Quantum revivals in two degrees of freedom integrable systems : the torus case

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QUANTUM REVIVALS IN TWO DEGREES OF FREEDOM INTEGRABLE SYSTEMS : THE TORUS CASE

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ABSTRACT. The paper deals with the semi-classical behaviour of quantum dynamics for a semi-classical completely integrable system with two degrees of freedom near Liouville regular torus. The phenomomenon of wave packet revivals is demonstrated in this article. The framework of this paper is semi-classical analysis (limit : $h \to 0$). For the proofs we use standard tools of real analysis, Fourier analysis and basic analytic number theory.

1. INTRODUCTION

1.1. **Motivation.** In quantum physics, on a Riemannian manifold (M, g) the evolution of an initial state $\psi_0 \in L^2(M)$ is given by the famous Schrödinger equation :

$$ih\frac{\partial \psi(t)}{\partial t} = P_h\psi(t);\ \psi(0) = \psi_0.$$

Here $h > 0$ is the semi-classical parameter and the operator $P_h : D(P_h) \subset L^2(M) \to L^2(M)$ is h-pseudo-differential operator (for example $P_h = -\frac{h^2}{2}\Delta_g + V$). In the case of dimension 1 or for completely integrable systems, we can describe the semi-classical eigenvalues of the Hamiltonian P_h and by linearity we can write the solutions of the Schrödinger equation. Nevertheless, the behaviour of the solutions when the times t evolves in larges times scales remains quite mysterious.

In dimension 1, the dynamics in the regular case and for elliptic non-degenerate singularity have been the subject of many research in physics **[Av-Pe], [LAS], [Robi1]**, **[Robi2], [BKP]**, **[Bl-Ko]** and, more recently in mathematics **[Co-Ro]**, **[Rob], [Pau1]**, **[Pau2], [Lab2]**. The strategy to understand the long times behaviour of dynamics is to use the spectrum of the operator P_h. In the regular case, the spectrum of P_h is given by the famous Bohr-Sommerfeld rules (see for example **[He-Ro], [Ch-VuN]**, **[Col]**) : in first approximation, the spectrum of P_h in a compact set is a sequence of real numbers with a gap of size h. The classical trajectories are periodic and supported on elliptic curves. Always in dimension 1, in the case of hyperbolic singularity we have a non-periodic trajectory. The spectrum near this singularity is more complicated than in the regular case. In **[Lab3]** we have an explicit description of the spectrum for an one-dimesional pesudo-differential operator near a hyperbolic non-degenerate singularity. The article **[Lab4]** deals with the quantum dynamics for the hyperbolic case. So, in dimension 1, we get the full and fractionnals revivals phenomenon (see **[Av-Pe], [LAS], [Robi1]**, **[Robi2], [BKP]**, **[Bl-Ko], [Co-Ro]**, **[Rob], [Pau1]** for the elliptic case and see **[Lab4]** or **[Pau2]** for the the hyperbolic case). For an initial wave packets localized in energy, the dynamics follows the classical motion during short time, and, for large time, a new period T_{rev} for the quantum dynamics appears : the initial wave packets form again at $t = T_{rev}$.

Date: 1 September 2010.

Physicists R. Blhum, A. Kostelecky and B. Tudose are interested in the case of the dimension 2 (see **[BKT]**). Our paper presents some accurate results on the time evolution for a generical semi-classical completely integrable system of dimension 2 with mathematical proofs.

1.2. **Results and paper organization.** Here the quantum Hamiltonian is of the type $P_h = F(P_1, P_2)$ where F is a real polynomial of two variables and P_1, P_2 are semi-classical one dimensional harmonic oscillators (see section 2 for details). By a diffeomorphism this Hamiltonian is less particular than it seems to be, since it gives the spectrum of any completely integrable system with two degrees of freedom near regular torus or around elliptic singularity **[VuN]**. Therefore, the Hamiltonian study leads to a study more or less general but which is not obvious in dimension 2. In this paper, we consider an initial state ψ_0 localized near some regular Liouville torus of energies (E_1, E_2) and we study the associated quantum dynamics. To understand the behaviour of dynamics, we interested in the evolution of the autocorrelation :

$$\mathbf{a}(t) = \left| \langle \psi(t), \psi_0 \rangle_{L^2(\mathbb{R}^2)} \right|.$$

Due to the simple nature of the Hamiltonian operator the autocorrelation function can be write as a serie :

$$\mathbf{a}(t) = \left| \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} |a_{n,m}|^2 \, e^{-i\frac{t}{h}F(\tau_n,\mu_m)} \right|,$$

where $\tau_n = \omega_1 h\left(n + \frac{1}{2}\right)$, $\mu_m = \omega_2 h\left(m + \frac{1}{2}\right)$ are eigenvalues of the one-dimensionnal harmonic oscillators P_1, P_2. The sequence $(a_{n,m})_{n,m}$ is just the decomposition of the initial vector ψ_0 on the Hermitte's eigenbasis of $L^2(\mathbb{R}^2)$.

Most of the paper (section 3 and 4) consists in estimating and analyzing the function $\mathbf{a}(t)$ for large times scales ($t \leq 1/h^s$ with various $s > 0$). We use Taylor's formula to expand the phase term $tF(\tau_n, \mu_m)/h$ in the variables (n, m); first in linear order (section 3), then to quadratic order (section 4).

In the section 3, we study the linear approximation $\mathbf{a_1}(t)$ (see definition 3.6) of the autocorrelation function, valid up on a time scale $[0, 1/h^\alpha]$ where $0 < \alpha < 1$. The dynamics depends strongly on the diophantin properties of the classical periods T_{cl_1}, T_{cl_2}. If the fraction T_{cl_1}/T_{cl_2} is commensurate (in this case the classical Hamiltonian flow is T_{cl}-periodic) we can describe accurately the behaviour of the dynamics on a classical period $[0, T_{cl}]$ (see theorem 3.12). In opposite, if the fraction T_{cl_1}/T_{cl_2} is a bad approximation by rationals (we suppose T_{cl_1}/T_{cl_2} is Roth number) the autocorrelation function collapse in the set $]0, T_s]$ where T_s is order of $1/h^s$ (see theorem 3.24). For large time we use the continuous fraction expansion of T_{cl_1}/T_{cl_2} to analyze some possible periods for linear approximation $\mathbf{a_1}(t)$ (see theorem 3.35).

In the last section, we use the quadradic approximation $\mathbf{a_2}(t)$ (see definition 4.6) of the autocorrelation function, valid up on a time scale $\left[0, 1/h^\beta\right]$ where $\beta > 1$. In this quadradic approximation appear three revivals periods T_{rev_1}, T_{rev_2} and $T_{rev_{12}}$ of order $1/h$ depending on the Hessian matrix of the function F at the point (E_1, E_2). If we suppose T_{rev_1}, T_{rev_2} and $T_{rev_{12}}$ are commensurate, we can proove and analyze the revivals phenomenon (see theorem 4.16 and corollary 4.17). In the last subsection we compute the modulus of the revival coefficients (see theorem 4.19).

2. General points

2.1. **Some basic facts on semi-classical analysis.** To explain quickly the philosophy of semi-classical analysis, starts by an example : for a real number $E > 0$; the equation

$$-\frac{h^2}{2}\Delta_g \varphi = E\varphi$$

(where Δ_g denotes the Laplace-Beltrami operator on a Riemaniann manifold (M, g)) admits the eigenvectors φ_k as solution if

$$-\frac{h^2}{2}\lambda_k = E.$$

Hence if $h \to 0^+$ then $\lambda_k \to +\infty$. So there exists a correspondence between the semi-classical limit ($h \to 0^+$) and large eigenvalues.

The asympotic of large eigenvalues for the Laplace-Beltrami operator Δ_g on a Riemaniann manifold (M, g), or more generally for a pseudo-differential operator P_h, is linked to a symplectic geometry : the phase space geometry. This is the same phenomenon between quantum mechanics (spectrum, operators algebra) and classical mechanics (length of periodic geodesics, symplectic geometry). For more details see for example the survey **[Lab1]**.

2.2. **Quantum dynamics and autocorrelation function.** For a quantum Hamiltonian $P_h : D(P_h) \subset \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, the Schrödinger dynamics is governed by the Schrödinger equation :

$$ih\frac{\partial \psi(t)}{\partial t} = P_h \psi(t).$$

With the functional calculus, we can reformulate this equation with the unitary group $U(t) = \left\{e^{-i\frac{t}{h}P_h}\right\}_{t\in\mathbb{R}}$. Indeed, for a initial state $\psi_0 \in \mathcal{H}$, the evolution is given by :

$$\psi(t) = U(t)\psi_0 \in \mathcal{H}.$$

We now introduce a simple tool to understand the behaviour of the vector $\psi(t)$: this tool is a quantum analog of the Poincaré return function :

Definition. The quantum return functions of the operator P_h and for an initial state ψ_0 is defined by :

$$\mathbf{r}(t) := \langle \psi(t), \psi_0 \rangle_{\mathcal{H}};$$

and the autocorrelation function is defined by :

$$\mathbf{a}(t) := |\mathbf{r}(t)| = |\langle \psi(t), \psi_0 \rangle_{\mathcal{H}}|.$$

The previous function measures the return on the initial state ψ_0. This function is the overlap of the time dependent quantum state $\psi(t)$ with the initial state ψ_0. Since the initial state ψ_0 is normalized, the autocorrelation function takes values in the compact set $[0, 1]$.

2.3. **The Hamiltonian of our model.** For our study, the quantum Hamiltonian is the operator :

$$P_h := F(P_1, P_2)$$

where F is a polynomial of $\mathbb{R}[X, Y]$ which does not depend on the paramater h; P_1 and P_2 are the Weyl-quantization of the classical one dimensional harmonic oscillator :

$$p_j(x_1, \xi_1, x_2, \xi_2) = \omega_j \left(x_j^2 + \xi_j^2\right)/2$$

with $\omega_1, \omega_2 > 0$. It is well know that the Hermitte functions $(e_{n,m})_{n,m} := (e_n \otimes e_m)_{n,m\in\mathbb{N}^2}$ is a Hilbert basis of the space $L^2(\mathbb{R}^2)$. Let us consider for all integers (n, m) the eigenvalues of P_1 and P_2 :

$$\tau_n := \omega_1 h\left(n + \frac{1}{2}\right),\ \mu_m := \omega_2 h\left(m + \frac{1}{2}\right);$$

so, we get immediatly that for all integers (n, m)

$$F\left(P_1, P_2\right)\left(e_n \otimes e_m\right) = F\left(\tau_n, \mu_m\right)\left(e_n \otimes e_m\right).$$

2.4. **The autocorrelation function rewritten in a eigenbasis.** Now, for a initial vector $\psi_0 = \sum_{n,m\in\mathbb{N}^2} a_{n,m}e_{n,m}$ we have for all $t \geq 0$

$$\psi(t) = \left(e^{-i\frac{t}{h}F(P_1,P_2)}\right)\left(\sum_{n,m\in\mathbb{N}^2} a_{n,m}e_{n,m}\right) = \sum_{n,m\in\mathbb{N}^2} a_{n,m}e^{-i\frac{t}{h}F(\tau_n,\mu_m)}e_{n,m}$$

so, for all $t \geq 0$ we obtain

$$\mathbf{r}(t) = \sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} |a_{n,m}|^2\, e^{-i\frac{t}{h}F(\tau_n,\mu_m)}$$

and

$$\mathbf{a}(t) = \left|\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty} |a_{n,m}|^2\, e^{-i\frac{t}{h}F(\tau_n,\mu_m)}\right|.$$

The aim of this paper is to study this sum, but unfortunately this function is too difficult to be understood immediatly.

2.5. **Strategy to study the autocorrelation function.** The strategy for simplify the sum function $t \mapsto \mathbf{a}(t)$, performed by the physicists (**[Av-Pe], [LAS], [Robi1]**, **[Robi2], [BKP]**, **[Bl-Ko]**) is the following :

(1) we define a initial vector $\psi_0 = \sum_{n,m\in\mathbb{N}^2} a_{n,m}e_{n,m}$ localized near some regular Liouville torus of energies (E_1, E_2) : consequently the sequence $(a_{n,m})_{n,m\in\mathbb{N}^2}$ is localized close to a pair of quantum numbers n_0, m_0 (depends on h and on the Liouville torus (E_1, E_2).

