



Assessing Symmetry using Quantiles and L-Moments

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Abstract:	Measures of distributional symmetry based on quantiles, L-moments and trimmed L-moments are briefly reviewed. Sampling properties of commonly used estimators are considered. Approximate standard errors are obtained when sampling from some known distributions. Simulation is used to assess the approximations and indicate the form of the sampling distributions. For sample data, bootstrapped resampling distributions are used to estimate standard errors and make other inferences about the unknown symmetry measure. It is found that symmetry measures based on 2- or 3-trimmed L-moments have some advantages over other measures in terms of their existence and statistical properties of the corresponding estimators.
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ASSESSING SYMMETRY USING QUANTILES AND L-MOMENTS

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ABSTRACT

Measures of distributional symmetry based on quantiles, L-moments and trimmed L-moments are briefly reviewed, and (asymptotic) sampling properties of commonly used estimators considered. Standard errors are estimated using both analytical and computer-intensive methods. Simulation is used to assess results when sampling from some known distributions; bootstrapping is used on sample data to estimate standard errors, construct confidence intervals and test a hypothesis of distributional symmetry. Symmetry measures based on 2- or 3-trimmed L-moments have some advantages over other measures in terms of their existence. Their estimators are generally well-behaved, even in relatively small samples.

1. INTRODUCTION

Symmetry measures based on a distribution’s quantiles, for example the quartiles or the octiles, have a long history, and go back at least to Bowley (1937). These measures have a number of advantages; for instance, they exist for all probability distributions, they are easy to define and simple to estimate from sample data. More recently measures based on L-moments and trimmed L-moments have been proposed. There are some superficial similarities between these measures and the quantile measures, but they are more sophisticated in terms of existence, definition and estimation.

This paper investigates properties of these types of symmetry measure, including sampling properties of their estimators. Quantile measures are considered in Section 2, and L-moments, trimmed L-moments and measures based on them in Section 3. For both types of measure straightforward sample estimators are introduced, and their sampling distributions considered, one aspect being methods for determining their standard errors at least approximately. The properties of the estimators are investigated for a number of standard distributions, viz. the standard normal, the uniform on the interval (0, 1), the exponential with mean 1, and where possible, the standard Cauchy. Calculated standard errors are compared with values obtained by simulation. The methods are also applied to data sets which come from an unknown probability distribution, and in these cases bootstrap methods are used to determine a confidence interval for the distribution’s symmetry measure, and to test for distributional symmetry. Coverage properties for these confidence intervals are assessed by means of a small scale simulation study. Normal Q-Q plots have been used to give an informal indication of how close simulated distributions of estimators are to normality; they are not always included when they indicate normality is reasonable. Computing was carried out using the R system for statistical computing (R Development Core Team (2008)).

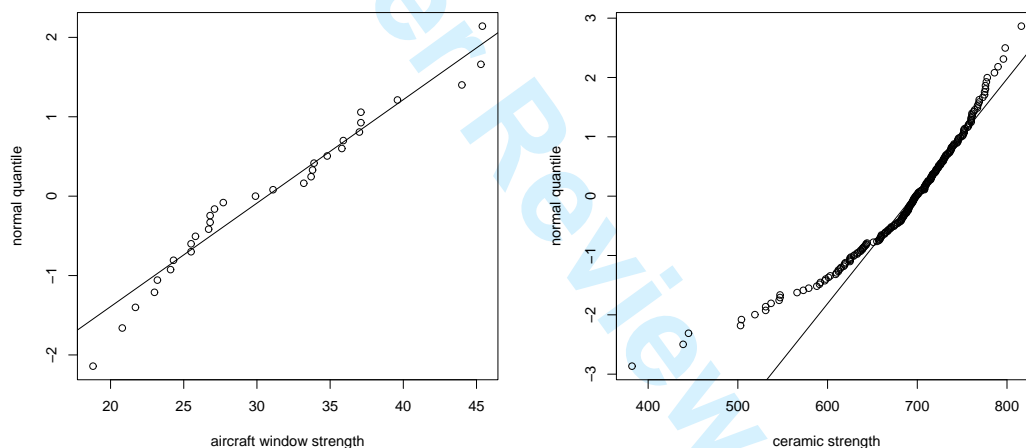
DATA SETS

Two data sets, each regarded as a single sample from an unknown population, are used throughout to illustrate aspects of the methods. They are the aircraft window strength data set (sample size $n = 31$) and the ceramic strength data set (sample size $n = 240$) taken from NIST/SEMATEC (2006) e-Handbook of Statistical Methods. Five-number summaries of these data sets consisting of the two extremes and the three quartiles are as follows:

data set	n	5 number summary				
aircraft window strength	31	18.8	25.5	29.9	35.9	45.4
ceramic strength	240	382	660	698.5	731.5	816

Normal Q-Q plots are given in Figure 1.1. The ceramic strength data set is clearly not from a normal distribution, but from a distribution which is strongly asymmetric with a long lower tail. On the other hand, the aircraft window strength data set may come from a symmetric distribution, possibly a normal distribution. Indeed, the standard Kolmogorov-Smirnov test for normality has p -value 0.074.

Figure 1.1: Normal Q-Q plots for the aircraft window strength and ceramic strength data sets.



2. MEASURES OF SKEWNESS BASED ON QUANTILES

One of the earliest symmetry measures for distributions based on quantiles is the quartile-based measure often referred to as Bowley's coefficient (e.g. Bowley (1937)). It compares the distance of the upper quartile ($F^{-1}(0.75) = Q_3$, say) from the median ($F^{-1}(0.5) = M$, say) with the distance of the lower quartile ($F^{-1}(0.25) = Q_1$, say) from the median relative to the interquartile range, $Q_3 - Q_1$, and is given by

$$QS = \frac{Q_3 - 2M + Q_1}{Q_3 - Q_1} = \frac{F^{-1}(0.75) - 2F^{-1}(0.5) + F^{-1}(0.25)}{F^{-1}(0.75) - F^{-1}(0.25)}, \quad (1)$$

where F^{-1} is the inverse of the cumulative distribution function.

Further quantile measures can be defined in a similar way; a generalisation of Bowley's

coefficient is to define for fixed α , $0 < \alpha < 0.5$, an α -quantile symmetry coefficient by

$$S(\alpha) = \frac{F^{-1}(1 - \alpha) - 2F^{-1}(0.5) + F^{-1}(\alpha)}{F^{-1}(1 - \alpha) - F^{-1}(\alpha)}. \quad (2)$$

Bowley's coefficient is given by $QS = S(0.25)$; the octile symmetry coefficient $OS = S(0.125)$ has also received some attention. $S(\alpha)$ takes the value 0 for any symmetric distribution; otherwise it lies in the interval $[-1, +1]$. As an illustration, for the exponential distribution with mean 1, $S(0.25) = \ln(4/3)/\ln(3) = 0.2619$ and $S(0.125) = \ln(16/7)/\ln(7) = 0.4248$. However, it is possible to find non-symmetric distributions for which $S(\alpha) = 0$.

An obvious drawback of $S(\alpha)$ as a measure of symmetry is that it ignores symmetry in the tails of the distribution outside the upper and lower α -quantiles; a smaller value of α may be preferred to a larger one. An alternative is to average the numerator and denominator in Equation (2) separately over α , $0 < \alpha < 0.5$. If a uniform weighting for α is used, it is straightforward to show that the resulting symmetry measure will be given by

$$\Psi = \frac{\mu - M}{E(|X - M|)}, \quad -1 \leq \Psi \leq +1, \quad (3)$$

where $\mu = E(X)$ is the mean of the distribution. However, this measure can only be defined for a distribution whose mean exists, unlike $S(\alpha)$ in Equation (2) which always exists. For a symmetric distribution, $\Psi = 0$, and for the exponential distribution with mean 1, $\Psi = (1 - \ln 2)/\ln 2 = 0.4427$.

2.1 A SAMPLE ESTIMATOR OF $S(\alpha)$

When a sample of size n is taken from a parent distribution with cumulative distribution function $F(x)$, a commonly used estimator of $S(\alpha)$ is the sample equivalent form of Equation (2), that is

$$\tilde{S}(\alpha) = \frac{\tilde{F}^{-1}(1 - \alpha) - 2\tilde{F}^{-1}(0.5) + \tilde{F}^{-1}(\alpha)}{\tilde{F}^{-1}(1 - \alpha) - \tilde{F}^{-1}(\alpha)}. \quad (4)$$

where $\tilde{F}^{-1}(\alpha)$ is the inverse of the sample cumulative distribution function, i.e. the sample quantile function, so $\tilde{F}^{-1}(0.25)$ is the sample lower quartile.

For a symmetric parent distribution, $\tilde{S}(\alpha)$ will be unbiased, but typically will be biased otherwise.

