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#### AN EXPONENTIAL MODEL FOR DAMAGE ACCUMULATION

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#### Abstract

In this paper, we present a general model for predicting the fatigue behavior for any stress level and amplitude using the exponential model. Based on the Wöhler field for fixed stress level, a compatibility functional equation enables us to derive the general model with eight parameters. The problem of parameter estimation is then discussed and some methods are described. Some examples are finally presented to illustrate the derived model and the proposed methods of estimation.

**Key Words**: Wöhler field, damage accumulation, interpolation of fatigue results, testing strategies, exponential model, maximum likelihood method, constrained maximum likelihood method.

#### 1 Introduction

In fatigue analysis that the stress ratio  $R = \sigma_{max}/\sigma_{min}$  is an important factor, next only to the stress level ( $\sigma_{min}$ ,  $\sigma_{max}$ ,  $\sigma_{mean}$ ), which needs to be taken into account while dealing with fatigue behavior. This becomes particularly relevant while considering real stress histories instead of constant stress level situations. However, most of the existing laboratory test strategies and fatigue models are applicable for a given stress level but can not predict the fatigue behavior for different stress levels; see, for example, Coleman (1958), Bastenaire (1972), ASTM (1981), Spindel and Haibach (1981), Fernández-Canteli (1982), Castillo *et al.* (1985), Castillo and Galambos (1987), Castillo and Hadi (1995), Pascual and Meeker (1999), and Castillo and Fernández-Canteli (2001).

Though there are some models that take mean stress effects into account [see Conway and Sjodahlo (1991) and Dowling and Thangjitham (2000)], such simplified models are far from satisfactory. Castillo and Fernández-Canteli (2006) and Castillo *et al.* (2006) presented a general model that enables the prediction of the fatigue behavior based on two groups of tests run at two different stress levels. This was a significant development as other models did not facilitate this prediction and also that the required testing strategy for this model was very simple.

In this paper, we consider this general model for the case of the exponential distribution and describe the prediction of the fatigue behavior for any stress level and amplitude, and discuss some methods of estimation of the parameters underlying the model. In Section 2, we describe first the exponential parameter fatigue model. In Section 3, we describe how this model can be extrapolated to different test conditions, i.e., different pairs of  $\sigma_{max}$  and  $\sigma_{min}$ , and then discuss the consequences of forcing the compatibility or independence of the results on the selected  $\sigma_{max}$  and  $\sigma_{min}$  test levels. In Section 4, we present a reparameterization of the model that is convenient for the estimation of the model parameters. In Section 5, we discuss some methods of estimation such as the maximum likelihood and constrained maximum likelihood methods. In Section 6, we present some examples to illustrate the derived model and the proposed methods of estimation. Finally, some concluding remarks are made in Section 7.

## 2 Derivation of the Fatigue Model

Let us consider a series of tests with constant  $\sigma_{min}$  and variable  $\sigma_{max}$ , but in each fatigue test we fix  $\sigma_{max}$  and repeat cycles in which  $\sigma$  is varied from  $\sigma_{min}$  to  $\sigma_{max}$  until the failure of the specimen occurs. In other words,  $\sigma_{min}$  is constant for all tests, and different tests in the series can have identical or possibly different  $\sigma_{max}$ . The resulting lifetimes N (the number of cycles to failure) are observed. Since  $\sigma_{min}$  is held constant, we use as variables the stress ratio  $R = \sigma_{max}/\sigma_{min}$  and N.

It is generally accepted that there are five variables initially involved in the fatigue problem:  $P, N, N_0, R$  and  $R_0$ , where P is the probability of fatigue failure of a piece when subjected to N cycles,  $N_0$  is the threshold value for N corresponding to the minimum observable lifetime for any R, and  $R_0$  is the endurance stress ratio limit below which fatigue failures can not occur. This means that there exists a relationship among the five initially independent variables, of the form

$$r(N, N_0, R, R_0, P) = 0, (1)$$

where  $r(\cdot)$  is an initially unknown function. However, using the  $\Pi$ -Theorem [see, for example, Buckingham (1914, 1915a,b) and Castillo and Fernández-Canteli (2001)], these initial five variables can be reduced to three non-dimensional variables, viz.,  $N/N_0$ ,  $R/R_0$  and P. Consequently, (1) can be written in terms of these three non-dimensional variables as

$$r(N, N_0, R, R_0, P) = 0 \quad \Longleftrightarrow \quad s\left(\frac{N}{N_0}, \frac{R}{R_0}, P\right) = 0; \tag{2}$$

since we are interested in P, we can express

$$P = q\left(\frac{N}{N_0}, \frac{R}{R_0}\right),\tag{3}$$

where  $s(\cdot)$  and  $q(\cdot)$  are functions to be determined. Thus, only the non-dimensional quotients  $N/N_0$  and  $R/R_0$  exert influence on the probability of failure P, and so either  $N/N_0$  and  $R/R_0$ , or some monotone functions of them, say  $h(N/N_0)$  and  $g(R/R_0)$ , need to be considered.

In this paper, we have chosen the h and g functions to be the logarithms of  $N/N_0$  and  $R/R_0$ , respectively. For simplicity in notation, we denote throughout this paper

$$N^* = \log(N/N_0), \quad R^* = \log(R/R_0), \quad N \ge N_0, \quad R \ge R_0.$$
(4)

The selection of the exponential model is then based on the following important considerations:

- 1. Weakest link principle: This principle states that the fatigue lifetime of a longitudinal element is the minimum fatigue life of its constituent pieces. In other words, the selected family of distributions must hold (be valid) for different specimen lengths.
- 2. Limit behavior: To include the extreme case of the size of the supposed pieces constituting the element going to zero, or the number of pieces going to infinity, it is necessary for the distribution function family to be an asymptotic family; see Galambos (1987), Castillo (1988), Castillo *et al.* (2004), and Castillo *et al.* (2006).
- 3. Limited range: Experience shows that the selected non-dimensional variables,  $N^*$  and  $R^*$ , have a finite lower end, which must coincide with the theoretical lower end of the selected cumulative distribution function (cdf).
- 4. Compatibility: In the S-N field, the cdf  $G(N^*; R^*)$  of the lifetime given stress range should be compatible with the cdf of the stress range given lifetime  $F(R^*; N^*)$ , i.e.,

$$G(N^*; R^*) = F(R^*; N^*).$$
(5)

Though in standard tests  $R^*$  is fixed and the associated random lifetime  $N^*$  is determined,  $R^*$  is interpreted here as the random stress that needs to be applied in order to produce failure at  $N^*$ .

