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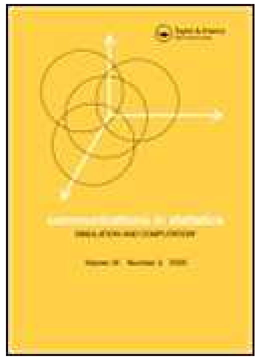

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An Exponential Model for Damage Accumulation

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Abstract

In this paper, we present a general model for predicting the fatigue behavior for any stress level and amplitude using the exponential model. Based on the Wöhler field for fixed stress level, a compatibility functional equation enables us to derive the general model with eight parameters. The problem of parameter estimation is then discussed and some methods are described. Some examples are finally presented to illustrate the derived model and the proposed methods of estimation.

Key Words: Wöhler field, damage accumulation, interpolation of fatigue results, testing strategies, exponential model, maximum likelihood method, constrained maximum likelihood method.

1 Introduction

In fatigue analysis that the stress ratio $R = \sigma_{max}/\sigma_{min}$ is an important factor, next only to the stress level (σ_{min}, σ_{max}, σ_{mean}), which needs to be taken into account while dealing with fatigue behavior. This becomes particularly relevant while considering real stress histories instead of constant stress level situations. However, most of the existing laboratory test strategies and fatigue models are applicable for a given stress level but can not predict the fatigue behavior for different stress levels; see, for example, Coleman (1958), Bastenaire (1972), ASTM (1981), Spindel and Haibach (1981), Fernández-Canteli (1982), Castillo *et al.* (1985), Castillo and Galambos (1987), Castillo and Hadi (1995), Pascual and Meeker (1999), and Castillo and Fernández-Canteli (2001).

Though there are some models that take mean stress effects into account [see Conway and Sjodahlo (1991) and Dowling and Thangjitham (2000)], such simplified models are far from satisfactory. Castillo and Fernández-Canteli (2006) and Castillo *et al.* (2006) presented a general model that enables the prediction of the fatigue behavior based on two groups of tests run at two different stress levels. This was a significant development as other models did not facilitate this prediction and also that the required testing strategy for this model was very simple.

In this paper, we consider this general model for the case of the exponential distribution and describe the prediction of the fatigue behavior for any stress level and amplitude, and discuss some methods of estimation of the parameters underlying the model. In Section 2, we describe first the exponential parameter fatigue model. In Section 3, we describe how this model can be extrapolated to different test conditions, i.e., different pairs of σ_{max} and σ_{min}, and then discuss the consequences of forcing the compatibility or independence of the results on the selected σ_{max} and σ_{min} test levels. In Section 4, we present a reparameterization of the model that is convenient for the estimation of the model parameters. In Section 5, we discuss some methods of estimation such as the maximum likelihood and constrained maximum likelihood methods. In Section 6, we present some examples to illustrate the derived model and the proposed methods of estimation. Finally, some concluding remarks are made in Section 7.

2 Derivation of the Fatigue Model

Let us consider a series of tests with constant σ_{min} and variable σ_{max}, but in each fatigue test we fix σ_{max} and repeat cycles in which σ is varied from σ_{min} to σ_{max} until the failure of the specimen occurs. In other words, σ_{min} is constant for all tests, and different tests in the series can have identical or possibly different σ_{max}. The resulting lifetimes N (the number of cycles to failure) are observed. Since σ_{min} is held constant, we use as variables the stress ratio $R = \sigma_{max}/\sigma_{min}$ and N.

It is generally accepted that there are five variables initially involved in the fatigue problem: P, N, N_0, R and R_0, where P is the probability of fatigue failure of a piece when subjected to N cycles, N_0 is the threshold value for N corresponding to the minimum observable lifetime for any R, and R_0 is the endurance stress ratio limit below which fatigue failures can not occur. This means that there exists a relationship among the five initially independent variables, of the form

$$r(N, N_0, R, R_0, P) = 0, \tag{1}$$

where $r(\cdot)$ is an initially unknown function. However, using the Π-Theorem [see, for example, Buckingham (1914, 1915a,b) and Castillo and Fernández-Canteli (2001)], these initial five variables can be reduced to three non-dimensional variables, viz., N/N_0, R/R_0 and P. Consequently, (1) can be written in terms of these three non-dimensional variables as

$$r(N, N_0, R, R_0, P) = 0 \quad \Longleftrightarrow \quad s\left(\frac{N}{N_0}, \frac{R}{R_0}, P\right) = 0; \tag{2}$$

since we are interested in P, we can express

$$P = q\left(\frac{N}{N_0}, \frac{R}{R_0}\right), \tag{3}$$

where $s(\cdot)$ and $q(\cdot)$ are functions to be determined. Thus, only the non-dimensional quotients N/N_0 and R/R_0 exert influence on the probability of failure P, and so either N/N_0 and R/R_0, or some monotone functions of them, say $h(N/N_0)$ and $g(R/R_0)$, need to be considered.

In this paper, we have chosen the h and g functions to be the logarithms of N/N_0 and R/R_0, respectively. For simplicity in notation, we denote throughout this paper

$$N^* = \log(N/N_0), \quad R^* = \log(R/R_0), \quad N \geq N_0, \quad R \geq R_0. \tag{4}$$

The selection of the exponential model is then based on the following important considerations:

1. **Weakest link principle**: This principle states that the fatigue lifetime of a longitudinal element is the minimum fatigue life of its constituent pieces. In other words, the selected family of distributions must hold (be valid) for different specimen lengths.

2. **Limit behavior**: To include the extreme case of the size of the supposed pieces constituting the element going to zero, or the number of pieces going to infinity, it is necessary for the distribution function family to be an asymptotic family; see Galambos (1987), Castillo (1988), Castillo *et al.* (2004), and Castillo *et al.* (2006).

3. **Limited range**: Experience shows that the selected non-dimensional variables, N^* and R^*, have a finite lower end, which must coincide with the theoretical lower end of the selected cumulative distribution function (cdf).

4. **Compatibility**: In the S-N field, the cdf $G(N^*; R^*)$ of the lifetime given stress range should be compatible with the cdf of the stress range given lifetime $F(R^*; N^*)$, i.e.,

$$G(N^*; R^*) = F(R^*; N^*). \tag{5}$$

Though in standard tests R^* is fixed and the associated random lifetime N^* is determined, R^* is interpreted here as the random stress that needs to be applied in order to produce failure at N^*.

It is important to note here that the exponential family satisfies all the conditions stated above.

