study \mathfrak{G}_m in more detail. Recall that

$$(2.27) \quad \partial_j |\Pi\rangle_m(\tau, x) = |\partial_j \Pi\rangle_m(\tau, x) + \sum_{k=1}^d m \partial_{jk}^2 \varphi(\tau, x) |\partial_{\xi_k} \Pi\rangle_m(\tau, x).$$

One sees, in the formula (2.24) defining \mathfrak{G}_m where $\partial_j |\Pi\rangle_m$ is replaced as indicated in (2.27), the quantities $|\partial_j \Pi\rangle_m$ for $j \in \{1, \dots, d\}$, which would not appear if the matrices Λ^0 and S_j^0 were *constant*. One also notices in (2.27) the presence of second order derivatives of φ , which would not appear if the phase φ was *linear*. Under conditions *i*) and *ii*) of the Introduction, those contributions would therefore disappear, and we would simply have to deal with $\mathfrak{G}_m \equiv 0$ for all $m \in \mathbb{Z}$.

Due to (2.27), one has $\mathfrak{G}_0 \equiv -i |G\rangle_0$ with G as in (2.21). When defining \mathfrak{G}_0 , the contributions $\partial_{jk}^2 \varphi(\tau, x)$, which are multiplied by the factor $m = 0$, play no role. Although that is not the case at first sight for \mathfrak{G}_m when $m \in \mathbb{Z}^*$, it turns out that they also vanish. This fact is pointed out in the next statement, where it appears that \mathfrak{G}_m only depends on $\nabla \varphi(\tau, x)$, and can easily be deduced from G . As the proof of that statement (which relies on algebraic computations) is rather technical, we postpone it to the appendix 5, paragraph 5.3.

Lemma 2.3. *Consider a smooth phase φ satisfying (1.8). Then*

$$(2.28) \quad i \mathfrak{G}_m \equiv i \sum_{j=1}^d |\Pi\rangle_m S_j^0 (\partial_j |\Pi\rangle_m) |\Pi\rangle_m \equiv |G\rangle_m, \quad \forall m \in \mathbb{Z}$$

where G is defined in (2.21).

2.4.4. *Long-time hamiltonians, and the eikonal equation.* In this section we shall concentrate on the equation (2.26) in the case when $m = 1$. This will allow to deduce an equation on the phase φ . Lemma 2.3 implies that one is considering the equation

$$(2.29) \quad \left\{ i \partial_\tau \varphi |\Pi S_0^0 \Pi\rangle_1 + |\Pi (P^1 - iG) \Pi\rangle_1 \right\} U_1^0 = 0.$$

In view of (1.2), the matrix $\Pi S_0^0 \Pi$ is positive definite on $\Pi(\mathbb{C}^n)$. Therefore

$$(2.30) \quad \exists M \in \mathcal{S}_n; \quad M \equiv \Pi M \Pi \equiv M^*, \quad M^* M = \Pi S_0^0 \Pi,$$

and one has

$$\exists c \in \mathbb{R}_+^*; \quad (M - c \Pi)(\tau, x; \xi) \geq 0, \quad \forall (\tau, x, \xi) \in [0, \mathcal{T}] \times T^*.$$

In particular, the map M is invertible as an operator from $\Pi(\mathbb{C}^n)$ to itself. For $(\tau, x, \xi) \in T^*$, we can introduce the matrix

$$(2.31) \quad H := (\Pi M \Pi)^{-1} \Pi (G + iP^1) \Pi (\Pi M \Pi)^{-1} = H^* \equiv \Pi H \Pi.$$

In view of the definition of H and due to the Lemma 2.2, the matrix H is hermitian. Hence, it is diagonalizable with real eigenvalues. Let us denote by $\text{spec} H \subset \mathbb{R}$ the set of its eigenvalues. We assume that

$$(H7) \quad \left[\begin{array}{l} \text{There is an eigenvalue } h \text{ of } H \text{ whose multiplicity} \\ \mu(\tau, x, \xi) \equiv \mu \in \mathbb{N}^* \text{ does not depend on } (\tau, x, \xi) \in [0, \mathcal{T}] \times T_{\setminus 0}^*. \end{array} \right.$$

From now on, we assume (H7) and we select accordingly some eigenvalue h of H which is thus defined on $[0, \mathcal{T}] \times T_{\setminus 0}^*$. For $(\tau, x, \xi) \in [0, \mathcal{T}] \times T_{\setminus 0}^*$, we denote by $\Pi^h(\tau, x; \xi)$ the unitary projector onto the kernel of $(H - hI)(\tau, x; \xi)$. For $(\tau, x, \xi) \in [0, \mathcal{T}] \times T_{\setminus 0}^*$, we denote by $Q^h(\tau, x; \xi)$ the unitary projector onto the kernel of $(\Pi - \Pi^h)(\tau, x; \xi)$.

The Assumption (H7) implies that the maps h and Π^h are \mathcal{C}^∞ on $[0, \mathcal{T}] \times T_{\setminus 0}^*$. The field $H(\tau, \cdot)$ is in fact defined on the whole of T^* . It is continuous on $T_{\setminus 0}^*$. However, when $p_0 > p$, it is possible that $H(\tau, \cdot)$ is not continuous on T^* since the behaviour of $\Pi(\tau, \cdot)$ near T_0^* is not known.

The unitary projector onto the kernel of $|H\rangle_0(\tau, x) \equiv H(\tau, x; 0)$ is denoted by $\Pi_0^h(\tau, x)$. The spectrum of $H(\tau, x; 0)$ may have nothing to do with that of the matrices $H(\tau, x; \xi)$ for $\xi \neq 0$. This is the reason why the function h has not been defined on T_0^* . However, when $p_0 = p$, both maps H and h may be continuously extended from $T_{\setminus 0}^*$ to T^* , in which case $h(\tau, x; 0)$ may be defined without ambiguity. Before going further, we put aside the following result which will be proved in the Appendix 5, paragraph 5.4

Lemma 2.4. *Suppose (2.22) and that $p_0 = p = 1$. Then, the function h is continuous on $[0, \mathcal{T}] \times T^*$ and it is odd with respect to the variable ξ . In particular, it satisfies*

$$(2.32) \quad h(\tau, x; 0) = 0, \quad \forall (\tau, x) \in [0, \mathcal{T}] \times \mathbb{R}^d.$$

From now on, the phase φ is required to satisfy the Cauchy problem (1.12) which has a smooth, \mathcal{C}^∞ solution locally in time. Up to shrinking $\mathcal{T} \in \mathbb{R}_+^*$, the function φ satisfies (2.6). The Proposition 2.2 is proved.

2.4.5. *The extra polarization condition.* Remark that equation (1.12) implies that (2.12) and (2.29) become

$$U_1^0 \equiv |\Pi\rangle_1 U_1^0, \quad |H - h\Pi\rangle_1 |M\rangle_1 U_1^0 \equiv 0.$$

Let us comment on those equations. In fast times $t \simeq 1$, one recovers the first polarization constraint, along $|\Pi\rangle_1(\mathbb{C}^n)$. To go beyond that time, up to the slow time $\tau \simeq 1$, a new, intermediate polarization is required, along $|\Pi^h\rangle_1(\mathbb{C}^n)$. To summarize, one has

$$(2.33) \quad U_1^0 \equiv |\Pi\rangle_1 U_1^0 \equiv |(\Pi M \Pi)^{-1} \Pi^h (\Pi M \Pi)\rangle_1 U_1^0.$$

Solving (2.26) therefore reduces to imposing (1.12) and (2.33). The scalar, nonlinear evolution equation (1.12) can be interpreted as an *eikonal equation* corresponding to a *long time* propagation ($\tau \simeq 1$) of oscillatory quantities of the type $e^{i\varphi(\tau,x)/\varepsilon}$, polarized according to (2.33).

One can consider the function $h(\tau, x; \xi)$ to be some *long-time hamiltonian* associated with the eigenvalue $\lambda \equiv 0$. There are, with this formulation, as many long-time hamiltonians as there are eigenvalues (counted without their multiplicity) in the spectrum of H . These are at most p .

When $p = 1$, The discussion is easier. The matrix H can be viewed as a scalar real-valued function and we can talk about **the** long-time hamiltonian. Besides, the assumption **(H7)** is necessarily verified (with $\tilde{p} = m = 1$).

When $p = 1$, we have simply

$$\Pi^h \equiv \Pi, \quad X^h \equiv X, \quad M \equiv ({}^t \bar{X} S_0^1 X)^{\frac{1}{2}} \in \mathbb{R}_+^*.$$

Let us define $Q^h := \Pi - \Pi^h$. Then, retain the following relations

$$I \equiv Q + Q^h + \Pi^h, \quad \Pi^h \circ \Pi \equiv \Pi \circ \Pi^h \equiv \Pi^h.$$

The application $(H - hI)(\tau, x; \xi)$ is linear and bijective if we look at it as acting on the vector space $Q^h(\tau, x; \xi)(\mathbb{C}^n)$. It has a partial (left and right) inverse which is denoted by $(Q^h (H - hI) Q^h)^{-1}$ and which is characterized through the identities

$$(Q^h (H - hI) Q^h)^{-1} (H - hI) \equiv (H - hI) (Q^h (H - hI) Q^h)^{-1} \equiv Q^h.$$

2.4.6. *Study of the harmonics.* Recall that the phase φ has been determined through (1.12). In the present dispersive context, the harmonics $m\varphi$ with $m \neq 1$ are not sure to be still solutions to (1.12). Nothing guarantees that

$$(2.34) \quad m \partial_\tau \varphi(\tau, x) = h(\tau, x; m \nabla \varphi(\tau, x)), \quad \forall (\tau, x) \in [0, \mathcal{T}] \times \mathbb{R}^d.$$

Let us define

$$(2.35) \quad \mathcal{HA} := \{0\} \cup \{m \in \mathbb{Z}^*; \text{relation (2.34) is satisfied}\}.$$

Due to (1.12), one has $1 \in \mathcal{HA}$. In (2.35), one imposes also $0 \in \mathcal{HA}$. This convention must be commented. Recall that $p_0 \geq p \geq 1$ meaning that $\det P^0(\tau, x; 0) = \det \Lambda^0(\tau, x) = 0$, implying that the trivial phase $\varphi \equiv 0$ is characteristic. Thus, it is natural to incorporate the mode $m = 0$ inside \mathcal{HA} . On the other hand, in the context of Lemma 2.4, the relation (2.34) is obvious for $m = 0$.

Since $U^0(0, \cdot)$ is real valued, assumption (2.13) requires that at time $\tau = 0$ both harmonics $m = -1$ and $m = 1$ have nontrivial contributions. If the matrices S_j and Λ are real valued, one expects that oscillations in $e^{-i\varphi/\varepsilon}$ and $e^{i\varphi/\varepsilon}$ will propagate and interact due to the nonlinearity of the equation. Generically, even if $U_0^0(0, \cdot) \equiv 0$, an average mode $U_0^0(\tau, \cdot) \not\equiv 0$ will therefore be produced for $\tau \in \mathbb{R}_+^*$. The question of the creation and propagation of that mode (and the others) is a delicate matter. To deal with it, we introduce the following assumptions:

(HN3) *There is a constant $c \in \mathbb{R}_+^*$ such that for every $m \in \mathcal{HA}$:*

$$c < \inf_{(\tau, x) \in [0, T] \times \mathbb{R}^d} \min_{h \neq \mu \in \text{spec } H} |(\mu - h)(\tau, x; m \nabla \varphi(\tau, x))|,$$

as well as:

(HN4) *There is a constant $c \in \mathbb{R}_+^*$ such that for every $m \notin \mathcal{HA}$:*

$$c < \inf_{(\tau, x) \in [0, T] \times \mathbb{R}^d} \min_{\mu \in \text{spec } H} |m \partial_\tau \varphi - \mu(\tau, x; m \nabla \varphi(\tau, x))|.$$

Remark 2.2. *The assumption (HN3) is necessarily verified when $p = 1$. Indeed, when $p = 1$, there is no spectral value $\mu \in \text{spec } H$ such that $\mu \neq h$. Therefore, there is nothing to check concerning (HN3).*

Remark 2.3. *Fix any $m \in \mathcal{HA}$. Due to (1.8) and (H5), there is a constant $c_m \in \mathbb{R}_+^*$ such that*

$$c_m < \inf_{(\tau, x) \in [0, T] \times \mathbb{R}^d} \min_{h \neq \mu \in \text{spec } H} |(\mu - h)(\tau, x; m \nabla \varphi(\tau, x))|.$$

Assumption (HN3) always holds when the cardinal of \mathcal{HA} is finite. If it is not, the problem lies for large values of $|m|$. Besides, one is certain to have (HN3) if there is an asymptotic spectral gap near h , in the sense that

$$0 < \inf_{(\tau, x) \in [0, T] \times \mathbb{R}^d} \min_{h \neq \mu \in \text{spec } H} \liminf_{|\xi| \rightarrow +\infty} |(\mu - h)(\tau, x; \xi)|.$$

Remark 2.4. *According to the condition (HN4), if the function $m\varphi$ is not totally characteristic along the mode h , then it can be characteristic at no point $(\tau, x) \in [0, T] \times \mathbb{R}^d$ and for no mode $\mu \in \text{spec } H$. Such a separation between characteristic and non characteristic harmonics is very classical in geometrical optics. This information will later play an important role when it comes to letting some operators act continuously in H^s .*

2.4.7. *Polarization constraints on U^0 and U^1 .* For $(\tau, x) \in [0, T] \times \mathbb{R}^d$, note $\Pi_0^h(\tau, x)$ or $|\Pi^h\rangle_0(\tau, x)$ the unitary projector onto the kernel of $|H\rangle_0(\tau, x)$. Define also $Q_0^h := |\Pi\rangle_0 - \Pi_0^h$ and the following action (as a formal series):

$$|(Q P^0 Q)^{-1}\rangle U := \sum_{m \in \mathbb{Z}} |(Q P^0 Q)^{-1}\rangle_m (\mathcal{F}_m U)(\tau, x) e^{im\theta}.$$

Introduce the projectors

$$\Pi_M^h := \Pi^h (\Pi M \Pi)^{-1}, \quad (\Pi_M^h)^* \equiv \Pi_M^{h*} = (\Pi M \Pi)^{-1} \Pi^h.$$

In this section, we shall prove the following proposition.