ASYMPTOTIC VARIANCE OF $\tilde{S}(\alpha)$

Kendall, Stuart and Ord (1987, Chapter 10) give a method for obtaining the asymptotic (large sample) variance of $\tilde{S}(\alpha)$, and they derive an expression for this variance in the case $\alpha = 0.25$. Writing $Q_\alpha = F^{-1}(\alpha)$ and $M = F^{-1}(0.5)$, and denoting the probability density function of the parent distribution evaluated at Q_α by f_α and at M by f_M , in the case of general α , the asymptotic variance takes the form:

$$\begin{aligned} \text{var}(\tilde{S}(\alpha)) = & \frac{1}{n(Q_{1-\alpha} - Q_\alpha)^4} \left\{ \left[(M - Q_\alpha) \left(\frac{1}{f_M} - \frac{2\alpha}{f_{1-\alpha}} \right) \right. \right. \\ & + (Q_{1-\alpha} - M) \left(\frac{1}{f_M} - \frac{2\alpha}{f_\alpha} \right) \left. \right]^2 \\ & + 4\alpha(1 - 2\alpha) \left[\left(\frac{M - Q_\alpha}{f_{1-\alpha}} \right)^2 + \left(\frac{Q_{1-\alpha} - M}{f_\alpha} \right)^2 \right] \left. \right\}. \end{aligned} \quad (5)$$

For the symmetric case, $f_\alpha = f_{1-\alpha}$, $M - Q_\alpha = Q_{1-\alpha} - M$ and $Q_{1-\alpha} - Q_\alpha = 2(Q_{1-\alpha} - M)$, whence Equation (5) reduces to

$$\text{var}(\tilde{S}(\alpha)) = \frac{1}{4n(Q_{1-\alpha} - M)^2} \left(\frac{1}{f_M^2} + \frac{2\alpha}{f_\alpha^2} - \frac{4\alpha}{f_\alpha f_M} \right). \quad (6)$$

Table 2.1 shows the value of $n\text{var}(\tilde{S}(\alpha))$ for $\alpha = 0.25, 0.125$ and 0.05 for some standard distributions. Note that for the Cauchy distribution, in contrast to the other distributions, variability in $\tilde{S}(\alpha)$ increases as α decreases.

Table 2.1: Value of $n\text{var}(\tilde{S}(\alpha))$ from Equations (5) or (6), as appropriate, for several standard probability distributions.

	$\alpha = 0.25$	$\alpha = 0.125$	$\alpha = 0.05$
standard normal	1.8390	1.1514	1.0002
uniform on (0, 1)	2	$\frac{4}{3}$	$\frac{10}{9}$
exponential, mean 1	1.7824	0.9838	0.7260
standard Cauchy	2.4674	3.9128	9.8916

SAMPLING DISTRIBUTION OF $\tilde{S}(\alpha)$

The exact sampling distribution of $\tilde{S}(\alpha)$ for a sample of size n is difficult to determine in general. The numerator and denominator defining $\tilde{S}(\alpha)$ in Equation (4) are both linear combinations of sample quantiles, and hence of some of the central order statistics. From Serfling

(1980, p91) asymptotically they have a bivariate normal distribution. Application of Theorem 3.3.A in Serfling(1980, p122) shows that $\tilde{S}(\alpha)$ is asymptotically normally distributed, with mean $S(\alpha)$ and variance given by Equations (5) or (6), as appropriate. Depending on the form of the parent distribution, the sample size n may have to be exceedingly large for this result to hold, even approximately.

Simulation may be used to give an indication of the form of the sampling distribution of $\tilde{S}(\alpha)$ when sampling from a particular distribution. Some results for the estimators $\tilde{S}(0.25)$ and $\tilde{S}(0.125)$ with samples of sizes $n = 30, 300$ for the distributions in Table 2.1 are presented in Tables 2.2 to 2.5 and Figures 2.1 to 2.4. These results are based on simulations of 10,000 samples. These simulated distributions are summarised by means of quartiles, mean and standard deviation, and a normal Q-Q plot. When n is large enough the standard deviation of the simulated distribution and the asymptotic standard error of $\tilde{S}(\alpha)$ derived from Equations (5) or (6), as appropriate, should be close in value.

From Tables 2.2 to 2.5 and Figures 2.1 to 2.4 the asymptotic standard error appears to overstate the standard error of the estimator for all the distributions, although the bias is small when $n = 300$. For the symmetric distributions, the sampling distributions have shorter tails than a normal distribution when $n = 30$. This feature is not unexpected as quantile measures of symmetry are constrained to lie in the interval $[-1, +1]$. In the large sample ($n = 300$) case for these distributions, the sampling distributions of both estimators are well described by the normal distribution apart from some extreme tail discrepancies.

For the exponential case the bias of the estimators is evident when $n = 30$, but is not so noticeable when $n = 300$. The sampling distributions for all the estimators are negatively skewed with short upper tails, although the skew is not particularly strong when $n = 300$.

2.2 ESTIMATING QUANTILE SYMMETRY MEASURES FROM DATA

In many applications the form of the distribution from which a data set has been sampled will not be known, in particular whether or not the parent distribution is symmetric. For a given value of α , $\tilde{S}(\alpha)$ defined in Equation (4) can be used to estimate $S(\alpha)$. Conclusions about $S(\alpha)$ can then be based on confidence intervals for $S(\alpha)$, or on testing a hypothesis about $S(\alpha)$, for example that $S(\alpha) = 0$.

Table 2.2: Statistics from simulated sampling distributions for three quantile symmetry estimators $\tilde{S}(\alpha)$, $\alpha = 0.25, 0.125$ and $\tilde{\Psi}$ using samples of sizes $n = 30, 300$ from a standard normal parent distribution; asymptotic SD for $\tilde{S}(\alpha)$ from Table 2.1.

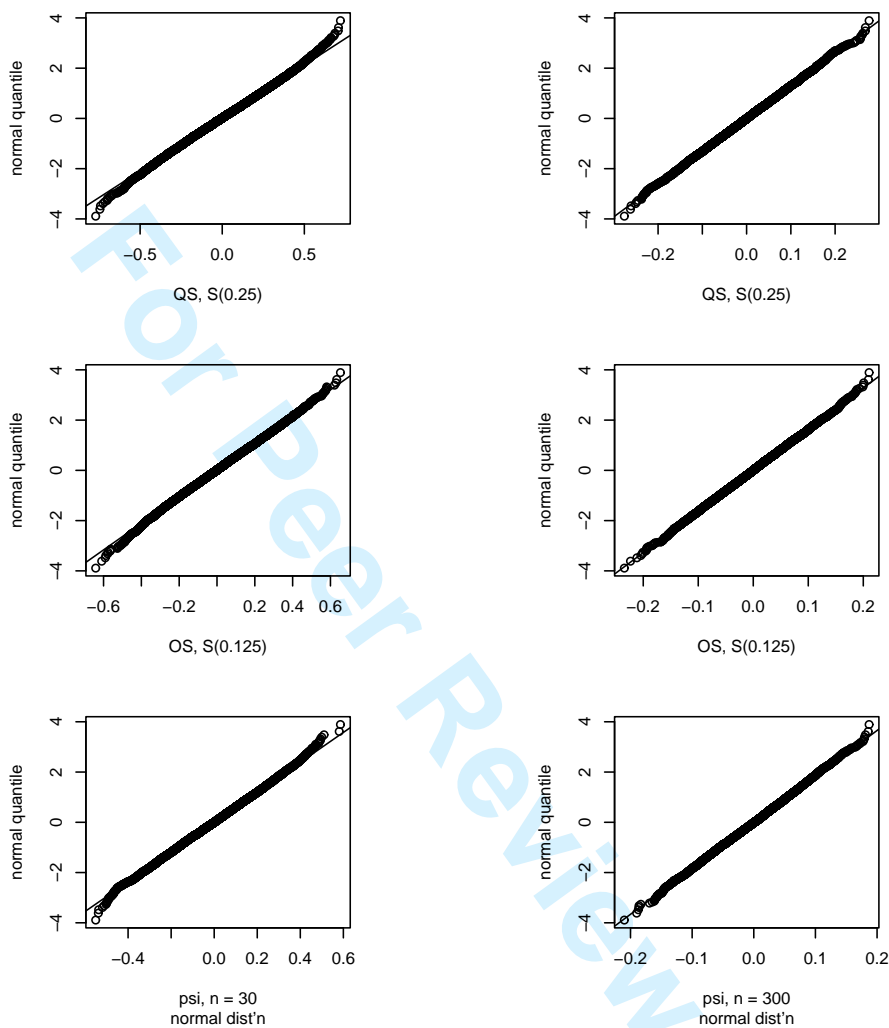
	$\alpha = 0.25$		$\alpha = 0.125$		$\tilde{\Psi}$	
	$n = 30$	$n = 300$	$n = 30$	$n = 300$	$n = 30$	$n = 300$
lower quartile	-0.1595	-0.0528	-0.1287	-0.0426	-0.113	-0.037
median	0.0018	0.0005	0.0010	-0.0006	-0.001	0.001
upper quartile	0.1679	0.0529	0.1282	0.0417	0.112	0.038
mean	0.0025	-0.0001	-0.0002	-0.0008	-0.001	-0.001
SD	0.2345	0.0777	0.1873	0.0619	0.164	0.054
asymptotic SD	0.2476	0.0783	0.1959	0.0620	–	–

Table 2.3: Statistics from simulated sampling distributions for three quantile symmetry estimators $\tilde{S}(\alpha)$, $\alpha = 0.25, 0.125$ and $\tilde{\Psi}$ using samples of sizes $n = 30, 300$ from a uniform parent distribution; asymptotic SD for $\tilde{S}(\alpha)$ from Table 2.1.