2.1 Compatibility

The compatibility condition in (5) for the exponential model requires

$$1 - \exp[-\delta(N^*)(R^* - \lambda(N^*))] = 1 - \exp[-\rho(R^*)(N^* - \eta(R^*))], \tag{6}$$

which leads to the functional equation

$$R^*\delta(N^*) - \delta(N^*)\lambda(N^*) - \rho(R^*)N^* + \rho(R^*)\eta(R^*) = 0 \tag{7}$$

whose solution is [see Castillo and Ruiz-Cobo (1992, p. 52) and Castillo *et al.* (2004, pp. 60–61)]

$$\rho(R^*) = aR^* + b, \tag{8}$$

$$\eta(R^*) = \frac{cR^* + d}{aR^* + b}, \tag{9}$$

$$\delta(N^*) = aN^* - c, \tag{10}$$

$$\lambda(N^*) = \frac{-bN^* + d}{aN^* - c}. \tag{11}$$

This leads to the model

$$G(N^*; R^*) = F(R^*; N^*) = 1 - \exp[(c - aN^*)R^* - bN^* + d], \tag{12}$$

where a, b, c and d are arbitrary constants; now, upon reparameterizing and subsuming two parameters into N_0 and R_0, we obtain

$$G(N^*; R^*) = F(R^*; N^*) = 1 - \exp\left[-\frac{R^*N^* - \lambda}{\delta}\right], \tag{13}$$

where λ and δ are some constants.

Once the model has been established in non-dimensional terms as in (13), as we need to recover the initial variables, we can reexpress the model in (13) as

$$F(\log N; \log R) = 1 - \exp\left\{-\left[\frac{(\log N - B)(\log R - C) - E}{D}\right]\right\}, \tag{14}$$

where $B = \log N_0, C = \log R_0$, and E and D are non-dimensional model parameters. Their physical meanings are as follows (see Figure 1):

B: Threshold value of log-lifetime (logarithm of N_0),

C: Endurance limit (logarithm of R_0),

E: Parameter defining the position of the corresponding zero-percentile hyperbola,

D: Scale factor.

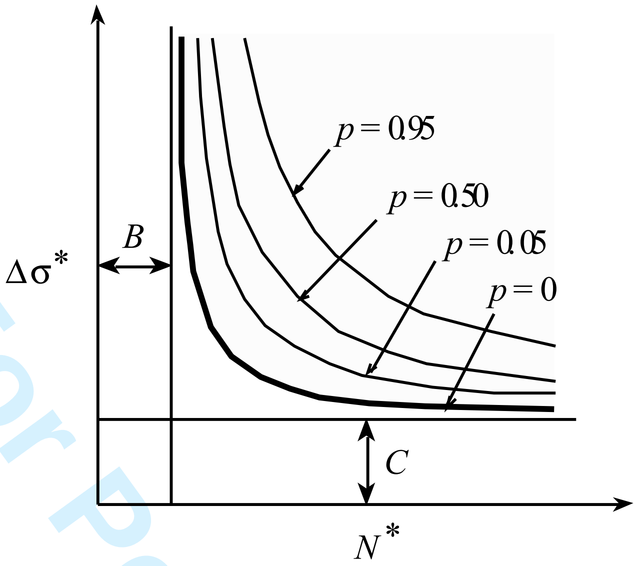

Figure 1: Percentiles curves representing the relationship between lifetime, N^*, and stress range, $\Delta\sigma^*$, in the S-N field for the fatigue model in (14).

The percentile curves are shown in Figure 1. The zero-percentile curve represents the minimum possible required number of cycles of fatigue failure for different values of R. All the remaining percentiles happen to be the positive branches of a hyperbola, i.e.,

$$N^* R^* = E + D\left[-\log(1-p)\right],$$

where $p \in [0,1]$ defines the percentile. For such curves, the minimum number of cycles to fatigue failure decreases with increasing R^*, which is in agreement with experimental results. The parameters E and D determine the p-percentile, and hence define a unique curve on the plot.

It can be seen from (13) that

$$N^* R^* \sim Exp(\lambda, \delta) \quad \Leftrightarrow \quad N^* \sim Exp\left(\frac{E}{R^*}, \frac{D}{R^*}\right), \tag{15}$$

for given R^*, where $Exp(\lambda, \delta)$ denotes a two-parameter exponential distribution with λ as the threshold parameter and δ as the scale parameter. It is of interest to note that (15) has a non-dimensional form and reveals that the probability of failure of a piece subject to a stress ratio R^* during N^* cycles depends only on the product $N^* R^*$. Thus, $V = N^* R^*$ is useful for comparing fatigue strength at different stress levels that are maintained constant, and can be considered as a normalizing variable.

3 Physical and compatibility considerations

Since for each constant σ_{min}, a model as in (14) is obtained, we can consider its parameters B, C, D and E to be functions of σ_{min}. Figure 2(a) shows the Wöhler curves for two groups of tests run at constant $\sigma_{min} = 0.8$ and at constant $\sigma_{min} = 0.4$. Note that larger values of σ_{min} lead to lower percentile curves. If we consider the Wöhler curves for two groups of tests run at constant σ_{max}, say $\sigma_{max} = 1.5$, we obtain the model in (14) once again and a set of percentiles, for example $\{0.01, 0.05, 0.5, 0.95, 0.99\}$, are shown in Figure 2(b). However, if the tests are run at constant $\sigma_{max} = 1$, the resulting percentiles change as displayed in the same figure. Note that larger values of σ_{max} lead to lower percentile curves.