Proposition 2.3. *Under the assumptions of Theorem 1 and with the above notation, the equation $\Gamma^{-1} \equiv 0$ implies the following polarization constraints:*

- If $m \notin \mathcal{HA}$, then $U_m^0 \equiv 0$;
- If $m \in \mathcal{HA} \setminus \{0\}$, then $U_m^0 \equiv |\Pi\rangle_m U_m^0 \equiv |\Pi_M^{h*}\rangle (\Pi M \Pi)\rangle_m U_m^0$;
- If $m = 0$, then $|\Pi\rangle_0 U_0^0 \equiv |\Pi_M^{h*}\rangle_0 |\Pi M \Pi\rangle_0 U_0^0 + G^{0,h}$ with:

$$(2.36) \quad \begin{aligned} G^{0,h} &:= i |\Pi M \Pi\rangle_0^{-1} (Q_0^h |H\rangle_0 Q_0^h)^{-1} |\Pi M \Pi\rangle_0^{-1} \mathfrak{K}^1, \\ \mathfrak{K}^1 &:= F^1 - |P^1\rangle_0 G^0 - \sum_{j=1}^d S_j^0 |Q\rangle_0 \partial_j G^0. \end{aligned}$$

- And finally

$$(2.37) \quad |Q\rangle U^1 = |(Q P^0 Q)^{-1}\rangle \left\{ F^1 - \partial_\tau \varphi S_0^0 \partial_\theta U^0 - \Lambda^1 U^0 - \sum_{j=1}^d \partial_j \varphi S_j^1 \partial_\theta U^0 - \sum_{j=1}^d S_j^0 \partial_j U^0 \right\}.$$

Combining the Propositions 2.1 and 2.3, we can see that the mean value U_0^0 of the principal profile U^0 must be adjusted according to

$$(2.38) \quad U_0^0 \equiv |\Pi_M^{h*}\rangle_0 |\Pi M \Pi\rangle_0 U_0^0 + G^{0,h} + |(Q P^0 Q)^{-1}\rangle_0 F^0.$$

Proof. Let us go back to equation (2.26), which for the moment has only been studied in the case when $m = 1$. Using (2.12) and the previous notation (2.31), one gets

$$(2.39) \quad (|H\rangle_m - m \partial_\tau \varphi I) |\Pi M \Pi\rangle_m (|\Pi\rangle_m U_m^0) \equiv 0, \quad \forall m \in \mathbb{Z}^*.$$

If $m \notin \mathcal{HA}$, assumption (HN4) implies that the matrix $|H\rangle_m - m \partial_\tau \varphi I$ is invertible. In view of (2.12), the condition (2.39) reduces to:

$$(2.40) \quad U_m^0 \equiv |\Pi\rangle_m U_m^0 \equiv 0, \quad \forall m \notin \mathcal{HA}.$$

On the opposite if $m \in \mathcal{HA} \setminus \{0\}$, one gets $|H\rangle_m - m \partial_\tau \varphi I = |H - h I\rangle_m$ and (2.39) becomes the polarization condition

$$(2.41) \quad U_m^0 \equiv |(\Pi M \Pi)^{-1} \Pi^h (\Pi M \Pi)\rangle_m U_m^0, \quad \forall m \in \mathcal{HA} \setminus \{0\}.$$

The case $m = 0$ must be dealt with separately. Taking into account the notation (2.36), the relation (2.25) is the same as

$$|H\rangle_0 |\Pi M \Pi\rangle_0 U_0^0 = i (|\Pi M \Pi\rangle_0)^{-1} |\Pi\rangle_0 \mathfrak{K}^1.$$

This enforces the compatibility condition

$$(H8) \quad \Pi_0^h (|\Pi M \Pi\rangle_0)^{-1} \left\{ F^1 - |P^1\rangle_0 G^0 - \sum_{j=1}^d S_j^0 |Q\rangle_0 \partial_j G^0 \right\} \equiv 0.$$

Under condition (C1), one has the expected result on U_0^0 . The remaining relation (2.37) comes from studying the equation $|Q\rangle \Gamma^{-1} \equiv 0$.

□

2.4.8. *Smoothness of the profiles.* More is needed than just identifying the coefficients $|Q\rangle_m U_m^1$ through (2.37). To complete the analysis, one needs to give a meaning in $H^s([0, T] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$ to the profiles $U^j(t, x, \theta)$, and this requires understanding how the various operators introduced in this construction act on H^s . To deal with $|(Q P^0 Q)^{-1}\rangle$, we introduce the following assumption:

$$(HN5) \quad \sup_{m \in \mathbb{Z}} \| |(Q P^0 Q)^{-1}\rangle_m \|_{H^s([0, T] \times \mathbb{R}^d)} < \infty, \quad \forall s \in \mathbb{R},$$

which is enough to ensure the boundedness of the linear map

$$|(Q P^0 Q)^{-1}\rangle : H^s([0, T] \times \mathbb{R}^d \times \mathbb{T}) \rightarrow H^s([0, T] \times \mathbb{R}^d \times \mathbb{T}).$$

Remark 2.5. *Assumption (HN5) amounts to the same thing as requiring the existence of a constant $c \in \mathbb{R}_+^*$ such that for every $m \in \mathbb{Z}$:*

$$(2.42) \quad c < \inf_{(\tau, x) \in [0, T] \times \mathbb{R}^d} \min_{0 \neq \mu \in \text{spec } P^0} |\mu(\tau, x; m \nabla \varphi(\tau, x))|.$$

For $m \in \mathbb{Z}$ fixed, the corresponding minoration with a constant $c_m \in \mathbb{R}_+^*$ is a consequence of (H5), (H4) and (HN1). Thus, the problem lies for large values of $|m|$, where (2.42) may be difficult to check.

Remark 2.6. *We can substitute (HN5) to the more restrictive assumption*

$$(2.43) \quad \dim(\ker \tilde{P}^0(\tau, x; \tilde{\xi})) = p, \quad \forall (\tau, x, \tilde{\xi}) \in [0, T] \times T_{\setminus 0}^* \times \mathbb{R},$$

where $\tilde{P}^0(\tau, x; \tilde{\xi}) := \sum_{j=1}^d i \xi_j S_j^0(\tau, x) + \xi_{d+1} \Lambda^0(\tau, x)$ and $\tilde{\xi} := (\xi, \xi_{d+1})$.

The condition (2.43) is clearly an extension of (H3). It gives additional information on the structure of $|\xi|^{-1} P^0(\tau, x; \xi)$ when $|\xi| \rightarrow +\infty$. Define $\tilde{\mu}(\tau, x; \tilde{\xi})$ the eigenvalues of $\tilde{P}^0(\tau, x; \tilde{\xi})$. One has (for $m \in \mathbb{Z}^*$)

$$(2.44) \quad \mu(\tau, x; m \nabla \varphi(\tau, x)) = \tilde{\mu}(\tau, x; \nabla \varphi(\tau, x), m^{-1}).$$

Due to (H5), (2.6) and (2.44), computing (2.42) for large values of $|m|$ needs only to look at directions $\tilde{\xi}$ which are located in a compact set of $T_{\setminus 0}^* \times \mathbb{R}$. Exploiting (H5) and the continuity on $[0, T] \times T_{\setminus 0}^* \times \mathbb{R}$ of the nonzero eigenvalues of \tilde{P}^0 , we can deduce (2.42) and therefore (HN5).

Remark 2.7. *Propositions 2.1 and 2.3 imply that at this stage, the phase is known, as well as a large part of the profile U^0 (it remains to find $|\Upsilon^h\rangle U^0$), and part of the profile U^1 (namely $|\Pi\rangle U^1$).*

2.5. The transport equation. In this Section 2.5, we determine $|\Pi\rangle U^0$ and part of the expression $|\Pi\rangle U^1$. We start by translating the equation $\mathcal{F}_m(\Gamma^0) \equiv 0$. We get

$$(2.45) \quad \begin{aligned} |P^0\rangle_m U_m^2 + i m \partial_\tau \varphi S_0^0 U_m^1 + |P^1\rangle_m U_m^1 + \sum_{j=1}^d S_j^0 \partial_j U_m^1 \\ + i m \partial_\tau \varphi S_0^1 U_m^0 + S_0^0 \partial_\tau U_m^0 + \sum_{j=1}^d S_j^1 \partial_j U_m^0 \\ + \mathcal{F}_m(\mathfrak{NL}(\tau, x, U^0)) = 0. \end{aligned}$$

Applying the partial inverse $|(\Pi M \Pi)^{-1}\rangle_m$ to (2.45) allows to find that

$$\begin{aligned}
(2.46) \quad & i [m \partial_\tau \varphi |\Pi\rangle_m - |H\rangle_m] |\Pi M \Pi\rangle_m U_m^1 \\
& + |(\Pi M \Pi)^{-1}\rangle_m [i m \partial_\tau \varphi S_0^0 + |P^1\rangle_m] |Q\rangle_m U_m^1 \\
& + |(\Pi M \Pi)^{-1}\rangle_m \sum_{j=1}^d S_j^0 \partial_j (|Q\rangle_m U_m^1) + |\Pi M \Pi\rangle_m \partial_\tau U_m^0 \\
& + |(\Pi M \Pi)^{-1}\rangle_m \left\{ \sum_{j=1}^d S_j^1 \partial_j U_m^0 + i m \partial_\tau \varphi S_0^1 U_m^0 \right\} \\
& + |(\Pi M \Pi)^{-1}\rangle_m \mathcal{F}_m(\mathfrak{NL}(\tau, x, U^0)) = 0.
\end{aligned}$$

The analysis of (2.46) is done in two steps. In Paragraphs 2.5.1 to 2.5.4, we study the constraint obtained by considering $|\Pi^h\rangle_m$ (2.46) when m belongs to \mathcal{HA} . Then, in Paragraph 2.5.5, we study the other situations.

2.5.1. *Preliminaries.* For $m \in \mathcal{HA}$, one has by definition

$$|\Pi^h\rangle_m [m \partial_\tau \varphi |\Pi\rangle_m - |H\rangle_m] = (m \partial_\tau \varphi - |h\rangle_m) |\Pi^h\rangle_m = 0.$$

For any $m \in \mathcal{HA}$, we define $\tilde{U}_m^0 := |\Pi^h (\Pi M \Pi)\rangle_m U_m^0 \equiv |\Pi^h\rangle_m \tilde{U}_m^0$ which is the part of U^0 still unknown to us. We shall define it by computing the equation it satisfies. Apply $|\Pi^h\rangle_m$ to (2.46) and use (2.37) to replace $|Q\rangle_m U_m^1$ accordingly. By developing the induced expression, that is

$$\begin{aligned}
- |\Pi_M^h\rangle_m [i m \partial_\tau \varphi S_0^0 + |P^1\rangle_m + S^0(\tau, x; \partial_x)] |(Q P^0 Q)^{-1}\rangle_m \\
\quad [i m \partial_\tau \varphi S_0^0 + |P^1\rangle_m + S^0(\tau, x; \partial_x)] [|\Pi_M^{h*}\rangle_m \tilde{U}_m^0].
\end{aligned}$$

we get the following system of constraints (indexed by $m \in \mathcal{HA}$):

$$\begin{aligned}
(2.47) \quad & |\Pi^h\rangle_m \partial_\tau \tilde{U}_m^0 + \sum_{j=1}^d |\Pi_M^h\rangle_m S_j^1 |\Pi_M^{h*}\rangle_m \partial_j \tilde{U}_m^0 \\
& + \mathfrak{D}_m(\tau, x; \partial_x) \tilde{U}_m^0 + |\Pi^h\rangle_m \mathfrak{L}_m(\tau, x) \tilde{U}_m^0 \\
& + |\Pi_M^h\rangle_m \mathcal{F}_m(\mathfrak{NL}(\tau, x, U^0)) + |\Pi^h\rangle_m \mathfrak{F}_m(\tau, x) = 0,
\end{aligned}$$

where $\mathfrak{D}_m(\tau, x; \partial_x)$ denotes the second order differential operator

$$\mathfrak{D}_m(\tau, x; \partial_x) := |\Pi_M^h\rangle_m \left\{ \sum_{i,j=1}^d \mathfrak{D}_m^{ij}(\tau, x) \partial_{ij}^2 + \sum_{k=1}^d \mathfrak{D}_m^k(\tau, x) \partial_k \right\} |\Pi_M^{h*}\rangle_m.$$

The operator involved above is anti-selfadjoint, hence the matrices \mathfrak{D}_m^{ij} and \mathfrak{D}_m^k are hermitian. One has precisely

$$\begin{aligned}
\mathfrak{D}_m^{ij} & := - |\Pi\rangle_m S_i^0 |(Q P^0 Q)^{-1}\rangle_m S_j^0 |\Pi\rangle_m, \\
\mathfrak{D}_m^k & := - |\Pi\rangle_m \left\{ [i m \partial_\tau \varphi S_0^0 + |P^1\rangle_m] |(Q P^0 Q)^{-1}\rangle_m S_k^0 \right\} |\Pi\rangle_m \\
& - |\Pi\rangle_m \left\{ S_k^0 |(Q P^0 Q)^{-1}\rangle_m [i m \partial_\tau \varphi S_0^0 + |P^1\rangle_m] \right\} |\Pi\rangle_m \\
& - |\Pi\rangle_m \left\{ S_k^0 |(Q P^0 Q)^{-1}\rangle_m S^0(\tau, x, \partial_x) (\Pi_M^{h*}) \right\} |\Pi M \Pi\rangle_m \\
& - |\Pi\rangle_m \left\{ S^0(\tau, x, \partial_x) (|(Q P^0 Q)^{-1}\rangle_m S_k^0 |\Pi_M^{h*}\rangle_m) \right\} |\Pi M \Pi\rangle_m.
\end{aligned}$$

The first line in (2.47) is clearly compatible with energy estimates (it has the structure of a quasilinear symmetric system), however that is much less apparent for the second line (due to the presence of \mathfrak{D}_m). That question is examined in the next Paragraphs 2.5.2 and 2.5.3.

2.5.2. *Erasing the second order terms.* In fact, the influence of the second order terms is reduced to zero, as is apparent in the following statement.