(2) Next, the idea is to expand by a Taylor formula's the eigenvalues $F\left(\tau_n, \mu_m\right)$ around the Liouville torus (E_1, E_2) :

$$F\left(\tau_n, \mu_m\right) =$$

$$F\left(\tau_{n_0}, \mu_{m_0}\right) + h\omega_1\left(n - n_0\right)\frac{\partial F}{\partial X}\left(\tau_{n_0}, \mu_{m_0}\right) + h\omega_2\left(m - m_0\right)\frac{\partial F}{\partial Y}\left(\tau_{n_0}, \mu_{m_0}\right)$$

$$+\frac{1}{2}\omega_1^2h^2\left(n - n_0\right)^2\frac{\partial^2 F}{\partial X^2}\left(\tau_{n_0}, \mu_{m_0}\right) + \frac{1}{2}\omega_2^2h^2\left(m - m_0\right)^2\frac{\partial^2 F}{\partial Y^2}\left(\tau_{n_0}, \mu_{m_0}\right)$$

$$+\omega_1\omega_2h^2\left(n - n_0\right)\left(m - m_0\right)\frac{\partial^2 F}{\partial X\partial Y}\left(\tau_{n_0}, \mu_{m_0}\right) + \cdots$$

(here τ_{n_0}, μ_{m_0} is the closest pair of eigenvalue to the pair E_1, E_2). As a consequence we get for all $t \geq 0$

$$\mathbf{a}(t) = \left|\sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2\, e^{-it\left[\omega_1(n-n_0)\frac{\partial F}{\partial X}\left(\tau_{n_0},\mu_{m_0}\right)+\cdots+\omega_1\omega_2h(n-n_0)(m-m_0)\frac{\partial^2 F}{\partial X\partial Y}\left(\tau_{n_0},\mu_{m_0}\right)+\cdots\right]}\right|.$$

(3) And, for small values of t, the first approximation of the autocorrelation function $\mathbf{a}(t)$ is the function

$$\mathbf{a}_1(t) := \left| \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \, e^{-it\left[\omega_1(n-n_0)\frac{\partial F}{\partial X}\left(\tau_{n_0},\mu_{m_0}\right)+\omega_2(m-m_0)\frac{\partial F}{\partial Y}\left(\tau_{n_0},\mu_{m_0}\right)\right]} \right|;$$

and for larger values of t, the order 2-approximation is given by

$$\mathbf{a}_2(t) := \left| \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \, e^{-it\left[\omega_1(n-n_0)\frac{\partial F}{\partial X}\left(\tau_{n_0},\mu_{m_0}\right)+\cdots+\omega_1\omega_2 h(n-n_0)(m-m_0)\frac{\partial^2 F}{\partial X\partial Y}\left(\tau_{n_0},\mu_{m_0}\right)\right]} \right|.$$

In section 3, we study in details the function $t \mapsto \mathbf{a}_1(t)$ and $t \mapsto \mathbf{a}_2(t)$ in section 4.

2.6. **Choice of an initial state.** Let us define an initial vector $\psi_0 = \sum_{n,m\in\mathbb{N}^2} a_{n,m} e_{n,m}$ localized near a regular Liouville torus of energies $E := (E_1, E_2)$ where $E_1 \in [0,1]$ and $E_2 \in [0,1]$.

Definition 2.1. Let us consider the quantum integers $n_0 = n_0(h, E_1)$ and $m_0 = m_0(h, E_2)$ defined by

$$n_0 := \arg\min_n |\tau_n - E_1| \,;\; m_0 := \arg\min_m |\mu_m - E_2| \,.$$

Remark 2.2. Without loss of generality, we may suppose that the integers n_0 and m_0 are unique.

The integer n_0 (resp. m_0) is the eigenvalues index of the operator from the family P_1 (resp. P_2) the closest to the real number E_1 (resp. E_2). Since the spectral gap of P_1 (resp. P_2) is equal to $\omega_1 h$ (resp. $\omega_2 h$) we have, for $h \to 0$: $n_0 \sim \frac{E_1}{\omega_1 h}$; $m_0 \sim \frac{E_2}{\omega_2 h}$.

Now, we can give definition of our initial state :

Definition 2.3. Let us consider the sequence $(a_{n,m})_{n,m\in\mathbb{Z}^2} = (a_{n,m}(h))_{n,m\in\mathbb{Z}^2}$ defined by :

$$a_{n,m} := K_h \chi\left(\frac{\tau_n - \tau_{n_0}}{h^{\delta_1'}}, \frac{\mu_m - \mu_{m_0}}{h^{\delta_2'}}\right) = K_h \chi\left(\omega_1 \frac{n-n_0}{h^{\delta_1'-1}}, \omega_2 \frac{m-m_0}{h^{\delta_2'-1}}\right);$$

where the function χ is non null, non-negative and belong ot the space $\mathcal{S}(\mathbb{R}^2)$. The parameters $(\delta_1', \delta_2') \in]0,1[^2$. We also denote

$$K_h := \left\| \chi\left(\frac{\tau_n - \tau_{n_0}}{h^{\delta_1'}}, \frac{\mu_m - \mu_{m_0}}{h^{\delta_2'}}\right) \right\|_{\ell^2(\mathbb{N}^2)}.$$

Let us detail this choice :

(1) the term $\chi\left(\frac{\tau_n-\tau_{n_0}}{h^{\delta_1'}}, \frac{\mu_m-\mu_{m_0}}{h^{\delta_2'}}\right)$ localize around the torus (E_1, E_2) (for technical reason we localize around the closest eigenvalues to (E_1, E_2).
(2) Constants δ_1' and δ_2' are coefficients for dilate the function χ (the reason to take $0 < \delta_j' < 1$ is the following : it is the unique way to have a non-trivial localization (not tend to $\{0\}$) and a localization larger the spectral $h^{\delta_j'} \gg h$).

So, clearly the sequence $(a_{n,m})_{n,m} \in \ell^2(\mathbb{Z}^2)$. Now, let us evaluate the constant of normalization K_h; start by the :

Lemma 2.4. *For a function $\varphi \in \mathcal{S}(\mathbb{R}^2)$ and $(\varepsilon_1, \varepsilon_2) \in]0,1]^2$ then we have uniformly for $(u_1, u_2) \in \mathbb{R}^2$:*

$$\sum_{\ell,s\in\mathbb{Z}^2,\, |\ell+u_1|\geq\frac{1}{2},\, |s+u_2|\geq\frac{1}{2}} \left|\varphi\left(\frac{\ell+u_1}{\varepsilon_1}, \frac{s+u_2}{\varepsilon_2}\right)\right| = O(\varepsilon_1^\infty + \varepsilon_2^\infty).$$

Proof. We see easily that, uniformly for $(u_1, u_2) \in \mathbb{R}^2$ we have

$$\sum_{\ell,s\in\mathbb{Z}^2,\,|\ell+u_1|\geq\frac{1}{2},\,|s+u_2|\geq\frac{1}{2}} \left|\varphi\left(\frac{\ell+u_1}{\varepsilon_1},\frac{s+u_2}{\varepsilon_2}\right)\right| = O(1).$$

Next

$$\sum_{\ell,s\in\mathbb{Z}^2,\,|\ell+u_1|\geq\frac{1}{2},\,|s+u_2|\geq\frac{1}{2}} \left|\varphi\left(\frac{\ell+u_1}{\varepsilon_1},\frac{s+u_2}{\varepsilon_2}\right)\right|$$

$$\leq \sum_{\ell,s\in\mathbb{Z}^2,\,|\ell+u_1|\geq\frac{1}{2},\,|s+u_2|\geq\frac{1}{2}} \left(\frac{\ell+u_1}{\varepsilon_1}\right)^{2N} \left|\varphi\left(\frac{\ell+u_1}{\varepsilon_1},\frac{s+u_2}{\varepsilon_2}\right)\right| \frac{\varepsilon_1^{2N}}{(\ell+u_1)^{2N}}$$

$$\leq \varepsilon_1^{2N} 4^N \sum_{\ell,s\in\mathbb{Z}^2} \left(\frac{\ell+u_1}{\varepsilon_1}\right)^{2N} \left|\varphi\left(\frac{\ell+u_1}{\varepsilon_1},\frac{s+u_2}{\varepsilon_2}\right)\right|.$$

And, similary we have

$$\sum_{\ell,s\in\mathbb{Z}^2,\,|\ell+u_1|\geq\frac{1}{2},\,|s+u_2|\geq\frac{1}{2}} \left|\varphi\left(\frac{\ell+u_1}{\varepsilon_1},\frac{s+u_2}{\varepsilon_2}\right)\right|$$

$$\leq \varepsilon_2^{2N} 4^N \sum_{\ell,s\in\mathbb{Z}^2} \left(\frac{s+u_2}{\varepsilon_2}\right)^{2N} \left|\varphi\left(\frac{\ell+u_1}{\varepsilon_1},\frac{s+u_2}{\varepsilon_2}\right)\right|.$$

To conclude the proof, we apply that to the functions $\psi(x,y) := x^{2N}\varphi(x,y)$ and $\psi(x,y) := y^{2N}\varphi(x,y)$. □

An obvious consequence of this lemma is the following result :

Proposition 2.5. *We get*

$$K_h = \frac{1}{\sqrt{\mathfrak{F}\left(\chi^2\right)(0,0)}h^{\frac{\delta_1'+\delta_2'-2}{2}}} + O\left(h^\infty\right);$$

hence $\|a_{n,m}\|_{\ell^2(\mathbb{N}^2)} = 1 + O(h^\infty)$.

Proof. By the Poisson formula and the lemma above we get the equality :

$$\sum_{n,m\in\mathbb{Z}^2} \chi^2\left(\omega_1\frac{n-n_0}{h^{\delta_1'-1}},\omega_2\frac{m-m_0}{h^{\delta_2'-1}}\right) = h^{\delta_1'+\delta_2'-2}\sum_{\ell,s\in\mathbb{Z}^2}\mathfrak{F}\left(\chi^2\right)\left(-\ell\frac{h^{\delta_1'-1}}{\omega_1},-s\frac{h^{\delta_2'-1}}{\omega_2}\right)$$

$$= h^{\delta_1'+\delta_2'-2}\left[\mathfrak{F}\left(\chi^2\right)(0,0) + \sum_{\ell,s\in\mathbb{Z}^2,\,|\ell|+|s|\geq 1}\mathfrak{F}\left(\chi^2\right)\left(-\ell\frac{h^{\delta_1'-1}}{\omega_1},-s\frac{h^{\delta_2'-1}}{\omega_2}\right)\right]$$

$$= h^{\delta_1'+\delta_2'-2}\mathfrak{F}\left(\chi^2\right)(0,0) + O\left(h^\infty\right).$$

Now, with the basic equality

$$\sum_{n,m\in\mathbb{N}^2} \chi^2\left(\frac{n-n_0}{h^{\delta_1'-1}},\frac{m-m_0}{h^{\delta_2'-1}}\right) = h^{\delta_1'+\delta_2'-2}\mathfrak{F}\left(\chi^2\right)(0,0) + O\left(h^\infty\right)$$

$$-\sum_{n=-\infty}^{-1}\sum_{m=-\infty}^{+\infty}\chi^2\left(\frac{n-n_0}{h^{\delta_1'-1}},\frac{m-m_0}{h^{\delta_2'-1}}\right) - \sum_{n=0}^{+\infty}\sum_{m=-\infty}^{-1}\chi^2\left(\frac{n-n_0}{h^{\delta_1'-1}},\frac{m-m_0}{h^{\delta_2'-1}}\right).$$

and with the lemma above we see easily that

$$\sum_{n=-\infty}^{-1}\sum_{m=-\infty}^{+\infty}\chi^2\left(\frac{n-n_0}{h^{\delta_1'-1}},\frac{m-m_0}{h^{\delta_2'-1}}\right) = O(h^\infty),$$

$$\sum_{n=0}^{+\infty}\sum_{m=-\infty}^{-1}\chi^2\left(\frac{n-n_0}{h^{\delta_1'-1}},\frac{m-m_0}{h^{\delta_2'-1}}\right)=O(h^\infty).$$

Finally we get :

$$\left\|\chi\left(\frac{n-n_0}{h^{\delta_1'-1}},\frac{m-m_0}{h^{\delta_2'-1}}\right)\right\|^2_{\ell^2(\mathbb{N}^2)}=\mathfrak{F}\left(\chi^2\right)(0,0)h^{\delta_1'+\delta_2'-2}+O\left(h^\infty\right);$$

hence

$$K_h=\frac{1}{\sqrt{\mathfrak{F}\left(\chi^2\right)(0,0)}h^{\frac{\delta_1'+\delta_2'-2}{2}}}+O\left(h^\infty\right).$$

For finish, we write

$$\begin{aligned}\|a_{n,m}\|^2_{\ell^2(\mathbb{N}^2)}&=K_h^2\sum_{n,m\in\mathbb{Z}^2}\left|\chi\left(\omega_1\frac{n-n_0}{h^{\delta_1'-1}},\omega_2\frac{m-m_0}{h^{\delta_2'-1}}\right)\right|^2\\&=K_h^2h^{\delta_1'+\delta_2'-2}\left[\mathfrak{F}\left(\chi^2\right)(0,0)+O(h^\infty)\right]=1+O(h^\infty).\end{aligned}$$

□

2.7. **Technical interlude : the set Δ.** In this subsection, we introduce the set $\Delta\subset\mathbb{N}^2$, this set is useful for making approximation for autocorrelation function. Start by the definition :

Definition 2.6. Let us define the set of integers $\Delta=\Delta(h,E_1,E_2)$ by :

$$\begin{aligned}\Delta&:=\left\{(n,m)\in\mathbb{N}^2;\,|\tau_n-\tau_{n_0}|\leq\omega_1h^{\delta_1}\text{ and }|\mu_m-\mu_{m_0}|\leq\omega_2h^{\delta_2}\right\}\\&=\left\{(n,m)\in\mathbb{N}^2;\,|n-n_0|\leq h^{\delta_1-1}\text{ and }|m-m_0|\leq h^{\delta_2-1}\right\}\end{aligned}$$

where $0<\delta_i<1$; and we define the set $\Gamma=\Gamma(h,E_1,E_2)$ by :

$$\Gamma:=\mathbb{N}^2-\Delta.$$

We have the following usefull lemma :