It is important to note here that the exponential family satisfies all the conditions stated above.

#### 2.1Compatibility

The compatibility condition in (5) for the exponential model requires

$$1 - \exp[-\delta(N^*)(R^* - \lambda(N^*))] = 1 - \exp[-\rho(R^*)(N^* - \eta(R^*))],$$
(6)

which leads to the functional equation

$$R^*\delta(N^*) - \delta(N^*)\lambda(N^*) - \rho(R^*)N^* + \rho(R^*)\eta(R^*) = 0$$
(7)

whose solution is [see Castillo and Ruiz-Cobo (1992, p. 52) and Castillo et al. (2004, pp. 60-61)]

$$\rho(R^*) = aR^* + b, \tag{8}$$

$$\eta(R^*) = \frac{cR^* + d}{aR^* + b},$$
(9)

$$(N^*) = aN^* - c, (10)$$

$$\delta(N^*) = aN^* - c,$$
(10)  
$$\lambda(N^*) = \frac{-bN^* + d}{aN^* - c}.$$
(11)

This leads to the model

$$G(N^*; R^*) = F(R^*; N^*) = 1 - \exp[(c - aN^*)R^* - bN^* + d],$$
(12)

where a, b, c and d are arbitrary constants; now, upon reparameterizing and subsuming two parameters into  $N_0$  and  $R_0$ , we obtain

$$G(N^*; R^*) = F(R^*; N^*) = 1 - \exp\left[-\frac{R^*N^* - \lambda}{\delta}\right],$$
(13)

where  $\lambda$  and  $\delta$  are some constants.

Once the model has been established in non-dimensional terms as in (13), as we need to recover the initial variables, we can reexpress the model in (13) as

$$F(\log N; \log R) = 1 - \exp\left\{-\left[\frac{(\log N - B)(\log R - C) - E}{D}\right]\right\},\tag{14}$$

where  $B = \log N_0, C = \log R_0$ , and E and D are non-dimensional model parameters. Their physical meanings are as follows (see Figure 1):

- B: Threshold value of log-lifetime (logarithm of  $N_0$ ),
- C: Endurance limit (logarithm of  $R_0$ ),
- E: Parameter defining the position of the corresponding zero-percentile hyperbola,
- D: Scale factor.



Figure 1: Percentiles curves representing the relationship between lifetime,  $N^*$ , and stress range,  $\Delta \sigma^*$ , in the S-N field for the fatigue model in (14).

The percentile curves are shown in Figure 1. The zero-percentile curve represents the minimum possible required number of cycles of fatigue failure for different values of R. All the remaining percentiles happen to be the positive branches of a hyperbola, i.e.,

$$N^*R^* = E + D\left[-\log(1-p)\right],$$

where  $p \in [0, 1]$  defines the percentile. For such curves, the minimum number of cycles to fatigue failure decreases with increasing  $R^*$ , which is in agreement with experimental results. The parameters E and D determine the *p*-percentile, and hence define a unique curve on the plot.

It can be seen from (13) that

$$N^*R^* \sim Exp(\lambda, \delta) \quad \Leftrightarrow \quad N^* \sim Exp\left(\frac{E}{R^*}, \frac{D}{R^*}\right),$$
(15)

for given  $R^*$ , where  $Exp(\lambda, \delta)$  denotes a two-parameter exponential distribution with  $\lambda$  as the threshold parameter and  $\delta$  as the scale parameter. It is of interest to note that (15) has a non-dimensional form and reveals that the probability of failure of a piece subject to a stress ratio  $R^*$  during  $N^*$  cycles depends only on the product  $N^*R^*$ . Thus,  $V = N^*R^*$  is useful for comparing fatigue strength at different stress levels that are maintained constant, and can be considered as a normalizing variable.

## **3** Physical and compatibility considerations

Since for each constant  $\sigma_{min}$ , a model as in (14) is obtained, we can consider its parameters B, C, D and E to be functions of  $\sigma_{min}$ . Figure 2(a) shows the Wöhler curves for two groups of tests run at constant  $\sigma_{min} = 0.8$  and at constant  $\sigma_{min} = 0.4$ . Note that larger values of  $\sigma_{min}$  lead to lower percentile curves. If we consider the Wöhler curves for two groups of tests run at constant  $\sigma_{max}$ , say  $\sigma_{max} = 1.5$ , we obtain the model in (14) once again and a set of percentiles, for example {0.01, 0.05, 0.5, 0.95, 0.99}, are shown

in Figure 2(b). However, if the tests are run at constant  $\sigma_{max} = 1$ , the resulting percentiles change as displayed in the same figure. Note that larger values of  $\sigma_{max}$  lead to lower percentile curves.



Figure 2: Wöhler curves for percentiles 0.01, 0.05, 0.5, 0.95, 0.99: (a) for constant  $\sigma_{min} = 0.4$  and  $\sigma_{min} = 0.8$ , and (b) for constant  $\sigma_{max} = 1$  and  $\sigma_{max} = 1.5$ .