Lemma 2.5. *For all $m \in \mathcal{HA}$, one has*

$$(2.48) \quad \mathfrak{D}_m^{ij} + \mathfrak{D}_m^{ji} \equiv 0, \quad \forall (i, j) \in \{1, \dots, d\}^2.$$

Proof. This is an adaptation of arguments which are classical in diffractive nonlinear geometric optics, see for instance [7]. Let us explain how the general procedure can be adapted in the current context. One differentiates the relation $\Pi S_j^0 \Pi \equiv 0$ in the direction ξ_i and one applies the projector Π on both sides. This leads to

$$(2.49) \quad \Pi (\partial_{\xi_i} \Pi) S_j^0 \Pi + \Pi S_j^0 (\partial_{\xi_i} \Pi) \Pi \equiv 0.$$

On the other hand one has (recalling that $Q \equiv I - \Pi$)

$$(Q P^0 Q)^{-1} P^0 \equiv I - \Pi, \quad P^0 (Q P^0 Q)^{-1} \equiv I - \Pi,$$

which one also differentiates in direction ξ_i and composes with Π (right or left). This gives

$$(2.50) \quad (Q P^0 Q)^{-1} S_i^0 \Pi \equiv i (\partial_{\xi_i} \Pi) \Pi, \quad \Pi S_i^0 (Q P^0 Q)^{-1} \equiv i \Pi (\partial_{\xi_i} \Pi).$$

By definition, one has

$$\mathfrak{D}_m^{ij} + \mathfrak{D}_m^{ji} \equiv -|\Pi S_i^0 (Q P^0 Q)^{-1} S_j^0 \Pi + \Pi S_j^0 (Q P^0 Q)^{-1} S_i^0 \Pi\rangle_m.$$

Use (2.50) in order to recognize (2.49), giving rise to (2.48). □

2.5.3. *More about the structure of one order terms.* Consider the matrix

$$\mathfrak{S}_{m0} := |\Pi_M^h\rangle_m S_0^0 |(\Pi_M^h)^*\rangle_m = (\mathfrak{S}_{m0})^*$$

and, for all $j \in \{1, \dots, d\}$, the matrices

$$\begin{aligned} \mathfrak{S}_{mj} &:= |\Pi_M^h\rangle_m S_j^1 |(\Pi_M^h)^*\rangle_m \\ &\quad - |\Pi_M^h\rangle_m \left\{ [i \ m \ \partial_\tau \varphi \ S_0^0 + |P^1\rangle_m] |(Q P^0 Q)^{-1}\rangle_m S_j^0 \right\} |(\Pi_M^h)^*\rangle_m \\ &\quad - |\Pi_M^h\rangle_m \left\{ S_j^0 |(Q P^0 Q)^{-1}\rangle_m [i \ m \ \partial_\tau \varphi \ S_0^0 + |P^1\rangle_m] \right\} |(\Pi_M^h)^*\rangle_m \\ &\quad - |\Pi_M^h\rangle_m \left\{ S_j^0 |(Q P^0 Q)^{-1}\rangle_m S^0(\tau, x, \partial_x) (|(\Pi_M^h)^*\rangle_m) \right\} |\Pi^h\rangle_m \\ &\quad - |\Pi_M^h\rangle_m \left\{ S^0(\tau, x, \partial_x) (|(Q P^0 Q)^{-1}\rangle_m S_j^0 |(\Pi_M^h)^*\rangle_m) \right\} |\Pi^h\rangle_m. \end{aligned}$$

It is clear that the matrix \mathfrak{S}_{m0} is hermitian and that it is positive definite when restricted to $|\Pi^h\rangle_m(\mathbb{C}^n)$. On the other hand, the matrices \mathfrak{S}_{mj} with $j \in \{1, \dots, d\}$ are by construction hermitian on $|\Pi^h\rangle_m(\mathbb{C}^n)$.

With the preceding conventions, the equation (2.47) becomes

$$(2.51) \quad \mathfrak{S}_{m0} \partial_\tau \tilde{U}_m^0 + \sum_{j=1}^d \mathfrak{S}_{mj} \partial_j \tilde{U}_m^0 + \mathfrak{L}_m(\tau, x) \tilde{U}_m^0 + \mathfrak{F}_m(\tau, x) + |\Pi_M^h\rangle_m \mathcal{F}_m(\mathfrak{NL}(\tau, x, U^0)) = 0.$$

The first line of (2.51) is a quasilinear symmetric hyperbolic system. Thus, it is compatible with energy estimates in H^s . We can obtain more information about it, when assuming the following simplified setting (implying that only one eigenvalue h of H is at play and that $\Pi_M^h \equiv \Pi^h \equiv \Pi \equiv M$) where we recall that the constant μ is the one appearing in (H7):

$$(2.52) \quad \mu = p, \quad \Pi S_0^1 \equiv S_0^1 \Pi \equiv \Pi S_0^1 \Pi \equiv \Pi.$$

Lemma 2.6. *Assume (2.52). Then, the energy (meaning the L^2 norm in θ of the profile \tilde{U}_m^0) is propagated along the group velocity associated with the Hamiltonian $h(\tau, x; \xi)$. This is due to the fact that*

$$(2.53) \quad \mathfrak{S}_{m0} \partial_\tau + \cdots + \mathfrak{S}_{md} \partial_d = [\partial_\tau - |\nabla_\xi h\rangle_m \cdot \nabla_x] |\Pi\rangle_m.$$

Proof. Let us differentiate (2.18) (considered for the index $j = k$) in the direction ξ_j . One gets

$$(2.54) \quad \Pi S_k^0 (\partial_{\xi_j} \Pi) + (\partial_{\xi_j} \Pi) S_k^0 \Pi = 0, \quad \forall (j, k) \in \{1, \dots, d\}^2.$$

Relation (2.17) can be written

$$(2.55) \quad (Q P^0 Q)^{-1} S_j^0 \Pi = i (\partial_{\xi_j} \Pi) \Pi, \quad \forall j \in \{1, \dots, d\}$$

or taking the adjoint $\Pi S_j^0 (Q P^0 Q)^{-1} = i \Pi (\partial_{\xi_j} \Pi)$. One can use (2.52), (5.2) and (2.55) to simplify \mathfrak{S}_{mj} into

$$\begin{aligned} \mathfrak{S}_{mj} &= -i |\Pi\rangle_m \left\{ |P^1 (\partial_{\xi_j} \Pi)\rangle_m + |(\partial_{\xi_j} \Pi) P^1\rangle_m + i S_j^1 \right\} |\Pi\rangle_m \\ &\quad - i |\Pi\rangle_m \left\{ |\partial_{\xi_j} \Pi\rangle_m S^0(\tau, x, \partial_x) |\Pi\rangle_m \right\} |\Pi\rangle_m \\ &\quad - i |\Pi\rangle_m \left\{ S^0(\tau, x, \partial_x) |(\partial_{\xi_j} \Pi) \Pi\rangle_m \right\} |\Pi\rangle_m. \end{aligned}$$

In the last line, the derivatives contained in $S^0(\tau, x, \partial_x)$ can act either on $|\partial_{\xi_j} \Pi\rangle_m$ or on $|\Pi\rangle_m$. Using (2.16) and (5.5), we obtain

$$|\Pi\rangle_m \left\{ S^0(\tau, x, \partial_x) |(\partial_{\xi_j} \Pi) \Pi\rangle_m \right\} |\Pi\rangle_m = \sum_{k=1}^d |\Pi\rangle_m S_k^0 \partial_k (|\partial_{\xi_j} \Pi\rangle_m) |\Pi\rangle_m.$$

So \mathfrak{S}_{mj} becomes

$$\begin{aligned} \mathfrak{S}_{mj} &= -i |\Pi\rangle_m \left\{ |P^1 (\partial_{\xi_j} \Pi) + (\partial_{\xi_j} \Pi) P^1\rangle_m + i S_j^1 \right\} |\Pi\rangle_m \\ &\quad - i |\Pi\rangle_m \left\{ \sum_{k=1}^d |(\partial_{\xi_j} \Pi) S_k^0 \partial_k \Pi + S_k^0 (\partial_{k\xi_j}^2 \Pi)\rangle_m \right\} |\Pi\rangle_m \\ &\quad - i |\Pi\rangle_m \left\{ \sum_{i,k} \partial_{ki}^2 \varphi |\partial_{\xi_j} \Pi S_k^0 \partial_{\xi_i} \Pi + S_k^0 \partial_{\xi_i \xi_j}^2 \Pi\rangle_m \right\} |\Pi\rangle_m. \end{aligned}$$

Using (2.52), the relation $H \Pi \equiv \Pi H \equiv h \Pi$ can be written

$$i \Pi P^1 \Pi + i \sum_{k=1}^d \Pi S_k^0 (\partial_k \Pi) \equiv h \Pi.$$

Taking an ξ_j derivative of that relation and composing on both sides by Π gives, using again (5.2),

$$\begin{aligned} & i \Pi P^1 (\partial_{\xi_j} \Pi) \Pi + i \Pi (\partial_{\xi_j} \Pi) P^1 \Pi - \Pi S_j^1 \Pi \\ & + i \sum_{k=1}^d \Pi (\partial_{\xi_j} \Pi) S_k^0 (\partial_k \Pi) \Pi + i \sum_{k=1}^d \Pi S_k^0 (\partial_{k\xi_j}^2 \Pi) \Pi \equiv (\partial_{\xi_j} h) \Pi. \end{aligned}$$

Replacing ξ by $m \nabla \varphi$, one sees that the two first lines in \mathfrak{S}_{mj} are reduced to $-|\partial_{\xi_j} h\rangle_m |\Pi\rangle_m$. To recover (2.53), it is therefore enough to show that the last line in \mathfrak{S}_{mj} vanishes. But separating in the sum the index couples (i, k) for which $i \leq k$ and $k \leq i$, that line is nothing but

$$-i |\Pi\rangle_m \left\{ \sum_{1 \leq k \leq i \leq d} \partial_{ki}^2 \varphi |\partial_{\xi_j} [\Pi S_k^0 (\partial_{\xi_i} \Pi) + \Pi S_i^0 (\partial_{\xi_k} \Pi)]\rangle_m \right\} |\Pi\rangle_m$$

which is equal to zero since $P^0 \Pi \equiv 0$ implies that

$$0 \equiv \Pi \partial_{\xi_i \xi_k}^2 (P^0 \Pi) \equiv i [\Pi S_k^0 (\partial_{\xi_i} \Pi) + \Pi S_i^0 (\partial_{\xi_k} \Pi)].$$

The lemma 2.6 is proved. □

2.5.4. *Solving the equation on the unknown part of U^0 .* It remains to identify the expressions \tilde{U}_m^0 with $m \in \mathcal{HA}$. Introduce the auxiliary function

$$\tilde{U}^0(\tau, x, \theta) := \sum_{m \in \mathcal{HA}} \tilde{U}_m^0(\tau, x) e^{im\theta} \equiv |\Pi^h\rangle \tilde{U}^0 := \sum_{m \in \mathcal{HA}} |\Pi^h\rangle_m \tilde{U}_m^0(\tau, x) e^{im\theta}.$$

Consider also the actions $|\Pi_M^h\rangle$, $|\Pi_M^{h*}\rangle$, \mathfrak{S}_j , \mathfrak{L} and \mathfrak{F} which are defined on $L^2(\mathbb{T})$ through the Fourier multipliers $|\Pi_M^h\rangle_m$, $|\Pi_M^{h*}\rangle_m$, \mathfrak{S}_{mj} , \mathfrak{L}_m and \mathfrak{F}_m indexed only by $m \in \mathcal{HA}$. For instance

$$\mathfrak{S}_j \tilde{U}^0(\tau, x, \theta) := \sum_{m \in \mathcal{HA}} \mathfrak{S}_{mj}(\tau, x) \tilde{U}_m^0(\tau, x) e^{im\theta}.$$

With these conventions, taking into account (2.38), we have

$$(2.56) \quad U^0 = \Upsilon^0 \tilde{U}^0 := |\Pi_M^{h*}\rangle \tilde{U}^0 + G^{0,h} + G^0.$$

The equations (2.51) indexed by $m \in \mathcal{HA}$ are coupled together. They form a system which can be abbreviated to

$$(2.57) \quad \begin{aligned} & \mathfrak{S}_0 \partial_\tau \tilde{U}^0 + \sum_{j=1}^d \mathfrak{S}_j \partial_j \tilde{U}^0 + \mathfrak{L} \tilde{U}^0 + \mathfrak{F} \\ & + |\Pi_M^h\rangle \left[\sum_{j=1}^d \partial_j \varphi S_j^2(0, \tau, x, \Upsilon^0 \tilde{U}^0) \right] |\Pi_M^h\rangle \partial_\theta \tilde{U}^0 \\ & + |\Pi_M^h\rangle \Lambda^2(0, \tau, x, \Upsilon^0 \tilde{U}^0) |\Pi_M^h\rangle \tilde{U}^0 + F^2(0, \tau, x, \Upsilon^0 \tilde{U}^0) = 0. \end{aligned}$$

It is supplemented by an initial condition:

$$(2.58) \quad \tilde{U}^0(0, x, \theta) = \tilde{U}^0(x, \theta) \equiv |\Pi^h\rangle \tilde{U}^0(x, \theta), \quad \tilde{U}^0 \in H^\infty(\mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n).$$

The equation (2.57) is a quasilinear hyperbolic system which can be viewed as acting on the functional space $|\Pi^h\rangle H^s([0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$. In this framework, all operators involving derivatives are antiselfadjoint. Moreover, \mathfrak{S}_0 is definite positive. It follows that the Cauchy problem (2.57)-(2.58) can be solved by applying the standard theory. We can find some $\mathcal{T} \in \mathbb{R}_*^+$ and a unique solution $\tilde{U}^0 \in H^\infty([0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$ to (2.57)-(2.58).

2.5.5. *Report on the constraint $\Gamma^0 \equiv 0$.* The discussion about $\Gamma^0 \equiv 0$ can be divided in four intermediate steps.

i) The determination of \tilde{U}^0 (and therefore of $U^0 = \Upsilon^0 \tilde{U}^0$) which has been performed in the Paragraph 2.5.4.

ii) The identification of $|Q\rangle U^1$ through (2.37).

At this stage, we can write

$$(2.59) \quad U^k = |Q\rangle U^k + \check{U}^k + |(\Pi M \Pi)^{-1}\rangle \tilde{U}^k, \quad \check{U}^k = \sum_{m \in \mathbb{Z}} \check{U}_m^k e^{i m \theta}$$

with

$$\begin{aligned} \check{U}^k &:= \sum_{m \in \mathcal{HA}} |(\Pi M \Pi)^{-1} Q^h (\Pi M \Pi)\rangle_m U_m^k e^{i m \theta} + \sum_{m \notin \mathcal{HA}} |\Pi\rangle_m U_m^k e^{i m \theta}, \\ \tilde{U}^k &:= \sum_{m \in \mathbb{Z}} \tilde{U}_m^k e^{i m \theta}, \quad \tilde{U}_m^k := \begin{cases} 0 & \text{if } m \in \mathcal{HA}, \\ |\Pi^h (\Pi M \Pi)\rangle_m U_m^k & \text{if } m \notin \mathcal{HA}. \end{cases} \end{aligned}$$

Observe that, due to the spectral assumptions (HN3) and (HN4), the map $U^k \mapsto \check{U}^k$ is continuous in $H^s([0, T] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$.

iii) The obtention of \check{U}^1 . When $m \in \mathcal{HA}$, the relation (2.34) along with the definition of h imply that the linear map $|H\rangle_m - m \partial_\tau \varphi |\Pi\rangle_m$ is not one-to-one on the vector space $|\Pi\rangle_m(\mathbb{C}^n)$. However, it has a right and left partial inverse $|H - h I\rangle_m^{-1}$. Applying $|H - h I\rangle_m^{-1}$ to (2.46), we can obtain all components $|Q^h (\Pi M \Pi)\rangle_m U_m^1$ with $m \in \mathcal{HA}$.