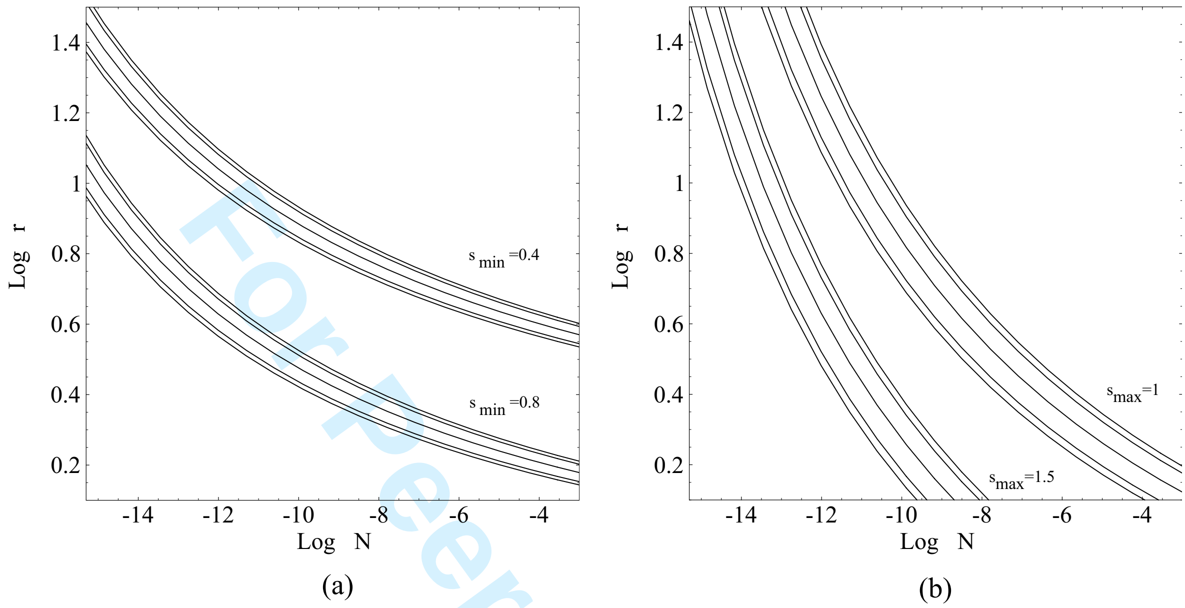

Figure 2: Wöhler curves for percentiles $0.01, 0.05, 0.5, 0.95, 0.99$: (a) for constant $\sigma_{min} = 0.4$ and $\sigma_{min} = 0.8$, and (b) for constant $\sigma_{max} = 1$ and $\sigma_{max} = 1.5$.

An important property to be observed is that the same percentile curves never intersect if they are associated with two different σ_{max} or with two different σ_{min}, but they intersect if one set of percentiles is associated with constant σ_{max} and another with constant σ_{min} as shown in Figure 3, where the four sets of percentiles have been plotted together, with dashed lines corresponding to Wöhler curves for constant σ_{min} and continuous lines corresponding to Wöhler curves for constant σ_{max}. Note that the line joining the intersections of associated percentiles for one constant σ_{max} and one constant σ_{min} must be straight and horizontal since both fields, i.e., the associated cdf's, must coincide. This is a very strong condition indeed and it assists greatly in deriving a simple model which is able to deal with different stress levels. Though in Figure 3 we have these four sets, we realize that there is an infinite set of them. More precisely, two families of S-N curves exist, one for different constant values of σ_{max} and another for different constant values of σ_{min}.

In this paper, we consider the case when the test experiments are conducted for constant $\sigma_{max} = \sigma_M$ and constant $\sigma_{min} = \sigma_m$. As mentioned above, the model in (14) is valid only if all tests are conducted at the same stress level, and in particular, for given values of $\sigma_{max} \equiv \sigma_M$ or $\sigma_{min} \equiv \sigma_m$. Thus, if two sample data do not coincide at either their σ_M or σ_m, the model can not be applied. Since there are two different cases, we must consider two fatigue models of the type in (14), i.e., we initially have two sets of parameters B_m, C_m, D_m, E_m and B_M, C_M, D_M, E_M, where the subindices m and M have been used for the cases of constant $\sigma_{min} = \sigma_m$ and constant $\sigma_{max} = \sigma_M$, respectively. However, for the same stress amplitude and level, i.e., when both σ_M and σ_m coincide, for the compatibility condition to hold, we must have the same model for both cases, i.e., for all N we must have

$$\frac{(\log N - B_m(\sigma_m))(\log R - C_m(\sigma_m)) - E_m(\sigma_m)}{D_m(\sigma_m)} = \frac{(\log N - B_M(\sigma_M))(\log R - C_M(\sigma_M) - E_M(\sigma_M)}{D_M(\sigma_M)}. \quad (16)$$

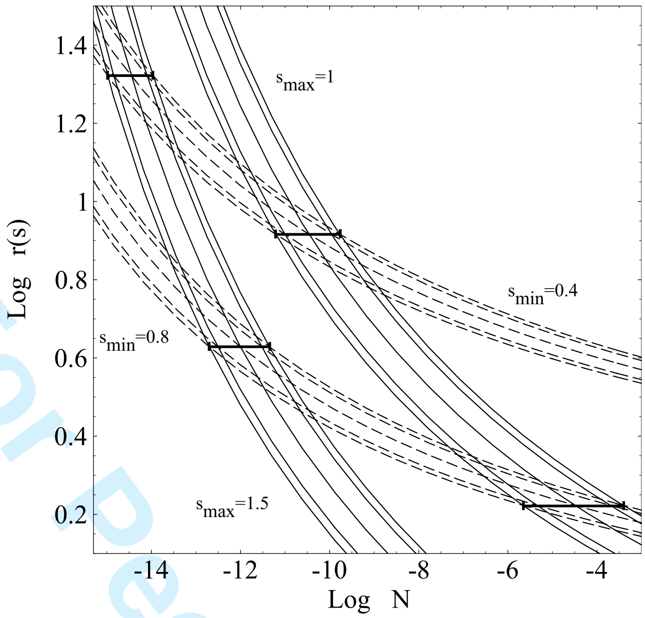

Figure 3: Wöhler curves for percentiles $\{0.01, 0.05, 0.5, 0.95, 0.99\}$ for $\sigma_{max} = 1$ and $\sigma_{max} = 1.5$, and $\sigma_{min} = 0.4$ and $\sigma_{min} = 0.8$. Dashed lines refer to Wöhler curves for constant σ_{min}, and continuous lines refer to Wöhler curves for constant σ_{max}.

For the functional equation in (16) to be satisfied for any N, σ_m and σ_M, both models must have the same parameters. Rewriting now the model in (16) as

$$\frac{\log N - \left[B_m(\sigma_m) + \dfrac{E_m(\sigma_m)}{\log R - C_m(\sigma_m)}\right]}{\dfrac{D_m(\sigma_m)}{\log R - C_m(\sigma_m)}} = \frac{\log N - \left[B_M(\sigma_M) + \dfrac{E_M(\sigma_M)}{\log R - C_M(\sigma_M)}\right]}{\dfrac{D_M(\sigma_M)}{\log R - C_M(\sigma_M)}} \quad \forall\ N \tag{17}$$

and forcing the exponential parameters to coincide, we obtain

$$\frac{D_m(\sigma_m)}{D_M(\sigma_M)} = \frac{\log \sigma_M - \log \sigma_m - C_m(\sigma_m)}{\log \sigma_M - \log \sigma_m - C_M(\sigma_M)} \quad \forall\ \sigma_m, \sigma_M, \tag{18}$$

$$B_M(\sigma_M) = B_m(\sigma_m) - \frac{E_M(\sigma_M)}{\log \sigma_M - \log \sigma_m - C_M(\sigma_M)} + \frac{E_m(\sigma_m)}{\log \sigma_M - \log \sigma_m - C_m(\sigma_m)} \quad \forall\ \sigma_m, \sigma_M. \tag{19}$$