An important property to be observed is that the same percentile curves never intersect if they are associated with two different  $\sigma_{max}$  or with two different  $\sigma_{min}$ , but they intersect if one set of percentiles is associated with constant  $\sigma_{max}$  and another with constant  $\sigma_{min}$  as shown in Figure 3, where the four sets of percentiles have been plotted together, with dashed lines corresponding to Wöhler curves for constant  $\sigma_{min}$  and continuous lines corresponding to Wöhler curves for constant  $\sigma_{max}$ . Note that the line joining the intersections of associated percentiles for one constant  $\sigma_{max}$  and one constant  $\sigma_{min}$  must be straight and horizontal since both fields, i.e., the associated cdf's, must coincide. This is a very strong condition indeed and it assists greatly in deriving a simple model which is able to deal with different stress levels. Though in Figure 3 we have these four sets, we realize that there is an infinite set of them. More precisely, two families of S-N curves exist, one for different constant values of  $\sigma_{max}$  and another for different constant values of  $\sigma_{min}$ .

In this paper, we consider the case when the test experiments are conducted for constant  $\sigma_{max} = \sigma_M$ and constant  $\sigma_{min} = \sigma_m$ . As mentioned above, the model in (14) is valid only if all tests are conducted at the same stress level, and in particular, for given values of  $\sigma_{max} \equiv \sigma_M$  or  $\sigma_{min} \equiv \sigma_m$ . Thus, if two sample data do not coincide at either their  $\sigma_M$  or  $\sigma_m$ , the model can not be applied. Since there are two different cases, we must consider two fatigue models of the type in (14), i.e., we initially have two sets of parameters  $B_m, C_m, D_m, E_m$  and  $B_M, C_M, D_M, E_M$ , where the subindices m and M have been used for the cases of constant  $\sigma_{min} = \sigma_m$  and constant  $\sigma_{max} = \sigma_M$ , respectively. However, for the same stress amplitude and level, i.e., when both  $\sigma_M$  and  $\sigma_m$  coincide, for the compatibility condition to hold, we must have the same model for both cases, i.e., for all N we must have

$$\frac{(\log N - B_m(\sigma_m))(\log R - C_m(\sigma_m)) - E_m(\sigma_m)}{D_m(\sigma_m)} = \frac{(\log N - B_M(\sigma_M))(\log R - C_M(\sigma_M) - E_M(\sigma_M))}{D_M(\sigma_M)}.$$
 (16)





Figure 3: Wöhler curves for percentiles  $\{0.01, 0.05, 0.5, 0.95, 0.99\}$  for  $\sigma_{max} = 1$  and  $\sigma_{max} = 1.5$ , and  $\sigma_{min} = 0.4$  and  $\sigma_{min} = 0.8$ . Dashed lines refer to Wöhler curves for constant  $\sigma_{min}$ , and continuous lines refer to Wöhler curves for constant  $\sigma_{max}$ .

For the functional equation in (16) to be satisfied for any N,  $\sigma_m$  and  $\sigma_M$ , both models must have the same parameters. Rewriting now the model in (16) as

$$\frac{\log N - \left[B_m(\sigma_m) + \frac{E_m(\sigma_m)}{\log R - C_m(\sigma_m)}\right]}{\frac{D_m(\sigma_m)}{\log R - C_m(\sigma_m)}} = \frac{\log N - \left[B_M(\sigma_M) + \frac{E_M(\sigma_M)}{\log R - C_M(\sigma_M)}\right]}{\frac{D_M(\sigma_M)}{\log R - C_M(\sigma_M)}} \quad \forall N$$
(17)

and forcing the exponential parameters to coincide, we obtain

$$\frac{D_m(\sigma_m)}{D_M(\sigma_M)} = \frac{\log \sigma_M - \log \sigma_m - C_m(\sigma_m)}{\log \sigma_M - \log \sigma_m - C_M(\sigma_M)} \quad \forall \sigma_m, \sigma_M, \tag{18}$$

$$B_M(\sigma_M) = B_m(\sigma_m) - \frac{E_M(\sigma_M)}{\log \sigma_M - \log \sigma_m - C_M(\sigma_M)} + \frac{E_m(\sigma_m)}{\log \sigma_M - \log \sigma_m - C_m(\sigma_m)} \quad \forall \ \sigma_m, \sigma_M.$$
(19)

The functional equations in (18) and (19) deserve a careful attention to get a deeper understanding of our problem; they are not simple equalities, but each a full collection of equalities as they hold for any feasible pair  $\sigma_m, \sigma_M$ . Eqs. (18) and (19), considered as functions of  $\sigma_m$ , must be independent of  $\sigma_M$ . This simple but powerful condition allows us to derive the structure of the functions  $D_m(\sigma_m), D_M(\sigma_M)$ and  $B_M(\sigma_M)$ . The complete derivation, as given by Castillo *et al.* (2006), leads to the final model as

$$F(N;\sigma_m,\sigma_M) = 1 - \exp\left\{-\left[C_0 + C_1\log N + C_2\log\sigma_m + C_3\log\sigma_M + C_4\log N\log\sigma_m + C_5\log N\log\sigma_M + C_6\log\sigma_m\log\sigma_M + C_7\log N\log\sigma_m\log\sigma_M\right]\right\}$$
(20)

(24)

$$= 1 - \exp\{-\left[(C_0 + C_2 \log \sigma_m + C_3 \log \sigma_M + C_6 \log \sigma_m \log \sigma_M) + \log N(C_1 + C_4 \log \sigma_m + C_5 \log \sigma_M + C_7 \log \sigma_m \log \sigma_M)\right]\}.$$
(21)

Observe that this model involves eight parameters  $C_i$  (i = 0, 1, ..., 7).