On the other hand, when $m \in \mathcal{HA}$, the hypothesis (HN4) says that the matrix $|H\rangle_m - m \partial_\tau \varphi |\Pi\rangle_m$ is invertible on the whole space $|\Pi\rangle_m(\mathbb{C}^n)$. Thus, applying the corresponding inverse, we can get $|\Pi\rangle_m U_m^1$.

iii) The link between $|Q\rangle U^2$ and \tilde{U}^1 . Applying the map $|((Q P^0 Q)^{-1})\rangle_m$ to (2.45) yields (for $k = 1$ in our case)

$$(2.60) \quad \begin{aligned} |Q\rangle_m U_m^{k+1} &= |(Q P^0 Q)^{-1}\rangle_m \{ K_m^k(\tau, x, \theta) \\ &\quad - [i m \partial_\tau \varphi S_0^0 + |P^1\rangle_m + \sum_{j=1}^d S_j^0 \partial_j] |(\Pi M \Pi)^{-1} \Pi^h\rangle_m \tilde{U}_m^k \} \end{aligned}$$

where K_m^1 is known. When $m \in \mathcal{HA}$, this relation (2.60) does not allow to conclude to the value of $|Q\rangle_m U_m^2$ (because the term \tilde{U}_m^1 is still unknown).

2.6. The induction. To pursue the analysis, we shall resort to an induction procedure. Let us define the following property, indexed by $k \in \mathbb{N}$:

- (\mathcal{P}_k)
- i) The profiles U^ℓ are known for all $\ell \in \mathbb{N}$ with $\ell < k$;*
 - ii) The function $|Q\rangle U^k$ is known;*
 - iii) The function \check{U}^k is known;*
 - iv) The relation (2.60) holds (with K_m^k known) for all $m \in \mathcal{HA}$.*

The property (\mathcal{P}_1) is exactly what has been obtained in Paragraph 2.5.5. Let us suppose that the properties (\mathcal{P}_l) hold for $l \leq k$. We shall show that it is possible to deduce step (\mathcal{P}_{k+1}). This amounts to studying the following system of equations: $\mathcal{F}_m(\Gamma^k(\tau, x; U^0, \dots, U^{k+2})) \equiv 0$ with $m \in \mathbb{Z}$.

In other words, the equations under study are

$$\begin{aligned}
(2.61) \quad & |P^0\rangle_m U_m^{k+2} \\
& + i m \partial_\tau \varphi S_0^0 U_m^{k+1} + |P^1\rangle_m U_m^{k+1} + \sum_{j=1}^d S_j^0 \partial_j U_m^{k+1} \\
& + i m \partial_\tau \varphi S_0^1 U_m^k + S_0^0 \partial_\tau U_m^k + \sum_{j=1}^d S_j^1 \partial_j U_m^k \\
& + \mathcal{F}_m \left([(U^k \cdot \nabla_u) \mathfrak{R}\mathfrak{L}] (\tau, x, U^0) \right) \\
& + \mathcal{F}_m (\mathfrak{B}^k (\tau, x, U^0, \dots, U^{k-1})) = 0, \quad m \in \mathbb{Z},
\end{aligned}$$

where the contribution $\mathcal{F}_m(\mathfrak{B}^k)$ is known and can be handled as a source term. The analysis of (2.61) takes place along the same lines as that developed in Paragraph 2.5 so we shall not write all the details but rather point out the new aspects to be taken into account.

Fix $m \in \mathcal{H}\mathcal{A}$ and apply $|\Pi_M^h\rangle_m$ to (2.61). This operation eliminates the term $|P^0\rangle_m U_m^{k+2}$. In the second line of (2.61), decompose U_m^{k+1} into $|\Pi\rangle_m U_m^{k+1}$ plus $|Q\rangle_m U_m^{k+1}$. By construction, the contributions coming from $|\Pi\rangle_m U_m^{k+1}$ disappear. Replace $|Q\rangle_m U_m^{k+1}$ as it is indicated in the point *iv*) of (\mathcal{P}_k) . It remains an expression involving \tilde{U}_m^k .

In the third and fourth line of (2.61), decompose U_m^k as in (2.59). The informations coming from *ii*) and *iii*) of (\mathcal{P}_k) allow not to be concerned with the parts $|Q\rangle_m \check{U}_m^k$ and \check{U}_m^k . In fact, we have $U^k = \Upsilon^k \tilde{U}^k$ for some smooth (known) map Υ^k . Finally, we get the following system

$$\begin{aligned}
(2.62) \quad & \mathfrak{S}_0 \partial_\tau \tilde{U}^k + \sum_{j=1}^d \mathfrak{S}_j \partial_j \tilde{U}^k + \mathfrak{L}_m \tilde{U}^k + \tilde{H}^k \\
& + |\Pi_M^h\rangle \left[(\Upsilon^k U^k \cdot \nabla_U) \mathfrak{R}\mathfrak{L} \right] (\tau, x, U^0) = 0
\end{aligned}$$

where \tilde{H}^k is known. Let us complete (2.62) with any initial data

$$(2.63) \quad \tilde{U}^k(0, \cdot) \equiv \tilde{U}_0^k(\cdot) \equiv |\Pi^h\rangle \tilde{U}_0^k(\cdot), \quad \tilde{U}_0^k \in H^\infty(\mathbb{R}^d; \mathbb{C}^n).$$

The system (2.62) is issued from (2.57) by a linearization procedure. It is still hyperbolic symmetric on the functional space $|\Pi^h\rangle H^s([0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$. Therefore, it is locally wellposed in time τ . In other words, we can find some $\mathcal{T} \in \mathbb{R}_*^+$ and a unique solution $\tilde{U}^k \in H^\infty([0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$ to the Cauchy problem (2.62)-(2.63). Knowing $|Q\rangle U^k$, \check{U}^k and \tilde{U}^k , we can piece together again U^k through (2.59). The point *i*) of (\mathcal{P}_{k+1}) is verified.

The line *iv*) of (\mathcal{P}_k) furnishes the relation (2.60) for the index k . Since the profile \tilde{U}^k has just been identified, we have the assertion *ii*) of (\mathcal{P}_{k+1}) .

Compose (2.61) with $|\Pi\rangle_m$ to get

$$i \left[m \partial_\tau \varphi |\Pi\rangle_m - |H\rangle_m \right] |\Pi M \Pi\rangle_m U_m^{k+1} = S_m^{k+1}$$

where S_m^{k+1} is known. The matrix on the left is partly invertible. By exploiting this fact, we can have access to \check{U}^{k+1} . This is *iii*) of (\mathcal{P}_{k+1}) .

Finally, we apply $|(QP^0Q)^{-1}\rangle_m$ to (2.61). By grouping all known expressions inside K_m^{k+1} , we just recover (2.60) written with $k+1$, which is precisely the point *iv*) of (\mathcal{P}_{k+1}) .

To conclude, properties (\mathcal{P}_l) with $l \leq k$ allow to recover (\mathcal{P}_{k+1}) by solving $\Gamma^k \equiv 0$. It is sufficient to stop at step $N - 1$ to obtain (2.9). The profiles U^j with $0 \leq j \leq N + 1$, are in H^∞ if the initial data belong to H^∞ . Note that one can also work with restricted (large) regularity but in that case one expects a loss of derivatives at each step of the procedure.

In conclusion, one has

$$QS(\varepsilon, \tau, x, u_\varepsilon^a; \partial) u_\varepsilon^a = \varepsilon^N R(\varepsilon, \tau, x, \frac{\varphi(\tau, x)}{\varepsilon}), \quad (\varepsilon, \tau, x) \in]0, 1] \times [0, T] \times \mathbb{R}^d$$

where the remainder R is a C^∞ function in $(\varepsilon, \tau, x, \theta)$. Since the integer N may be chosen arbitrary, this implies that u_ε^a is an approximate solution to (1.1) in the sense given in Theorem 1.

2.7. Exact solutions. In this Paragraph 2.7, we prove Theorem 3. We are looking in diffractive times $\tau \simeq 1$ for families $\{u_\varepsilon\}_{\varepsilon \in]0, 1]}$ of solutions to the oscillatory Cauchy problem (1.13). The aim is to show that u_ε remains close to u_ε^a , in the sense of (1.16). To this end, we decompose QS into two parts:

$$QS(\varepsilon, \tau, x, u; \partial) u = QS^\ell(\varepsilon, \tau, x; \partial) u + QS^n(\varepsilon, \tau, x, u; \partial) u$$

where QS^ℓ collects all (main) linear parts:

$$\begin{aligned} QS^\ell(\varepsilon, \tau, x; \partial) &:= \frac{1}{\varepsilon^2} \Lambda^0(\tau, x) + \frac{1}{\varepsilon} \left\{ \sum_{j=1}^d S_j^0(\tau, x) \partial_j + \Lambda^1(\tau, x) \right\} \\ &\quad + \varepsilon^0 \left\{ S_0^0(\tau, x) \partial_\tau + \sum_{j=1}^d S_j^1(\tau, x) \partial_j \right\}. \end{aligned}$$

while one finds in QS^n all the other contributions, and in particular the nonlinear ones. One has

$$\begin{aligned} QS^n(\varepsilon, \tau, x, u; \partial) u &= -\frac{1}{\varepsilon^2} F^0(\tau, x) - \frac{1}{\varepsilon} F^1(\tau, x) + \Lambda^2(\varepsilon, \tau, x, u) \\ &\quad + \varepsilon S_0^1(\tau, x) \partial_\tau u + \varepsilon^2 S_0^2(\varepsilon, \tau, x, u) \partial_\tau u \\ &\quad + \varepsilon \sum_{j=0}^d S_j^2(\varepsilon, \tau, x, u) \partial_j u - F^2(\varepsilon, \tau, x, u). \end{aligned}$$

In the following we shall first concentrate on the easier situation, that is when the contribution QS^n is linear. We will state an improved version of Theorem 3 in that case. The paragraph 2.7 deals with the general case. Note that we shall only give the functional setting and the main estimates necessary in each case to conclude to stability, as the arguments giving rise to the existence of an exact solution are very standard [4, 12, 13, 17, 18].

2.7.1. The linear case ($\nabla_u QS^n \equiv 0$). Since the matrices Λ^0 and Λ^1 are anti-hermitian, condition (H3) implies that the action of the operator QS^ℓ is compatible with L^2 -energy estimates, uniform in $\varepsilon \in]0, 1]$. In other words, there is therefore a constant C , independent of $\varepsilon \in]0, 1]$, such that for any $u \in C^\infty(\mathbb{R}_+; H^\infty(\mathbb{R}^d; \mathbb{C}^n))$ and for all times $\tau \in \mathbb{R}_+$, one has

$$(2.64) \quad \| u(\tau, \cdot) \|_{L^2} \lesssim \| u(0, \cdot) \|_{L^2} + \int_0^\tau \| QS^\ell(\varepsilon, \tau, x; \partial) u(s, \cdot) \|_{L^2} ds.$$

Notice that the advantage of the linear case is that assumptions (HN \star) are not required, and the matrix coefficients S_j^0 may depend on x .

The existence of a local (in time) solution u_ε is not a problem since the equation is linear. Knowing (1.11), the estimate (1.14) can be obtained just by applying (2.64) to the equation satisfied by the difference $u_\varepsilon - u_\varepsilon^a$. This completes the proof of Theorem 2.

2.7.2. The nonlinear case. The framework is described in Theorem 3. The discussion is here classical and inspired from [12, 13, 15].

Due to the nonlinearity, one needs to control uniformly u_ε and $\partial_j u_\varepsilon$ in the space $L^\infty([0, T] \times \mathbb{R}^d \times \mathbb{T}; \mathbb{C}^n)$. To get around this difficulty, one uses the weighted Sobolev spaces $H_{\varepsilon^\iota}^s$ which were introduced at the level of (1.17). The choice $\iota = 2$ is sufficient to handle the penalization term $\varepsilon^{-2} \Lambda^0(\tau, x)$ because we have (for $|\alpha| = 1$ below)

$$\begin{aligned} |\langle \varepsilon^2 \partial_j u_\varepsilon, \varepsilon^2 \partial_j (\varepsilon^{-2} \Lambda^0 u_\varepsilon) \rangle_{L^2 \times L^2}| &= |\langle \varepsilon^2 \partial_j u_\varepsilon, (\partial_j \Lambda^0) u_\varepsilon \rangle_{L^2 \times L^2}| \\ &\lesssim \|\varepsilon^2 \partial_j u_\varepsilon\|_{L^2} \|u_\varepsilon\|_{L^2} \lesssim \|u_\varepsilon\|_{H_{\varepsilon^2}^1}^2. \end{aligned}$$

When estimating $\varepsilon^2 \partial_k u_\varepsilon$, difficulties come from the matrices S_j^0 . There is indeed a loss in powers of $\varepsilon \in]0, 1]$ coming from the contributions

$$\begin{aligned} \varepsilon^{-1} [\varepsilon^2 \partial_k; S_j^0(\tau, x) \partial_j] &\equiv \varepsilon^{-1} (\partial_k S_j^0)(\tau, x) (\varepsilon^2 \partial_j), \quad (k, j) \in \{1, \dots, d\}^2, \\ \varepsilon^{-1} [\varepsilon^2 \partial_k; S_0^0(\tau, x) \partial_\tau] &\equiv \varepsilon^{-1} (\partial_k S_0^0)(\tau, x) (\varepsilon^2 \partial_\tau), \quad k \in \{1, \dots, d\}. \end{aligned}$$

This is the reason why (1.15) is needed. It is to obtain contributions of the order $O(1)$ instead of being $O(\varepsilon^{-1})$. Just use (1.1) and (1.2) in order to replace the time derivative $\varepsilon^2 \partial_\tau$ accordingly.