The functional equations in (18) and (19) deserve a careful attention to get a deeper understanding of our problem; they are not simple equalities, but each a full collection of equalities as they hold for any feasible pair σ_m, σ_M. Eqs. (18) and (19), considered as functions of σ_m, must be independent of σ_M. This simple but powerful condition allows us to derive the structure of the functions $D_m(\sigma_m), D_M(\sigma_M)$ and $B_M(\sigma_M)$. The complete derivation, as given by Castillo *et al.* (2006), leads to the final model as

$$\begin{aligned} F(N; \sigma_m, \sigma_M) = \ & 1 - \exp\{-[C_0 + C_1 \log N + C_2 \log \sigma_m + C_3 \log \sigma_M + C_4 \log N \log \sigma_m \\ & + C_5 \log N \log \sigma_M + C_6 \log \sigma_m \log \sigma_M + C_7 \log N \log \sigma_m \log \sigma_M]\} \end{aligned} \tag{20}$$

$$
\begin{aligned}
&= 1-\exp\{-[(C_0+C_2\log\sigma_m+C_3\log\sigma_M+C_6\log\sigma_m\log\sigma_M) \\
&\quad +\log N(C_1+C_4\log\sigma_m+C_5\log\sigma_M+C_7\log\sigma_m\log\sigma_M)]\}\,. \qquad (21)
\end{aligned}
$$

Observe that this model involves eight parameters C_i $(i=0,1,\ldots,7)$.

3.1 Physical conditions

For the model to be physically feasible, the constants must satisfy some constraints. In particular:

- The cdf in (21) must be increasing in $\log N$;
- The cdf in (21) must be non-increasing in σ_m;
- The cdf in (21) must be non-decreasing in σ_M;
- The curvature of the zero-percentile of $(\log N, \log R)$ for constant σ_{min} must be non-negative;
- The curvature of the zero-percentile of $(\log N, \log R)$ for constant σ_{max} must be non-negative.

3.2 Resulting models

In summary, we can obtain some final set of feasible submodels of (21) as follows:

Linear Model: The simplest model with no asymptotes is given by

$$
\begin{aligned}
F(N;\sigma_m,\sigma_M) &= 1-\exp\{-[C_0+C_1\log N+C_2\log\sigma_m+C_3\log\sigma_M]\}\,, \\
&\qquad C_1, C_2\le 0,\ C_3\le 0, \qquad (22)
\end{aligned}
$$

which in log-log scale leads to a Wöhler field made of straight lines.

Model with asymptotes independent on σ_m and σ_M: The model with $\log N$ asymptotes independent on σ_m and σ_M is given by

$$
\begin{aligned}
F(N;\sigma_m,\sigma_M) &= 1-\exp\{-[C_0+C_2\log\sigma_m+C_3\log\sigma_M+C_4\log N\log\sigma_m+C_5\log N\log\sigma_M]\}\,, \\
&\qquad C_2, C_4\le 0,\ C_3, C_5\ge 0,\ C_4+C_5\ge 0, \qquad (23)
\end{aligned}
$$

which is obtained when $C_1=C_6=C_7=0$.

The $\log N$ asymptotes are $\log N=-\frac{C_3}{C_5}$ for the case of constant σ_{min}, and $\log N=-\frac{C_2}{C_4}$ for the case of constant σ_{max}, and the $\log R$ asymptotes are $\log R=-(C_4+C_5)\log\sigma_m/C_5$ for the case of constant σ_{min}, and $\log R=(C_4+C_5)\log\sigma_M/C_4$ for the case of constant σ_{max}.

Model with fixed asymptotes: The model with $\log R$ asymptotes independent on σ_m and σ_M is given by

$$
\begin{aligned}
F(N;\sigma_m,\sigma_M) &= 1-\exp\{-[C_0+C_1\log N+C_2\log\sigma_m+C_3\log\sigma_M+C_4\log N(\log\sigma_m-\log\sigma_M) \\
&\quad +C_6\log\sigma_m\log\sigma_M]\}\,, \qquad (24)
\end{aligned}
$$

which is obtained when $C_7=0$ and $C_4+C_5=0$. The $\log R$ asymptotes are $\log R=-C_1/C_5$ for the case of constant σ_{min}, and $\log R=C_1/C_4$ for the case of constant σ_{max}.

The $\log N$ asymptotes are $\log N=-\frac{C_3+\log\sigma_M C_6}{C_5}$ for the case of constant σ_{min}, and $\log N=-\frac{C_2+\log\sigma_M C_6}{C_4}$ for the case of constant σ_{max}, so that they are constant if, in addition, $C_6=0$.

General Model: The general model with $\log R$ and $\log N$ asymptotes dependent on σ_m and σ_M is given by

$$\begin{aligned} F(N;\sigma_m,\sigma_M) &= 1-\exp\{-[C_0+C_1\log N+C_2\log\sigma_m+C_3\log\sigma_M+C_4\log N\log\sigma_m \\ &\quad +C_5\log N\log\sigma_M+C_6\log\sigma_m\log\sigma_M+C_7\log N\log\sigma_m\log\sigma_M]\}\,, \end{aligned} \tag{25}$$

subject to adequate constraints on the parameters.

These constraints are very important in order to have a physically meaningful model.

3.3 Testing strategies

The aim of any testing strategy is to estimate the model parameters in (21). For this purpose, a testing strategy involving one single group of tests with constant σ_{max} or one single group with constant σ_{min} or two groups one with constant σ_{max} and one with constant σ_{min} are not sufficient, because of the linear combination of the parameters that are involved in the model. In contrast, two groups of tests one with constant σ_{max} and one with constant σ_{min} are sufficient for estimating the parameters. Many other alternatives are also possible by combining different constant (more than one) levels of σ_{max} or constant σ_{min}. A very efficient testing strategy, in particular, consists of selecting two different values of σ_{max} and two different values of σ_{min} and combine them to obtain 4 groups of tests, for which several specimens must be tested.