Lemma 2.7. *If we suppose for all $i\in\{1,2\}$, $\delta_i'>\delta_i$ then we have*

$$\sum_{n,m\in\Gamma}|a_{n,m}|^2=O(h^\infty).$$

Proof. The starting point is the following inequality :

$$\sum_{n,m\in\Gamma}|a_{n,m}|^2\leq\sum_{n,m\in\mathbb{Z}^2,\,|n-n_0|>h^{\delta_1-1}}|a_{n,m}|^2+\sum_{n,m\in\mathbb{Z}^2,\,|m-m_0|>h^{\delta_2-1}}|a_{n,m}|^2.$$

Since the function χ^2 is in the space $\mathcal{S}(\mathbb{R}^2)$, for all integer $N\geq1$ we have

$$\sum_{n,m\in\mathbb{Z}^2}\left(\frac{n-n_0}{h^{\delta_1'-1}}\right)^{2N}|a_{n,m}|^2+\sum_{n,m\in\mathbb{Z}^2}\left(\frac{m-m_0}{h^{\delta_2'-1}}\right)^{2N}|a_{n,m}|^2=O(1).$$

Without loss generality, we may suppose that $n_0=m_0=0$. Next we write

$$\begin{aligned}\sum_{n,m\in\mathbb{Z}^2,\,|n|>h^{\delta_1-1}}|a_{n,m}|^2&=h^{2N(\delta_1'-1)}\sum_{n,m\in\mathbb{Z}^2,\,|n|>h^{\delta_1-1}}|a_{n,m}|^2\left(\frac{n}{h^{\delta_1'-1}}\right)^{2N}\frac{1}{n^{2N}}\\&=O\left(h^{2N(\delta_1'-\delta_1)}\right).\end{aligned}$$

In a similar way, we get

$$\sum_{n,m\in\mathbb{Z}^2,\,|m|>h^{\delta_2-1}} |a_{n,m}|^2 = O\left(h^{2N(\delta_2'-\delta_2)}\right);$$

because $\delta_i' > \delta_i$, this implies $\sum_{n,m\in\Gamma} |a_{n,m}|^2 = O(h^\infty)$, so we prove the lemma. □

3. Order 1 approximation : classical periods

3.1. **Introduction.** In this section, we use a Taylor's formula to expand the phase term $tF(\tau_n,\mu_m)/h$ in the variables (n,m) in linear order. In this approximation appear two periods T_{cl_1} and T_{cl_2} of order $O(1)$ depending on the gradient of the function F at the point (E_1,E_2).

3.2. **Linear approximation and classical periods.**

Assumption 3.1. *Here, we suppose that* $\frac{\partial F}{\partial X}(E_1,E_2)\neq 0$, $\frac{\partial F}{\partial Y}(E_1,E_2)\neq 0$.

3.2.1. *Semi-classical and classical periods.*

Definition 3.2. We define semi-classical periods $T_{scl_1}$and T_{scl_2} by :

$$T_{scl_1} := \frac{2\pi}{\frac{\partial F}{\partial X}(\tau_{n_0},\mu_{m_0})\,\omega_1} \text{ and } T_{scl_2} := \frac{2\pi}{\frac{\partial F}{\partial Y}(\tau_{n_0},\mu_{m_0})\,\omega_2}.$$

So, in linear order approximation, we have :

Proposition 3.3. *Let α a real number such that $\alpha > 1-2\min\delta_i$. Then, uniformly for all $t\in[0,h^\alpha]$:*

$$\mathbf{r}(t) = e^{-itF(\tau_{n_0},\mu_{m_0})/h}\sum_{n,m\in\mathbb{N}^2}|a_{n,m}|^2\,e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}\right)} + O\left(h^{\alpha+2\min\delta_i-1}\right).$$

Proof. Let us introduce the difference $\varepsilon(t) := \varepsilon(t,h)$ defined by

$$\varepsilon(t) := \left|\sum_{n,m\in\mathbb{N}^2}|a_{n,m}|^2\,e^{-i\frac{t}{h}F(\tau_n,\mu_m)} - e^{-itF(\tau_{n_0},\mu_{m_0})/h}\sum_{n,m\in\mathbb{N}^2}|a_{n,m}|^2\,e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}\right)}\right|.$$

For all integers $(n,m)\in\mathbb{N}^2$ the Taylor-Lagrange's formula (at order 2) around (τ_{n_0},μ_{m_0}) on the function F gives the existence of a real number $\theta=\theta(n,m,n_0,m_0)\in\,]0,1[$ such that

$$\begin{aligned}F(\tau_n,\mu_m) = F(\tau_{n_0},\mu_{m_0}) + \frac{2\pi h(n-n_0)}{T_{scl_1}} + \frac{2\pi h(m-m_0)}{T_{scl_2}}\\ +\frac{1}{2}\frac{\partial^2F(\rho_{n,m})}{\partial X^2}\omega_1^2h^2(n-n_0)^2 + \frac{1}{2}\frac{\partial^2F(\rho_{n,m})}{\partial Y^2}\omega_2^2h^2(m-m_0)^2\\ +\frac{\partial^2F}{\partial X\partial Y}(\rho_{n,m})\,\omega_1\omega_2h^2(n-n_0)(m-m_0),\end{aligned}$$

with $\rho_{n,m}=\rho(n,m,n_0,m_0,h) := (\tau_{n_0}+\theta(\tau_n-\tau_{n_0}),\mu_{m_0}+\theta(\mu_m-\mu_{m_0}))$.

So, we get

$$\varepsilon(t) = \left|\sum_{n,m\in\mathbb{N}^2}|a_{n,m}|^2\,e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}\right)}\left[e^{-i2\pi tR_{n,m}(h)}-1\right]\right|$$

where we have used the notation

$$R_{n,m}(h) := \frac{h\omega_1^2(n-n_0)^2}{4\pi}\frac{\partial^2F(\rho_{n,m})}{\partial X^2} + \frac{h\omega_2^2(m-m_0)^2}{4\pi}\frac{\partial^2F(\rho_{n,m})}{\partial Y^2}$$

$$+\frac{h\omega_1\omega_2(n-n_0)(m-m_0)}{2\pi}\frac{\partial^2 F(\rho_{n,m})}{\partial X\partial Y}.$$

With the sets Γ,Δ and by triangular inequality, we obtain for all $t \geq 0$

$$\varepsilon(t) \leq \left| \sum_{n,m\in\Delta} |a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}\right)} \left[e^{-i2\pi t R_{n,m}(h)} - 1\right] \right| + 2\sum_{n,m\in\Gamma} |a_{n,m}|^2 .$$

For all $t \geq 0$, for h small enough and for all integers $(n,m) \in \Delta$, we observe that

$$\frac{th\omega_1^2(n-n_0)^2}{4\pi}\frac{\partial^2 F(\rho_{n,m})}{\partial X^2} \leq tK_1 h^{2\delta_1-1};$$

$$\frac{th\omega_2^2(m-m_0)^2}{4\pi}\frac{\partial^2 F(\rho_{n,m})}{\partial Y^2} \leq tK_2 h^{2\delta_2-1};$$

$$\frac{th\omega_1\omega_2(n-n_0)(m-m_0)}{2\pi}\frac{\partial^2 F(\rho_{n,m})}{\partial X\partial Y} \leq tK_{12} h^{\delta_1+\delta_2-1};$$

where $K_1, K_2, K_{12} > 0$ are constants which does not depend on h. Indeed : let us denotes by $B((E_1,E_2),r)$ the Euclidian ball of dimension 2 with center (E_1,E_2) and radius r; since $\lim_{h\to 0}(\tau_{n_0},\mu_{m_0}) = (E_1,E_2)$ we obtain that $\forall \varepsilon > 0$, $\exists h_0 > 0$, such that for all $h \leq h_0$, $(\tau_{n_0},\mu_{m_0}) \in B((E_1,E_2),\varepsilon)$; next for all integers $(n,m) \in \Delta$, we have $|\theta(\tau_n - \tau_{n_0})| = h\omega_1\theta|n-n_0| \leq \omega_1 h^{\delta_1}$ and $|\theta(\mu_m - \mu_{n_0})| = h\omega_2\theta|m-m_0| \leq \omega_2 h^{\delta_2}$, this means that for h small enough ($h \leq h_0$) we have

$$\rho_{n,m} \in B((E_1,E_2),\varepsilon);$$

therefore we obtain for all $h \leq h_0$,

$$\left|\frac{\partial^2 F}{\partial X^2}(\rho_{n,m})\right| \leq \sup_{(x,y)\in B((E_1,E_2),\varepsilon)} \left|\frac{\partial^2 F}{\partial X^2}(x,y)\right|$$

and this quantity is > 0 and does not depend on h. Next we have for all $t \in [0,h^\alpha]$

$$t|R_{n,m}(h)| \leq K_1 h^{\alpha+2\delta_1-1} + K_2 h^{\alpha+2\delta_2-1} + K_{1,2} h^{\alpha+\delta_1+\delta_2-1}$$

$$\leq Mh^{\alpha-1}\left(h^{2\delta_1} + h^{2\delta_2} + h^{\delta_1+\delta_2}\right) = 3Mh^{2\min\delta_i+\alpha-1};$$

where $M := \max(K_1,K_2,K_{12})$; with (by hypothesis) $2\min\delta_i + \alpha - 1 > 0$. This implies that for all $t \in [0,h^\alpha]$ and for all integers $(n,m) \in \Delta$ we get

$$e^{-i2\pi t R_{n,m}(h)} - 1 = O\left(h^{2\min\delta_i+\alpha-1}\right);$$

and consequently we have for all $t \in [0,h^\alpha]$

$$\left| \sum_{n,m\in\Delta} |a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}\right)} \left[e^{-i2\pi t R_{n,m}(h)} - 1\right] \right|$$

$$= O\left(h^{2\min\delta_i+\alpha-1}\right) \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 = O\left(h^{2\min\delta_i+\alpha-1}\right).$$

Finally, for all $t \in [0,h^\alpha]$ we have $\varepsilon(t) = O\left(h^{2\min\delta_i+\alpha-1}\right)$. □

The semi-classical periods T_{scl_i} depend on the parameter h. Later we consider two cases : $T_{scl_1}/T_{scl_2} \in \mathbb{Q}$ or not. Consequently we don't make commensurability hypothesis on the number T_{scl_1}/T_{scl_2} valid up for all $h > 0$, so we prefer introduce two quantities which does not depend on h to make latter commensurability hypothesis. So we replace semi-classical periods T_{scl_i} by semi-classical periods T_{cl_i} .

Definition 3.4. We define classical periods T_{cl_1} and T_{cl_2} by :

$$T_{cl_1} := \frac{2\pi}{\frac{\partial F}{\partial X}(E_1, E_2)\,\omega_1} \text{ and } T_{cl_2} := \frac{2\pi}{\frac{\partial F}{\partial Y}(E_1, E_2)\,\omega_2}.$$

An obvious remark is that for all $j \in \{1, 2\}$ we have $\lim_{h\to 0} T_{scl_j} = T_{cl_j}$.

Proposition 3.5. *Let τ be a real number such that $\tau > -\min \delta_i$. Then, uniformly for all $t \in [0, h^\tau]$:*

$$\sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}\right)} = \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{cl_1}}+\frac{m-m_0}{T_{cl_2}}\right)} + O\left(h^{\tau+\min\delta_i}\right).$$

Proof. We observe that

$$\left|\sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \left[e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}\right)} - e^{-2i\pi t\left(\frac{n-n_0}{T_{cl_1}}+\frac{m-m_0}{T_{cl_2}}\right)}\right]\right| \leq \sum_{n,m\in\Gamma} 2\,|a_{n,m}|^2$$

$$+2\sum_{n,m\in\Delta} |a_{n,m}|^2 \left[\left|2\pi t(n-n_0)\left(\frac{1}{T_{scl_1}}-\frac{1}{T_{cl_1}}\right)\right| + \left|2\pi t(m-m_0)\left(\frac{1}{T_{scl_2}}-\frac{1}{T_{cl_2}}\right)\right|\right],$$

because $\left|e^{iX_1}e^{iX_2} - e^{iY_1}e^{iY_2}\right| \leq 2\,|X_1 - Y_1| + 2\,|X_2 - Y_2|$.

Next for all $t \geq 0$ we have

$$\left|2\pi t(n-n_0)\left(\frac{1}{T_{scl_1}}-\frac{1}{T_{cl_1}}\right)\right| = \left|2\pi t(n-n_0)\left(\frac{T_{cl_1}-T_{scl_1}}{T_{scl_1}T_{cl_1}}\right)\right|,$$

and we know that

$$T_{cl_1} - T_{scl_1} = \frac{2\pi}{\omega_1}\frac{\frac{\partial F}{\partial X}(\tau_{n_0}, \mu_{m_0}) - \frac{\partial F}{\partial X}(E_1, E_2)}{\frac{\partial F}{\partial X}(E_1, E_2)\,\frac{\partial F}{\partial X}(\tau_{n_0}, \mu_{m_0})}:$$

first, applying the inequality of Lagrange we obtain :

$$\left|\frac{\partial F}{\partial X}(\tau_{n_0}, \mu_{m_0}) - \frac{\partial F}{\partial X}(E_1, E_2)\right|$$

$$\leq \sup_{x,y\in B((E_1,E_2),1)} \left\|\nabla\left(\frac{\partial F}{\partial X}\right)(x, y)\right\|_{\mathbb{R}^2} \|(\tau_{n_0}, \mu_{m_0}) - (E_1, E_2)\|_{\mathbb{R}^2}$$

$$\leq M\sqrt{(\tau_{n_0} - E_1)^2 + (\mu_{m_0} - E_2)^2} \leq Mh\frac{\sqrt{2}}{2}.$$

where $M > 0$ and does not depend on h.