#### 3.1 Physical conditions

For the model to be physically feasible, the constants must satisfy some constraints. In particular:

- The cdf in (21) must be increasing in  $\log N$ ;
- The cdf in (21) must be non-increasing in  $\sigma_m$ ;
- The cdf in (21) must be non-decreasing in  $\sigma_M$ ;
- The curvature of the zero-percentile of  $(\log N, \log R)$  for constant  $\sigma_{min}$  must be non-negative;
- The curvature of the zero-percentile of  $(\log N, \log R)$  for constant  $\sigma_{max}$  must be non-negative.

#### 3.2 Resulting models

In summary, we can obtain some final set of feasible submodels of (21) as follows:

**Linear Model:** The simplest model with no asymptotes is given by

$$F(N;\sigma_m,\sigma_M) = 1 - \exp\{-[C_0 + C_1 \log N + C_2 \log \sigma_m + C_3 \log \sigma_M]\}, C_1, C_2 \le 0, C_3 \le 0,$$
(22)

which in log-log scale leads to a Wöhler field made of straight lines.

Model with asymptotes independent on  $\sigma_m$  and  $\sigma_M$ : The model with  $\log N$  asymptotes independent on  $\sigma_m$  and  $\sigma_M$  is given by

$$F(N;\sigma_m,\sigma_M) = 1 - \exp\left\{-\left[C_0 + C_2\log\sigma_m + C_3\log\sigma_M + C_4\log N\log\sigma_m + C_5\log N\log\sigma_M\right]\right\} \\ C_2, C_4 \le 0, \ C_3, C_5 \ge 0, \ C_4 + C_5 \ge 0,$$
(23)

which is obtained when  $C_1 = C_6 = C_7 = 0$ .

The log N asymptotes are log  $N = -\frac{C_3}{C_5}$  for the case of constant  $\sigma_{min}$ , and log  $N = -\frac{C_2}{C_4}$  for the case of constant  $\sigma_{max}$ , and the log R asymptotes are log  $R = -(C_4 + C_5) \log \sigma_m / C_5$  for the case of constant  $\sigma_{min}$ , and log  $R = (C_4 + C_5) \log \sigma_M / C_4$  for the case of constant  $\sigma_{max}$ .

Model with fixed asymptotes: The model with  $\log R$  asymptotes independent on  $\sigma_m$  and  $\sigma_M$  is given by

$$F(N;\sigma_m,\sigma_M) = 1 - \exp\left\{-\left[C_0 + C_1 \log N + C_2 \log \sigma_m + C_3 \log \sigma_M + C_4 \log N (\log \sigma_m - \log \sigma_M) + C_6 \log \sigma_m \log \sigma_M\right]\right\},$$

which is obtained when  $C_7 = 0$  and  $C_4 + C_5 = 0$ . The log R asymptotes are log  $R = -C_1/C_5$  for the case of constant  $\sigma_{min}$ , and log  $R = C_1/C_4$  for the case of constant  $\sigma_{max}$ .

The log N asymptotes are log  $N = -\frac{C_3 + \log \sigma_M C_6}{C_5}$  for the case of constant  $\sigma_{min}$ , and  $\log N = -\frac{C_2 + \log \sigma_M C_6}{C_4}$  for the case of constant  $\sigma_{max}$ , so that they are constant if, in addition,  $C_6 = 0$ .

**General Model:** The general model with  $\log R$  and  $\log N$  asymptotes dependent on  $\sigma_m$  and  $\sigma_M$  is given by

$$F(N;\sigma_m,\sigma_M) = 1 - \exp\left\{-\left[C_0 + C_1\log N + C_2\log \sigma_m + C_3\log \sigma_M + C_4\log N\log \sigma_m + C_5\log N\log \sigma_M + C_6\log \sigma_m\log \sigma_M + C_7\log N\log \sigma_m\log \sigma_M\right]\right\},$$
(25)

subject to adequate constraints on the parameters.

These constraints are very important in order to have a physically meaningful model.

#### 3.3 Testing strategies

The aim of any testing strategy is to estimate the model parameters in (21). For this purpose, a testing strategy involving one single group of tests with constant  $\sigma_{max}$  or one single group with constant  $\sigma_{min}$  or two groups one with constant  $\sigma_{max}$  and one with constant  $\sigma_{min}$  are not sufficient, because of the linear combination of the parameters that are involved in the model. In contrast, two groups of tests one with constant  $\sigma_{max}$  and one with constant for estimating the parameters. Many other alternatives are also possible by combining different constant (more than one) levels of  $\sigma_{max}$  or constant  $\sigma_{min}$ . A very efficient testing strategy, in particular, consists of selecting two different values of  $\sigma_{max}$  and two different values of  $\sigma_{min}$  and combine them to obtain 4 groups of tests, for which several specimens must be tested.

### 4 Convenient Reparameterized Model

In order to estimate the cumulative distribution function, the model in (21) is reparameterized in such a way that it can be viewed as a two-parameter exponential distribution in which case the required estimation becomes simple. So, the model in (21) is expressed as

$$F(\log N; \sigma_{m_i}, \sigma_{M_j}) = 1 - \exp\left\{-\left[C_{0ij} + C_{1ij}\log N\right]\right\},\tag{26}$$

for i, j = 1, 2, where

$$C_{0ij} = C_0 + C_2 \log \sigma_{m_i} + C_3 \log \sigma_{M_j} + C_6 \log \sigma_{m_i} \log \sigma_{M_j}$$
(27)

and

$$C_{1ij} = C_1 + C_4 \log \sigma_{m_i} + C_5 \log \sigma_{M_j} + C_7 \log \sigma_{m_i} \log \sigma_M M_j.$$
(28)