4 Convenient Reparameterized Model

In order to estimate the cumulative distribution function, the model in (21) is reparameterized in such a way that it can be viewed as a two-parameter exponential distribution in which case the required estimation becomes simple. So, the model in (21) is expressed as

$$F(\log N;\sigma_{m_i},\sigma_{M_j})=1-\exp\{-[C_{0ij}+C_{1ij}\log N]\}\,, \tag{26}$$

for $i,j=1,2$, where

$$C_{0ij}=C_0+C_2\log\sigma_{m_i}+C_3\log\sigma_{M_j}+C_6\log\sigma_{m_i}\log\sigma_{M_j} \tag{27}$$

and

$$C_{1ij}=C_1+C_4\log\sigma_{m_i}+C_5\log\sigma_{M_j}+C_7\log\sigma_{m_i}\log\sigma_M M_j. \tag{28}$$

We then recognize that $\log N$, with $\log N\geq -C_{0ij}/C_{1ij}$ and $C_{1ij}>0$, follows a two-parameter exponential distribution with $\gamma_{ij}=-C_{0ij}/C_{1ij}$ and $\beta_{ij}=1/C_{1ij}$ as the location and scale parameters, respectively. We now use the expression in (26) for the purpose of estimation. Once the parameters C_{0ij} and C_{1ij} are all estimated, the expressions in (27) and (28) evaluated at these estimates for the four conditions produced by two different values for σ_m (σ_{m_1}, σ_{m_2}) and σ_M (σ_{M_1}, σ_{M_2}), provide a system of eight equations, using which the estimates of the eight initial parameters C_i ($i=0,1,\ldots,7$) can be achieved. Solving this system of eight equations, we get

$$C_7 = \frac{C_{111}-C_{112}-C_{121}+C_{122}}{(\log\sigma_{m_1}-\log\sigma_{m_2})(\log\sigma_{M_1}-\log\sigma_{M_2})}, \tag{29}$$

$$C_5 = \frac{C_{111}-C_{112}}{\log\sigma_{M_1}-\log\sigma_{M_2}}-C_7\log\sigma_{m_1}, \tag{30}$$

$$C_4 = \frac{C_{111}-C_{121}}{\log\sigma_{m_1}-\log\sigma_{m_2}}-C_7\log\sigma_{M_1}, \tag{31}$$

$$C_1 = C_{111}-C_4\log\sigma_{m_1}-C_5\log\sigma_{M_1}-C_7\log\sigma_{m_1}\log\sigma_{M_1}, \tag{32}$$

$$C_6 = \frac{C_{011} - C_{012} - C_{021} + C_{022}}{(\log\sigma_{m_1} - \log\sigma_{m_2})(\log\sigma_{M_1} - \log\sigma_{M_2})}, \tag{33}$$

$$C_3 = \frac{C_{011} - C_{012}}{\log\sigma_{M_1} - \log\sigma_{M_2}} - C_6\log\sigma_{m_1}, \tag{34}$$

$$C_2 = \frac{C_{011} - C_{021}}{\log\sigma_{m_1} - \log\sigma_{m_2}} - C_6\log\sigma_{M_1}, \tag{35}$$

$$C_0 = C_{011} - C_2\log\sigma_{m_1} - C_3\log\sigma_{M_1} - C_6\log\sigma_{m_1}\log\sigma_{M_1}. \tag{36}$$

5 Parameter Estimation

In this section, we describe two methods of estimating the parameters of the model.

1. **Method 1 (based on the maximum likelihood on 4 stress level tests and least-squares)** From the model in (26), the sample log-likelihood, taking the run-outs into account, turns out to be

$$L = -\sum_{i,j=1}^{2}\sum_{k=1}^{n_{ij}}(C_{0ij} + C_{1ij}\log N_{ijk}) + \sum_{i,j=1}^{2} n^*_{ij}\log C_{1ij}, \tag{37}$$

where N_{ijk}, n_{ij} and n^*_{ij} are the lifetime (or the censored value) of the specimen k, the total number of specimens, and the number of uncensored specimens, respectively, tested at stress levels σ_{m_i} and σ_{M_j}. Then, by standard results for the two-parameter exponential distribution, we have the maximum likelihood estimates (MLEs) to be

$$\hat{C}_{1ij} = \frac{n^*_{ij}}{\sum\limits_{k=1}^{n_{ij}}\left[\log N_{ijk} - \log N_{min_{ij}}\right]} \tag{38}$$

and

$$\hat{C}_{0ij} = -\log N_{min_{ij}}\hat{C}_{1ij}, \quad i,j = 1,2, \tag{39}$$

where $N_{min_{ij}} = \min(N_{ij1}, N_{ij2}, \ldots, N_{ijk_{ij}})$ denotes the smallest lifetime observed for stress levels σ_{m_i} and σ_{M_j}. The MLEs of the initial parameters in (21) are then obtained by replacing in Eqs. (29) – (36) C_{0ij} and C_{1ij} by their MLEs in (39) and (38), respectively. However, if one uses any of the submodels presented in Section 3.2, then there will be more equations than unknown parameters. In this case, we propose to get the estimates of the initial parameters by minimizing the sum of squares based on the MLEs C_{0ij} and C_{1ij} given by

$$\begin{aligned} Q = \sum_{i,j}\Big[&\left\{C_{0ij} - (C_0 + C_2\log\sigma_{m_i} + C_3\log\sigma_{M_j} + C_6\log\sigma_{m_i}\log\sigma_{M_j})\right\}^2 \\ &+ \left\{C_{1ij} - (C_1 + C_4\log\sigma_{m_i} + C_5\log\sigma_{M_j} + C_7\log\sigma_{m_i}\log\sigma_M M_j)\right\}^2\Big], \end{aligned} \tag{40}$$

wherein the parameters constraints can also be included in the optimization process, if desired.

In this manner, we may avoid the constrained maximum likelihood estimation method used so far for these types of problems; see, for example, Castillo *et al.* (2006). At this stage, we can test whether the physical constraints of the model are satisfied by the estimates determined by the above method. In case they are, the computed estimates are indeed the desired MLEs. If not, the unconstrained MLEs do not belong to the restricted parameter space and in this case the constrained MLEs need to be computed by a numerical method. To this end, existing software such as GAMS enables not only to solve the problem but also to perform a sensitivity analysis; for details, see Castillo *et al.* (2001, 2004) and Conejo *et al.* (2006)).