On the other hand, since we suppose $\frac{\partial F}{\partial X}(E_1, E_2) \neq 0$, there exists $\varepsilon_1 > 0$ and $r_1 > 0$ such that for all $(x, y) \in B((E_1, E_2), r_1)$ we get

$$\left|\frac{\partial F}{\partial X}(x, y)\right| \geq \varepsilon_1.$$

We have seen that hence that there exists $h_1 > 0$ such that for all $h \in]0, h_1[$

$$(\tau_{n_0}, \mu_{m_0}) \in B((E_1, E_2), r_1);$$

as a consequence the application $h \mapsto \frac{1}{\frac{\partial F}{\partial X}(E_1,E_2)\frac{\partial F}{\partial X}(\tau_{n_0},\mu_{m_0})}$ is bounded on the open set $]0, h_1[$; indeed for all $h \in]0, h_1[$

$$\left|\frac{1}{\frac{\partial F}{\partial X}(E_1, E_2)\,\frac{\partial F}{\partial X}(\tau_{n_0}, \mu_{m_0})}\right| \leq \frac{1}{\varepsilon_1^2} < +\infty$$

hence, with $M' := \frac{2\pi}{\omega_1} Mh \frac{\sqrt{2}}{2} \frac{1}{\varepsilon_1^2}$, for all $h \in]0, h_1[$ we have $\left|T_{cl_1} - T_{scl_1}\right| \leq hM'$. Next, since

$$\left|\frac{1}{T_{scl_1} T_{cl_1}}\right| \leq \frac{\omega_1\omega_2}{4\pi^2} \left|\frac{\partial F}{\partial X}(E_1, E_2) \frac{\partial F}{\partial X}(\tau_{n_0}, \mu_{m_0})\right|$$

$$\leq \frac{\omega_1\omega_2}{4\pi^2} \left(\sup_{x,y \in B((E_1,E_2),1)} \left|\frac{\partial F}{\partial X}(x,y)\right|\right)^2 < \infty$$

there exists a constant $C_1 > 0$ which does not depend on h such that for all $h \in]0, h_1[$ we get $\left|1/T_{scl_1} - 1/T_{cl_1}\right| \leq C_1 h$. In a similar way there exists $C_2 > 0$ and $h_2 > 0$ such that for all $h \in]0, h_2[$ we get $\left|1/T_{scl_2} - 1/T_{cl_2}\right| \leq C_2 h$. As a consequence, for all $h \in]0, h^*[$ where $h^* := \min h_i$, for all $t \in [0, h^\tau]$ with $\tau \in \mathbb{R}$, and for all integers $(n, m) \in \triangle$ we have :

$$\left|t(n-n_0)\left(\frac{1}{T_{scl_1}} - \frac{1}{T_{cl_1}}\right)\right| \leq C_1 h^{v+\delta_1}, \left|t(m-m_0)\left(\frac{1}{T_{scl_2}} - \frac{1}{T_{cl_2}}\right)\right| \leq C_2 h^{v+\delta_2};$$

we thus obtain for all $t, (n, m) \in [0, h^v] \times \Delta$

$$\left|t(n-n_0)\left(\frac{1}{T_{scl_1}} - \frac{1}{T_{cl_1}}\right) + t(m-m_0)\left(\frac{1}{T_{scl_2}} - \frac{1}{T_{cl_2}}\right)\right| \leq Mh^{\tau+\min\delta_i}.$$

Therefore

$$\left|\sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \left[e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}} + \frac{m-m_0}{T_{scl_2}}\right)} - e^{-2i\pi t\left(\frac{n-n_0}{T_{cl_1}} + \frac{m-m_0}{T_{cl_2}}\right)}\right]\right|$$

$$\leq 4\pi Mh^{\tau+\min\delta_i} \sum_{n,m\in\Delta} |a_{n,m}|^2 + O\left(h^\infty\right) = O\left(h^{\tau+\min\delta_i}\right).$$

□

3.2.2. *Comparison between classical periods and the time scale* $[0, h^\alpha]$. In proposition 3.3 the hypothesis on α is that $\alpha > 1 - 2\min\delta_i$; therefore with $\delta_i \in]\frac{1}{2}, 1[$ we can make a "good choice" for α; i.e. to have $\alpha < 0$. Hence for h small enough we obtain :

$$\left[0, T_{cl_i}\right] \subset [0, h^\alpha].$$

Next, since $-\min\delta_i - (1 - 2\min\delta_i) = -1 + \min\delta_i \leq 0$ we get

$$h^{-\min\delta_i} \gg h^{1-2\min\delta_i};$$

this means that we can choose to take $\tau = \alpha$.

3.2.3. *The linear approximation* $\mathbf{a_1}$. In conclusion, the linear approximation of the autocorrelation function on the time scale $[0, h^\alpha]$ is :

Definition 3.6. The linear approximation of the autocorrelation function is

$$\mathbf{a_1} : t \mapsto \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{cl_1}} + \frac{m-m_0}{T_{cl_2}}\right)}.$$

3.3. **Geometrical interpretation of classical periods.** The periods T_{cl_i} have geometrical interpretation. For $E_1, E_2 > 0$ consider the energy level set $M_{E_1,E_2} := p_1^{-1}(E_1) \cap p_2^{-1}(E_2) \subset \mathbb{R}^4$, this manifold is isomorphic to the torus $\sqrt{\frac{2E_1}{\omega_1}}\mathbb{S}^1 \times \sqrt{\frac{2E_2}{\omega_2}}\mathbb{S}^1$, here $\mathbb{S}^1$ is the one-dimension circle. Start with the calculus of the Hamiltonian flow of $p = F(p_1, p_2)$ with an initial point $m_0 \in M_{E_1,E_2}$. So the Hamilton's equations are

$$\begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \\ \dot{\xi_1}(t) \\ \dot{\xi_2}(t) \end{pmatrix} = \begin{pmatrix} a\xi_1(t) \\ b\xi_2(t) \\ -ax_1(t) \\ -bx_2(t) \end{pmatrix}$$

where we have used the notation $a := \frac{\partial F}{\partial X}(E_1, E_2)\,\omega_1,\ b := \frac{\partial F}{\partial Y}(E_1, E_2)\,\omega_2$. For all $j \in \{1, 2\}$, let us consider the complex number $Z_j(t) := x_j(t) + i\xi_j(t)$; from the Hamilton equations we obtain the equalities $\dot{Z_1}(t) = -iaZ_1(t),\ \dot{Z_2}(t) = -ibZ_2(t)$. Therefore we get

$$Z_1(t) = Z_1(0)e^{-iat},\ Z_2(t) = Z_2(0)e^{-ibt}$$

and

$$|Z_1(0)|^2 = x_1^2(0) + \xi_1^2(0) = \frac{2E_1}{a},\ |Z_2(0)|^2 = x_2^2(0) + \xi_2^2(0) = \frac{2E_2}{b};$$

this means that the Hamiltonian's flow in complex coordinate is given by

$$\varphi_t : \begin{pmatrix} Z_1(0) \\ Z_2(0) \end{pmatrix} \mapsto \begin{pmatrix} Z_1(t) \\ Z_2(t) \end{pmatrix}.$$

In angular coordinate the flow is given by

$$\varphi_t : \begin{pmatrix} \theta_{1,0} \\ \theta_{2,0} \end{pmatrix} \mapsto \begin{pmatrix} \theta_{1,0} - t\frac{a}{2\pi} \\ \theta_{2,0} - t\frac{b}{2\pi} \end{pmatrix}$$

with $\theta_{j,0} \equiv \frac{\arg Z_j(0)}{2\pi}$ [1]. So we have exactly the classical periods of the Hamiltonian's flow :

$$\frac{2\pi}{a} = \frac{2\pi}{\frac{\partial F}{\partial X}(E_1, E_2)\,\omega_1} = T_{cl_1},\ \frac{2\pi}{b} = \frac{2\pi}{\frac{\partial F}{\partial Y}(E_1, E_2)\,\omega_2} = T_{cl_2}.$$

It's well know that if the periods are commensurate the flow is periodic on the torus. In opposite the flow is quasi-periodic on the torus.

3.4. **The principal part of the function $\mathbf{a_1}$.** Now, let us study in details the function $\mathbf{a_1}(t)$ on the time scale $\left[0, \max T_{cl_i}\right]$. Start by a technical proposition :

Proposition 3.7. *For all $t \geq 0$ we have*

$$\sum_{n,m\in\mathbb{Z}^2} |a_{n,m}|^2\, e^{-2i\pi t\left(\frac{n-n_0}{T_{cl_1}} + \frac{m-m_0}{T_{cl_2}}\right)}$$

$$= \frac{1}{\mathfrak{F}(\chi^2)(0,0)} \sum_{\ell,s\in\mathbb{Z}^2} \mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}\left(\ell + \frac{t}{T_{cl_1}}\right), -\frac{h^{\delta_2'-1}}{\omega_2}\left(s + \frac{t}{T_{cl_2}}\right)\right).$$

Proof. The trick here is just to use the Poisson formula, so let us consider the function Ω_t defined by

$$\Omega_t : \begin{cases} \mathbb{R}^2 \to \mathbb{C} \\ (x_1, x_2) \mapsto |a_{x_1,x_2}|^2\, e^{-2i\pi t\frac{(x_1-n_0)}{T_{cl_1}}}\, e^{-2i\pi t\frac{(x_2-m_0)}{T_{cl_2}}} \end{cases}$$

where $t \in \mathbb{R}$ is a parameter. For all integers $(n,m) \in \mathbb{Z}^2$ we have

$$|a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{cl_1}}+\frac{m-m_0}{T_{cl_2}}\right)} = \Omega_t(n,m).$$

So clearly, the function $\Omega_t \in \mathcal{S}(\mathbb{R}^2)$, then the Fourier transform $\mathfrak{F}(\Omega_t)$ is equal, for all $\zeta_1, \zeta_2 \in \mathbb{R}^2$

$$\mathfrak{F}(\Omega_t)(\zeta_1,\zeta_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \Omega_t(x_1,x_2)\, e^{-2i\pi x_1\zeta_1} e^{-2i\pi x_2\zeta_2}\, dx_1 dx_2;$$

therefore for all $\zeta_1, \zeta_2 \in \mathbb{R}^2$ we get

$$\mathfrak{F}(\Omega_t)(\zeta_1,\zeta_2) = \frac{e^{-2i\pi(n_0\zeta_1+m_0\zeta_2)}}{\mathfrak{F}(\chi^2)(0,0)} \mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}\left(\zeta_1+\frac{t}{T_{cl_1}}\right), -\frac{h^{\delta_2'-1}}{\omega_2}\left(\zeta_2+\frac{t}{T_{cl_2}}\right)\right).$$

It comes from the Poisson formula the equality

$$\sum_{n,m\in\mathbb{Z}^2} \Omega_t(n,m) = \sum_{\ell,s\in\mathbb{Z}^2} \mathfrak{F}(\Omega_t)(\ell,s)$$

$$= \frac{1}{\mathfrak{F}(\chi^2)(0,0)} \sum_{\ell,s\in\mathbb{Z}^2} \mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}\left(\ell+\frac{t}{T_{cl_1}}\right), -\frac{h^{\delta_2'-1}}{\omega_2}\left(s+\frac{t}{T_{cl_2}}\right)\right)$$

which gives the proposition. □

Since the function $\mathfrak{F}\left(\chi^2\right) \in \mathcal{S}(\mathbb{R}^2)$, we observe that only index $\ell, s \in \mathbb{Z}^2$ such that $\ell + \frac{t}{T_{cl_1}}$ or $s + \frac{t}{T_{cl_2}}$ are close to zero are important in the sum. More precisely :

Definition 3.8. For all $t \geq 0$, let us define the integers $\ell_i(t) = \ell_i(t,h,E)$ as the closest integers to the real numbers $-t/T_{cl_i}$; i.e :

$$\ell_i(t) + \frac{t}{T_{cl_i}} = d\left(t, T_{cl_i}\mathbb{Z}\right);$$

where $d(.,.)$ denote the Euclidiean distance on $\mathbb{R}$.