We then recognize that  $\log N$ , with  $\log N \ge -C_{0ij}/C_{1ij}$  and  $C_{1ij} > 0$ , follows a two-parameter exponential distribution with  $\gamma_{ij} = -C_{0ij}/C_{1ij}$  and  $\beta_{ij} = 1/C_{1ij}$  as the location and scale parameters, respectively. We now use the expression in (26) for the purpose of estimation. Once the parameters  $C_{0ij}$  and  $C_{1ij}$  are all estimated, the expressions in (27) and (28) evaluated at these estimates for the four conditions produced by two different values for  $\sigma_m$  ( $\sigma_{m_1}$ ,  $\sigma_{m_2}$ ) and  $\sigma_M$  ( $\sigma_{M_1}$ ,  $\sigma_{M_2}$ ), provide a system of eight equations, using which the estimates of the eight initial parameters  $C_i$  (i = 0, 1, ..., 7) can be achieved. Solving this system of eight equations, we get

$$C_7 = \frac{C_{111} - C_{112} - C_{121} + C_{122}}{(\log \sigma_{m_1} - \log \sigma_{m_2})(\log \sigma_{M_1} - \log \sigma_{M_2})},\tag{29}$$

$$C_5 = \frac{C_{111} - C_{112}}{\log \sigma_{M_1} - \log \sigma_{M_2}} - C_7 \log \sigma_{m_1}, \tag{30}$$

$$C_4 = \frac{C_{111} - C_{121}}{\log \sigma_{m_1} - \log \sigma_{m_2}} - C_7 \log \sigma_{M_1}, \tag{31}$$

$$C_1 = C_{111} - C_4 \log \sigma_{m_1} - C_5 \log \sigma_{M_1} - C_7 \log \sigma_{m_1} \log \sigma_{M_1}, \tag{32}$$

$$C_6 = \frac{C_{011} - C_{012} - C_{021} + C_{022}}{(\log \sigma_{m_1} - \log \sigma_{m_2})(\log \sigma_{M_1} - \log \sigma_{M_2})},$$
(33)

$$C_3 = \frac{C_{011} - C_{012}}{\log \sigma_{M_1} - \log \sigma_{M_2}} - C_6 \log \sigma_{m_1}, \tag{34}$$

$$C_2 = \frac{C_{011} - C_{021}}{\log \sigma_{m_1} - \log \sigma_{m_2}} - C_6 \log \sigma_{M_1}, \tag{35}$$

$$C_0 = C_{011} - C_2 \log \sigma_{m_1} - C_3 \log \sigma_{M_1} - C_6 \log \sigma_{m_1} \log \sigma_{M_1}.$$
(36)

#### 5 Parameter Estimation

In this section, we describe two methods of estimating the parameters of the model.

1. Method 1 (based on the maximum likelihood on 4 stress level tests and least-squares) From the model in (26), the sample log-likelihood, taking the run-outs into account, turns out to be

$$L = -\sum_{i,j=1}^{2} \sum_{k=1}^{n_{ij}} (C_{0ij} + C_{1ij} \log N_{ijk}) + \sum_{i,j=1}^{2} n_{ij}^* \log C_{1ij},$$
(37)

where  $N_{ijk}$ ,  $n_{ij}$  and  $n_{ij}^*$  are the lifetime (or the censored value) of the specimen k, the total number of specimens, and the number of uncensored specimens, respectively, tested at stress levels  $\sigma_{m_i}$ and  $\sigma_{M_j}$ . Then, by standard results for the two-parameter exponential distribution, we have the maximum likelihood estimates (MLEs) to be

$$\hat{C}_{1ij} = \frac{n_{ij}^*}{\sum_{k=1}^{n_{ij}} \left[ \log N_{ijk} - \log N_{min_{ij}} \right]}$$
(38)

and

$$\hat{C}_{0ij} = -\log N_{min_{ij}} \hat{C}_{1ij}, \quad i, j = 1, 2,$$
(39)

where  $N_{min_{ij}} = \min(N_{ij1}, N_{ij2}, \ldots, N_{ijk_{ij}})$  denotes the smallest lifetime observed for stress levels  $\sigma_{m_i}$  and  $\sigma_{M_j}$ . The MLEs of the initial parameters in (21) are then obtained by replacing in Eqs. (29) – (36)  $C_{0ij}$  and  $C_{1ij}$  by their MLEs in (39) and (38), respectively. However, if one uses any of the submodels presented in Section 3.2, then there will be more equations than unknown parameters. In this case, we propose to get the estimates of the initial parameters by minimizing the sum of squares based on the MLEs  $C_{0ij}$  and  $C_{1ij}$  given by

$$Q = \sum_{i,j} \left[ \left\{ C_{0ij} - (C_0 + C_2 \log \sigma_{m_i} + C_3 \log \sigma_{M_j} + C_6 \log \sigma_{m_i} \log \sigma_{M_j}) \right\}^2 + \left\{ C_{1ij} - (C_1 + C_4 \log \sigma_{m_i} + C_5 \log \sigma_{M_j} + C_7 \log \sigma_{m_i} \log \sigma_M M_j) \right\}^2 \right], \quad (40)$$

wherein the parameters constraints can also be included in the optimization process, if desired.

In this manner, we may avoid the constrained maximum likelihood estimation method used so far for these types of problems; see, for example, Castillo *et al.* (2006). At this stage, we can test whether the physical constraints of the model are satisfied by the estimates determined by the above method. In case they are, the computed estimates are indeed the desired MLEs. If not, the unconstrained MLEs do not belong to the restricted parameter space and in this case the constrained MLEs need to be computed by a numerical method. To this end, existing software such as GAMS enables not only to solve the problem but also to perform a sensitivity analysis; for details, see Castillo *et al.* (2001, 2004) and Conejo *et al.* (2006)).