	$\alpha = 0.25$		$\alpha = 0.125$		$\tilde{\Psi}$	
	$n = 30$	$n = 300$	$n = 30$	$n = 300$	$n = 30$	$n = 300$
lower quartile	-0.1720	-0.0534	-0.1443	-0.0460	-0.141	-0.045
median	-0.0041	0.0005	-0.0015	0.0008	-0.002	-0.001
upper quartile	0.1639	0.0560	0.1358	0.0459	0.137	0.044
mean	-0.0032	0.0011	-0.0034	0.0006	-0.001	-0.001
SD	0.2382	0.0807	0.1986	0.0664	0.196	0.067
asymptotic SD	0.2582	0.0816	0.2108	0.0667	–	–

In general, it would not be safe to assume that the sampling distribution of $\tilde{S}(\alpha)$ is even approximately normal. It may be for a very large sample from a parent distribution which is not heavily skewed, in which case an estimate of the standard error based on Equation (5) would be needed. This itself requires estimates of the density at the median and at the upper and lower α quantiles. It therefore seems appropriate to use nonparametric bootstrap methodology to obtain confidence intervals for $S(\alpha)$ and to test the hypothesis that $S(\alpha) = 0$.

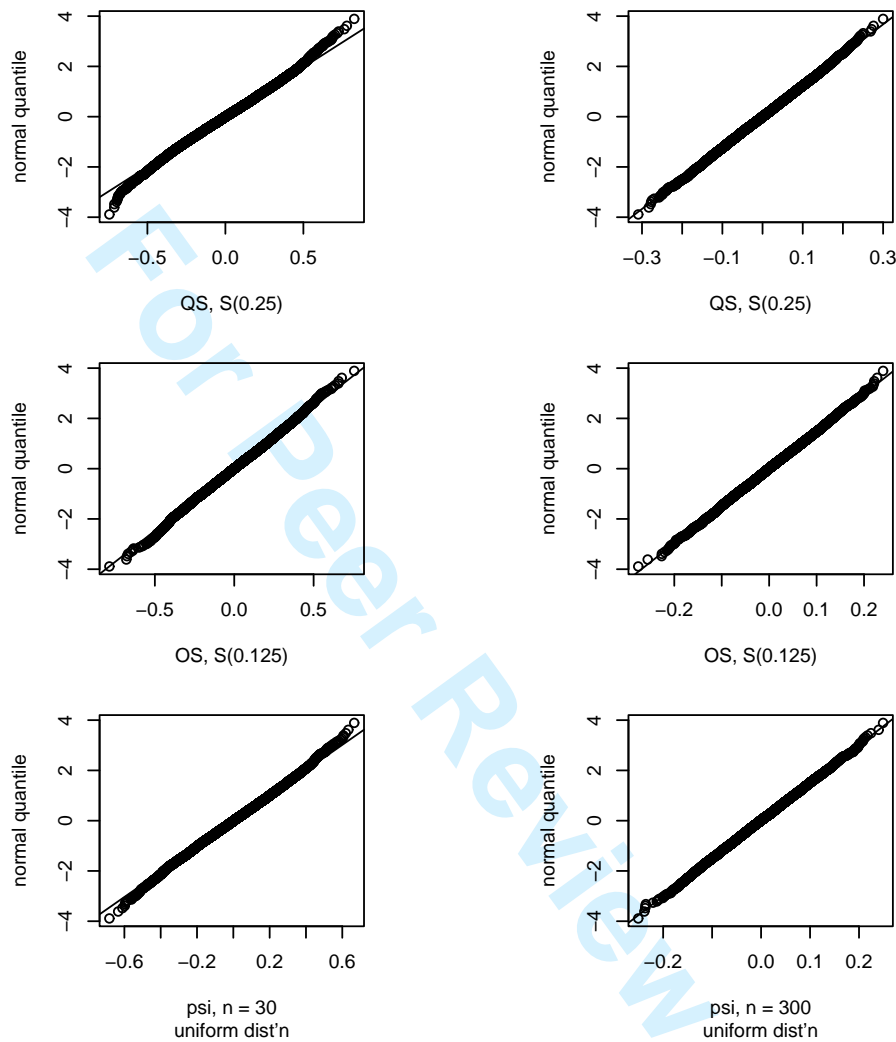
Figure 2.1: Normal Q-Q plots for $\tilde{S}(0.25)$, $\tilde{S}(0.125)$ and $\tilde{\Psi}$ for samples of size $n = 30, 300$ from the standard normal distribution (10,000 simulations).



BOOTSTRAP STANDARD ERRORS FOR $\tilde{S}(\alpha)$ AND CONFIDENCE INTERVALS FOR $S(\alpha)$

Bootstrap sampling was carried out using the standard non-parametric bootstrap procedure in the R system (R Development Core Team (2008)) with 10,000 resamples to obtain standard errors of the skewness estimators and 95% confidence intervals (CIs) for $S(0.25)$ and $S(0.125)$. Confidence intervals were generated using the bias corrected and adjusted (BCa) bootstrap, the method advocated by Efron and Tibshirani (1993). Section 2.4 shows some results from a small scale simulation study of the coverage rates of BCa bootstrap CIs for $S(0.25)$ and $S(0.125)$ using the distributions in Table 2.1.

Figure 2.2: Normal Q-Q plots for $\tilde{S}(0.25)$, $\tilde{S}(0.125)$ and $\tilde{\Psi}$ for samples of size $n = 30, 300$ from the uniform distribution on $(0, 1)$ (10,000 simulations).



In addition, the numerator and denominator of $\tilde{S}(\alpha)$ in Equation (4) are both linear combinations of the sample order statistics, so are L -estimators. Hutson and Ernst (2000), abbreviated to HE, show how to determine the exact nonparametric bootstrap estimates of the mean, the variance and the covariances for L -estimators, from which an approximation to the bootstrap standard error of the ratio of two such estimators can be found using Fieller's Theorem; that is for random variables X and Y with $\mu_X = E(X)$ and $\mu_Y = E(Y) \neq 0$,

$$\text{var} \left(\frac{X}{Y} \right) \approx \frac{1}{\mu_Y^2} \left[\left(\frac{\mu_X}{\mu_Y} \right)^2 \text{var}(Y) - 2 \left(\frac{\mu_X}{\mu_Y} \right) \text{cov}(X, Y) + \text{var}(X) \right]. \quad (7)$$

Table 2.4: Statistics from simulated sampling distributions for three quantile symmetry estimators $\tilde{S}(\alpha)$, $\alpha = 0.25, 0.125$ and $\tilde{\Psi}$ using samples of sizes $n = 30, 300$ from an exponential (mean 1) parent distribution for which $S(0.25) = 0.2619, S(0.125) = 0.4248$ and $\Psi = 0.4427$; asymptotic SD for $\tilde{S}(\alpha)$ from Table 2.1.

	$\alpha = 0.25$		$\alpha = 0.125$		$\tilde{\Psi}$	
	$n = 30$	$n = 300$	$n = 30$	$n = 300$	$n = 30$	$n = 300$
lower quartile	0.0871	0.2092	0.3055	0.3838	0.319	0.408
median	0.2566	0.2611	0.4281	0.4240	0.434	0.442
upper quartile	0.4126	0.3127	0.5423	0.4632	0.539	0.477
mean	0.2433	0.2597	0.4161	0.4223	0.421	0.441
SD	0.2315	0.0767	0.1722	0.0572	0.160	0.052
asymptotic SD	0.2437	0.0771	0.1811	0.0573	–	–

Table 2.5: Statistics from simulated sampling distributions for three quantile symmetry estimators $\tilde{S}(\alpha)$, $\alpha = 0.25, 0.125$ and $\tilde{\Psi}$ using samples of sizes $n = 30, 300$ from a standard Cauchy parent distribution; asymptotic SD for $\tilde{S}(\alpha)$ from Table 2.1.