Even if the condition (1.15) is very restrictive, it does not imply that the matrix $P^0(\tau, x; \xi)$ is constant in $x \in \mathbb{R}^d$, since $\Lambda^0(\tau, x)$ may depend on x . The combination of the preceding arguments gives easily (for all $s \in \mathbb{N}^*$)

$$(2.65) \quad \|u(\tau, \cdot)\|_{H_{\varepsilon^2}^s} \lesssim \|u(0, \cdot)\|_{H_{\varepsilon^2}^s} + \int_0^\tau \|QS^\ell(\varepsilon, \tau, x; \partial)u(s, \cdot)\|_{H_{\varepsilon^2}^s} ds.$$

Observe that the approximate solution u_ε^a oscillates at frequency ε^{-1} , and not ε^{-2} (as it could be the case for an arbitrary polarization). On the other hand, the linearization of the system (1.1) along u_ε^a gives rise to functions of u_ε^a with ε in factor. These two remarks are crucial. They imply (for instance when $|\alpha| = 0$) that

$$|\langle u_\varepsilon, \varepsilon S_j^2(\varepsilon, \tau, x, u_\varepsilon^a) \partial_j u_\varepsilon \rangle_{L^2 \times L^2}| \lesssim \|\varepsilon \partial_j u_\varepsilon^a\|_{L^\infty} \|u_\varepsilon\|_{L^2}^2 \lesssim \|u_\varepsilon\|_{L^2}^2.$$

Another aspect of the analysis is to deduce L^∞ -bounds from $H_{\varepsilon^2}^s$ -estimates on u_ε . For $s > \frac{d}{2} + 1$, this point can be obtained by using the injections

$$(2.66) \quad \begin{aligned} \|u_\varepsilon\|_{W^{1, \infty}} &\lesssim \varepsilon^{-2} \|u_\varepsilon(\varepsilon^2 \cdot)\|_{W^{1, \infty}} \\ &\lesssim \varepsilon^{-2} \|u_\varepsilon(\varepsilon^2 \cdot)\|_{H^s} \lesssim \varepsilon^{-2-d} \|u_\varepsilon\|_{H_{\varepsilon^2}^s}. \end{aligned}$$

As usual [4, 12], the solution u^ε is decomposed into $u_\varepsilon^a + \varepsilon^{2+d} w_\varepsilon$. The equation on w_ε involves a source term which is of size $O(\varepsilon^{N-2-d})$ in $H_{\varepsilon^2}^s$. Since all nonlinear contributions are of the form $\varepsilon S_j^2(\varepsilon, \tau, x, u_\varepsilon^a + \varepsilon^{2+d} w_\varepsilon)$, and therefore can be uniformly controlled in the Lipschitz norm through (2.66), it is compatible with energy estimates in the space $H_{\varepsilon^2}^s$. The corresponding bound on w_ε leads directly to (1.16). This ends the proof of Theorem 3.

3. APPLICATION TO THE PROPAGATION OF ROSSBY WAVES.

Rossby waves can be found in the ocean. They are due to the variations of the Coriolis force. They propagate on very long time scales. For example, they can take months or even years to cross the Pacific. To see the underlying Physics, the reader can refer to [8, 11, 19, 21].

In Section 3.1 we present the various equations and scalings useful to our study. We explain how to relate the various terms of the equations to the general model (1.1) studied in Theorem 1. In Sections 3.2, we check that the *linear* Assumptions (H \star) of Theorem 1 are indeed satisfied. The eikonal equation is computed at the level of Section 3.3. The non linear aspects linked with Assumptions (HN \star) are considered in Section 3.4. Something special happens in the present situation. As explained in Section 3.5, due to transparencies, the expected non linear effects are not present. Combining the preceding informations, we can prove Theorem 4.

3.1. The equations. The number of state variables is $n = 3$. The vector $\tilde{u} = {}^t(\tilde{u}_1, \tilde{u}_2, \tilde{u}_3) \in \mathbb{R}^3$ symbolizing those variables is decomposed into pressure $\tilde{u}_1 \equiv \tilde{p} \in \mathbb{R}_+^*$ and the two velocity components $\tilde{u}_2 \equiv \tilde{v}_1 \in \mathbb{R}$ and $\tilde{u}_3 \equiv \tilde{v}_2 \in \mathbb{R}$. The velocity field is $\tilde{v} \equiv {}^t(\tilde{v}_1, \tilde{v}_2) \in \mathbb{R}^2$. The space dimension is $d = 2$. We work with

$$(\tau, x, \xi) = (\tau, x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2$$

where the coordinates $(x_1, x_2) \in \mathbb{R}^2$ represent respectively the longitude and latitude, in a Cartesian approximation. We are interested in describing the large-scale structure of the oceans, using the following equation of the compressible Euler type

$$(3.1) \quad \begin{cases} (\partial_t + \tilde{v} \cdot \nabla) \tilde{p} + f(\tilde{p}) \operatorname{div} \tilde{v} = F_1^s, \\ (\partial_t + \tilde{v} \cdot \nabla) \tilde{v}_1 + f(\tilde{p}) \partial_1 \tilde{p} - \varepsilon^{-1} b \tilde{v}_2 = F_2^s, \\ (\partial_t + \tilde{v} \cdot \nabla) \tilde{v}_2 + f(\tilde{p}) \partial_2 \tilde{p} + \varepsilon^{-1} b \tilde{v}_1 = F_3^s, \end{cases}$$

where $t \in \mathbb{R}_+$ denotes the fast time. The number $\varepsilon \in]0, 1]$ comes from writing the physical equations in adimensionalized form. It is supposed to be very small. The function $f \in C^\infty(\mathbb{R}; \mathbb{R})$ plays the role of a state law. The terms $F_j^s(\varepsilon, x)$ are the components of a field $F^s = {}^t(F_1^s, F_2^s, F_3^s)$ belonging to $C^\infty([0, 1] \times \mathbb{R}^2; \mathbb{R}^3)$ which represents exterior forcing (like wind for instance). Finally, the function $b(\varepsilon, x)$ satisfies

$$b \in C_b^\infty([0, 1] \times \mathbb{R}^2; \mathbb{R}), \quad b(\varepsilon, x) = b^0(x) + \varepsilon b^1(x) + \varepsilon^2 b^2(\varepsilon, x).$$

The contribution $\varepsilon^{-1} b \tilde{v}^\perp$ is considered as a term of *fast rotation*. It is due to the influence of the Coriolis force. The choice $b^0(x) = \sin x_2$ is adapted to applications in oceanography. Recall that the choice $b^0(x) = x_2$, with limited validity in the vicinity of the equator, corresponds to the so-called betaplane approximation. From now on, we restrict our attention to some connected domain $\mathcal{D} \subset \mathbb{R}^2$ satisfying (1.20). When $b^0(x) = \sin x_2$, this restriction means that we avoid the equatorial zone $\mathcal{E} := \{x; x_2 = 0\}$ while focussing on a region \mathcal{D} placed at midlatitudes. Moreover, we suppose that the flow under study is close to a *stationary* solution $u^s(\varepsilon, x)$ to (3.1), which we choose to be of the following form:

$$u^s(\varepsilon, x) = {}^t(\bar{p} + \varepsilon p^s(\varepsilon, x), \varepsilon v_1^s(\varepsilon, x), \varepsilon v_2^s(\varepsilon, x)), \quad \bar{p} \in \mathbb{R}_+^*$$

where

$${}^t(p^s, v_1^s, v_2^s) \in \mathcal{C}^\infty([0, 1]; H^\infty(\mathbb{R}^2; \mathbb{R}^3)).$$

This implies selecting the source term F^s as follows:

$$\begin{aligned} F_1^s &= \varepsilon^2 (v^s \cdot \nabla) p^s + \varepsilon f(\bar{p} + \varepsilon p^s) \operatorname{div} v^s, \\ F_2^s &= \varepsilon^2 (v^s \cdot \nabla) v_1^s + \varepsilon f(\bar{p} + \varepsilon p^s) \partial_1 p^s - b v_2^s, \\ F_3^s &= \varepsilon^2 (v^s \cdot \nabla) v_2^s + \varepsilon f(\bar{p} + \varepsilon p^s) \partial_2 p^s + b v_1^s, \end{aligned}$$

Up to a rescaling of the type $(x, \tilde{u}) \mapsto (\lambda x, \lambda \tilde{u})$ with $\lambda := f(\bar{p})^{-1}$, one can assume that $f(\bar{p}) = 1$. One then changes the time variable to a slow time $\tau = \varepsilon t \in \mathbb{R}_+$, and finally one modifies the source term $F^s(\varepsilon, x)$ into

$$F^s + \varepsilon^3 F^p, \quad F^p(\varepsilon, \tau, x) = {}^t(F_0^p, F_1^p, F_2^p) \in \mathcal{C}^\infty([0, 1] \times \mathbb{R}_+; H^\infty(\mathbb{R}^2; \mathbb{R}^3)).$$

One expects the new solution \tilde{u} , corresponding to the new source term, to be of the form

$$\tilde{u} = u^s + \varepsilon^2 u, \quad u = {}^t(p, v) = {}^t(p, v_1, v_2).$$

Considering the field $F^r = {}^t(F_1^r, F_2^r, F_3^r) \in \mathcal{C}^\infty([0, 1] \times \mathbb{R}_+; H^\infty(\mathbb{R}^2; \mathbb{R}^3))$ by

$$(3.2) \quad \begin{aligned} F_1^r &:= \varepsilon^2 F_1^p + g(\varepsilon, p) \operatorname{div} v^s - \varepsilon^2 (v \cdot \nabla) p^s, \\ F_2^r &:= \varepsilon^2 F_2^p + g(\varepsilon, p) \partial_1 p^s - \varepsilon^2 (v \cdot \nabla) v_1^s, \\ F_3^r &:= \varepsilon^2 F_3^p + g(\varepsilon, p) \partial_2 p^s - \varepsilon^2 (v \cdot \nabla) v_2^s. \end{aligned}$$

with $g(\varepsilon, p) := f(\bar{p} + \varepsilon p^s) - f(\bar{p} + \varepsilon p^s + \varepsilon^2 p) = O(\varepsilon^2)$, we find that u must solve the system (1.19). The next paragraphs are devoted to the proof of Theorem 4, which amounts principally to checking that the hypotheses made imply that the Assumptions of Theorem 1 are satisfied.

3.2. Assumptions (H \star). With the notation of the general system (1.1), one has of course here $F \equiv F^r$, $S_0^r(\varepsilon, \tau, x, u) \equiv \varepsilon I$ and

$$\begin{aligned} S_1^r(\varepsilon, \tau, x, u) &:= \begin{pmatrix} \varepsilon v_1^s + \varepsilon^2 v_1 & f(\bar{p} + \varepsilon p^s + \varepsilon^2 p) & 0 \\ f(\bar{p} + \varepsilon p^s + \varepsilon^2 p) & \varepsilon v_1^s + \varepsilon^2 v_1 & 0 \\ 0 & 0 & \varepsilon v_1^s + \varepsilon^2 v_1 \end{pmatrix}, \\ S_2^r(\varepsilon, \tau, x, u) &:= \begin{pmatrix} \varepsilon v_2^s + \varepsilon^2 v_2 & 0 & f(\bar{p} + \varepsilon p^s + \varepsilon^2 p) \\ 0 & \varepsilon v_2^s + \varepsilon^2 v_2 & 0 \\ f(\bar{p} + \varepsilon p^s + \varepsilon^2 p) & 0 & \varepsilon v_2^s + \varepsilon^2 v_2 \end{pmatrix}, \end{aligned}$$

$$\Lambda^r(\varepsilon, \tau, x, u) \equiv \Lambda^r(\varepsilon, x) := b(\varepsilon, x) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We emphasize the fact that we are working on a model for Rossby waves by adding a superscript r to all the operators. Since the functions p^s , v_1^s and v_2^s are given, by Taylor-expanding f and \bar{p} , we can obtain

$$\begin{aligned} S_1^{r0}(t, x) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, & S_1^{r1}(t, x) &= \begin{pmatrix} v_1^s & f'(\bar{p}) p^s & 0 \\ f'(\bar{p}) p^s & v_1^s & 0 \\ 0 & 0 & v_1^s \end{pmatrix}, \\ S_2^{r0}(t, x) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & S_2^{r1}(t, x) &= \begin{pmatrix} v_2^s & 0 & f'(\bar{p}) p^s \\ 0 & v_2^s & 0 \\ f'(\bar{p}) p^s & 0 & v_2^s \end{pmatrix}. \end{aligned}$$

One has of course (1.2). Just take $c = 1$. On the other hand, the condition (1.3) is due to the choice for b . Retain that

$$\Lambda^{rj}(t, x) = b^j(x) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad j \in \{0, 1\}.$$

Restriction (1.4) is a consequence of the construction of F^r . The nonlinearity only appears at order ε^2 . One has $F^{r0} \equiv F^p(0, \cdot)$ and $F^{r1} \equiv (\partial_\varepsilon F^p)(0, \cdot)$. One also can find the symbol

$$P^{r0}(\tau, x; \xi) \equiv P^{r0}(x; \xi) = \begin{pmatrix} 0 & i\xi_1 & i\xi_2 \\ i\xi_1 & 0 & -b^0 \\ i\xi_2 & b^0 & 0 \end{pmatrix} \in \mathcal{A}_3.$$

- The form required in (H1) is a consequence of the preceding expansions.
- The matrices S_1^{r0} and S_2^{r0} are constant, so (H3) follows immediately. On the other hand, notice that

$$\dim(\ker P^{r0}(\tau, x; \xi)) = p = 1, \quad \forall (\tau, x, \xi) \in [0, T] \times T_{\setminus 0}^*$$

which means that (H4) and therefore (H2) are satisfied.

- Consider (H5). Since the speed of propagation which is associated with the system (1.19) is clearly not uniformly controlled with respect to the parameter $\varepsilon \in]0, 1]$, we have to explain why such a localization process is possible. The reason is that our WKB analysis involves quantities which are polarized according to the 0 eigenvalue, where some uniform (in $\varepsilon \in]0, 1]$) finite speed of propagation is available (as can be seen by looking at the transport equations).

In practice, we can select any domain \mathcal{D} . Then, to guarantee (H5), it suffices to extend all coefficients by a constant outside \mathcal{D} . This manipulation does not affect what happens in a domain of propagation contained in \mathcal{D} .

- Observe that the conditions (H6) and (H8) are both satisfied because $F^r = O(\varepsilon^2)$ so that $F^{r0} \equiv 0$, $G^{r0} \equiv 0$ and $F^{r1} \equiv 0$.

◦ Since $p = 1$, the matrix H can be identified to a scalar. There is only one eigenvalue of H which is of constant multiplicity 1. The Assumption (H7) is obviously verified.

3.3. Description of the geometry. Rossby waves are by definition waves which are polarized along X^r . As announced in [3], one can identify their trajectories through semiclassical arguments as given by the integral curves of a Hamiltonian h^r .

Of course, our WKB analysis detects the same trajectories. It shows also that the *rays* under question are quantitatively associated with an energy transport, up to a damping and a source terms that can be computed (in the spirit of the Lemma 2.6).