2. Method 2 (based on constrained maximum likelihood) — In this case, we maximize the log-likelihood function

$$L = -\sum_{i,j=1}^{2} \sum_{k=1}^{n_{ij}} \left( C_0 + C_2 \log \sigma_{m_i} + C_3 \log \sigma_{M_j} + C_6 \log \sigma_{m_i} \log \sigma_{M_j} + (C_1 + C_4 \log \sigma_{m_i} + C_5 \log \sigma_{M_j} + C_7 \log \sigma_{m_i} \log \sigma_M M_j) \log N_{ijk} \right) + \sum_{i,j=1}^{2} n_{ij}^* \log C_1 + C_4 \log \sigma_{m_i} + C_5 \log \sigma_{M_j} + C_7 \log \sigma_{m_i} \log \sigma_M M_j$$
(41)

with respect to the parameters  $C_i$  (i = 0, 1, ..., 7) subject to the constraints

$$C_0 + C_2 \log \sigma_m + C_3 \log \sigma_M + C_4 \log N_{ijk} \log \sigma_m + C_5 \log N_{ijk} \log \sigma_M \ge 0, \quad \forall \quad i, j, k,$$
(42)

and additional constraints to be satisfied by the parameters for the different models as presented earlier in Section 3.2.

#### 6 Illustrative Examples

The proposed estimation methods are illustrated in this section with some data sets. For the set of parameter values  $C_0 = -407.905$ ,  $C_1 = 0$ ,  $C_2 = -1.272$ ,  $C_3 = 53.132$ ,  $C_4 = -1.755$ ,  $C_5 = 2.107$ ,  $C_6 = 0$ ,  $C_7 = 0$  and for each of the four combinations of the two values of  $\sigma_{min}$  ( $\sigma_{min} = 510,714$ ) and two values of  $\sigma_{max}$  ( $\sigma_{max} = 1054,1190$ ), two samples were simulated, one of small size (n = 7) and another of moderate size (n = 20). The log N values of these samples are presented in Table 1.

The model parameters were then estimated by using the two methods described in Section 5.

Method 1: The MLEs  $\hat{C}_{0ij}$  and  $\hat{C}_{1ij}$  (i, j = 1, 2) and the corresponding MLEs  $\hat{C}_i$  (i = 0, 1, ..., 7) estimates are provided in Columns 3 and 4 of Table 2 as "ML unconstrained estimates", for the cases when n = 7 and 20 without runouts. The true values of the parameters of the model, viz.,  $C_{0ij}, C_{1ij}$  (i, j = 1, 2) and  $C_i$  (i = 0, 1, ..., 7), are also presented in Column 2 for the purpose of comparison. The parameters  $C_{0ij}$  and  $C_{1ij}$  are the ones that are crucial for modelling and prediction purposes. We observe that even though the estimation of the initial parameters is not accurate, the estimates of the parameters  $C_{0ij}$  and  $C_{1ij}$  parameters (which are functions of them) are quite good.

Further, in order to illustrate the influence of run-outs, we also considered additional threshold values. For each condition, we considered the corresponding 90-th percentile  $Q_{90}(\sigma_m, \sigma_M)$  as threshold value, which are  $Q_{90}(510, 1054) = 12.97520$ ,  $Q_{90}(510, 1190) = 10.5213$ ,  $Q_{90}(714, 1054) = 15.5570$  and  $Q_{90}(714, 1190) = 12.4807$ . Thus, for the samples presented in Table 1, we observe the number of run-outs to be 1, 1, 1, 1 for the case when n = 7, and 3, 3, 3, 2 for the case when n = 20. The estimates of the parameters of the original model as well as those of the reparameterized model are presented in Columns 5 and 6 of Table 2. Moreover, the quartiles of the true parameter values and the estimated values are presented in Table 3, while the cdf's as well as their estimates are presented in Figure 4. From the values of  $\hat{C}_{0ij}$  and  $\hat{C}_{1ij}$ , as well as from the plots, it is clear that increasing sample size results in more precision while the presence of run-outs result in loss of precision in the estimation.

**Method 2**: We fitted the model in (23) using the constrained maximum likelihood method described in Section 5. The parameter estimates so obtained are presented in Table 2 as "Constrained estimates". It is clear that the fitted values of the original  $C_i$  parameters are better than those

Table 1: Simulated log N ordered values for the four different conditions of (a) 7 and (b) 20 specimens at each condition.

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18	a.)
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Condition		Simulated $log N$ values					
$\sigma_m = 510$							
$\sigma_M = 1054$	12.4235	12.4396	12.5503	12.6968	12.8483	12.854	13.0437
$\sigma_m = 510$							
$\sigma_M = 1190$	9.95172	9.98658	10.0196	10.1651	10.2444	10.2453	10.7026
$\sigma_m = 714$							
$\sigma_M = 1054$	14.8401	14.8985	14.9619	14.9907	15.3109	15.3237	15.7461
$\sigma_m = 714$							
$\sigma_M = 1190$	11.8056	11.8549	11.9707	11.9759	12.041	12.3971	12.7405

(b)

Condition		Simulated log N values								
$\sigma_m = 510$	12.3879	12.3945	12.4266	12.4364	12.4759	12.494	12.4955	12.5004	12.5329	12.6312
$\sigma_M = 1054$	12.643	12.7081	12.7257	12.7924	12.7981	12.8534	12.8892	12.9967	13.0073	13.0074
$\sigma_m = 510$	9.96423	9.98116	9.99038	10.016	10.0338	10.0425	10.0432	10.0482	10.0941	10.0995
$\sigma_M = 1190$	10.1064	10.1385	10.1428	10.2108	10.211	10.2116	10.2846	10.8629	10.8929	11.0735
$\sigma_m = 714$	14.8585	14.8848	14.8892	14.8893	14.8984	14.8998	14.9122	14.9212	14.9253	15.0038
$\sigma_M = 1054$	15.0649	15.0831	15.1102	15.2873	15.384	15.3964	15.5167	15.6858	15.8732	15.9394
$\sigma_m = 714$	11.8027	11.8065	11.8129	11.8224	11.8371	11.8609	11.9272	11.9326	11.9428	11.9901
$\sigma_M = 1190$	12.0414	12.0539	12.0895	12.090	12.1639	12.2018	12.2985	12.2991	12.8947	13.1203

obtained by Method 1. The quantile estimates are also presented in Table 3 as "Constrained estimates" which are also better than those obtained by Method 1.