2. **Method 2 (based on constrained maximum likelihood)** — In this case, we maximize the log-likelihood function

$$\begin{aligned} L &= -\sum_{i,j=1}^{2}\sum_{k=1}^{n_{ij}}\big(C_0 + C_2\log\sigma_{m_i} + C_3\log\sigma_{M_j} + C_6\log\sigma_{m_i}\log\sigma_{M_j} \\ &\quad +(C_1 + C_4\log\sigma_{m_i} + C_5\log\sigma_{M_j} + C_7\log\sigma_{m_i}\log\sigma_M M_j)\log N_{ijk}\big) \\ &\quad + \sum_{i,j=1}^{2} n^*_{ij}\log C_1 + C_4\log\sigma_{m_i} + C_5\log\sigma_{M_j} + C_7\log\sigma_{m_i}\log\sigma_M M_j \end{aligned} \tag{41}$$

with respect to the parameters C_i $(i = 0, 1, \ldots, 7)$ subject to the constraints

$$C_0 + C_2\log\sigma_m + C_3\log\sigma_M + C_4\log N_{ijk}\log\sigma_m + C_5\log N_{ijk}\log\sigma_M \geq 0, \;\; \forall \;\; i, j, k, \tag{42}$$

and additional constraints to be satisfied by the parameters for the different models as presented earlier in Section 3.2.

6 Illustrative Examples

The proposed estimation methods are illustrated in this section with some data sets. For the set of parameter values $C_0 = -407.905, C_1 = 0, C_2 = -1.272, C_3 = 53.132, C_4 = -1.755, C_5 = 2.107, C_6 = 0, C_7 = 0$ and for each of the four combinations of the two values of σ_{min} ($\sigma_{min} = 510, 714$) and two values of σ_{max} ($\sigma_{max} = 1054, 1190$), two samples were simulated, one of small size ($n = 7$) and another of moderate size ($n = 20$). The *log N* values of these samples are presented in Table 1.

The model parameters were then estimated by using the two methods described in Section 5.

Method 1: The MLEs $\hat{C}_{0ij}$ and $\hat{C}_{1ij}$ $(i, j = 1, 2)$ and the corresponding MLEs $\hat{C}_i$ $(i = 0, 1, \ldots, 7)$ estimates are provided in Columns 3 and 4 of Table 2 as "ML unconstrained estimates", for the cases when $n = 7$ and 20 without runouts. The true values of the parameters of the model, viz., C_{0ij}, C_{1ij} $(i, j = 1, 2)$ and C_i $(i = 0, 1, \ldots 7)$, are also presented in Column 2 for the purpose of comparison. The parameters C_{0ij} and C_{1ij} are the ones that are crucial for modelling and prediction purposes. We observe that even though the estimation of the initial parameters is not accurate, the estimates of the parameters C_{0ij} and C_{1ij} parameters (which are functions of them) are quite good.

Further, in order to illustrate the influence of run-outs, we also considered additional threshold values. For each condition, we considered the corresponding 90-th percentile $Q_{90}(\sigma_m, \sigma_M)$ as threshold value, which are $Q_{90}(510, 1054) = 12.97520$, $Q_{90}(510, 1190) = 10.5213$, $Q_{90}(714, 1054) = 15.5570$ and $Q_{90}(714, 1190) = 12.4807$. Thus, for the samples presented in Table 1, we observe the number of run-outs to be $1, 1, 1, 1$ for the case when $n = 7$, and $3, 3, 3, 2$ for the case when $n = 20$. The estimates of the parameters of the original model as well as those of the reparameterized model are presented in Columns 5 and 6 of Table 2. Moreover, the quartiles of the true parameter values and the estimated values are presented in Table 3, while the cdf's as well as their estimates are presented in Figure 4. From the values of $\hat{C}_{0ij}$ and $\hat{C}_{1ij}$, as well as from the plots, it is clear that increasing sample size results in more precision while the presence of run-outs result in loss of precision in the estimation.

Method 2: We fitted the model in (23) using the constrained maximum likelihood method described in Section 5. The parameter estimates so obtained are presented in Table 2 as "Constrained estimates". It is clear that the fitted values of the original C_i parameters are better than those

Table 1: Simulated $log\ N$ ordered values for the four different conditions of (a) 7 and (b) 20 specimens at each condition.

(a)

Condition	Simulated $log\ N$ values						
$\sigma_m = 510$, $\sigma_M = 1054$	12.4235	12.4396	12.5503	12.6968	12.8483	12.854	13.0437
$\sigma_m = 510$, $\sigma_M = 1190$	9.95172	9.98658	10.0196	10.1651	10.2444	10.2453	10.7026
$\sigma_m = 714$, $\sigma_M = 1054$	14.8401	14.8985	14.9619	14.9907	15.3109	15.3237	15.7461
$\sigma_m = 714$, $\sigma_M = 1190$	11.8056	11.8549	11.9707	11.9759	12.041	12.3971	12.7405

(b)

Condition	Simulated $log\ N$ values									
$\sigma_m = 510$	12.3879	12.3945	12.4266	12.4364	12.4759	12.494	12.4955	12.5004	12.5329	12.6312
$\sigma_M = 1054$	12.643	12.7081	12.7257	12.7924	12.7981	12.8534	12.8892	12.9967	13.0073	13.0074
$\sigma_m = 510$	9.96423	9.98116	9.99038	10.016	10.0338	10.0425	10.0432	10.0482	10.0941	10.0995
$\sigma_M = 1190$	10.1064	10.1385	10.1428	10.2108	10.211	10.2116	10.2846	10.8629	10.8929	11.0735
$\sigma_m = 714$	14.8585	14.8848	14.8892	14.8893	14.8984	14.8998	14.9122	14.9212	14.9253	15.0038
$\sigma_M = 1054$	15.0649	15.0831	15.1102	15.2873	15.384	15.3964	15.5167	15.6858	15.8732	15.9394
$\sigma_m = 714$	11.8027	11.8065	11.8129	11.8224	11.8371	11.8609	11.9272	11.9326	11.9428	11.9901
$\sigma_M = 1190$	12.0414	12.0539	12.0895	12.090	12.1639	12.2018	12.2985	12.2991	12.8947	13.1203

obtained by Method 1. The quantile estimates are also presented in Table 3 as "Constrained estimates" which are also better than those obtained by Method 1.

We thus observe that the two methods of estimation proposed in Section 5 offer convenient methods of modeling the fatigue behavior, and that Method 1 presents a simpler alternative to the computationally involved constrained maximum likelihood method.

7 Conclusions

In this paper, we have presented a general model for the prediction of the fatigue behavior for any stress level and amplitude using an exponential model and discuss some important considerations such as weakest link principle, limit behavior, limited range, and compatibility. Using the compatibility condition, an exponential model is obtained with some non-dimensional model parameters whose physical interpretations are given in terms of threshold value of log-lifetime, endurance limit, parameter defining the position of the corresponding zero-percentile hyperbola, and scale factor. Next, based on physical and compatibility considerations and on the Wöhler field for fixed stress level, a compatibility functional equation yields a general exponential model with eight parameters. We have then reparametrized this general model into a two-parameter exponential distribution and have explained how the estimation of its parameters can be used to set-up a system of eight equations for the estimation of the eight parameters of the original model. For the estimation method, we have described two methods, one based on the maximum likelihood on four stress level tests and least-squares, and the other based on the constrained

Table 2: True and estimated values of the parameters, based on the two samples in Table 1, with and without run-outs.