Since $p = 1$, the matrix $H^r(\tau, x; \xi)$ can be identified to a real scalar function $h^r(\tau, x; \xi)$. In order to find $h^r : T_{\mathbb{0}}^* \rightarrow \mathbb{R}$, we shall simply compute $|h^r\rangle_1$ in terms of $\nabla\varphi$, and extrapolate the values of $h^r(\tau, x; \xi)$ by replacing everywhere $\nabla\varphi$ by ξ . Equation (2.19) for $m = 1$ translates into

$$(3.3) \quad i \partial_\tau \varphi U_1^{r0} + S_1^{r0} \partial_1 U_1^{r0} + S_2^{r0} \partial_2 U_2^{r0} + |P^{r0}\rangle_1 U_1^{r1} + (i \partial_1 \varphi S_1^{r1} + i \partial_2 \varphi S_2^{r1} + \Lambda^{r1}) U_1^{r0} = 0.$$

The polarization constraint (2.12) for $m = 1$ leads to

$$U_1^{r0}(\tau, x, \theta) = u_1^{r0}(\tau, x) |X^r\rangle_1(\tau, x), \quad u_1^{r0} \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^d; \mathbb{C}).$$

By Lemma 2.1 we know that

$$|{}^t\bar{X}^r\rangle_1 S_1^{r0} |X^r\rangle_1 \equiv 0, \quad |{}^t\bar{X}^r\rangle_1 S_2^{r0} |X^r\rangle_1 \equiv 0.$$

Replacing S_j^{r1} and Λ^{r1} , for $j \in \{1, 2\}$, by their values one gets

$$|{}^t\bar{X}^r\rangle_1 S_j^{r1} |X^r\rangle_1 \equiv v_j^s ||X^r\rangle_1|^2, \quad |{}^t\bar{X}^r\rangle_1 \Lambda^{r1} |X^r\rangle_1 \equiv 0.$$

To get equation (2.29) one now multiplies (3.3) to the left by the vector $|{}^t\bar{X}^r\rangle_1$, which gives

$$\begin{aligned} i \partial_\tau \varphi ||X^r\rangle_1|^2 u_1^{r0} + [{}^t\bar{X}^r\rangle_1 \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_1 |X^r\rangle_1] u_1^{r0} \\ + [{}^t\bar{X}^r\rangle_1 \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \partial_2 |X^r\rangle_1] u_1^{r0} \\ + i (v_1^s \partial_1 \varphi + v_2^s \partial_2 \varphi) ||X^r\rangle_1|^2 u_1^{r0} = 0. \end{aligned}$$

Let us collect the following identities:

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_1 |X^r\rangle_1 + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \partial_2 |X^r\rangle_1 = \begin{pmatrix} 0 \\ -i \partial_1 b^0 \\ -i \partial_2 b^0 \end{pmatrix}.$$

$$||X^r\rangle_1|^2 = (b^0)^2 + (\partial_1 \varphi)^2 + (\partial_2 \varphi)^2.$$

One gets directly

$$(3.4) \quad \partial_\tau \varphi = |h^r\rangle_1(\tau, x) = -v^s \cdot \nabla \varphi + \frac{\partial_1 \varphi \partial_2 b^0 - \partial_2 \varphi \partial_1 b^0}{(b^0)^2 + (\partial_1 \varphi)^2 + (\partial_2 \varphi)^2}.$$

Remark 3.1. *One can also see what formula (2.21) becomes here. Along with Lemma 2.1, one gets*

$$\begin{aligned} G^r(\tau, x; \xi) &\equiv i \Pi [S_1^{r0} \partial_1 \Pi + S_2^{r0} \partial_2 \Pi] \Pi \\ &\equiv i \frac{t \bar{X}^r}{|X^r|^2} \left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \partial_1 X^r + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \partial_2 X^r \right\} \\ &\equiv \frac{\xi_1 \partial_2 b^0 - \xi_2 \partial_1 b^0}{(b^0)^2 + \xi_1^2 + \xi_2^2}. \end{aligned}$$

Since $S_0^{r1} \equiv I$, the formula (2.21) giving H leads simply to (1.21), that is

$$(3.5) \quad h^r(\tau, x; \xi) = \Pi^r (G^r + i P^{r1}) \Pi^r = -v^s \cdot \xi + \frac{\xi_1 \partial_2 b^0 - \xi_2 \partial_1 b^0}{(b^0)^2 + \xi_1^2 + \xi_2^2}.$$

Remark 3.2. *When b^0 does not depend on x_1 and when $v^s \equiv 0$, one can recognize the hamiltonian $h^r(\tau, x; \xi)$ exhibited in [2] and [3].*

Let us now choose a function $\varphi_0(x)$ satisfying an estimate of the type (1.8) and whose gradient $\nabla \varphi_0(x)$ is constant outside a compact set. We can solve locally in time on $[0, T] \times \mathbb{R}^2$ the Cauchy problem (3.4) with data $\varphi(0, \cdot) \equiv \varphi_0$, and this produces a phase $\varphi \in C^\infty([0, T] \times \mathbb{R}^d; \mathbb{R})$ still satisfying (1.8).

3.4. The nonlinear assumptions (HN*). In this paragraph 3.4, we go over the assumptions (HN*) in the framework of the system (1.19).

◦ We start with (HN1). The situation can be delicate. To understand why, look at what happens when $b^0(x) = \sin x_2$. In this special case, one has

$$P^{r0}(\tau, x_1, 0; 0, 0) \equiv 0, \quad \dim(\ker P^{r0}(\tau, x_1, 0; 0, 0)) = 3, \quad \forall x_1 \in \mathbb{R}$$

while for $x_2 \neq 0$, one gets

$$\dim(\ker P^{r0}(\tau, x_1, x_2; 0, 0)) = 1, \quad \forall x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^*.$$

Thus, there is a jump (from 1 to 3) in the multiplicity of the 0 eigenvalue along the equatorial zone $\mathcal{E} := \{x; x_2 = 0\}$. The purpose of the restriction (1.20) is precisely to avoid the related difficulties. Basically, it requires a localization far from equatorial zones. Since b^0 is supposed to be non zero on \mathcal{D} , it may be extended by some non zero constant outside \mathcal{D} . Following the same localization argument as before, we can always assume that

$$(3.6) \quad \exists c \in \mathbb{R}_+^*; \quad c \leq |b^0(x)|, \quad \forall x \in \mathbb{R}^2.$$

Under that assumption, the smooth vector field

$$X^r(\tau, x; \xi) \equiv X^r(x; \xi) := {}^t(-i b^0(x), -\xi_2, \xi_1), \quad X^r \in C^\infty(T^*; \mathbb{R}^3)$$

is for all $(\tau, x, \xi) \in [0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$ a basis of $\ker P^r(\tau, x; \xi)$. In other words, we have $p = p_0 = 1$. In particular, the Assumption (HN1) is satisfied.

◦ Since $p = p_0 = 1$, condition (HN2) follows simply from Remark 2.1.

◦ Since $\text{spec } H \equiv \{h^r\}$, one recovers (HN3) following Remark 2.2.

◦ To be able to ensure assumption (HN4), one first needs to identify the set \mathcal{HA}^r given by (2.34) and (2.35), using the explicit formula (3.5) obtained for $h^r(\tau, x; \xi)$. More precisely, we have to identify the elements $m \in \mathbb{Z}$ such that the following expression is equal to zero:

$$(3.7) \quad \begin{aligned} & |m \partial_\tau \varphi - h^r(\tau, x; m \nabla \varphi(\tau, x))| \\ &= |m| |m^2 - 1| |\nabla \varphi|^2 \frac{|\partial_1 \varphi \partial_2 b^0 - \partial_2 \varphi \partial_1 b^0|}{[(b^0)^2 + |\nabla \varphi|^2] [(b^0)^2 + m^2 |\nabla \varphi|^2]}. \end{aligned}$$

One can distinguish the two following situations:

- i) The phase φ is constant on the level lines of the function b^0 . This can happen in particular when we impose the condition **(Hi)** of Theorem 4. Formula (3.4) then gives access to $\varphi(\tau, \cdot) \equiv \varphi_0$. In that case, one has $\mathcal{HA}_i^r := \mathbb{Z}$. Since there are no $m \in \mathbb{Z}$ outside \mathcal{HA}_i^r , there is nothing to check concerning (HN4).
- ii) The level sets of b^0 are a foliation of \mathbb{R}^2 (or of the domain \mathcal{D} where the solutions are localized) by curves on which φ_0 is monotonous. More precisely, we require the condition **(Hii)** of Theorem 4. Notice that the phase φ is \mathcal{C}^1 on $[0, \mathcal{T}] \times \mathbb{R}^2$ with $\nabla \varphi$ bounded because of (1.8). In view of (3.4), the derivative $\partial_\tau \varphi$ is also bounded. So up to shrinking \mathcal{T} , we can deduce that

$$(3.8) \quad 0 < \inf_{(\tau, x) \in [0, \mathcal{T}] \times \mathbb{R}^2} |(\partial_1 \varphi \partial_2 b^0 - \partial_2 \varphi \partial_1 b^0)(\tau, x)|.$$

Hence the set of harmonics is reduced to $\mathcal{HA}_{ii}^r := \{-1, 0, 1\}$. Now, combining (3.6), (3.7) and (3.8), we can obtain

$$\exists c \in \mathbb{R}_+^*; \quad c |m| \leq |m \partial_\tau \varphi - h^r(\tau, x; m \nabla \varphi(\tau, x))|, \quad \forall m \notin \mathcal{HA}_{ii}^r.$$

It follows that (HN4) is satisfied.

◦ Recall that, for $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$, we have defined the auxiliary matrix

$$\tilde{P}^{r0}(\tau, x; \tilde{\xi}) := i \xi_1 S_1^{r0}(\tau, x) + i \xi_2 S_2^{r0}(\tau, x) + \xi_3 \Lambda^{r0}(\tau, x).$$

◦ Now, consider (HN5). To this end, we adopt the criterion (2.43) involving

$$\tilde{P}^{r0}(\tau, x; \tilde{\xi}) := \begin{pmatrix} 0 & i \xi_1 & i \xi_2 \\ i \xi_1 & 0 & -\xi_3 b^0 \\ i \xi_2 & \xi_3 b^0 & 0 \end{pmatrix}, \quad \tilde{\xi} = (\xi, \xi_3) = (\xi_1, \xi_2, \xi_3).$$

When $\xi \neq 0$, the matrix $\tilde{P}^{r0}(\tau, x; \tilde{\xi})$ has three distinct eigenvalues. These are $\mu \equiv 0$ (associated with *Rossby* waves) and $\mu \equiv \pm i (\xi_1^2 + \xi_2^2 + \xi_3^2 (b^0)^2)^{1/2}$ (associated with *Poincaré* waves [2, 3, 10]). Hence (2.43) holds with $p = 1$. In particular, there is no jump of p near the value $\xi_3 = 0$. Applying the remark 2.6, we are sure that condition (HN5) is verified.

Finally, since the two matrices S_1^{r0} and S_2^{r0} are constant, the preliminary (non linear) stability condition (1.15) is obviously satisfied.

3.5. About transparencies. As already explained in the Paragraph 2.5.4, the transport equation on the component \tilde{U}^0 of the main profile is in general quasilinear, see (2.47). The nonlinearity comes from the contribution \mathfrak{NL} which is defined line (2.8) with U^0 given by (2.56).

Lemma 3.1. *In the case of the model (1.19), the transport equation (2.57) is linear.*

Proof. This property is due to *transparency* relations which we exhibit below. In the present case (1.19), denoting $U = {}^t(P, V)$ with $V = {}^t(V_1, V_2)$ and introducing $h(P) := f'(\bar{p}) P + \frac{1}{2} f''(\bar{p}) (p^s)^2$, we find

$$\begin{aligned} \mathfrak{NL}^r(\tau, x, U) &= (V \cdot \nabla \varphi) \partial_\theta U + h(P) \begin{pmatrix} \nabla \varphi \cdot \partial_\theta V \\ \partial_1 \varphi \partial_\theta P \\ \partial_2 \varphi \partial_\theta P \end{pmatrix} \\ &+ b^2(0, x) \begin{pmatrix} 0 \\ -V_2 \\ +V_1 \end{pmatrix} + f'(\bar{p}) P \begin{pmatrix} \operatorname{div} v^s \\ \partial_1 p^s \\ \partial_2 p^s \end{pmatrix} + (V \cdot \nabla) \begin{pmatrix} p^s \\ v_1^s \\ v_2^s \end{pmatrix}. \end{aligned}$$

On the other hand, we have here $\Pi^{h^r} \equiv \Pi^r \equiv M^r$ so that $\Pi_{M^r}^{h^r} \equiv \Pi^r$. In what follows, the discussion depends on the symbol \star which may be i or ii , depending on the choice of the set \mathcal{HA}_i^r or \mathcal{HA}_{ii}^r . Recall the convention

$$|\Pi_{M^r}^{h^r}\rangle U(t, x, \theta) \equiv |\Pi^r\rangle U(t, x, \theta) := \sum_{m \in \mathcal{HA}_\star^r} |\Pi^r\rangle_m U_m(\tau, x) e^{im\theta}$$

where $U_m := \mathcal{F}_m U$. To prove the Lemma 3.1, it suffices to show that, for all choice of m in \mathcal{HA}_\star^r (with the symbol \star being i or ii) and for all profile $U \in \mathcal{C}^\infty([0, T] \times \mathbb{R}^2 \times \mathbb{T}; \mathbb{C}^3)$, we have

$$(3.9) \quad |\Pi^r\rangle_m \mathcal{F}_m(\mathfrak{NL}^r(\tau, x, |\Pi^r\rangle U + G^{0, h^r} + G^{r0})) \equiv 0.$$

Because $p = 1$, Π^r is the unitary projector onto the direction X^r . We find

$$|\Pi^r\rangle_m U_m := \alpha_m \begin{pmatrix} -i b^0 \\ -m \partial_2 \varphi \\ +m \partial_1 \varphi \end{pmatrix}, \quad \alpha_m := \frac{{}^t|\bar{X}^r\rangle_m \cdot U_m}{\| |X^r\rangle_m \|^2}, \quad m \in \mathbb{Z}.$$

It follows that we can find functions $f_j^r(\tau, x)$ with $j \in \{0, 1, 2\}$ such that

$$\begin{aligned} |\Pi^r\rangle U + G^{0, h^r} + G^{r0} &= \begin{pmatrix} f_0^r(\tau, x) \\ f_1^r(\tau, x) \\ f_2^r(\tau, x) \end{pmatrix} + \alpha_0(\tau, x) \begin{pmatrix} -i b^0(x) \\ 0 \\ 0 \end{pmatrix} \\ &+ \sum_{m \in \mathcal{HA}_\star^r \setminus \{0\}} \alpha_m(\tau, x) e^{im\theta} \begin{pmatrix} -i b^0(x) \\ -m \partial_2 \varphi \\ +m \partial_1 \varphi \end{pmatrix}. \end{aligned}$$