	$\alpha = 0.25$		$\alpha = 0.125$		$\tilde{\Psi}$	
	$n = 30$	$n = 300$	$n = 30$	$n = 300$	$n = 30$	$n = 300$
lower quartile	−0.1892	−0.0590	−0.2397	−0.0758	−0.344	−0.238
median	0.0013	0.0020	0.0007	0.0024	−0.003	−0.006
upper quartile	0.1889	0.0629	0.2395	0.0785	0.340	0.226
mean	0.0004	0.0016	−0.0003	0.0015	–	–
SD	0.2661	0.0895	0.3296	0.1121	–	–
asymptotic SD	0.2868	0.0907	0.3611	0.1142	–	–

The HE computations require the evaluation of the exact bootstrap estimates of the covariances between all the order statistics which occur in the numerator and in the denominator of $\tilde{S}(\alpha)$ in Equation (4). Expressions for such covariances are given in HE Equations (3.1) and (3.2). For larger data sets, e.g. when n is in excess of 50, computations using the fully expanded expressions for the coefficients in the covariances may become numerically unreliable when they are formed from the sums of terms of opposite signs of increasing magnitude.

To avoid this, a simple remedy is to use a standard numerical quadrature procedure as, for example, found in the R system, to evaluate the integral expressions for these coefficients which are given in HE Equations (3.3) and (3.4).

Some numerical results for the two data sets are presented in Table 2.6. The table gives both estimated and ‘exact’ HE bootstrap standard errors for $\tilde{S}(\alpha)$, $\alpha = 0.25, 0.125$, along with the bootstrap 95% CIs for the corresponding $S(\alpha)$. In all cases, the estimated and the HE standard errors are close to each other. For the window strength data, the CIs contain 0, indicating no evidence of asymmetry with either of these measures. From the 95% CIs for the ceramic strength data there is no evidence of asymmetry with $S(0.25)$, but there is some evidence of asymmetry with $S(0.125)$.

Figure 2.3: Normal Q-Q plots for $\tilde{S}(0.25)$, $\tilde{S}(0.125)$ and $\tilde{\Psi}$ for samples of size $n = 30, 300$ from the exponential distribution with mean 1 (10,000 simulations).

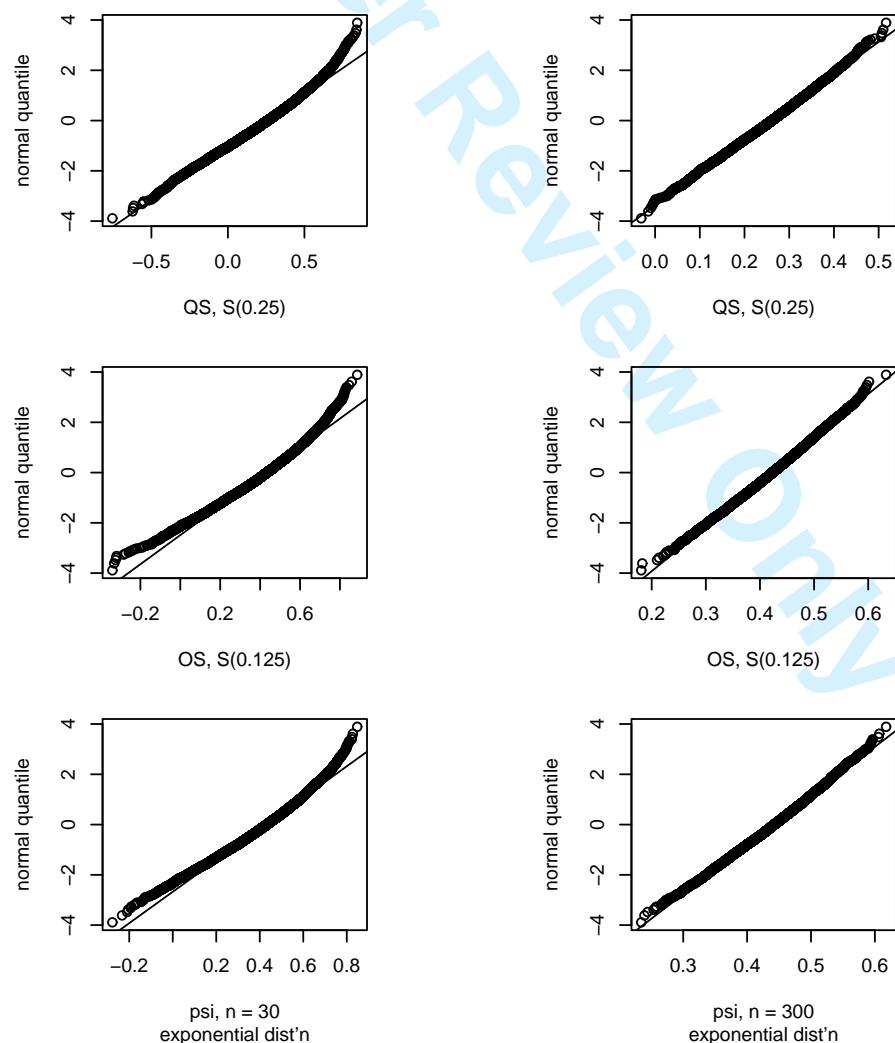


Figure 2.4: Normal Q-Q plots for $\tilde{S}(0.25)$ and $\tilde{S}(0.125)$ for samples of size $n = 30, 300$ from the standard Cauchy distribution (10,000 simulations).

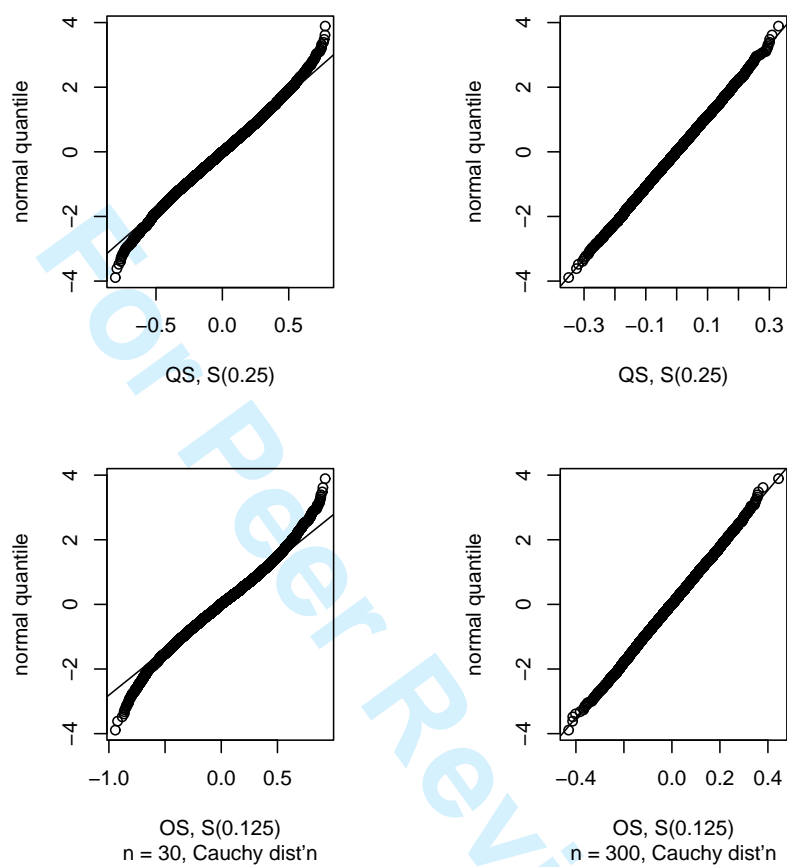


Table 2.6: Bootstrap standard errors (estimated and HE) for $\tilde{S}(\alpha)$, $\alpha = 0.25, 0.125$ and $\tilde{\Psi}$, and 95% bootstrap confidence intervals for $S(\alpha)$ and Ψ for the aircraft window strength ($n = 31$) and ceramic strength ($n = 240$) data sets.

	window strength			ceramic strength		
	$\alpha = 0.25$	$\alpha = 0.125$	$\tilde{\Psi}$	$\alpha = 0.25$	$\alpha = 0.125$	$\tilde{\Psi}$
$\tilde{S}(\alpha)$	0.154	0.169	0.150	-0.077	-0.173	-0.201
est'd SE	0.421	0.308	0.330	0.102	0.064	0.062
HE SE	0.419	0.298	–	0.103	0.065	–
95% CI	-0.60, 0.75	-0.52, 0.64	-0.44, 0.63	-0.29, 0.11	-0.30, -0.05	-0.32, -0.08

BOOTSTRAP HYPOTHESIS TESTS ABOUT $S(\alpha)$

Bootstrap methods were also used to test the null hypothesis $H_0 : S(\alpha) = 0$, loosely the symmetry hypothesis. First, for the data set y_1, \dots, y_n with median \tilde{M} , a symmetric version was constructed comprising the $2n$ values $y_i, 2\tilde{M} - y_i, i = 1, \dots, n$ (Davison and Hinkley (1997), p78). Then 10,000 bootstrap samples were taken to estimate the sampling distribution of $\tilde{S}(\alpha)$ under H_0 . The estimated p -value for the test is the proportion of simulated values whose absolute value is at least as large as $|\tilde{S}(\alpha)|$. For the window strength and ceramic strength data sets, the p -values for testing H_0 with $\alpha = 0.25, 0.125$ are:

	window strength	ceramic strength
$\alpha = 0.25$	0.829	0.386
$\alpha = 0.125$	0.666	0.007

The p -values are, of course, subject to bootstrap sampling error, but confirm the conclusions made about symmetry from the 95% CIs for these data sets.