Parameter	True value	Estimated value[1] ($n=7$)	Estimated value[1] ($n=20$)	Estimated value[2] ($n=7$)	Estimated value[2] ($n=20$)
		ML unconstrained estimates			
C_{011}	-46.018	-45.972	-45.554	-39.404	-38.721
C_{111}	3.724	3.700	3.677	3.172	3.126
C_{012}	-39.570	-42.140	-38.593	-36.120	-32.804
C_{112}	3.980	4.234	3.873	3.630	3.292
C_{021}	-46.446	-47.405	-47.512	-40.633	-40.386
C_{121}	3.134	3.194	3.198	2.738	2.718
C_{022}	-39.998	-38.499	-39.767	-32.999	-35.791
C_{122}	3.389	3.261	3.369	2.795	3.032
C_0	-407.905	-239.195	-408.472	-205.024	-347.201
C_1	0.000	0.000	0.000	0.000	0.000
C_2	-1.272	-4.258	-5.821	-3.650	-4.948
C_3	53.132	31.575	57.355	27.064	48.752
C_4	-1.755	-1.504	-1.425	-1.289	-1.212
C_5	2.107	4.400	1.614	3.772	1.372
C_6	0.000	0.000	0.000	0.000	0.000
C_7	0.000	0.000	0.000	0.000	0.000
		ML and least-squares			
C_0	-407.905	-412.142	-438.344	-353.264	-296.642
C_1	0.000	0.000	0.000	0.000	0.000
C_2	-1.272	0.000	-4.673	0.000	-6.927
C_3	53.132	52.505	60.590	45.004	43.308
C_4	-1.755	-2.193	-1.426	-1.879	-1.060
C_5	2.107	2.512	1.803	2.153	1.400
C_6	0.000	0.000	0.000	0.000	0.000
C_7	0.000	0.000	0.000	0.000	0.000
		Constrained estimates			
C_0	-407.905	-420.957	-410.061	-360.820	-353.680
C_1	0.000	0.000	0.000	0.000	0.000
C_2	-1.272	-2.779	-0.305	-2.382	-0.263
C_3	53.132	56.300	52.586	48.257	45.355
C_4	-1.755	-1.621	-1.812	-1.389	-1.563
C_5	2.107	1.988	2.156	1.704	1.860
C_6	0.000	0.000	0.000	0.000	0.000
C_7	0.000	0.000	0.000	0.000	0.000

[1] without run-outs [2] with run-outs

* In these cases, no run-outs occurred and the estimates are therefore the same.

maximum likelihood. Finally, we have presented two examples to illustrate the exponential model derived as well as to demonstrate the two methods of estimation proposed here.

Moreover, it is also important to mention the following points:

1. The method will enable one to extrapolate laboratory results obtained from any combination of values of σ_{min} or σ_{max} to any possible combination of values of σ_{min} or σ_{max} arising in a practical case;

Table 3: True and estimated quartiles, based on the two samples in Table 1, with and without run-outs.

Condition	Quartile	True value	Estimated value[1] $(n=7)$	Estimated value[1] $(n=20)$	Estimated value[2] $(n=7)$	Estimated value[2] $(n=20)$
			ML unconstrained estimates			
$\sigma_m = 510$, $\sigma_M = 1054$	Q_{25}	12.4342	12.5012	12.4661	12.5142	12.4799
	Q_{50}	12.5431	12.6108	12.5764	12.6420	12.6096
	Q_{75}	12.7292	12.7981	12.7649	12.8606	12.8314
	Q_{90}	12.9752	13.0458	13.0140	13.1495	13.1245
$\sigma_m = 510$, $\sigma_M = 1190$	Q_{25}	10.0150	10.0197	10.0385	10.0310	10.0516
	Q_{50}	10.1169	10.1154	10.1432	10.1427	10.1748
	Q_{75}	10.2911	10.2791	10.3222	10.3337	10.3853
	Q_{90}	10.5213	10.4955	10.5587	10.5861	10.6636
$\sigma_m = 714$, $\sigma_M = 1054$	Q_{25}	14.9139	14.9301	14.9484	14.9452	14.9643
	Q_{50}	15.0433	15.0571	15.0752	15.0932	15.1135
	Q_{75}	15.2645	15.2741	15.2920	15.3464	15.3685
	Q_{90}	15.5570	15.5609	15.5785	15.6810	15.7056
$\sigma_m = 714$, $\sigma_M = 1190$	Q_{25}	11.8862	11.8938	11.8880	11.9085	11.8957
	Q_{50}	12.0059	12.0181	12.0084	12.0536	12.0312
	Q_{75}	12.2104	12.2307	12.2141	12.3015	12.2598
	Q_{90}	12.4807	12.5117	12.4860	12.6294	12.5620
			ML and least-squares			
$\sigma_m = 510$, $\sigma_M = 1054$	Q_{25}	12.4342	12.3172	12.5789	12.3298	12.3338
	Q_{50}	12.5431	12.4235	12.6896	12.4538	12.4631
	Q_{75}	12.7292	12.6052	12.8790	12.6658	12.6841
	Q_{90}	12.9752	12.8455	13.1294	12.9461	13.9763
$\sigma_m = 510$, $\sigma_M = 1190$	Q_{25}	10.0150	9.8585	9.9735	9.8701	10.1102
	Q_{50}	10.1169	9.9569	10.0781	9.9850	10.2328
	Q_{75}	10.2911	10.1252	10.2568	10.1813	10.4425
	Q_{90}	10.5213	10.3477	10.4930	10.4408	10.7197
$\sigma_m = 714$, $\sigma_M = 1054$	Q_{25}	14.9139	15.2713	14.9710	15.2869	14.7548
	Q_{50}	15.0433	15.4031	15.0985	15.4407	15.9007
	Q_{75}	15.2645	15.6285	15.3165	15.7036	15.1501
	Q_{90}	15.5570	15.9263	15.6046	16.0511	15.4797
$\sigma_m = 714$, $\sigma_M = 1190$	Q_{25}	11.8862	12.0097	11.8439	12.0239	12.1229
	Q_{50}	12.0059	12.1297	12.9632	12.1638	12.2603
	Q_{75}	12.2104	12.3347	12.1671	12.4030	12.4954
	Q_{90}	12.4807	12.6057	12.4367	12.7192	12.8060
			Constrained estimates			
$\sigma_m = 510$, $\sigma_M = 1054$	Q_{25}	12.4342	12.5005	12.4654	12.5133	12.4778
	Q_{50}	12.5431	12.6090	12.574	12.6399	12.6045
	Q_{75}	12.7292	12.7945	12.7616	12.8564	12.8212
	Q_{90}	12.9752	13.0398	13.0087	13.1425	13.1077
$\sigma_m = 510$, $\sigma_M = 1190$	Q_{25}	10.0150	10.0240	10.0367	10.0361	10.0482
	Q_{50}	10.1169	10.1260	10.1388	10.1550	10.1666
	Q_{75}	10.2911	10.3003	10.3133	10.3583	10.3690
	Q_{90}	10.5213	10.5306	10.5441	10.6271	10.6365
$\sigma_m = 714$, $\sigma_M = 1054$	Q_{25}	14.9139	14.9302	14.9513	14.9453	14.9661
	Q_{50}	15.0433	15.0573	15.0821	15.0935	15.1178
	Q_{75}	15.2645	15.2746	15.3058	15.3470	15.3771
	Q_{90}	15.5570	15.5618	15.6014	15.6820	15.7199
$\sigma_m = 714$, $\sigma_M = 1190$	Q_{25}	11.8862	11.8894	11.8883	11.9034	11.9019
	Q_{50}	12.0059	12.0075	12.0089	12.0412	12.0418
	Q_{75}	12.2104	12.2095	12.2152	12.2768	12.2809
	Q_{90}	12.4807	12.4765	12.4878	12.5883	12.5970