We thus observe that the two methods of estimation proposed in Section 5 offer convenient methods of modeling the fatigue behavior, and that Method 1 presents a simpler alternative to the computationally involved constrained maximum likelihood method.

# 7 Conclusions

In this paper, we have presented a general model for the prediction of the fatigue behavior for any stress level and amplitude using an exponential model and discuss some important considerations such as weakest link principle, limit behavior, limited range, and compatibility. Using the compatibility condition, an exponential model is obtained with some non-dimensional model parameters whose physical interpretations are given in terms of threshold value of log-lifetime, endurance limit, parameter defining the position of the corresponding zero-percentile hyperbola, and scale factor. Next, based on physical and compatibility considerations and on the Wöhler field for fixed stress level, a compatibility functional equation yields a general exponential model with eight parameters. We have then reparametrized this parameters can be used to set-up a system of eight equations for the estimation of the eight parameters of the original model. For the estimation method, we have described two methods, one based on the maximum likelihood on four stress level tests and least-squares, and the other based on the constrained

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Table 2: True and estimated values of the parameters, based on the two samples in Table 1, with and without run-outs.

Paramotor	True value Estimated value <sup>1</sup>			Fetimat	Estimated value <sup>2</sup>		
1 arameter	The value	(n - 7)	(n - 20)	(n - 7)	(n - 20)		
		(n-1)	(n = 20)	(n-1)	(n = 20)		
C	46.019	45.079		20 404	20 701		
$C_{011}$	-40.018	-40.972	-40.004	-59.404	-30.721		
$C_{111}$	3.724 20.570	3.700	3.077	0.172 26 100	3.120		
$C_{012}$	-39.570	-42.140	-38.593	-30.120	-32.804		
$C_{112}$	3.980	4.234	3.8/3	3.630	3.292		
$C_{021}$	-46.446	-47.405	-47.512	-40.633	-40.386		
$C_{121}$	3.134	3.194	3.198	2.738	2.718		
$C_{022}$	-39.998	-38.499	-39.767	-32.999	-35.791		
$C_{122}$	3.389	3.261	3.369	2.795	3.032		
$C_0$	-407.905	-239.195	-408.472	-205.024	-347.201		
$C_1$	0.000	0.000	0.000	0.000	0.000		
$C_2$	-1.272	-4.258	-5.821	-3.650	-4.948		
$C_3$	53.132	31.575	57.355	27.064	48.752		
$C_4$	-1.755	-1.504	-1.425	-1.289	-1.212		
$C_5$	2.107	4.400	1.614	3.772	1.372		
$C_6$	0.000	0.000	0.000	0.000	0.000		
$C_7$	0.000	0.000	0.000	0.000	0.000		
			ML and le	east-square	s		
$C_0$	-407.905	-412.142	-438.344	-353.264	-296.642		
$C_1$	0.000	0.000	0.000	0.000	0.000		
$C_2$	-1.272	0.000	-4.673	0.000	-6.927		
$C_3$	53.132	52.505	60.590	45.004	43.308		
$C_4$	-1.755	-2.193	-1.426	-1.879	-1.060		
$C_5$	2.107	2.512	1.803	2.153	1.400		
$C_6$	0.000	0.000	0.000	0.000	0.000		
$C_7$	0.000	0.000	0.000	0.000	0.000		
			Constraine	ed estimate	es		
$C_0$	-407.905	-420.957	-410.061	-360.820	-353.680		
$C_1$	0.000	0.000	0.000	0.000	0.000		
$C_2$	-1.272	-2.779	-0.305	-2.382	-0.263		
$C_3$	53.132	56.300	52.586	48.257	45.355		
$\tilde{C_4}$	-1.755	-1.621	-1.812	-1.389	-1.563		
$C_5$	2.107	1.988	2.156	1.704	1.860		
$\tilde{C_6}$	0.000	0.000	0.000	0.000	0.000		
$C_7$	0.000	0.000	0.000	0.000	0.000		

<sup>1</sup> without run-outs <sup>2</sup> with run-outs

\* In these cases, no run-outs occurred and the estimates are therefore the same.

maximum likelihood. Finally, we have presented two examples to illustrate the exponential model derived as well as to demonstrate the two methods of estimation proposed here. Moreover, it is also important to mention the following points:

1. The method will enable one to extrapolate laboratory results obtained from any combination of values of  $\sigma_{min}$  or  $\sigma_{max}$  to any possible combination of values of  $\sigma_{min}$  or  $\sigma_{max}$  arising in a practical case;