Remark 3.9. Without loss of generality, we may suppose the integers $\ell_i(t)$ are unique. On the other hand, for all integer $\ell \in \mathbb{Z}$ such that $\ell \neq \ell_i(t)$ we get :

$$\left|\ell + \frac{t}{T_{cl_i}}\right| \geq \frac{1}{2}.$$

Lemma 3.10. *Uniformly for $t \geq 0$ we have :*

$$\mathbf{a_1}(t) = \frac{1}{\mathfrak{F}(\chi^2)(0,0)} \mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1} d\left(T_{cl_1}\mathbb{Z}, t\right), -\frac{h^{\delta_2'-1}}{\omega_2} d\left(T_{cl_2}\mathbb{Z}, t\right)\right) + O\left(h^\infty\right).$$

Proof. Since $\mathfrak{F}\left(\chi^2\right) \in \mathcal{S}(\mathbb{R}^2)$ we have

$$\forall k,d \in \mathbb{N}^{*2},\ \exists B_{k,d} > 0,\ \forall \zeta_1,\zeta_2 \in \mathbb{R}^2,\ \left|\mathfrak{F}\left(\chi^2\right)(\zeta_1,\zeta_2)\right| \leq \frac{B_{k,d}}{\left(1+|\zeta_1|\right)^k \left(1+|\zeta_2|\right)^d}.$$

Next, it then follow from the proposition above and from the lemma 2.4 that for all $t \geq 0$

$$\mathbf{a_1}(t) = \frac{1}{\mathfrak{F}(\chi^2)(0)} \sum_{\ell,s\in\mathbb{Z}^2} \mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}\left(\ell+\frac{t}{T_{cl_1}}\right), -\frac{h^{\delta_2'-1}}{\omega_2}\left(s+\frac{t}{T_{cl_2}}\right)\right)$$

$$= \frac{1}{\mathfrak{F}(\chi^2)(0,0)} \mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1} d\left(t, T_{cl_1}\mathbb{Z}\right), -\frac{h^{\delta_2'-1}}{\omega_2} d\left(t, T_{cl_2}\mathbb{Z}\right)\right) + O\left(h^\infty\right).$$

Next, for all $t \geq 0$

$$\mathbf{a_1}(t) = \sum_{n,m\in\mathbb{Z}^2} |a_{n,m}|^2 \, e^{-2i\pi t\left(\frac{n-n_0}{T_{cl_1}}+\frac{m-m_0}{T_{cl_2}}\right)} - \sum_{n,m\in\mathbb{Z}^2-\mathbb{N}^2} |a_{n,m}|^2 \, e^{-2i\pi t\left(\frac{n-n_0}{T_{cl_1}}+\frac{m-m_0}{T_{cl_2}}\right)};$$

thus

$$\left|\mathbf{a_1}(t) - \frac{1}{\mathfrak{F}(\chi^2)(0,0)}\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z},t\right), -\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z},t\right)\right)\right|$$

$$\leq \sum_{n,m\in\mathbb{Z}^2-\mathbb{N}^2} |a_{n,m}|^2 + O\left(h^\infty\right).$$

For finish, we observe

$$\sum_{n,m\in\mathbb{Z}^2-\mathbb{N}^2} |a_{n,m}|^2$$

$$= \sum_{n=0}^{+\infty}\sum_{m=-\infty}^{-1} |a_{n,m}|^2 + \sum_{n=-\infty}^{-1}\sum_{m=-\infty}^{-1} |a_{n,m}|^2 + \sum_{n=-\infty}^{-1}\sum_{m=0}^{+\infty} |a_{n,m}|^2;$$

and an obvious consequence of the lemma 2.4 is that $\sum_{n,m\in\mathbb{Z}^2-\mathbb{N}^2} |a_{n,m}|^2 = O\left(h^\infty\right)$. □

3.5. **Behaviour of the function $\mathbf{a_1}$: case $\frac{T_{cl_1}}{T_{cl_2}} \in \mathbb{Q}$.** In this subsection we suppose $\frac{T_{cl_1}}{T_{cl_2}} = \frac{b}{a} \in \mathbb{Q}$; hence $aT_{cl_1} = bT_{cl_2}$.

Definition 3.11. If the classical periods T_{cl_1}, T_{cl_2} are commensurate the classical period of the global system is defined by $T_{cl} := aT_{cl_1} = bT_{cl_2}$.

Now, we can formulate an important result of this section :

Theorem 3.12. We have :

(i) for t real such that $t \in T_{cl}\mathbb{Z}$ we get (i.e. for all $i \in \{1,2\}$, $d\left(t, T_{cl_i}\mathbb{Z}\right) = 0$)

$$\mathbf{a_1}(t) = 1.$$

(ii) If there exists $i \in \{1,2\}$ such that $d\left(T_{cl_i}\mathbb{Z},t\right) > h^{1-\delta_i'}$ then :

$$\mathbf{a_1}(t) = O(h^\infty).$$

Proof. The first point (i) is clear. For the second : it follows from the lemma 3.10 that

$$\mathbf{a_1}(t) = \frac{1}{\mathfrak{F}(\chi^2)(0,0)}\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z},t\right), -\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z},t\right)\right) + O\left(h^\infty\right);$$

since the function $\mathfrak{F}\left(\chi^2\right) \in \mathcal{S}(\mathbb{R}^2)$ we have

$$\forall q \in \mathbb{N},\ \exists D_q > 0,\ \forall \zeta_1,\zeta_2 \in \mathbb{R}^2,\ \left|\mathfrak{F}\left(\chi^2\right)(\zeta_1,\zeta_2)\right| \leq \frac{D_q}{(1+|\zeta_1|+|\zeta_2|)^q}$$

and therefore

$$\left|\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z},t\right), -\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z},t\right)\right)\right|$$

$$\leq \frac{D_q}{\left(1+\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z},t\right)+\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z},t\right)\right)^q}.$$

Thus, if there exists $i \in \{1,2\}$ such that $d\left(T_{cl_i}\mathbb{Z},t\right) > \omega_i h^{1-\delta_i'}$ then there exists $\varepsilon > 0$ such that $d\left(T_{cl_i}\mathbb{Z},t\right) \geq \omega_i h^{1-\delta_i'-\varepsilon}$ and thus for all $q \in \mathbb{N}$ we obtain

$$\left|\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z},t\right),-\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z},t\right)\right)\right| \leq D_q h^{\varepsilon q};$$

hence we get

$$\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z},t\right),-\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z},t\right)\right) = O(h^{\infty}).$$

□

3.6. **Behaviour of the function $\mathbf{a_1}$: case $\frac{T_{cl_1}}{T_{cl_2}} \notin \mathbb{Q}$.** Let us now tackle an important case : the case $\frac{b}{a} = \frac{T_{cl_1}}{T_{cl_2}}$ is not a fraction of $\mathbb{Q}$. Here, there not exists classical common period, the Hamiltonian flow is not periodic on the torus.

First, we note that, in view of lemma 3.10, the behaviour of the function $\mathbf{a_1}$ is given by the function :

$$t \mapsto \frac{1}{\mathfrak{F}\left(\chi^2\right)(0,0)}\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z},t\right),-\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z},t\right)\right).$$

Therefore, since the function $\mathfrak{F}\left(\chi^2\right)$ belongs to the space $\mathcal{S}(\mathbb{R}^2)$, we need to explain simultaneously the evolutions of the distances $d\left(T_{cl_j}\mathbb{Z},t\right)$ depending on time. In another formulation, we want to analyze the behaviour of the Euclidian distance between the segment line (OM_t) where $O := (0,0)$, $M_t := (t,t)$ and the lattice $T_{cl_1}\mathbb{Z} \times T_{cl_2}\mathbb{Z}$ depending on the time t and the number $\frac{T_{cl_1}}{T_{cl_2}} = \frac{b}{a}$. Precisely, we want to compare the distance $d\left((OM_t), T_{cl_1}\mathbb{Z} \times T_{cl_2}\mathbb{Z}\right)$ with the real number $h^{\delta_j'-1}$. For example, if for a time t^* the distance is larger than $h^{1-\delta_j'}$ we get $\mathbf{a_1}(t^*) = O(h^{\infty})$.

Start this new subsection by some geometrical results and latter we explain the study of the autocorrelation function $\mathbf{a_1}(t)$.

3.6.1. *Some general points.* In angular coordinates the Hamiltonian flow is :

$$\varphi_t : \begin{cases} [0,1]^2 \to [0,1]^2 \\ \begin{pmatrix} \theta_{1,0} \\ \theta_{2,0} \end{pmatrix} \mapsto \begin{pmatrix} -\frac{at}{2\pi} + \theta_{1,0} \\ -\frac{bt}{2\pi} + \theta_{2,0} \end{pmatrix} \end{cases}.$$

Without loss of generality we may suppose that the initial data is $\begin{pmatrix} \theta_{1,0} \\ \theta_{2,0} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and that $a, b < 0$. Therefore the Hamiltonian flow is given by $\varphi_t = \begin{pmatrix} \mathbf{a}t \\ \mathbf{b}t \end{pmatrix}$, where we have used the notation $\mathbf{a} := -\frac{a}{2\pi} > 0$ and $\mathbf{b} := -\frac{b}{2\pi} > 0$. Recall that $T_{cl_1} = \left|\frac{2\pi}{a}\right| = \left|\frac{1}{\mathbf{a}}\right|$ and $T_{cl_2} = \left|\frac{2\pi}{b}\right| = \left|\frac{1}{\mathbf{b}}\right|$. So, to understand the behaviour of the function

$$t \mapsto \frac{1}{\mathfrak{F}\left(\chi^2\right)(0,0)}\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z},t\right),-\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z},t\right)\right)$$

we need to explain the evolution of the Euclidian distance $d\left(\varphi_t, \mathbb{Z}_*^2\right)$ depending on time t and on the real number $\frac{\mathbf{b}}{\mathbf{a}}$.

3.6.2. *Suppose $\frac{b}{a}$ verify diophantin condition.* J. Liouville proved in 1884 the following theorem :

Theorem 3.13. (Liouville). For all algebraic irrational number θ with degree $d \geq 2$ there exists a constant $C = C(\theta) > 0$ such that the inequality

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C}{q^d}$$

holds for all rationals $\frac{p}{q}$.

In other words, algebraic numbers are bad approximation by rationals. Finally, in 1955 K. F. Roth has considerably improved this result (he was awarded the Field medal in 1958).

Definition 3.14. We say an irrational number θ satisfy a ε-diophantin condition ($\varepsilon \geq 0$) if and only if there exists a constant $C_\varepsilon > 0$ such that

$$\left|\theta - \frac{p}{q}\right| \geq \frac{C_\varepsilon}{q^{2+\varepsilon}}.$$

holds for all $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$. We denote by $\mathcal{C}_\varepsilon$ the set of irrationals θ that holds ε-diophantin condition. We say that an θ irrational θ is a Roth number if and only if $\theta \in \bigcap_{\varepsilon>0} \mathcal{C}_\varepsilon$; i.e

$$\forall \varepsilon > 0,\ \exists C_\varepsilon > 0;\ \forall (p, q) \in \mathbb{Z} \times \mathbb{N}^*;\ \left|\theta - \frac{p}{q}\right| \geq \frac{C_\varepsilon}{q^{2+\varepsilon}}.$$

There is a lot of Roth numbers examples :

Theorem 3.15. (Thue-Siegel-Roth). Every real algebraic irrational number of degree $d \geq 2$ is a Roth number.

We have also the (see for example **[Cas]**) :

Theorem 3.16. The Lebesgue measure of Roth's numbers is infinite.

Remark 3.17. Let θ a ε-diophantin number. Since for all p we have $\left|\theta - \frac{p}{1}\right| \geq C_\varepsilon$ and $|\theta - p| \leq \frac{1}{2} \leq \frac{\sqrt{2}}{2}$; thus we obtain that $0 < C_\varepsilon \leq \frac{1}{2}$ holds for all $\varepsilon > 0$

Now, we estimate the Euclidian distance between the set $\{\varphi_t\}_{t\in[0,T]}$ and $\mathbb{Z}^2_* := \mathbb{Z}^2 - \{(0, 0)\}$.

Notation 3.18. Let us denote by Δ and Γ the following orthogonal lines

$$\Delta := \text{Vect}(\mathbf{a}e_1 + \mathbf{b}e_2),\ \Gamma := \text{Vect}(-\mathbf{b}e_1 + \mathbf{a}e_2)$$

where (e_1, e_2) is the canonical basis of the vector space $\mathbb{R}^2$. Let us also considers π_Δ the orthogonal projector on the line Δ and π_Γ the orthogonal projector on the line Γ.

Here we suppose $\theta = \frac{b}{a}$ is a Roth number.

Lemma 3.19. *For all $\varepsilon > 0$ there exists $0 < K_\varepsilon \leq C_\varepsilon$, here C_ε denotes the Roth constant of $\theta = \frac{b}{a}$, such that*

$$\|\pi_\Delta (ne_1 + me_2)\|_{\mathbb{R}^2} \geq \frac{K_\varepsilon}{\|ne_1 + me_2\|^{1+\varepsilon}_{\mathbb{R}^2}};\ \|\pi_\Gamma (ne_1 + me_2)\|_{\mathbb{R}^2} \geq \frac{K_\varepsilon}{\|ne_1 + me_2\|^{1+\varepsilon}_{\mathbb{R}^2}}$$

holds for all $(n, m) \in \mathbb{Z}^2_*$.