2.3 ESTIMATING Ψ

Similar simulation and bootstrap approaches can be applied to the estimation of Ψ , the average quantile symmetry measure defined in Equation (3). As mentioned before, this measure only exists for distributions with a mean, so cannot be defined for the Cauchy distribution.

For a random sample of size n , the corresponding sample statistic

$$\tilde{\Psi} = \frac{\bar{X} - \tilde{M}}{\tilde{V}},$$

where \bar{X} is the sample mean, \tilde{M} is the sample median and $\tilde{V} = \frac{1}{n} \sum_{i=1}^n |X_i - \tilde{M}|$, is an estimator of Ψ . The sampling distribution of $\tilde{\Psi}$ will not be known in general, so is most easily investigated by simulation. Similarly, given sample data, bootstrapping can be used to make inferences about Ψ .

Tables 2.2-2.5 and Figures 2.1-2.3 also include some results for $\tilde{\Psi}$ based on the same sample sizes for the distributions considered before. For the normal and uniform cases, $\tilde{\Psi}$ is an unbiased estimator of Ψ , and its sampling distribution is symmetric. In both these cases when $n = 300$ the distribution is close to being normal, but when $n = 30$ the distributions are slightly short-tailed. For the exponential distribution, $\tilde{\Psi}$ appears biased, the bias reducing

with increasing n . The sampling distribution of $\tilde{\Psi}$ approaches normality much more slowly than for either of the two symmetric distributions. When $n = 300$, the distribution still has a noticeable negative skew. As the Cauchy distribution has no moments, neither does $\tilde{\Psi}$, and its sampling distribution is very heavy tailed relative to the normal distribution.

Bootstrapping was used in the same way as for estimating the quantile measures of skewness both to estimate the standard error of $\tilde{\Psi}$ and to obtain CIs for Ψ . Results for the two data sets, i.e. aircraft window strength data ($n = 31$) and ceramic strength data ($n = 240$), are given in Table 2.6.

A bootstrap test of the null hypothesis $H_0 : \Psi = 0$ was carried out in the same way as before. The resulting p -values are: for the window strength data 0.763, and for the ceramic strength data 0.002. The bootstrap confidence interval and test results indicate that it would not be unreasonable to assume that the window strength data come from a symmetric distribution, but the ceramic strength data come from a distribution with a negative skew.

2.4 COVERAGE RATES FOR BOOTSTRAP CONFIDENCE INTERVALS

Coverage rates for 90% and 95% BCa bootstrap CIs for $S(0.25)$, $S(0.125)$ and Ψ from the four distributions in Table 2.1 were estimated from 200 simulations each using 2000 bootstrap samples. Results are presented in Table 2.7 for the sample size $n = 30$. The standard errors of the coverage rate estimators for the 90% and 95% CIs are 2.1% and 1.5%, respectively. Generally, the estimated coverage rates are in line with the corresponding nominal values, and in a similar study using a sample size $n = 300$, all the estimates were in line.

Table 2.7: Estimated coverage rates for BCa bootstrap 90% and 95% CIs for $S(0.25)$, $S(0.125)$ and Ψ for four distributions using a sample size $n = 30$. Results from 200 simulations using 2000 bootstrap samples.

	$S(0.25)$		$S(0.125)$		Ψ	
	90%	95%	90%	95%	90%	95%
normal	88.5	94.5	91.0	96.0	92.5	97.5
uniform	92.0	96.5	90.5	95.5	87.0	90.5
exponential	90.5	95.5	91.0	98.5	88.0	95.5
Cauchy	93.0	98.0	89.5	96.0	–	–

3. MEASURES OF SKEWNESS BASED ON L-MOMENTS

L-moments and trimmed L-moments are discussed by Hosking (1990) and Elamir and Seheult (2003), respectively. These quantities naturally give rise to skewness measures for distributions which bear superficial similarities to $S(\alpha)$ in Equation (2).

The r^{th} L-moment of a distribution, λ_r , is defined in terms of the expected values of the order statistics in a random sample of size r , $r = 1, 2, \dots$, if they exist. In particular, using the notation $X_{k:m}$ for the k^{th} order statistic in a random sample of size m , $k = 1, \dots, m$,

$$\lambda_1 = E(X_{1:1}) = \mu, \lambda_2 = \frac{1}{2}E(X_{2:2} - X_{1:2}) \text{ and } \lambda_3 = \frac{1}{3}E(X_{3:3} - 2X_{2:3} + X_{1:3}), \quad (8)$$

as long as the expectations exist. λ_1 is clearly the distribution's mean; λ_2 is a measure of spread or scale, and is half of Gini's mean difference; λ_3 is a measure of skewness in that for a random sample of size 3, it gives a multiple of the expectation of the difference between the distances of the largest and the smallest values from the median, so is 0 for symmetric distributions. A unitless measure of skewness can then be defined as $\tau_3 = \lambda_3/\lambda_2$, with $-1 \leq \tau_3 \leq +1$. For a symmetric distribution $\tau_3 = 0$ as long as the required expectations exist. Elamir and Seheult (2003) call the sample sizes used in these definitions 'conceptual sample sizes' to distinguish them from the number of observations taken, i.e. the sample size n , in order to make inferences.

Now if $E(|X|^a)$, $a > 0$, exists, then $E(|X_{k:m}|^r)$ also exists when r, a, k and m satisfy $r < a \min(k, m - k + 1)$ (David (1981), p34). Thus, for example, with the standard Cauchy distribution, as $E(|X|^a)$ exists for $a = 1 - \epsilon$, $\epsilon > 0$, $E(|X_{k:m}|^r)$ exists for $r < (1 - \epsilon) \min(k, m - k + 1)$. In particular for the Cauchy example, $E(X_{k:m})$ exists for $k = 2, \dots, m - 1$, but not for $k = 1$ or $k = m$, and so none of the L-moments, λ_r , exist as each one involves the expectations of the most extreme of the order statistics in a sample of size r . Similarly, $E(X_{k:m}^2)$ only exists for $k = 3, \dots, m - 2$.

The r^{th} t -trimmed L-moment, $\lambda_r^{(t)}$, is defined in terms of the expected values of the order statistics from a random sample of conceptual size $r + 2t$, and it does not involve either the t largest or the t smallest order statistics. For $r = 1, 2$, and 3 , $\lambda_r^{(t)}$ are measures of central location, spread and skewness, respectively, as for the untrimmed L-moments. The t -trimmed L-moments of order $r \leq 3$ and the corresponding unitless skewness coefficients are defined by:

$$\begin{aligned}\lambda_1^{(t)} &= E(X_{t+1:2t+1}); & \lambda_2^{(t)} &= \frac{1}{2}E(X_{t+2:2t+2} - X_{t+1:2t+2}) \\ \lambda_3^{(t)} &= \frac{1}{3}E(X_{t+3:2t+3} - 2X_{t+2:2t+3} + X_{t+1:2t+3}); & \tau_3^{(t)} &= \lambda_3^{(t)}/\lambda_2^{(t)}\end{aligned}$$

as long as the expectations exist. In this case, Hosking (2007) shows that $|\tau_3^{(t)}| \leq 2(2t+3)/3(t+2)$, and gives an example of a distribution for which $|\tau_3^{(t)}| > 1$ is attained, $t > 0$. Note that $\lambda_1^{(t)}$ is the expectation of the median in a conceptual sample of size $2t+1$. The above definitions also apply to the case $t = 0$, giving rise to the L-moments $\lambda_r = \lambda_r^{(0)}$, $r = 1, 2, 3$, in Equation (8) and $\tau_3 = \tau_3^{(0)}$.