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Table 3: True and estimated quartiles, based on th	e two samples in Table 1, with and without run-outs.
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Condition	Quartile	True value	Estima	ited value <sup>1</sup>	Estima	ted value <sup>2</sup>
	•		(n = 7)	(n = 20)	(n = 7)	(n = 20)
			Ń	L unconstr	ained estir	nates
$\sigma_m = 510$	$Q_{25}$	12.4342	12.5012	12.4661	12.5142	12.4799
$\sigma_M = 1054$	$\dot{Q}_{50}$	12.5431	12.6108	12.5764	12.6420	12.6096
	$Q_{75}$	12.7292	12.7981	12.7649	12.8606	12.8314
	$Q_{90}$	12.9752	13.0458	13.0140	13.1495	13.1245
$\sigma_m = 510$	$Q_{25}$	10.0150	10.0197	10.0385	10.0310	10.0516
$\sigma_M = 1190$	$Q_{50}$	10.1169	10.1154	10.1432	10.1427	10.1748
	$Q_{75}$	10.2911	10.2791	10.3222	10.3337	10.3853
	$Q_{90}$	10.5213	10.4955	10.5587	10.5861	10.6636
$\sigma_m = 714$	$Q_{25}$	14.9139	14.9301	14.9484	14.9452	14.9643
$\sigma_M = 1054$	$Q_{50}$	15.0433	15.0571	15.0752	15.0932	15.1135
	$Q_{75}$	15.2645	15.2741	15.2920	15.3464	15.3685
	$Q_{90}$	15.5570	15.5609	15.5785	15.6810	15.7056
$\sigma_m = 714$	$Q_{25}$	11.8862	11.8938	11.8880	11.9085	11.8957
$\sigma_M = 1190$	$Q_{50}$	12.0059	12.0181	12.0084	12.0536	12.0312
	$Q_{75}$	12.2104	12.2307	12.2141	12.3015	12.2598
	$Q_{90}$	12.4807	12.5117	12.4860	12.6294	12.5620
				ML and l	east-squar	es
$\sigma_m = 510$	$Q_{25}$	12.4342	12.3172	12.5789	12.3298	12.3338
$\sigma_M = 1054$	$Q_{50}$	12.5431	12.4235	12.6896	12.4538	12.4631
	$Q_{75}$	12.7292	12.6052	12.8790	12.6658	12.6841
	$Q_{90}$	12.9752	12.8455	13.1294	12.9461	13.9763
$\sigma_m = 510$	$Q_{25}$	10.0150	9.8585	9.9735	9.8701	10.1102
$\sigma_M = 1190$	$Q_{50}$	10.1169	9.9569	10.0781	9.9850	10.2328
	$Q_{75}$	10.2911	10.1252	10.2568	10.1813	10.4425
	$Q_{90}$	10.5213	10.3477	10.4930	10.4408	10.7197
$\sigma_m = 714$	$Q_{25}$	14.9139	15.2713	14.9710	15.2869	14.7548
$\sigma_M = 1054$	$Q_{50}$	15.0433	15.4031	15.0985	15.4407	15.9007
	$Q_{75}$	15.2645	15.6285	15.3165	15.7036	15.1501
714	$Q_{90}$	15.5570	15.9263	15.6046	16.0511	15.4797
$\sigma_m = 114$	$Q_{25}$	11.8862	12.0097	11.8439	12.0239	12.1229
$\sigma_M = 1190$	$Q_{50}$	12.0059	12.1297	12.9632	12.1038	12.2603
	$Q_{75}$	12.2104	12.3347	12.1071	12.4030 19.7109	12.4954
	$Q_{90}$	12.4807	12.0057	12.4307	12.7192	12.8000
	0	10 4949	19 5005	Constrain	$\frac{10}{10}$ $\frac{5122}{10}$	es
$\sigma_m = 510$	$Q_{25}$	12.4342 19 5421	12.0000	12.4004	12.3133	12.4778
$O_M = 1034$	$\tilde{Q}_{50}$	12.0401 12.7202	12.0090 12.7045	12.074	12.0399 12.8564	12.0040 12.8010
	Q75	12.7292 12.0752	12.7945	12.7010 13.0087	-12.0004	12.0212 12.1077
$\sigma = 510$	Q90	12.9752	10.0398	10.0367	10.1425	10.1077
$\sigma_m = 510$	$Q_{25}^{25}$	10.0150 10.1160	10.0240 10.1260	10.0307	10.0501 10.1550	10.0462
$O_M = 1190$	Q 50	10.1109 10.2011	10.1200	10.1300 10.3133	10.1550 10.3583	10.1000
	Q75	10.2911 10.5913	10.5005 10.5306	10.5155 10.5441	10.3363 10.6271	10.3090
$\sigma = 714$	Q90	14.0120	14 0302	14 0512	14 0453	14.0661
$\sigma_m = 114$ $\sigma_{14} = 1054$	$\tilde{Q}_{25}^{25}$	14.9139	14.9502 15.0573	14.9313 15.0821	14.9400 15.0035	15 1178
$0_M = 1004$	Q 50	15 2645	15.0075 15.076	15.0021 15.3058	15.0355 15.3470	15.1170 15.3771
	0°75	15.2045 15.5570	15.2740 15.5618	15.5058 15.6014	15 6820	15 7199
$\sigma_{\rm m} = 714$	0 90	11 8862	11 880/	11 8883	11 003/	11 9010
$\sigma_{M} = 1190$	0±0	12 0059	120075	12 0089	12 0412	12.0418
$S_{M} = 1150$	07E	12.2104	12.2095	12.2000	12.2768	12.2809
		12.4807	12.4765	12.4878	12.5883	12.5970
	~ <b>C</b> 30	12.100,	<sup>1</sup> without	run cute	$\frac{2}{2}$ with	run outs

2. Testing strategies (test designs) suggested for producing enough data will enable the above extrapolation possible. This will result in a substantial reduction as compared to traditional testing

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Figure 4: Cdf of lifetimes  $\log N$  under the four conditions (from top to bottom: ( $\sigma_m = 510; \sigma_M = 1054$ ), ( $\sigma_m = 510; \sigma_M = 1190$ ), ( $\sigma_m = 714; \sigma_M = 1054$ ) and ( $\sigma_m = 714; \sigma_M = 1190$ )). Estimated cdf based on the sample of size n = 7 and n = 20 are plotted in green and red, respectively, for the case without run-outs [figures in the left column (a)] and with run-outs [in the right column], for the unconstrained method.

procedures; in fact, four levels are normally used in standard tests of constant  $\sigma_{min}$  or  $\sigma_{max}$ ;

- 3. Explicit simple formulas derived will facilitate the extrapolation (or interpolation) from lab results;
- 4. The normalization principle, together with the model proposed in this paper, would enable the analysis of damage accumulation when one knows the history of  $\sigma_{min}$  or  $\sigma_{max}$  as a function of time or cycle in a practical case;
- 5. The unconstrained and constrained methods of estimation proposed will facilitate the model fitting in a simple and efficient manner.

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