Proof. Let us denotes by $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \frac{1}{\sqrt{\mathbf{a}^2+\mathbf{b}^2}} \begin{pmatrix} \mathbf{a} \\ \mathbf{b} \end{pmatrix}$ the unitary vector of the line Δ, so we have

$$\begin{aligned} \|\pi_\Delta (ne_1 + me_2)\|_{\mathbb{R}^2} &= |\langle u, ne_1 + me_2\rangle_{\mathbb{R}^2}| \\ &= |nu_1 + mu_2| = |u_1|\,|n + m\theta|\,; \end{aligned}$$

since θ is a Roth number, for all $\varepsilon > 0$ there exist $C_\varepsilon > 0$ such that

$$\begin{aligned} \|\pi_\Delta (ne_1 + me_2)\|_{\mathbb{R}^2} &\geq |u_1| \frac{C_\varepsilon}{|m|^{1+\varepsilon}} \\ &\geq \frac{|u_1|\, C_\varepsilon}{\|ne_1 + me_2\|_{\mathbb{R}^2}^{1+\varepsilon}}. \end{aligned}$$

In a similar way we ge

$$\|\pi_\Gamma (ne_1 + me_2)\|_{\mathbb{R}^2} \geq |u_2| \frac{C_\varepsilon}{|n|^{1+\varepsilon}} \geq \frac{|u_2|\, C_\varepsilon}{\|ne_1 + me_2\|_{\mathbb{R}^2}^{1+\varepsilon}};$$

therefore with $K_\varepsilon := \min(|u_1|\, C_\varepsilon, |u_2|\, C_\varepsilon) \leq C_\varepsilon$ we obtain the lemma. □

A consequence of this lemma is :

Theorem 3.20. For all $\varepsilon > 0$ there $K_\varepsilon \leq C_\varepsilon$, here C_ε denotes the Roth constant of $\theta = \frac{b}{a}$, such that for all $t \geq 0$

$$d\left(\varphi_t, \mathbb{Z}_*^2\right) \geq \frac{K_\varepsilon}{\left(\frac{\sqrt{2}}{2} + t\sqrt{\mathbf{a}^2 + \mathbf{b}^2}\right)^{1+\varepsilon}}.$$

Proof. We observe that for all $t \geq 0$ the point φ_t belongs to the line Δ, thus there exists a pair $(n_t, m_t) \in \mathbb{Z}_*^2$ such that

$$\begin{aligned} d\left(\varphi_t, \mathbb{Z}_*^2\right) &= \left\|\overrightarrow{O\varphi_t} - (n_t e_1 + m_t e_2)\right\|_{\mathbb{R}^2} \\ &\geq \|\pi_\Gamma (n_t e_1 + m_t e_2)\|_{\mathbb{R}^2}\,; \end{aligned}$$

applying the lemma above we get

$$d\left(\varphi_t, \mathbb{Z}_*^2\right) \geq \frac{K_\varepsilon}{\|n_t e_1 + m_t e_2\|_{\mathbb{R}^2}^{1+\varepsilon}}.$$

On the other hand we have the majorization

$$\left\|\overrightarrow{O\varphi_t} - (n_t e_1 + m_t e_2)\right\|_{\mathbb{R}^2} \leq \frac{\sqrt{2}}{2},$$

and, by triangular inequality we obtain

$$\|n_t e_1 + m_t e_2\|_{\mathbb{R}^2} \leq \left\|\overrightarrow{O\varphi_t}\right\|_{\mathbb{R}^2} + \frac{\sqrt{2}}{2}.$$

Therefore, since $\left\|\overrightarrow{O\varphi_t}\right\|_{\mathbb{R}^2} = t\sqrt{\mathbf{a}^2 + \mathbf{b}^2}$, we get for all $t \geq 0$, $\varepsilon > 0$

$$d\left(\varphi_t, \mathbb{Z}_*^2\right) \geq \frac{K_\varepsilon}{\left(t\sqrt{\mathbf{a}^2 + \mathbf{b}^2} + \frac{\sqrt{2}}{2}\right)^{1+\varepsilon}}.$$

□

Corollary 3.21. *For all $\varepsilon > 0$ and for every $\eta \in \left]0, \frac{\sqrt{2}^{1+\varepsilon}}{2}\right[\subset \left]0, \frac{\sqrt{2}}{2}\right[$ we have :*

$$d\left(\varphi_t, \mathbb{Z}_*^2\right) < \eta \Rightarrow t > \frac{1}{\sqrt{\mathbf{a}^2 + \mathbf{b}^2}} \left(\left(\frac{K_\varepsilon}{\eta}\right)^{\frac{1}{1+\varepsilon}} - \frac{\sqrt{2}}{2}\right).$$

Proof. Suppose $d\left(\varphi_t(0), \mathbb{Z}_*^2\right) < \eta$, it then follows from the theorem above that for all $\varepsilon > 0$ there exists a constant $K_\varepsilon \in \left]0, \frac{1}{2}\right[$ such that

$$\frac{K_\varepsilon}{\left(\frac{\sqrt{2}}{2} + t\sqrt{\mathbf{a}^2 + \mathbf{b}^2}\right)^{1+\varepsilon}} < \eta$$

holds for all $t \geq 0$; i.e.

$$\left(\frac{K_\varepsilon}{\eta}\right)^{\frac{1}{1+\varepsilon}} < \frac{\sqrt{2}}{2} + t\sqrt{\mathbf{a}^2 + \mathbf{b}^2}.$$

□

Remark 3.22. Since $\eta \in \left]0, \frac{\sqrt{2}^{1+\varepsilon}}{2}\right[\subset \left]0, \frac{\sqrt{2}}{2}\right[$ we have $\left(\frac{K_\varepsilon}{\eta}\right)^{\frac{1}{1+\varepsilon}} \geq \frac{\sqrt{2}}{2}$.

Notation 3.23. For $\varepsilon > 0$ and $\eta > 0$, let us denote :

$$t_\eta(\varepsilon) := \frac{1}{\sqrt{\mathbf{a}^2 + \mathbf{b}^2}}\left(\left(\frac{K_\varepsilon}{\eta}\right)^{\frac{1}{1+\varepsilon}} - \frac{\sqrt{2}}{2}\right).$$

Theorem 3.24. For all $\varepsilon > 0$, for every $\eta \in \left]0, \frac{\sqrt{2}^{1+\varepsilon}}{2}\right[$ with η small enough such that $t_\eta(\varepsilon) \geq \max T_{cl_i}$ and for all $k \geq 1$; there exists a constant $D_k > 0$ which does not depend on h such that the inequality

$$|\mathbf{a_1}(t)| \leq D_k h^{k(1-\max \delta_i')} \eta^{-k}$$

holds for all $t \in [\max T_{cl_i}, t_\eta(\varepsilon)]$.

Proof. Our starting point is that for all $t \geq \max T_{cl_i}$ we have

$$d\left(t, T_{cl_i}\mathbb{Z}\right) = d\left(t, T_{cl_i}\mathbb{N}^*\right).$$

Next, since $T_{cl_1} = |\frac{2\pi}{a}| = \left|\frac{1}{\mathbf{a}}\right|$ and $T_{cl_2} = |\frac{2\pi}{b}| = \left|\frac{1}{\mathbf{b}}\right|$ we get

$$d\left(t, T_{cl_1}\mathbb{Z}\right) = d\left(\mathbf{a}t, \mathbb{N}^*\right),\ d\left(t, T_{cl_2}\mathbb{Z}\right) = d\left(\mathbf{b}t, \mathbb{N}^*\right).$$

Therefore, from the corrolary above (by contraposed) we obtain for all $t \in [\max T_{cl_i}, t_\eta(\varepsilon)]$

$$d\left((\mathbf{a}t, \mathbf{b}t), \mathbb{Z}_*^2\right) \geq \eta;$$

and since the norms $\|(x,y)\|_{\mathbb{R}^2}$ and $|x| + |y|$ are equivalent on $\mathbb{R}^2$, there exists a constant $C > 0$ such that

$$d\left(t, T_{cl_1}\mathbb{Z}\right) + d\left(t, T_{cl_2}\mathbb{Z}\right) \geq C\eta$$

holds for all $t \in [\max T_{cl_i}, t_\eta(\varepsilon)]$.

Next, since the function $\mathfrak{F}\left(\chi^2\right)$ belongs to the space $\mathcal{S}(\mathbb{R}^2)$ we have

$$\forall k \in \mathbb{N}^2,\ \exists M_k > 0,\ \forall \zeta_1, \zeta_2 \in \mathbb{R}^2,\ \left|\mathfrak{F}\left(\chi^2\right)(\zeta_1, \zeta_2)\right| \leq \frac{M_k}{\left(|\zeta_1| + |\zeta_2|\right)^k};$$

thus we obtain for all $t \in [\max T_{cl_i}, t_\eta(\varepsilon)]$

$$\left|\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1} d\left(t, T_{cl_1}\mathbb{Z}\right), -\frac{h^{\delta_2'-1}}{\omega_2} d\left(t, T_{cl_2}\mathbb{Z}\right)\right)\right|$$

$$\leq \frac{M_k}{\left(\frac{h^{\delta_1'-1}}{\omega_1} d\left(t, T_{cl_1}\mathbb{Z}\right) + \frac{h^{\delta_2'-1}}{\omega_2} d\left(t, T_{cl_2}\mathbb{Z}\right)\right)^k}$$

$$\leq \frac{\max(\omega_i)^k M_k}{h^{k\max(\delta_i')-k}\left(d\left(t,T_{cl_1}\mathbb{Z}\right)+d\left(t,T_{cl_2}\mathbb{Z}\right)\right)^k}$$

$$\leq \frac{\max(\omega_i)^k M_k}{h^{k\max(\delta_i')-k}}\frac{1}{C^k\eta^k} = \max(\omega_i)^k\frac{M^k}{C^k}h^{k(1-\max(\delta_i'))}\eta^{-k}.$$

□

Applying this theorem with $\eta = h^s$ where the real number s belongs to $\left]0, 1-\max\delta_i'\right[$, we have :

Corollary 3.25. *For all $\varepsilon > 0$, $s \in \left]0, 1-\max\delta_i'\right[$ and for h small enough such that $t_{h^s}(\varepsilon) \geq \max T_{cl_i}$; the following equality*

$$\mathbf{a}_1(t) = O(h^\infty)$$

holds uniformly for all $t \in [\max T_{cl_i}, t_{h^s}(\varepsilon)]$.

Notes on time scales. From a pratical point of view, we must verify that for all $\varepsilon \geq 0$

$$t_{h^s}(\varepsilon) \leq h^\alpha$$

where $\alpha > 1 - 2\min(\delta_i)$. Indeed we have :

Proposition 3.26. *Suppose $\min\delta_i > \frac{2}{3}$, for all $\varepsilon > 0$ and for all $s \in \left]0, 1-\max\delta_i'\right[$ we have*

$$t_{h^s}(\varepsilon) \leq \frac{1}{2\sqrt{\mathbf{a}^2+\mathbf{b}^2}}\left(h^{1-2\min\delta_i} - \sqrt{2}\right).$$

Proof. For all $s > 0$ and for all $\varepsilon > 0$, since

$$t_{h^s}(\varepsilon) = \frac{1}{\sqrt{\mathbf{a}^2+\mathbf{b}^2}}\left((K_\varepsilon)^{\frac{1}{1+\varepsilon}}h^{-\frac{s}{1+\varepsilon}} - \frac{\sqrt{2}}{2}\right)$$

for $h \to 0$ we have the equivalence

$$t_{h^s} \sim D_\varepsilon h^{-\frac{s}{1+\varepsilon}}$$

where $D_\varepsilon := \frac{1}{\sqrt{\mathbf{a}^2+\mathbf{b}^2}}(K_\varepsilon)^{\frac{1}{1+\varepsilon}} > 0$. On the other hand, we see that for all $\varepsilon > 0$ we have $1 - 2\min\delta_i \leq \frac{\max\delta_i'-1}{1+\varepsilon}$. Hence

$$h^{\frac{\max\delta_i'-1}{1+\varepsilon}} \leq h^{1-2\min\delta_i}.$$

Therefore, we obtain

$$t_{h^s}(\varepsilon) \leq \frac{1}{\sqrt{\mathbf{a}^2+\mathbf{b}^2}}\left((K_\varepsilon)^{\frac{1}{1+\varepsilon}}h^{\frac{\max\delta_i'-1}{1+\varepsilon}} - \frac{\sqrt{2}}{2}\right)$$

$$\leq \frac{1}{\sqrt{\mathbf{a}^2+\mathbf{b}^2}}\left((K_\varepsilon)^{\frac{1}{1+\varepsilon}}h^{1-2\min\delta_i} - \frac{\sqrt{2}}{2}\right) \leq \frac{1}{\sqrt{\mathbf{a}^2+\mathbf{b}^2}}\left(\frac{1}{2}h^{1-2\min\delta_i} - \frac{\sqrt{2}}{2}\right).$$

□

Use of continued fractions. Now, we can wonder what are the accurate times when $d\left(\varphi_t, \mathbb{Z}_*^2\right) < \eta$? To solve this problem we will use the continued fraction theory.

Some useful theorems. The continued fractions are essentially used for the approximation of real numbers. There exists two types of continued fractions: the finite continued fractions representing rational numbers and the infinite continued fractions representing irrational numbers. For all irrational number θ, there exists a pair sequence $(q_n, p_n) \in \mathbb{N}^2$ such that

$$\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}$$

holds for all $n \geq 0$. This sequence is given by the continued fractions algorithm (see **[Ro-Sz], [Khi]**). Geometrically speaking, the construction principle for this sequence is as follows (see **[Arn2]**) : consider $v_0 := (0,1)$ and $v_{-1} := (1,0)$. It is obvious that these points lie on different sides of the line $y = \theta x$. By induction : let the vectors v_{k-1} and v_k be constructed whereas to construct the new vector v_{k+1}, we add to the vector v_{k-1} the vector v_k as many times as we can in such a way the new vector v_{k+1} lies on the same side of the line $y = \theta x$ as the vector v_{k-1} :

$$v_{k+1} = a_k v_k + v_{k-1}$$

i.e

$$\begin{cases} q_{k+1} = a_k q_k + q_{k-1} \\ \\ p_{k+1} = a_k p_k + p_{k-1} \end{cases}$$

where $(a_k)_{k\geq 0}$ is a sequence of integers strictly > 0.