Hosking (1990) gives values of the first four untrimmed L-moments and of τ_3 for a number of standard distributions. Elamir and Seheult (2003) give values of the corresponding 1-trimmed L-moments and of $\tau_3^{(1)}$. For a symmetric distribution, $\lambda_3^{(t)}$ and $\tau_3^{(t)}$ are, of course, 0 as long as the necessary expectations exist. For the normal distribution $\lambda_1^{(t)} = 0$, and using symmetry about 0, $\lambda_2^{(t)} = E(X_{t+2:2t+2})$. The values of $\lambda_2^{(t)}$ are 0.56419, 0.29701, 0.20155 and 0.15251 for $t = 0, 1, 2, 3$, respectively; these values can also be found in Teichroew's (1956) tables. For the uniform distribution, $\lambda_1^{(t)} = 1/2$ and $\lambda_2^{(t)} = 1/(2(2t+3))$. For the exponential distribution $\lambda_1^{(t)} = E(X_{t+1:2t+1}) = \sum_{i=1}^{t+1} \frac{1}{2t-i+2}$, $\lambda_2^{(t)} = 1/(2(t+1))$ and $\lambda_3^{(t)} = 1/(3(t+1)(t+2))$. Hence $\tau_3^{(t)} = 2/(3(t+2))$, so that $\tau_3^{(0)} = \frac{1}{3}$, $\tau_3^{(1)} = \frac{2}{9}$, $\tau_3^{(2)} = \frac{1}{6}$ and $\tau_3^{(3)} = \frac{2}{15}$. For the standard Cauchy distribution, as long as $t \geq 1$ all the required expectations exist.

Skewness measures based on the t -trimmed moments, e.g. $\tau_3^{(1)}$, $\tau_3^{(2)}$ or $\tau_3^{(3)}$, may be preferred to the untrimmed skewness measure, i.e. $\tau_3^{(0)}$, as they exist for a wider class of distributions; indeed, the greater the level of trimming, the wider the class of distributions for which the corresponding trimmed skewness measure exists.

3.1 SAMPLE ESTIMATOR OF $\tau_3^{(t)}$

A natural estimator of $\tau_3^{(t)}$ from a random sample of size n is $t_3^{(t)} = l_3^{(t)}/l_2^{(t)}$, where $l_r^{(t)}$, $r = 2, 3$, are the sample t -trimmed L-moments. Elamir and Seheult (2003) use the work by Downton (1966) on estimators based on linear combinations of the order statistics to define $l_r^{(t)}$ as the following linear combination of the order statistics $X_{t+1:n}, \dots, X_{n-t:n}$

$$l_r^{(t)} = \frac{1}{r} \sum_{i=t+1}^{n-t} \left[\frac{\sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{i-1}{r+t-1-k} \binom{n-i}{t+k}}{\binom{n}{r+2t}} \right] X_{i:n}. \quad (9)$$

They show that this is an unbiased estimator of $\lambda_r^{(t)}$. Note that in Equation (9) the t largest and the t smallest order statistics have zero weight, i.e. they have been trimmed, and $l_r^{(t)}$ depends on all the remaining central order statistics. In contrast $\tilde{S}(\alpha)$ with a corresponding trimming level $\alpha \approx t/n$ does not depend on all these central order statistics.

In general where the required expectations exist $t_3^{(t)}$ will be a biased estimator of $\tau_3^{(t)}$, although not for symmetric distributions, and its exact distribution will be difficult to determine. As $t_3^{(t)} = l_3^{(t)}/l_2^{(t)}$ is a ratio of two random variables, Fieller's Theorem (Equation (7)) may again be used to obtain an approximation to $\text{var}(t_3^{(t)})$. The resulting approximation

$$\text{var}(t_3^{(t)}) \approx \frac{\text{var}(l_3^{(t)})}{\lambda_2^{(t)^2}} - 2\lambda_3^{(t)} \frac{\text{cov}(l_2^{(t)}, l_3^{(t)})}{\lambda_2^{(t)^3}} + \lambda_3^{(t)^2} \frac{\text{var}(l_2^{(t)})}{\lambda_2^{(t)^4}} \quad (10)$$

should be reasonable for large sample sizes, improving with increasing n . Further, Hosking (2007) shows that as long as $E(|X|^{2/(2t+1)+\epsilon})$ exists for some $\epsilon > 0$, $t_3^{(t)}$ is asymptotically normally distributed with mean $\tau_3^{(t)}$ and variance given by the right hand side of Equation (10). As for the quantile estimator $\tilde{S}(\alpha)$, simulation or the bootstrap may be used to investigate the sampling distribution of $t_3^{(t)}$.

ESTIMATING THE STANDARD ERROR OF $t_3^{(t)}$

Elamir and Seheult (2003) show how to obtain $\text{cov}(l_r^{(t)}, l_s^{(t)}) = E(l_r^{(t)} l_s^{(t)}) - E(l_r^{(t)}) E(l_s^{(t)}) = E(l_r^{(t)} l_s^{(t)}) - \lambda_r^{(t)} \lambda_s^{(t)}$ required in Equation (10). The expression they derive for $E(l_r^{(t)} l_s^{(t)})$ is in terms of the expectations of the squares of order statistics in samples of conceptual size up to $r + s + 4t - 1$ and of the expectations of the products of successive order statistics in samples of conceptual size up to $r + s + 4t$.

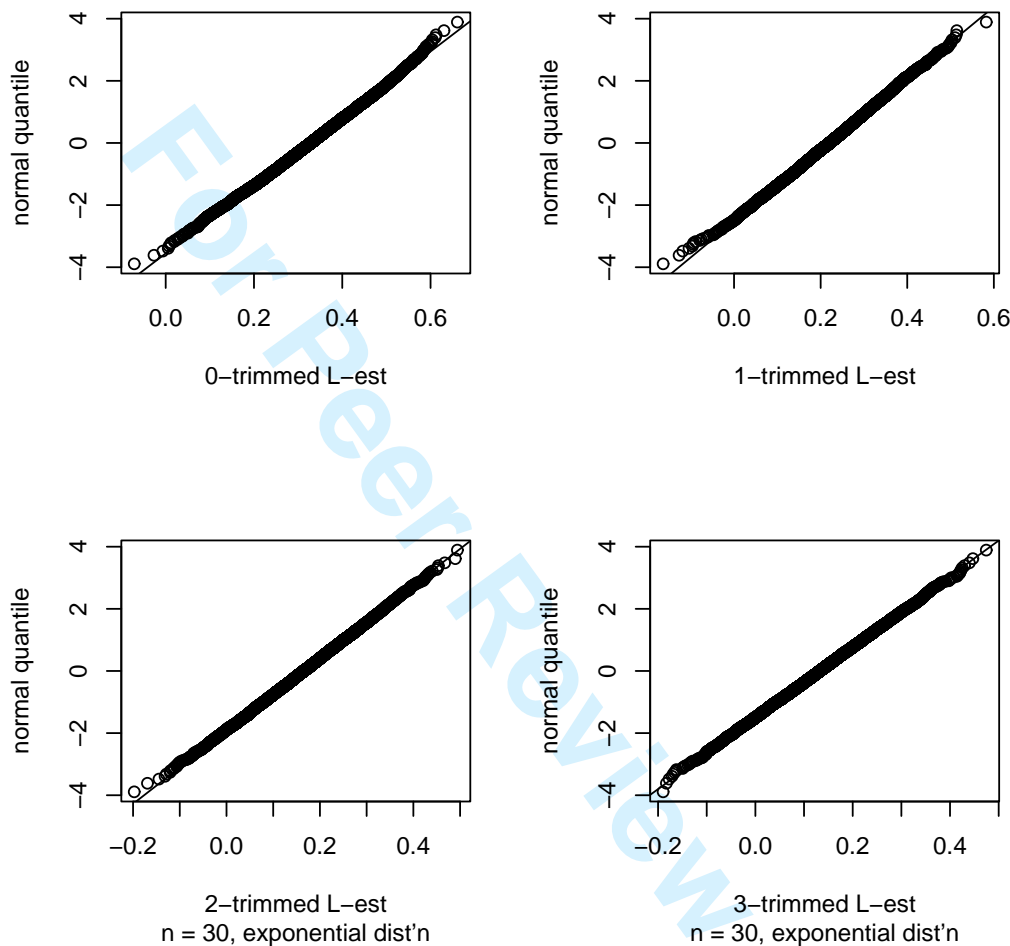
For the uniform and the exponential distributions expressions for the required expectations and covariances are given in the Appendix. In the case of the standard normal distribution, there are exact expressions for some, but possibly not all, the required expectations (see David (1981)). Additional expected values may be obtained using a single or a double numerical quadrature routine, as appropriate, as long as sufficient accuracy can be guaranteed. As this may not always be possible, in particular for any required double quadratures, values tabulated by Teichroew (1956) can be used. These tables give required expectations in samples of conceptual size up to 20, sufficient to determine $E(l_r^{(t)} l_s^{(t)})$ for r, s and $t = 0, 1, 2, 3$. For the Cauchy distribution, $\text{var}(l_r^{(t)})$, $r = 2, 3$ exists for $t \geq 2$.