[1] without run-outs [2] with run-outs

2. Testing strategies (test designs) suggested for producing enough data will enable the above extrapolation possible. This will result in a substantial reduction as compared to traditional testing

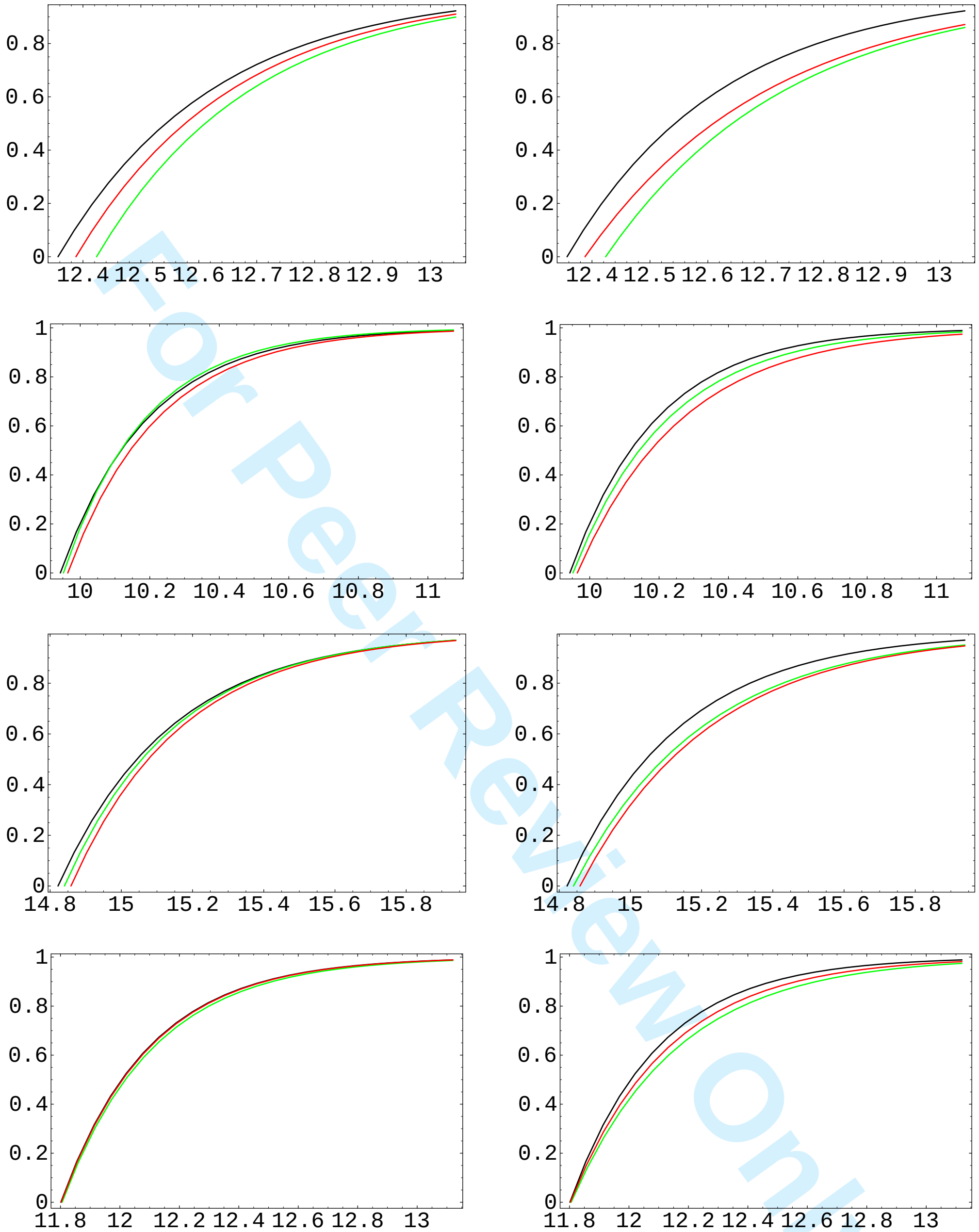

Figure 4: Cdf of lifetimes $\log N$ under the four conditions (from top to bottom: ($\sigma_m = 510; \sigma_M = 1054$), ($\sigma_m = 510; \sigma_M = 1190$), ($\sigma_m = 714; \sigma_M = 1054$) and ($\sigma_m = 714; \sigma_M = 1190$)). Estimated cdf based on the sample of size $n = 7$ and $n = 20$ are plotted in green and red, respectively, for the case without run-outs [figures in the left column (a)] and with run-outs [in the right column], for the unconstrained method.

procedures; in fact, four levels are normally used in standard tests of constant σ_{min} or σ_{max};

3. Explicit simple formulas derived will facilitate the extrapolation (or interpolation) from lab results;

4. The normalization principle, together with the model proposed in this paper, would enable the analysis of damage accumulation when one knows the history of σ_{min} or σ_{max} as a function of time or cycle in a practical case;

5. The unconstrained and constrained methods of estimation proposed will facilitate the model fitting in a simple and efficient manner.

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