We note that the sequence $(q_n)_n$ is strictly increasing. With the standard notation continued fractions we have

$$[a_0, a_1, \ldots, a_n, \ldots] := a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}$$

end we have (see for example **[Khi]**) the relation $[a_0, a_1, \ldots, a_n] = \frac{p_n}{q_n}$.

Example 3.27. The number π is given by : $\pi = [3, 7, 15, 1, 292, 1 \ldots]$.

Approach time. Let us denote by D the line $y = \frac{\mathbf{b}}{\mathbf{a}} x = \theta x$. Hence, for all $t \geq 0$ we have :

$$[O\varphi_t] \subset D.$$

For a fixed $n \geq 0$ we wish to find the point M_n of D such that $d(M_n, (q_n, p_n)) = d(D, (q_n, p_n))$. In other words, we wish to find the time τ_n such that $d(\varphi_{\tau_n}, (q_n, p_n)) = d(D, (q_n, p_n))$.

Proposition 3.28. *For all $n \geq 1$ the unique $\tau_n \geq 0$ such that $d(\varphi_{\tau_n}, (q_n, p_n)) = d(D, (q_n, p_n))$ is given by*

$$\tau_n = \frac{\mathbf{a} q_n + \mathbf{b} p_n}{\mathbf{a}^2 + \mathbf{b}^2}.$$

Moreover we have

$$d\left(\varphi_{\tau_n}, \mathbb{Z}^2_*\right) \leq \frac{\mathbf{a}}{\sqrt{\mathbf{a}^2 + \mathbf{b}^2}} \frac{1}{q_n} < \frac{1}{q_n}.$$

Proof. For $n \neq 0$ fixed, we want to find $t \geq 0$ such that $d(\varphi_t, (q_n, p_n)) = d(D, (q_n, p_n))$. This means that we want to find $t \geq 0$ such that

$$\overrightarrow{O\varphi_t} \perp \left((q_n e_1 + p_n e_2) - \overrightarrow{O\varphi_t}\right)$$

i.e. : to find $t \geq 0$ such that

$$\left\langle \overrightarrow{O\varphi_t}, (q_n e_1 + p_n e_2) - \overrightarrow{O\varphi_t} \right\rangle_{\mathbb{R}^2} = 0.$$

Consequently we solve the equation $\mathbf{a}t(q_n - \mathbf{a}t) + \mathbf{b}t(p_n - \mathbf{b}t) = 0$ and we find for non-null solution : $t = \frac{\mathbf{a}q_n + \mathbf{b}p_n}{\mathbf{a}^2+\mathbf{b}^2}$. Therefore, at time $t = \tau_n := \frac{\mathbf{a}q_n + \mathbf{b}p_n}{\mathbf{a}^2+\mathbf{b}^2}$ we obtain

$$d\left(\varphi_{\tau_n}, (q_n, p_n)\right) = d\left(\left(\frac{\mathbf{a}^2 q_n + \mathbf{ab}p_n}{\mathbf{a}^2+\mathbf{b}^2}, \frac{\mathbf{ab}q_n + \mathbf{b}^2 p_n}{\mathbf{a}^2+\mathbf{b}^2}\right), (q_n, p_n)\right)$$

$$= \frac{1}{\mathbf{a}^2+\mathbf{b}^2}\sqrt{\mathbf{b}^2\left(\mathbf{a}p_n - \mathbf{b}q_n\right)^2 + \mathbf{a}^2\left(\mathbf{b}q_n - \mathbf{a}p_n\right)^2}.$$

Since for all integer n we know that $\left|\theta - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^2}$, i.e. $|q_n\mathbf{b} - p_n\mathbf{a}| \leq \frac{\mathbf{a}}{q_n}$ holds for all integer n, so we deduce that

$$d\left(\varphi_{\tau_n}, (q_n, p_n)\right) \leq \frac{1}{\mathbf{a}^2+\mathbf{b}^2}\sqrt{\mathbf{b}^2\frac{\mathbf{a}^2}{q_n^2} + \mathbf{a}^2\frac{\mathbf{a}^2}{q_n^2}} \leq \frac{\mathbf{a}}{\sqrt{\mathbf{a}^2+\mathbf{b}^2}}\frac{1}{q_n} < \frac{1}{q_n}.$$

For conclude, we note that

$$d\left(\varphi_{\tau_n}, (q_n, p_n)\right) \geq d\left(\varphi_{\tau_n}, \mathbb{Z}_*^2\right)$$

holds for all integer n. □

We wish to generalize this result : we wish to analyze the behaviour of the distance between the set $\mathbb{Z}_*^2$ and the flow φ_t when t is in a neighbourhood of the time τ_n.

Notation 3.29. For $r > 0$ let us denotes by $B(\tau_n, r)$ the closed ball of center τ_n and radius $r > 0$:

$$B(\tau_n, r) := \left[\frac{\mathbf{a}q_n + \mathbf{b}p_n}{\mathbf{a}^2+\mathbf{b}^2} - r, \frac{\mathbf{a}q_n + \mathbf{b}p_n}{\mathbf{a}^2+\mathbf{b}^2} + r\right].$$

Proposition 3.30. *For all $r > 0$*

$$d\left(\varphi_t, \mathbb{Z}_*^2\right) \leq \frac{1}{\mathbf{a}^2+\mathbf{b}^2}\sqrt{\left(\left(\frac{\mathbf{ab}}{q_n}\right) + r\mathbf{a}(\mathbf{a}^2+\mathbf{b}^2)\right)^2 + \left(\frac{\mathbf{a}^2}{q_n} + r\mathbf{b}(\mathbf{a}^2+\mathbf{b}^2)\right)^2}$$

holds for all $t \in B(\tau_n, r)$.

Proof. We begin with the following inequality : for all $t \geq 0$ and for all n

$$d\left(\varphi_t, \mathbb{Z}_*^2\right) \leq d\left(\varphi_t, (q_n, p_n)\right).$$

Next, it's clear that for all $t \in B(\tau_n, r)$ we have

$$\varphi_t \in B\left(\frac{\mathbf{a}^2 q_n + \mathbf{ab}p_n}{\mathbf{a}^2+\mathbf{b}^2}, r\mathbf{a}\right) \times B\left(\frac{\mathbf{ab}q_n + \mathbf{b}^2 p_n}{\mathbf{a}^2+\mathbf{b}^2}, r\mathbf{b}\right).$$

Therefore, for all $t \in B(\tau_n, r)$ we have

$$d\left(\varphi_t, (q_n, p_n)\right)$$

$$\leq \sqrt{\left(\frac{\mathbf{a}^2 q_n + \mathbf{ab}p_n}{\mathbf{a}^2+\mathbf{b}^2} + r\mathbf{a} - q_n\right)^2 + \left(\frac{\mathbf{ab}q_n + \mathbf{b}^2 p_n}{\mathbf{a}^2+\mathbf{b}^2} + r\mathbf{b} - p_n\right)^2}$$

$$= \frac{1}{\mathbf{a}^2+\mathbf{b}^2}\sqrt{\left(\mathbf{ab}p_n - \mathbf{b}^2 q_n + r\mathbf{a}(\mathbf{a}^2+\mathbf{b}^2)\right)^2 + \left(\mathbf{ab}q_n - \mathbf{a}^2 p_n + r\mathbf{b}(\mathbf{a}^2+\mathbf{b}^2)\right)^2}$$

$$\leq \frac{1}{\mathbf{a}^2+\mathbf{b}^2}\sqrt{\left(\left(\frac{\mathbf{ab}}{q_n}\right) + r\mathbf{a}(\mathbf{a}^2+\mathbf{b}^2)\right)^2 + \left(\frac{\mathbf{a}^2}{q_n} + r\mathbf{b}(\mathbf{a}^2+\mathbf{b}^2)\right)^2}.$$

□

Remark 3.31. For $r = 0$ we obtain

$$d\left(\varphi_t, \mathbb{Z}_*^2\right) \leq \frac{1}{\mathbf{a}^2+\mathbf{b}^2}\sqrt{\left(\left(\frac{\mathbf{ab}}{q_n}\right)\right)^2 + \left(\frac{\mathbf{a^2}}{q_n}\right)^2} \leq \frac{\mathbf{a}}{\sqrt{\mathbf{a}^2+\mathbf{b}^2}}\frac{1}{q_n}.$$

and we obtain again the result of the proposition 3.28.

Now, let us give an asymptotic equivalent (for $n \to \infty$) of the real number τ_n :

Proposition 3.32. *For $n \to \infty$ we have $\tau_n \sim \Omega q_n$; where $\Omega := \frac{\mathbf{a}+\mathbf{b}\theta}{\mathbf{a}^2+\mathbf{b}^2} > 0$.*

Proof. We just write the fraction $\tau_n/\Omega q_n$:

$$\frac{\tau_n}{\Omega q_n} = \frac{\mathbf{a}q_n + \mathbf{b}p_n}{\mathbf{a}^2+\mathbf{b}^2}\frac{\mathbf{a}^2+\mathbf{b}^2}{q_n(\mathbf{a}+\mathbf{b}\theta)} = \frac{\mathbf{a}}{\mathbf{a}+\mathbf{b}\theta} + \frac{\mathbf{b}p_n}{q_n(\mathbf{a}+\mathbf{b}\theta)}$$

and since $\lim_{n\to\infty} p_n/q_n = \theta$ we obtain that $\lim_{n\to\infty} \tau_n/\Omega q_n = 1$. □

Now, let us come back to the autocorrelation function approximation $\mathbf{a_1}$. Start by a notation and a remark :

Notation 3.33. For $\mu > 0$ let us denotes by $\mathcal{A}_h = \mathcal{A}_h(\theta, \mu)$ the following set :

$$\mathcal{A}_h := \left\{q_n \in \mathbb{N},\, q_n \in \left[h^{\min \delta_i' - 1 + \mu}, h^{1-2\min\delta_i - \mu}\right]\right\}.$$

Remark 3.34. For $\mu > 0$, we have of course $\left[h^{\min\delta_i'-1-\mu}, h^{1-2\min\delta_i+\mu}\right] \subset \left[h^{\min\delta_i'-1}, h^{1-2\min\delta_i}\right]$.

If we suppose that the set $\mathcal{A}_h$ is non empty, we have some periods for the function $\mathbf{a_1}$, indeed we have :

Theorem 3.35. Suppose $\mathcal{A}_h \neq \emptyset$, then

$$\sup_{n\in\{m\in\mathbb{N},\, q_m(\theta)\in\mathcal{A}_h\}} |\mathbf{a_1}(\tau_n) - 1| = O(h^\mu).$$

Proof. Applying the Taylor-Lagrange formula on the function $(x,y) \mapsto \mathfrak{F}\left(\chi^2\right)(x,y)$ near the origin : for all $t \geq 0$ there exists $\theta = \theta\left(t, h, T_{cl_1}, T_{cl_2}\right) \in]0,1[$ such that

$$\mathfrak{F}\left(\chi^2\right)\left(-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z}, t\right), -\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z}, t\right)\right) = \mathfrak{F}\left(\chi^2\right)(0,0)$$
$$-\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z}, t\right)\frac{\partial\mathfrak{F}\left(\chi^2\right)}{\partial x}\left(\theta\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z}, t\right), \theta\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z}, t\right)\right)$$
$$-\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z}, t\right)\frac{\partial\mathfrak{F}\left(\chi^2\right)}{\partial y}\left(\theta\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z}, t\right), \theta\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z}, t\right)\right).$$