Table 3.1: Approximate (ES) and simulated standard errors for $t_3^{(t)}$, $t = 0, 1, 2, 3$ for sample sizes $n = 30, 300$ from the normal, uniform and exponential distributions. Simulations using 10,000 samples.

t		normal		uniform		exponential		Cauchy	
		$n = 30$	$n = 300$	$n = 30$	$n = 300$	$n = 30$	$n = 300$	$n = 30$	$n = 300$
0	ES	0.0842	0.0251	0.0798	0.0240	0.0943	0.0283	–	–
	sim'd	0.0849	0.0252	0.0809	0.0245	0.0927	0.0283	–	–
1	ES	0.0797	0.0226	0.0816	0.0234	0.0874	0.0248	–	–
	sim'd	0.0798	0.0227	0.0827	0.0248	0.0883	0.0248	–	–
2	ES	0.0809	0.0217	0.0833	0.0226	0.0872	0.0235	n/a	n/a
	sim'd	0.0809	0.0219	0.0840	0.0226	0.0857	0.0233	0.1560	0.0534
3	ES	0.0834	0.0210	0.0857	0.0218	0.0889	0.0225	n/a	n/a
	sim'd	0.0840	0.0212	0.0871	0.0218	0.0882	0.0227	0.1269	0.0394

The resulting approximate standard errors of $t_3^{(t)}$ for $t = 0, 1, 2, 3$ from Elamir and Seheult's method (the ES method) are presented in Table 3.1 for the distributions used earlier, apart from the Cauchy distribution, again with the two sample sizes $n = 30, 300$. The table also gives estimated standard errors found from 10,000 simulations for all these distributions. The approximate values and the simulated values for the standard errors agree well for all levels of trimming across all the distributions. For both the normal and the uniform distributions normal Q-Q plots of the simulated distributions of $t_3^{(t)}$ were close to linear for both sample sizes and all trimming levels. Figure 3.1 shows the normal Q-Q plots for the exponential case with $n = 30$; the distributions are slightly negatively skewed with trimming levels $t = 0$ and $t = 1$, but closer to normal with trimming levels $t = 2$ and $t = 3$. For $n = 300$, the normal Q-Q plots showed that the distributions are close to normal with all trimming levels. For the Cauchy distribution, only simulated standard errors are given in Table 3.1, and only for the trimming levels $t = 2$ and $t = 3$ for which the required variances and covariances of the trimmed L-moment estimators exist. For the 2-trimmed estimator, the normal Q-Q plots in Figure 3.2 indicate long-tailed distributions for both sample sizes, but the distributions are closer to normal with the 3-trimmed estimator.

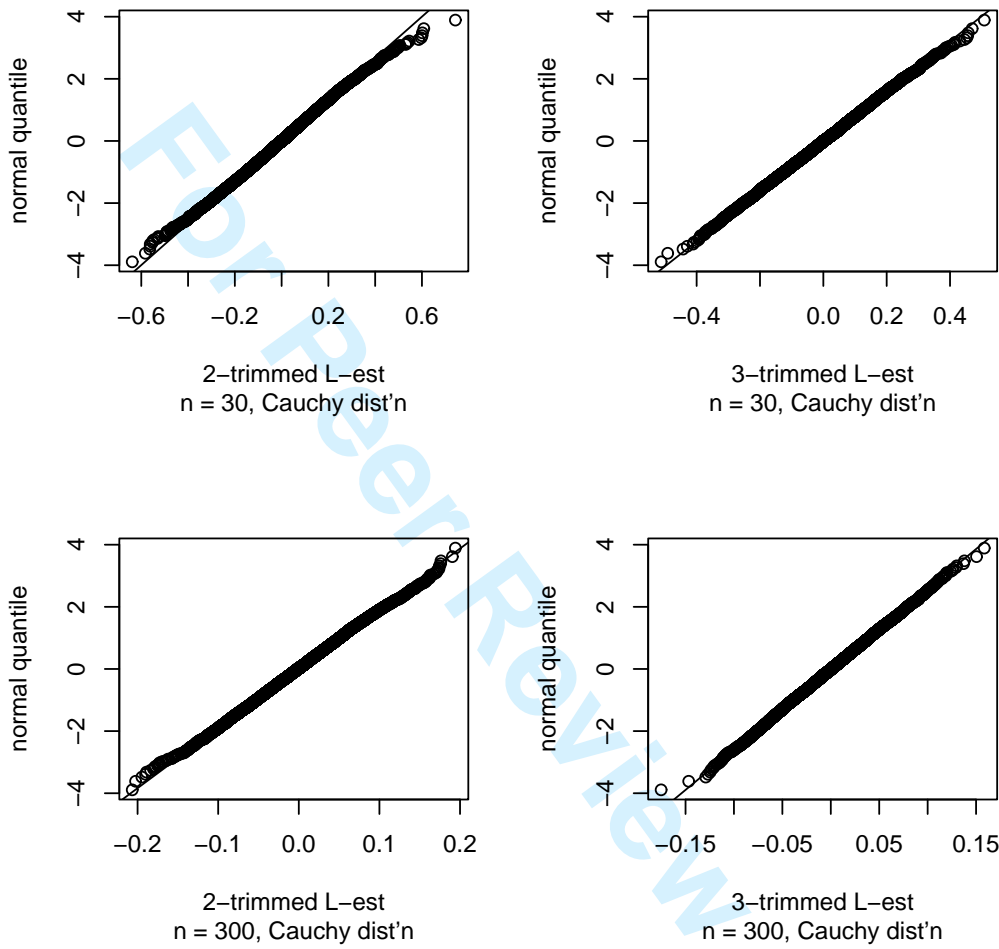
Figure 3.1: Normal Q-Q plots for the t -trimmed L-moment skewness estimator, $t = 0, 1, 2, 3$; sample of size $n = 30$ from the exponential distribution with mean 1 (10,000 simulations).



3.2 ESTIMATING L-MOMENT SYMMETRY MEASURES FROM DATA

For data from an unknown distribution, the t -trimmed L-moment estimator $t_3^{(t)}$ is used to estimate the corresponding symmetry measure $\tau_3^{(t)}$, $t = 0, 1, 2, 3$, as long as the measure exists. Also, the analytical methods of Elamir and Seheult (2003) and Hosking (2007) are used to determine a distribution free estimator of the standard error of $t_3^{(t)}$; the nonparametric bootstrap is also used to estimate the standard error of $t_3^{(t)}$ and further to make inferences about $\tau_3^{(t)}$. Coverage rates of bootstrap CIs are also considered.

Figure 3.2: Normal Q-Q plots for the t -trimmed L-moment skewness estimator, $t = 2, 3$ only, for sample sizes $n = 30, 300$ from the standard Cauchy distribution (10,000 simulations).



A DISTRIBUTION FREE ESTIMATOR OF THE STANDARD ERROR OF $t_3^{(t)}$

Elamir and Seheult (2003) indicate how to obtain an unbiased distribution free estimator of $\text{cov}(l_r^{(t)} l_s^{(t)}) = E(l_r^{(t)} l_s^{(t)}) - E(l_r^{(t)}) E(l_s^{(t)})$ from a random sample of size n . Hosking (2007), using results from Elamir and Seheult (2004), gives an equivalent expression which is simpler in form. Elamir and Seheult (2004) point out that although covariance estimators obtained in this way are unbiased, when a variance is being estimated there is no guarantee that a positive value will be obtained. This is unlikely to be a problem when n is large, but it may be when n is small. Given the estimated variances and covariance of $l_2^{(t)}$ and $l_3^{(t)}$, an estimated variance of $t_3^{(t)}$ can be found approximately using these estimates in Equation (10). Results

from the two data sets are considered later.

BOOTSTRAP STANDARD ERRORS FOR $t_3^{(t)}$, CONFIDENCE INTERVALS FOR $\tau_3^{(t)}$ AND HYPOTHESIS TESTING

An alternative approach to obtaining the standard error of $t_3^{(t)}$ for a data set for which the distribution is unknown is to use the nonparametric bootstrap in exactly the same way as for $\tilde{S}(\alpha)$ and $\tilde{\Psi}$. Note that $t_3^{(t)}$ is a ratio of L -estimators so the results of Hutson and Ernst (2000) may be applied as before to determine exact values of bootstrap means, variances and the covariance of the numerator and denominator L -estimators, and Equation (10) used again to obtain an approximation to the bootstrap standard error of $t_3^{(t)}$. The method requires the evaluation of covariances between all order statistics involved in Equation (9). When n is large, the necessary computations may prove to be excessive. Bootstrap hypothesis testing can again be used to test the null hypothesis that $\tau_3^{(t)} = 0$ for $t = 0, 1, 2$ and 3.

COVERAGE RATES FOR BOOTSTRAP CONFIDENCE INTERVALS

Table 3.2 shows the estimated coverage rates for 90% and 95% BCa bootstrap CIs for $\tau_3^{(t)}$, $t = 0, 1, 2, 3$ for the distributions in Table 2.1 using a sample size $n = 30$. The same simulation scheme was used as for the results given in Table 2.7. With the possible exception of estimating $\tau_3^{(0)}$ for the exponential distribution, estimated coverage rates are in line with the nominal rates. With a sample of size $n = 300$, all estimated coverage rates were in line.