One notices that the nonlinear contribution inside (3.9) can be decomposed into a quadratic part (denoted \mathfrak{Q}^r) and a linear part (denoted \mathfrak{L}^r):

$$\mathfrak{NL}^r(\tau, x, |\Pi^r\rangle U + G^{0, h^r} + G^{r0}) = \mathfrak{Q}^r(\tau, x, U) + \mathfrak{L}^r(\tau, x) U.$$

Observe that the velocity component of $|\Pi^r\rangle U$ is polarized in the direction $\nabla\varphi^\perp$. It follows simplifications when computing \mathfrak{Q}^r . One gets indeed

$$\mathfrak{Q}^r(\tau, x, U) := f'(\bar{p}) P \partial_\theta P^t(0, \partial_1\varphi, \partial_2\varphi).$$

Finally, we have to look at the quantity

$$|\Pi^r\rangle_m \mathcal{F}_m(\mathfrak{Q}^r(\tau, x, U)) = \beta_m(P) \left| |X^r\rangle_m \right|^{-2} |X^r\rangle_m$$

with (for all $m \in \mathbb{Z}$):

$$\beta_m(P) = \frac{i}{2} m f'(\bar{p}) \mathcal{F}_m(P^2) \times (i b^0, -m \partial_2\varphi, m \partial_1\varphi) \cdot \begin{pmatrix} 0 \\ \partial_1\varphi \\ \partial_2\varphi \end{pmatrix} \equiv 0.$$

It means that all terms coming from $\mathfrak{N}\mathfrak{L}^r$ are in fact linear in U . □

The transport equation (2.51) deals with the profile $\tilde{U}_m^{r0} \equiv U_m^{r0}$ which can be here identified with the scalar coefficient α_m . It follows that the equation (2.51) translates into a constraint on α_m . Lemmas 2.5 and 2.6 imply that the constraint under question is a linear transport equation of the form

$$\partial_\tau \alpha_m^{r0} = |\partial_{\xi_1} h^r\rangle_m \partial_1 \alpha_m^{r0} + |\partial_{\xi_2} h^r\rangle_m \partial_2 \alpha_m^{r0} + b_m \partial_\theta \alpha_m^{r0} + c_m \alpha_m^{r0} + d_m$$

involving known functions $b_m(\tau, x)$, $c_m(\tau, x)$ and $d_m(\tau, x)$. Moreover, the antisymmetric aspect of (2.51) imposes $(b_m, c_m, d_m) \in \mathcal{C}^\infty([0, \mathcal{T}] \times \mathbb{R}^2; \mathbb{C})^3$. Integrating the preceding equation with respect to the variable $\theta \in \mathbb{T}$, one recovers here a general principle in geometrical optics which states that the energy is propagated, up to a damping coefficient (c_m) and a source term (d_m), along rays resulting from the eikonal equation (3.4).

4. APPLICATION TO THE PROPAGATION OF ELECTROMAGNETIC WAVES.

Models coming from electrodynamics (Maxwell-Ampère, Lorentz, Bloch, \dots [17, 24]) yield quasilinear symmetric systems involving some skew-symmetric penalization term and having $\lambda \equiv 0$ as eigenvalue of high multiplicity. It is therefore natural to study in this context how to describe the propagation in long time of electromagnetic waves which are polarized along the eigenspace E_0 which is associated with $\lambda \equiv 0$.

The waves which, inside E_0 , correspond to conserved quantities (like the divergence \dots) lead often to a trivial geometry: the rays remain parallel even in diffractive times. What happens in the other directions depends on the model which is selected, together with the involved parameters such as the inhomogeneity of the medium or the presence of forcing terms.

For instance, the work of R. Sentis [23] on the interaction of three waves in plasma physics reveals the system of Boyd-Kadomtsev whose structure is very close to what has been studied in Section 3.1.

In this Paragraph 4, we study another typical situation. We explain the mechanisms underlying the propagation in a ferromagnetic medium.

4.1. The equations. In ferromagnetic models [22, 24], the relevant quantities are the electric field $E(t, y)$, the magnetic field $B(t, y)$ and the magnetization field $M(t, y)$ which are functions from $\mathbb{R} \times \mathbb{R}^3$ into \mathbb{R}^3 . The physical features at a point are described through the vector

$$V(t, y) := {}^t(E(t, y), H(t, y), M(t, y)) \in \mathbb{R}^9.$$

In the presence of a weak damping which is measured by ε^2 with $\varepsilon \in]0, 1]$, Maxwell-Landau-Lifshitz equations are the following

$$(4.1) \quad \begin{cases} \partial_t E - \nabla_y \times H = 0, \\ \partial_t H + \nabla_y \times E + \partial_t M = 0, \\ \partial_t M = -M \times H - \frac{\varepsilon^2}{|M|} M \times (M \times H). \end{cases}$$

We fix two stationary vector fields

$$\begin{aligned} E^s &\in \mathcal{C}^\infty([0, 1] \times \mathbb{R}^3; \mathbb{R}^3), & E^s(\varepsilon, y) &= E^{s0}(y) + \varepsilon E^{s1}(y) + \dots \\ H^s &\in \mathcal{C}^\infty([0, 1] \times \mathbb{R}^3; \mathbb{R}^3), & H^s(\varepsilon, y) &= H^{s0}(y) + \varepsilon H^{s1}(y) + \dots \end{aligned}$$

and a scalar function

$$\alpha \in \mathcal{C}^\infty([0, 1] \times \mathbb{R}^3; \mathbb{R}), \quad \alpha(\varepsilon, y) = \alpha^0(y) + \varepsilon \alpha^1(y) + \dots$$

We suppose that these functions are adjusted so that

$$\nabla_y \times E^s \equiv 0, \quad \nabla_y \times H^s \equiv 0.$$

We require moreover that

$$(4.2) \quad \alpha(\varepsilon, y) H^s(\varepsilon, y) \neq 0, \quad \forall (\varepsilon, y) \in [0, 1] \times \mathbb{R}^3.$$

Introduce

$$\Xi(\varepsilon, y) := \alpha(\varepsilon, y)^{-2} H^s(\varepsilon, y) = \Xi^0(y) + \varepsilon \Xi^1(y) + \dots$$

together with the expression

$$V_\varepsilon^s(t, y) \equiv V_\varepsilon^s(y) := {}^t(E^s(\varepsilon, \varepsilon y), \alpha(\varepsilon, \varepsilon y)^2 \Xi(\varepsilon, \varepsilon y), \Xi(\varepsilon, \varepsilon y)).$$

The expression V_ε^s is a special solution of the system (4.1). We want to understand how a small perturbation of V_ε^s (for instance at the time $t = 0$) can modify the long-time evolution (for times t of the order ε^{-2}) of the solution to (4.1). To this end, we consider

$$V := V_\varepsilon^s + \varepsilon^2 {}^t(E, H, \alpha^{-1} M), \quad \tau = \varepsilon^2 t, \quad x = \varepsilon y.$$

We introduce the field $F^r = {}^t(F_1^r, F_2^r, F_3^r) \in \mathcal{C}^\infty([0, 1]; H^\infty(\mathbb{R}^2; \mathbb{R}^9))$ which is given by $F_1^r \equiv 0$, $F_2^r \equiv \varepsilon^2 \tilde{F}$ and $F_3^r \equiv -\alpha F_2^r$. Define also

$$\begin{aligned} \tilde{F}(\varepsilon, x, H, M) &:= \alpha^{-1} M \times H \\ &+ \frac{\Xi + \varepsilon^2 \alpha^{-1} M}{|\Xi + \varepsilon^2 \alpha^{-1} M|} \times (\alpha M \times \Xi + \Xi \times H + \varepsilon^2 \alpha^{-1} M \times H). \end{aligned}$$

The new unknown $U := {}^t(E, H, M)$ must satisfy the semi-linear system

$$(4.3) \quad \begin{cases} \partial_\tau E - \varepsilon^{-1} \nabla_x \times H = \varepsilon^{-2} F_1^r, \\ \partial_\tau H + \varepsilon^{-1} \nabla_x \times E - \varepsilon^{-2} \Xi \times H + \varepsilon^{-2} \alpha \Xi \times M = \varepsilon^{-2} F_2^r, \\ \partial_\tau M + \varepsilon^{-2} \alpha \Xi \times H - \varepsilon^{-2} \alpha^2 \Xi \times M = \varepsilon^{-2} F_3^r. \end{cases}$$

4.2. Assumptions (H \star). The prerequisites exposed in the introduction are verified for the choices $F \equiv F^r$, $S_0^f(\varepsilon, \tau, x, U) \equiv \varepsilon I$ and

$$S^f(\varepsilon, \tau, x, U; \partial_x) \equiv \sum_{j=1}^3 S_j^f(\varepsilon, \tau, x, U) \partial_j := \begin{pmatrix} 0 & -\nabla_x \times & 0 \\ \nabla_x \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda^f(\varepsilon, \tau, x, U) := \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Xi \times & \alpha \Xi \times \\ 0 & \alpha \Xi \times & -\alpha^2 \Xi \times \end{pmatrix}.$$

The preliminary conditions (H1), (1.2), (1.3) and (1.4) are obviously verified. The assumption (H5) can be obtained by selecting applications α and Ξ which are constant outside a compact set. The matrices S_j^f being constant, we recover (H3). We compute the matrix

$$P^{f0}(\tau, x; \xi) \equiv P^{f0}(x; \xi) := \begin{pmatrix} 0 & -i \xi \times & 0 \\ +i \xi \times & -\Xi^0 \times & \alpha^0 \Xi^0 \times \\ 0 & \alpha^0 \Xi^0 \times & -(\alpha^0)^2 \Xi^0 \times \end{pmatrix}.$$

The restriction (4.2) means that the vector $\Xi^0(x)$ is nonzero on \mathbb{R}^3 . Thus, the kernel of $P^{f0}(x; \xi)$ is of constant dimension $p = 3$ on $T_{\setminus 0}^*$. It is spanned by the three following *orthogonal* vectors

$$Z_1^f := \begin{pmatrix} \xi \\ 0 \\ 0 \end{pmatrix}, \quad Z_2^f := \begin{pmatrix} 0 \\ \alpha^0 |\Xi^0|^2 \xi \\ |\Xi^0|^2 \xi - (\xi \cdot \Xi^0) \Xi^0 \end{pmatrix}, \quad Z_3^f := \begin{pmatrix} 0 \\ 0 \\ \Xi^0(x) \end{pmatrix}.$$

We can check (H4) and therefore (H2). By the way, we can observe that the restriction (HN1) is verified with $p_0 = 7 > p = 3$. As usual, the constraint (1.8) is guaranteed when the initial data φ_0 is adjusted as in (1.8).

In what follows, we explain why $H^f \equiv 0$. It follows that we have (H7) with $m = p = 3$ and with the spectral value $h^f \equiv 0$.

The direction Z_1^f is associated with the divergence-free condition. Thus, we expect that nothing happens on this side. We concentrate on the fields Z_2^f and Z_3^f which can depend on x . We keep in mind that the unitary projectors onto these directions are given by

$$\Pi_j^f(x, \xi) U := \frac{{}^t \bar{Z}_j^f(x, \xi) \cdot U}{|Z_j^f(x, \xi)|^2} Z_j^f(x, \xi), \quad j \in \{1, 2, 3\}.$$

Since the directions Z_j^f are orthogonal, the unitary projector Π^f onto the kernel of P^{f0} can be reconstituted through the formula $\Pi^f = \Pi_1^f + \Pi_2^f + \Pi_3^f$. We know that the application H^f can be put in the form of some hermitian matrix of size 3×3 . Since $T^f \equiv \Pi^f$, we find here $H^f := \Pi^f (G^f + i P^{f1}) \Pi^f$. Because $\Pi_1^f(x, \xi) \equiv \Pi_1^f(\xi)$ does not depend on the variable x , we recover

$$G^f := i \sum_{j=1}^3 \Pi^f S_j^{f0} (\partial_j \Pi_2^f + \partial_j \Pi_3^f) (\Pi_2^f + \Pi_3^f).$$

We remark that

$$(\partial_j \Pi_k^f) U \equiv \left(\partial_j \left(\frac{t \bar{Z}_k^f}{|Z_k^f|^2} \right) \cdot U \right) Z_k^f + \frac{t \bar{Z}_k^f \cdot U}{|Z_k^f|^2} \partial_j Z_k^f, \quad k \in \{2, 3\}.$$

We recall (2.18) which amounts here to the same thing as $\Pi^f S_j^{f0} \Pi^f \equiv 0$. This information allows to eliminate, when computing G^f , the contributions which (in the right hand side) are polarized according to Π^f . It remains

$$G^f \equiv i \sum_{j=1}^3 \sum_{k=2}^3 \frac{t \bar{Z}_k^f \cdot \Pi_k^f}{|Z_k^f|^2} (\Pi^f S_j^{f0} \partial_j Z_k^f).$$

We have the following block structures

$$S_j^{f0} \equiv \begin{pmatrix} 0 & \star & 0 \\ \star & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \partial_j Z_3^f \equiv \begin{pmatrix} 0 \\ 0 \\ \partial_j \Xi^0 \end{pmatrix}.$$

On the other hand, we have

$$\begin{aligned} \partial_j Z_2^f &:= \partial_j (\alpha^0 |\Xi^0|^2) \begin{pmatrix} 0 \\ \xi \\ (\alpha^0)^{-1} \xi \end{pmatrix} + \alpha^0 |\Xi^0|^2 \begin{pmatrix} 0 \\ 0 \\ \partial_j [(\alpha^0)^{-1}] \xi \end{pmatrix} \\ &\quad - \partial_j (\xi \cdot \Xi^0) \begin{pmatrix} 0 \\ 0 \\ \Xi^0 \end{pmatrix} - (\xi \cdot \Xi^0) \begin{pmatrix} 0 \\ 0 \\ \partial_j \Xi^0 \end{pmatrix}. \end{aligned}$$

The relation (2.18) allows to remove the terms which point in the directions of $\Pi^f(\mathbb{R}^9)$ (those implying derivatives of coefficients). The contributions which are likely to persist are in fact those which are polarized according to the magnetic component (those implying derivatives of the directions). However, these terms are not detected by the matrices S_j^{f0} .