Table 3.2: Estimated coverage rates for BCa bootstrap 90% and 95% CIs for $\tau_3^{(t)}$, $t = 0, 1, 2, 3$ for four distributions using a sample size $n = 30$. Results from 200 simulations using 2000 bootstrap samples.

	$\tau_3^{(0)}$		$\tau_3^{(1)}$		$\tau_3^{(2)}$		$\tau_3^{(3)}$	
	90%	95%	90%	95%	90%	95%	90%	95%
normal	87.0	94.5	88.5	94.5	92.0	97.0	92.5	97.0
uniform	90.0	94.5	90.5	94.5	90.5	95.0	89.5	96.5
exponential	83.0	92.0	91.5	95.5	90.5	95.0	88.0	93.5
Cauchy	—	—	—	—	87.0	93.0	89.0	96.5

RESULTS FOR DATA SETS

The unbiased distribution free method (ES using Hosking’s (2007) formulation) and the bootstrap methods (both HE and sampling) were applied to the two data sets, aircraft window strength ($n = 31$) and ceramic strength ($n = 240$); the results are given in Tables 3.3 and 3.4, respectively. As before, only BCa bootstrap CIs are given.

For the aircraft window strength data, the ES method leads to some negative variance estimates for all trimming levels considered, hence the ‘n/a’ entries in the table. The HE ‘exact’ bootstrap values are all somewhat smaller than the estimated bootstrap versions. The CIs all contain 0, and the corresponding p -values for testing a null hypothesis of distributional symmetry (i.e. $\tau_3^{(t)} = 0$) are large, indicating that this hypothesis would be a reasonable assumption in this case.

For the ceramic strength data, computations for the HE ‘exact’ bootstrap values proved to be excessive, and were not completed. The values from the ES method did not lead to negative variance estimates, and they agreed closely with the estimated bootstrap values for all trimming levels. None of the CI’s contained 0 and all the p -values were small, indicating that distributional symmetry would not be a reasonable assumption in this case.

Table 3.3: Approximate (ES) and bootstrap (HE and estimated) standard errors for $t_3^{(t)}$, 95% BCa bootstrap CIs for $\tau_3^{(t)}$ and p -values for testing $\tau_3^{(t)} = 0$, $t = 0, 1, 2, 3$, for the aircraft window strength ($n = 31$) data set.

t	0	1	2	3
$t_3^{(t)}$	0.1021	0.0758	0.0513	0.0472
ES	n/a	n/a	n/a	n/a
HE	0.0814	0.0984	0.1091	0.1193
est’d	0.0907	0.1021	0.1158	0.1303
b/s 95% CI	−0.063, 0.265	−0.131, 0.267	−0.188, 0.255	−0.201, 0.288
p -value	0.249	0.470	0.670	0.741

Table 3.4: Approximate (ES) and bootstrap (HE and estimated) standard errors for $t_3^{(t)}$, 95% BCa bootstrap CIs for $\tau_3^{(t)}$ and p -values for testing $\tau_3^{(t)} = 0$, $t = 0, 1, 2, 3$, for the ceramic strength ($n = 240$) data set.

t	0	1	2	3
$t_3^{(t)}$	-0.1980	-0.1209	-0.0869	-0.0676
ES	0.0326	0.0264	0.0247	0.0248
HE	n/a	n/a	n/a	n/a
est'd	0.0316	0.0271	0.0257	0.0255
b/s 95% CI	-0.264, -0.139	-0.175, -0.069	-0.135, -0.036	-0.115, -0.015
p -value	0.0001	0.0002	0.0005	0.0076

DISCUSSION

This paper has considered measures of skewness based on quantiles, which include the long-established Bowley's coefficient $S(0.25)$, and more recently introduced measures based on L-moments and on trimmed L-moments with low levels of trimming.

The quantile measure, $S(\alpha)$ for fixed α , $0 < \alpha < 1$, can be defined for any distribution. Choice of α is arbitrary; α close to 0.5 assesses symmetry towards the centre of the distribution, whereas α close to 0 assesses it towards the tails. Neither the averaged form of the quantile measure, Ψ , nor the L-moment measures, $\tau_3^{(t)}$, necessarily exist as they may depend on expected values which themselves do not exist. The greater the trimming level t used, the wider the class of distributions for which a trimmed measure is defined. For example, use of 1-trimmed L-moments allows the measure $\tau_3^{(1)}$ to be defined for distributions where the expected values of the two most extreme of order statistics in samples of size 2 and 3 may not exist. This measure exists for the Cauchy distribution, whereas $\tau_3^{(0)}$ does not. Similarly, measures based on 2- or 3-trimmed L-moments, i.e. $\tau_3^{(2)}$ and $\tau_3^{(3)}$, are defined for larger classes of distributions than the 1-trimmed measure $\tau_3^{(1)}$. Further trimming may be considered, but at the expense of assessing symmetry in an increasingly narrow central portion of the distribution.

Statistical properties of straightforward estimators in samples of size n have also been considered. Standard errors always exist for the estimators of $S(\alpha)$ and of $\tau_3^{(t)}$, $t \geq 2$, but not always for the estimators of Ψ , $\tau_3^{(0)}$ and $\tau_3^{(1)}$. Various analytical and computer-intensive methods

have been used to determine standard errors of estimators where they exist for a number of standard distributions, and for unknown distributions based on sample data. Where comparisons can be made, nonparametric bootstrap or simulated standard errors and standard errors from analytical methods are in reasonable agreement with each other, in particular for large sample sizes. However, neither of the two analytic methods for estimating standard errors from sample data proved to be entirely satisfactory. The ES method may lead to negative variance estimates which was evident in the smaller data set ($n = 31$). For larger sample sizes, the HE method applied to L-moment estimators could be computationally excessive compared with bootstrap estimation.

Where the estimators of the various symmetry measures are ratios of L -estimators they will be asymptotically normally distributed as long as the necessary moments exist. Simulations demonstrate this when sampling from symmetric distributions such as the normal and the uniform distributions; it is also in evidence when sampling from the exponential and the Cauchy distributions. Normality appears reasonable for all estimators, in particular for L-moment estimators, when using a sample of size $n = 300$ from any of the example distributions. It is also not unreasonable for some L-moment estimators, e.g. the 3-trimmed estimator, with a sample of size $n = 30$. However the distribution of quantile-based estimators for $n = 30$ is not close to normal.

The non-parametric bootstrap can be used to make inferences about the value of a particular symmetry measure using CIs or hypothesis testing. Bias corrected and adjusted (BCa) CIs were found to give adequate coverage rates when estimating a range of symmetry measures from the normal, the uniform and the Cauchy distributions using samples of sizes $n = 30, 300$. Coverage rates when sampling from the exponential distribution were similarly adequate with the exception of estimating the untrimmed L-moment measure, $\tau_3^{(0)}$, from a sample of size $n = 30$.

The normal approximation for the distribution of all estimators appears to be sufficiently close when $n = 300$ so that standard normal CIs can be used in place of BCa bootstrap CIs, with standard errors estimated from the bootstrap. When $n = 30$, standard normal CIs may be used in the same way for many of the trimmed L-moment measures, but not when estimating quantile-based symmetry measures; in this case BCa bootstrap CIs would be more reliable.

Similar comments apply if a test of distributional symmetry is required. For large samples, e.g. $n = 300$, the standard z -test with bootstrapped standard error in the denominator will give a similar p -value to the bootstrap test. This will also apply for small samples, e.g. $n = 30$, for testing many of the trimmed L-moment measures.

In conclusion, for ease of use across a wide range of situations, it is recommended that 2- or 3-trimmed L-moment symmetry measures are used, i.e. $\tau_3^{(2)}$ or $\tau_3^{(3)}$. Even for moderately sized samples, e.g. as small as $n = 30$, reliable CIs and tests of symmetry can be constructed based on the standard normal distribution and using a standard error most simply estimated from the nonparametric bootstrap. The arguments and methods used throughout the paper could be adapted to define and assess measures of kurtosis in data sets or distributions if they are required.

APPENDIX

1. Expected values of squares of order statistics and of products of successive order statistics in samples of size k from the uniform distribution on the interval $(0, 1)$.

$$E(Y_{r:k}^2) = \frac{r(r+1)}{(k+1)(k+2)}, \quad r = 1, \dots, k.$$

$$E(Y_{r:k} Y_{r+1:k}) = \frac{r(r+2)}{(k+2)(k+3)}, \quad r = 1, \dots, k-1.$$

2. Variance of order statistics and covariance between successive order statistics in samples of size k from the exponential distribution with mean 1.

$$\text{var}(Y_{r:k}) = \sum_{s=1}^r \frac{1}{(k-s+1)^2}, \quad r = 1, \dots, k$$

$$\text{cov}(Y_{r:k}, Y_{r+1:k}) = \text{var}(Y_{r:k}), \quad r = 1, \dots, k-1.$$

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