This last property expresses some kind of weakness of the coupling between the electric and magnetic field on the one hand, and the magnetization field on the other hand. Consequently, we find $G^f \equiv 0$. Of course a strengthening of the coupling would lead to a very different conclusion.

The matrices S_j^{f1} are all zero so that $P^{f1} \equiv \Lambda^{f1}$.

We write $\Lambda^{f1} \equiv \Lambda_0^{f1} + \Lambda_1^{f1}$ with

$$\begin{aligned} \Lambda_0^{f1} &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha^1 \Xi^0 \times \\ 0 & \alpha^1 \Xi^0 \times & -2 \alpha^0 \alpha^1 \Xi^0 \times \end{pmatrix}, \\ \Lambda_1^{f1} &:= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\Xi^1 \times & \alpha^0 \Xi^1 \times \\ 0 & \alpha^0 \Xi^1 \times & -(\alpha^0)^2 \Xi^1 \times \end{pmatrix}. \end{aligned}$$

It is easy to see that the terms involving Π_1^f and Λ_0^{f1} play no part when computing H^f . Indeed, we find

$$H^f \equiv (\Pi_2^f + \Pi_3^f) \Lambda_1^{f1} (\Pi_2^f + \Pi_3^f).$$

Other simplifications occur which are due to the way we have adjusted the stationary solution V_ε^s (manner which is in fact imposed by the skew-symmetry of Λ). They furnish

$${}^t \bar{Z}_k^f \Lambda_1^{f1} Z_j^f \equiv 0, \quad \forall (j, k) \in \{2, 3\}^2.$$

Finally, we obtain $H^f \equiv 0$ which means as expected that the geometry is *trivial* even in diffractive times ($\tau \simeq 1$). Note again that we have selected here a very basic model, just to illustrate the type of discussion which can happen. By taking into account other aspects (more inhomogeneities, more forcing terms, \dots), we can find $h^f \neq 0$.

5. APPENDIX.

In this Section 5, we shall first prove a lemma giving some general algebraic identities of frequent use in WKB analysis. Due to their generality we choose to state the lemma in an abstract framework. Then, we provide a proof to the Lemmas 2.2 and 2.3.

5.1. Algebraic identities. We denote by $\Pi : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ a linear mapping which depends smoothly on parameters chosen in an open subset Υ of \mathbb{R}^g with $g \in \mathbb{N}^*$. We denote these parameters by $v = (v_1, \dots, v_g) \in \Upsilon \subset \mathbb{R}^g$. Those operators are characterized by the property

$$(5.1) \quad \Pi \in \mathcal{C}^\infty(\mathbb{R}), \quad \Pi(v) \circ \Pi(v) \equiv \Pi(v), \quad \forall v \in \Upsilon.$$

In the context of this article for instance, one can choose Π equal to Π , $|\Pi\rangle_m$ or Q , and v equal to τ , x or ξ . One denotes by ∂_j the differentiation in direction v_j . The result is the following.

Lemma 5.1. *For any $j \in \{1, \dots, g\}$, one has*

$$(5.2) \quad (\partial_j \Pi) \Pi \equiv (I - \Pi) (\partial_j \Pi),$$

$$(5.3) \quad \Pi (\partial_j \Pi) \equiv (\partial_j \Pi) (I - \Pi),$$

$$(5.4) \quad \Pi (\partial_j \Pi) \Pi \equiv 0.$$

Besides, for any couple of integers $(j, k) \in \{1, \dots, g\}^2$, one has

$$(5.5) \quad (\partial_j \Pi) (\partial_k \Pi) \Pi = \Pi (\partial_j \Pi) (\partial_k \Pi).$$

Proof. To obtain (5.2) and (5.3), one just needs to apply ∂_j to (5.1). Then (5.4) follows. Similarly (5.5) is a combination of (5.2) (which is written for $j = k$ and composed to the left by $\partial_j \Pi$) and of (5.3). □

5.2. Proof of the Lemma 2.2. By construction, we have $G \equiv \Pi G \Pi$. The fact that G corresponds to the action of some matrix of size $p_0 \times p_0$ or $p \times p$ (depending on whether $\xi \in T_0^*$ or not) is a direct consequence of (H4) and of (HN1). The regularity of G on $[0, \mathcal{T}] \times T_0^*$ comes from the one of Π which itself is issued from (H4).

To see why the matrix G is hermitian, it is necessary to compute

$$G - G^* = i \sum_{j=1}^d (\Pi S_j^0 \partial_j \Pi \Pi + \Pi \partial_j \Pi S_j^0 \Pi).$$

Taking the derivative of (2.18) in the direction x_j and exploiting (H3) yield

$$G - G^* = -i \Pi (\operatorname{div} S^0) \Pi \equiv 0.$$

The constraint (2.22) implies that

$$P^0(\tau, x; -\xi) = -i \sum_{j=1}^d \xi_j S_j^0(\tau, x) + \Lambda^0(\tau, x) = \bar{P}^0(\tau, x; \xi).$$

Thus, we have the relation

$$\Pi(\tau, x; -\xi) = \bar{\Pi}(\tau, x; \xi), \quad \forall (\tau, x, \xi) \in [0, \mathcal{T}] \times \mathbb{R}^d \times \mathbb{R}^d.$$

The identity (2.23) can be now deduced from a direct computation. We have indeed

$$G(\tau, x; -\xi) = i \sum_{j=1}^d [\bar{\Pi} S_j^0 (\partial_j \bar{\Pi}) \bar{\Pi}] (\tau, x; \xi) = -\bar{G}(\tau, x; \xi).$$

5.3. Proof of the Lemma 2.3. In fact, one only needs to consider the case $m \in \mathbb{Z}^*$. Due to (1.8), one can also reduce the discussion to the set $T_{\setminus 0}^*$. Note incidently that the precise choice of $m \in \mathbb{Z}^*$ has no role to play since one can reduce the study to $m = 1$ simply by changing φ into $m\varphi$.

To really understand what happens, let us not suppose for the moment that $\lambda \equiv 0$. We start with the following relation (which defines $i\lambda$ to be an eigenvalue of P^0):

$$(5.6) \quad (P^0 \Pi)(\tau, x; \xi) = i (\lambda \Pi)(\tau, x; \xi), \quad \forall (\tau, x, \xi) \in [0, \mathcal{T}] \times T_{\setminus 0}^*.$$

Since P^0 is skew-symmetric, one has $\lambda(\tau, x; \xi) \in \mathbb{R}$. Differentiating (5.6) in the direction ξ_j yields (recalling the definition of P^0 in Paragraph 2.1)

$$(5.7) \quad i S_j^0 \Pi + P^0 (\partial_{\xi_j} \Pi) \equiv i (\partial_{\xi_j} \lambda) \Pi + i \lambda (\partial_{\xi_j} \Pi).$$

Replacing the variable ξ by $m \nabla \varphi$ gives

$$i S_j^0 |\Pi\rangle_m + |P^0\rangle_m |\partial_{\xi_j} \Pi\rangle_m \equiv i |\partial_{\xi_j} \lambda\rangle_m |\Pi\rangle_m + i |\lambda\rangle_m |\partial_{\xi_j} \Pi\rangle_m.$$

Taking a derivative in x_j then implies that

$$\begin{aligned} & i (\partial_j S_j^0) |\Pi\rangle_m + i S_j^0 \partial_j |\Pi\rangle_m + |\partial_j P^0\rangle_m |\partial_{\xi_j} \Pi\rangle_m \\ & + i \sum_{k=1}^d m \partial_{jk}^2 \varphi S_k^0 |\partial_{\xi_j} \Pi\rangle_m + |P^0\rangle_m \partial_j |\partial_{\xi_j} \Pi\rangle_m \equiv i \partial_j |\partial_{\xi_j} \lambda\rangle_m |\Pi\rangle_m \\ & + i |\partial_{\xi_j} \lambda\rangle_m \partial_j |\Pi\rangle_m + i \partial_j |\lambda\rangle_m |\partial_{\xi_j} \Pi\rangle_m + i |\lambda\rangle_m \partial_j |\partial_{\xi_j} \Pi\rangle_m. \end{aligned}$$

Now, let us project that identity right and left using $|\Pi\rangle_m$. Before that, we notice that according to Lemma 5.1, line (5.4), one has

$$|\Pi\rangle_m \partial_j |\Pi\rangle_m |\Pi\rangle_m \equiv 0, \quad |\Pi \partial_{\xi_j} \Pi \Pi\rangle_m \equiv 0.$$

Taking the adjoint of (5.6), one can see that $|\Pi\rangle_m |P^0\rangle_m \equiv i |\lambda\rangle_m |\Pi\rangle_m$. Those simplifications allow to write that

$$\begin{aligned} & \left(\sum_{k=1}^d m \partial_{jk}^2 \varphi |\Pi\rangle_m S_k^0 |\partial_{\xi_j} \Pi\rangle_m \right) |\Pi\rangle_m \equiv |\partial_{j\xi_j}^2 \lambda\rangle_m |\Pi\rangle_m \\ & + \left(\sum_{k=1}^d m \partial_{jk}^2 \varphi |\partial_{\xi_j \xi_k}^2 \lambda\rangle_m \right) |\Pi\rangle_m - |\Pi\rangle_m (\partial_j S_j^0) |\Pi\rangle_m \\ & - |\Pi\rangle_m S_j^0 \partial_j |\Pi\rangle_m |\Pi\rangle_m + i |\Pi\rangle_m |\partial_j P^0\rangle_m |\partial_{\xi_j} \Pi\rangle_m |\Pi\rangle_m. \end{aligned}$$

Let us take the sum in j . It is at this level that cancelations occur. We can notice that

$$\sum_{j=1}^d \sum_{k=1}^d m \partial_{jk}^2 \varphi |\Pi\rangle_m [S_k^0 |\partial_{\xi_j} \Pi\rangle_m - S_j^0 |\partial_{\xi_k} \Pi\rangle_m] |\Pi\rangle_m = 0.$$

Hence, we obtain

$$\begin{aligned} & \sum_{j=1}^d |\Pi\rangle_m S_j^0 \left(\sum_{k=1}^d m \partial_{jk}^2 \varphi |\partial_{\xi_k} \Pi\rangle_m \right) |\Pi\rangle_m \\ & \equiv \sum_{j=1}^d |\partial_{j\xi_j}^2 \lambda\rangle_m |\Pi\rangle_m + \left(\sum_{j=1}^d \sum_{k=1}^d m \partial_{jk}^2 \varphi |\partial_{\xi_j \xi_k}^2 \lambda\rangle_m \right) |\Pi\rangle_m \\ & \quad + |\Pi\rangle_m \sum_{j=1}^d (-\partial_j S_j^0 - S_j^0 \partial_j |\Pi\rangle_m + i |\partial_j P^0\rangle_m |\partial_{\xi_j} \Pi\rangle_m) |\Pi\rangle_m. \end{aligned}$$

We recall the identity (2.27) which we compose to the left with $|\Pi\rangle_m S_j^0$ and to the right with $|\Pi\rangle_m$ and sum over j . In view of the definition of \mathfrak{G}_m , see (2.24), this operation yields

$$\begin{aligned} i \mathfrak{G}_m & \equiv \frac{1}{2} |\Pi\rangle_m \sum_{j=1}^d [i S_j^0 |\partial_j \Pi\rangle_m - i \partial_j S_j^0 - |\partial_j P^0\rangle_m |\partial_{\xi_j} \Pi\rangle_m] |\Pi\rangle_m \\ & \quad + \frac{i}{2} \sum_{j=1}^d |\partial_{j\xi_j}^2 \lambda\rangle_m |\Pi\rangle_m + \frac{i}{2} \left(\sum_{j=1}^d \sum_{k=1}^d m \partial_{jk}^2 \varphi |\partial_{\xi_j \xi_k}^2 \lambda\rangle_m \right) |\Pi\rangle_m. \end{aligned}$$

Finally we differentiate (5.7) along x_j , project left and right with Π , and use (5.4) to obtain

$$\begin{aligned} & -i |\Pi\rangle_m \partial_j S_j^0 |\Pi\rangle_m - |\Pi\rangle_m |\partial_j P^0\rangle_m |\partial_{\xi_j} \Pi\rangle_m |\Pi\rangle_m \\ & = i |\Pi\rangle_m S_j^0 |\partial_j \Pi\rangle_m |\Pi\rangle_m - i |\partial_{j\xi_j}^2 \lambda\rangle_m |\Pi\rangle_m, \end{aligned}$$

which finally gives rise to

$$i \mathfrak{G}_m \equiv i \sum_{j=1}^d |\Pi\rangle_m S_j^0 |(\partial_j \Pi) \Pi\rangle_m + \frac{i}{2} \sum_{j=1}^d \sum_{k=1}^d m \partial_{jk}^2 \varphi |\partial_{\xi_j \xi_k}^2 \lambda\rangle_m |\Pi\rangle_m.$$

So (2.28) is obtained as soon as the Hessian in ξ of λ is zero. This is of course the case when $\lambda \equiv 0$ or, more generally, when $\lambda(\tau, x; \cdot)$ is linear in ξ . This concludes the proof of Lemma 2.3.

5.4. Proof of the Lemma 2.4. When $p_0 = p = 1$, one has (H7) with $\mu = 1$ and $X^h \in \mathcal{C}^\infty(T^*; \mathbb{C}_*^n)$. The matrix $H - hI$ has a unique eigenspace, which is given by a vector field $X^h \in \mathcal{C}^\infty(T_0^*; \mathbb{C}_*^n)$. One still has the relation (2.4), but with X^h and Π^h instead of X and Π . Taking into account (2.22), the relation (2.23) and the parity of Π , we can extract

$$(5.8) \quad H(\tau, x; -\xi) = -\bar{H}(\tau, x; \xi), \quad \forall (\tau, x, \xi) \in [0, \mathcal{T}] \times T^*.$$

Given $z = a + ib \in \mathbb{C}$, note $\sigma(z) := a - ib$.

Obviously, we have $\sigma(\text{spec } H) \equiv \text{spec } \sigma(H) \equiv \text{spec } \bar{H}$. Since the eigenvalues of an hermitian matrix are real-valued, we can deduce that $\text{spec } H \equiv \text{spec } \bar{H}$. Combined with (5.8), this yields $\text{spec } H(\tau, x; -\xi) \equiv -\text{spec } H(\tau, x; \xi)$. In other words, $-h(\tau, x; -\xi)$ is an eigenvalue of the matrix $H(\tau, x; \xi)$. Since there is a unique such eigenvalue ($p_0 = p = 1$), it must be equal to $h(\tau, x; \xi)$. The function h is continuous in the variable $\xi \in \mathbb{R}^d$ and odd. In particular, it must be zero when $\xi = 0$. This is precisely (2.32).

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