We know that for all $n \geq 1$ the distance between the part of the flow φ_{τ_n} and the set $\mathbb{Z}_*^2$ is strictly lower than $\frac{1}{q_n}$. Hence, if we suppose that from a certain point, like $n \geq N$, the sequence $\left(\frac{1}{q_n}\right)_n$ is strictly lower than h^s (with $s > 0$), then we obtain the majorization $d\left(\varphi_{\tau_n}, \mathbb{Z}_*^2\right) \leq h^s$. Therefore, for all $i \in \{1,2\}$ we have also $d\left(\tau_n, T_{cl_i}\mathbb{Z}\right) \leq h^s$. Consequently for all $i \in \{1,2\}$ we get $h^{\delta_i'-1}d\left(\tau_n, T_{cl_i}\mathbb{Z}\right) \leq h^{\delta_i'-1+s}$. Since we suppose $s \geq -\min\delta_i' + 1 + \mu$ with $\mu > 0$ we deduce that :

$$h^{\delta_i'-1}d\left(\tau_n, T_{cl_i}\mathbb{Z}\right) \leq h^\mu;$$

and, for h small enough, we obtain

$$\left|\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z}, \tau_n\right)\frac{\partial\mathfrak{F}\left(\chi^2\right)}{\partial x}\left(\theta\frac{h^{\delta_1'-1}}{\omega_1}d\left(T_{cl_1}\mathbb{Z}, \tau_n\right), \theta\frac{h^{\delta_2'-1}}{\omega_2}d\left(T_{cl_2}\mathbb{Z}, t\tau_n\right)\right)\right| \leq Mh^\mu,$$

$$\left| \frac{h^{\delta_2'-1}}{\omega_2} d\left(T_{cl_2}\mathbb{Z}, \tau_n\right) \frac{\partial \mathfrak{F}\left(\chi^2\right)}{\partial y} \left(\theta \frac{h^{\delta_1'-1}}{\omega_1} d\left(T_{cl_1}\mathbb{Z}, \tau_n\right), \theta \frac{h^{\delta_2'-1}}{\omega_2} d\left(T_{cl_2}\mathbb{Z}, t\tau_n\right) \right) \right| \leq N h^{\mu};$$

where M, $M' > 0$ are constant which does not depend on h. Next, it comes from the Taylor formula written above that

$$\sup_{n \in \{m \in \mathbb{N},\, q_m(\theta) \in \mathcal{A}_h\}} |\mathbf{a_1}(\tau_n) - 1| = \frac{1}{\mathfrak{F}\left(\chi^2\right)(0,0)}$$

$$\sup_{n \in \{m \in \mathbb{N},\, q_m(\theta) \in \mathcal{A}_h\}} \left| \frac{h^{\delta_1'-1}}{\omega_1} d\left(T_{cl_1}\mathbb{Z}, \tau_n\right) \frac{\partial \mathfrak{F}\left(\chi^2\right)}{\partial x} \left(\theta \frac{h^{\delta_1'-1}}{\omega_1} d\left(T_{cl_1}\mathbb{Z}, \tau_n\right), \theta \frac{h^{\delta_2'-1}}{\omega_2} d\left(T_{cl_2}\mathbb{Z}, \tau_n\right) \right) \right.$$
$$\left. \frac{h^{\delta_2'-1}}{\omega_2} d\left(T_{cl_2}\mathbb{Z}, \tau_n\right) \frac{\partial \mathfrak{F}\left(\chi^2\right)}{\partial y} \left(\theta \frac{h^{\delta_1'-1}}{\omega_1} d\left(T_{cl_1}\mathbb{Z}, \tau_n\right), \theta \frac{h^{\delta_2'-1}}{\omega_2} d\left(T_{cl_2}\mathbb{Z}, \tau_n\right) \right) \right|$$
$$\leq h^{\mu} M.$$

□

Counting of the sequence q_n. In view of the theorem above, let us now tackle an important problem : what is the cardinality of the set $\mathcal{A}_h$? Note that since the sequence $(q_n(\theta))_{n \in \mathbb{N}}$ is strictly increasing we have

$$\#\{\mathcal{A}_h\} = \#\{n \in \mathbb{N},\, q_n(\theta) \in \mathcal{A}_h\}.$$

Start by a simple majorization of the integer $\#\{\mathcal{A}_h\}$:

$$\#\{\mathcal{A}_h\} \leq \#\left\{\mathbb{N} \cap \left[h^{\min \delta_i'-1-\mu}, h^{1-2\min \delta_i+\mu}\right]\right\} \leq E\left[\delta(h)\right] + 1$$

where $\delta(h) := h^{1-2\min \delta_i+\mu} - h^{\min \delta_i'-1-\mu}$ and $E[x]$ denotes the integer part of x. Then, for $h \to 0$ we have the equivalence $\delta(h) \sim h^{1-2\min \delta_i+\mu}$. So we get a majorization of the integer $\#\{\mathcal{A}_h\}$ in order $h^{1-2\min \delta_i+\mu}$.

Nevertheless, find a minoration of the integer $\#\{\mathcal{A}_h\}$ is more difficult; but it's cleat that for all $n^* > 1$ there exists $h^* \in]0,1[$ such that :

$$\left[h^{* \min \delta_i'-1-\mu}, h^{*1-2\min \delta_i+\mu}\right] \bigcap \left(\bigcup_{n=0}^{+\infty} q_n(\theta) \right) = \{q_{n^*}(\theta)\}.$$

In order to estimate the integer $\#\{\mathcal{A}_h\}$, we must know the distribution of the sequence $(q_n(\theta))_{n \in \mathbb{N}}$ on the real axis (in particular on the compact set $\left[h^{\min \delta_i'-1-\mu}, h^{1-2\min \delta_i+\mu}\right]$) depending on the number θ. Let's try to give some distribution examples of the sequance $(q_n(\theta))_{n \in \mathbb{N}}$.

An exemple : the golden ratio. The golden ratio φ is the unique real roots of $X^2 - X - 1 = 0$, i.e. $\varphi = (1+\sqrt{5})/2$. The continued fraction of the golden ration is :

$$\varphi = [1, 1, \ldots, 1, \ldots] = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1+\ldots}}}.$$

Consequently the golden ratio is that one of the most difficult real number to approximate with rationals numbers. An another particularity of the golden ratio is that the sequence of the denominators $(q_n)_n$ from the continued fraction algorithm is equal to the Fibonacci sequence $(\mathbb{F}_n)_n$:

$$\mathbb{F}_n := \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

We note that

$$\lim_{n \to +\infty} \frac{\mathbb{F}_{n+1}}{\mathbb{F}_n} = \varphi.$$

Next for $n \to +\infty$ we have also :

$$\mathbb{F}_n \sim \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n .$$

We have also the following property :

Proposition 3.36. *Denote by $q_n(x)$ the sequence of denominators from the continued fraction algorithm of the number x; for all $\theta \in \mathbb{R}$, $n \geq 0$ we have*

$$q_n(\theta) \geq \mathbb{F}_n.$$

In the general case for any θ irrational number, we have the following theorem (see for example **[Khi]**) :

Theorem 3.37. (Khintchine-Lévy, 1952). Almost surely for $\theta \in \mathbb{R}$ we have

$$\lim_{n \to +\infty} q_n(\theta)^{\frac{1}{n}} = K;$$

where K denotes the Khintchine-Lévy constant $K := e^{\frac{\pi}{12 \ln(2)}} > 1$.

Thus for instance from a certain point we obtain :

$$\left(\frac{1}{2} K \right)^n \leq q_n(\theta) \leq \left(\frac{3}{2} K \right)^n .$$

The study of the distribution of the geometrical sequences $\left(\left(\frac{1}{2} K \right)^n \right)_{n \in \mathbb{N}}$ and $\left(\left(\frac{3}{2} K \right)^n \right)_{n \in \mathbb{N}}$ on the compact set $\left[h^{\min \delta_i' - 1 - \mu}, h^{1 - 2 \min \delta_i + \mu} \right]$ is easy; unfortunately it does not provide accurate informations on the distribution of the sequence $(q_n(\theta))_{n \in \mathbb{N}}$.

Open question. Do we know the denominator distribution of $(q_n(\theta))_{n \in \mathbb{N}}$ with the real axis depending on θ? More specifically, for a non-empty compact set of diameter $\delta > 0$ includes in $\mathbb{R}_+^*$ is it possible to estimate the number of elements of the sequence $(q_n(\theta))_{n \in \mathbb{N}}$ in this compact set depending on the numbers θ and δ ?

4. Second order approximation : revival periods

4.1. **Introduction.** Our next aim is to use a more accurate approximation of the function $t \mapsto \mathbf{a}(t)$. In this section, we use the quadradic approximation $\mathbf{a_2}(t)$ of the autocorrelation function, valid up on a time scale $\left[0, 1/h^{\beta} \right]$ where $\beta > 1$. This approximation is a consequence of a Taylor formula on the the term $tF\left(\tau_n, \mu_m\right)/h$ in order 2. In this quadradic approximation appear three revivals periods T_{rev_1}, T_{rev_2} and $T_{rev_{12}}$ (of order $1/h$).

Assumption 4.1. In this section, we suppose $\frac{\partial^2 F}{\partial X^2}\left(E_1, E_2\right) \neq 0$, $\frac{\partial^2 F}{\partial X \partial Y}\left(E_1, E_2\right) \neq 0$, $\frac{\partial^2 F}{\partial Y^2}\left(E_1, E_2\right) \neq 0$.

4.2. Quadradic approximation andc revival periods.

4.2.1. *Semi-classical revival and revival periods.*

Definition 4.2. Let us define the semi-classical revival periods T_{srev_1}, T_{srev_2} and $T_{srev_{12}}$ by :

$$T_{srev_1} := \frac{4\pi}{h \frac{\partial^2 F}{\partial X^2}\left(\tau_{n_0}, \mu_{m_0}\right) \omega_1^2}; \; T_{srev_2} := \frac{4\pi}{h \frac{\partial^2 F}{\partial Y^2}\left(\tau_{n_0}, \mu_{m_0}\right) \omega_2^2};$$

$$T_{srev_{12}} := \frac{4\pi}{h \frac{\partial^2 F}{\partial X \partial Y}\left(\tau_{n_0}, \mu_{m_0}\right) \omega_1 \omega_2}.$$

So we get the approximation :

Proposition 4.3. *Let β a real number such that $\beta > 1 - 3\min\delta_i$. Then we have uniformly for $t \in [0, h^\beta]$:*

$$e^{+itF\left(\tau_{n_0},\mu_{m_0}\right)/h}\mathbf{r}(t)$$
$$= \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}+\frac{(n-n_0)^2}{T_{srev_1}}+\frac{(m-m_0)^2}{T_{srev_2}}+\frac{(n-n_0)(m-m_0)}{T_{srev_{12}}}\right)}$$
$$+O\left(h^{\beta+3\min\delta_i-1}\right).$$

Proof. The principle is the same as in the proof of proposition 3.3. Here we use the Taylor-Lagrange formula at order 3 : for all pair $(n,m) \in \mathbb{N}^2$ there exists $\theta = \theta(n,m,n_0,m_0) \in]0,1[$ such that

$$F(\tau_n,\mu_m) = F(\tau_{n_0},\mu_{m_0}) + \frac{\partial F(\tau_{n_0},\mu_{m_0})}{\partial X}\omega_1 h(n-n_0) + \frac{\partial F(\tau_{n_0},\mu_{m_0})}{\partial y}\omega_2 h(m-m_0)$$
$$+\frac{1}{2}\frac{\partial^2 F(\tau_{n_0},\mu_{m_0})}{\partial X^2}\omega_1^2 h^2(n-n_0)^2 + \frac{1}{2}\frac{\partial^2 F(\tau_{n_0},\mu_{m_0})}{\partial Y^2}\omega_2^2 h^2(m-m_0)^2$$
$$+\frac{1}{2}\frac{\partial^2 F}{\partial X\partial Y}(\tau_{n_0},\mu_{m_0})\omega_1\omega_2 h^2(n-n_0)(m-m_0) + \frac{1}{6}\frac{\partial^3 F(\rho_{n,m})}{\partial X^3}\omega_1^3 h^3(n-n_0)^3$$
$$+\frac{1}{2}\frac{\partial^3 F(\rho_{n,m})}{\partial X^2\partial Y}\omega_1^2\omega_2 h^3(n-n_0)^2(m-m_0) + \frac{1}{2}\frac{\partial^3 F(\rho_{n,m})}{\partial X\partial Y^2}\omega_1\omega_2^2 h^3(n-n_0)(m-m_0)^2$$
$$+\frac{1}{6}\frac{\partial^3 F(\rho_{n,m})}{\partial X^3}\omega_2^3 h^3(m-m_0)^3,$$

with $\rho_{n,m} = \rho(n,m,n_0,m_0,h) := (\tau_{n_0} + \theta(\tau_n - \tau_{n_0}), \mu_{m_0} + \theta(\mu_m - \mu_{m_0}))$.

Next, we observe that for all pair $(n,m) \in \Delta$ and for all $t \in [0,h^\beta]$

$$\left|t(n-n_0)^3 h^2\right| \le h^{\beta+3\delta_1-1};\ \left|t(n-n_0)^2(m-m_0)h^2\right| \le h^{\beta+2\delta_1+\delta_2-1};$$
$$\left|t(n-n_0)(m-m_0)^2 h^2\right| \le h^{\beta+\delta_1+2\delta_2-1};\ \left|t(m-m_0)^3 h^2\right| \le h^{\beta+3\delta_2-1};$$

hence, since $\beta > 1 - \min(3\delta_1, 2\delta_1+\delta_2, \delta_1+2\delta_2, 3\delta_2) = 1 - 3\min\delta_i$, for all $t \in [0,h^\beta]$ and for all pair $(n,m) \in \mathbb{N}^2$ we get

$$e^{-2i\pi t\left((n-n_0)^3h^2+(n-n_0)^2(m-m_0)h^2+(n-n_0)(m-m_0)^2h^2+(m-m_0)^3h^2\right)}$$
$$= 1 + O\left(h^{\beta-1+3\min\delta_i}\right).$$

And the statement of the proposition is established. □

For the same reason as in definition 3.4 we introduce the revival periods :

Definition 4.4. Let us defines the revival periods T_{rev_1}, T_{rev_2} and $T_{rev_{12}}$ by :

$$T_{rev_1} := \frac{4\pi}{h\frac{\partial^2 F}{\partial X^2}(E_1,E_2)\omega_1^2};\ T_{rev_2} := \frac{4\pi}{h\frac{\partial^2 F}{\partial Y^2}(E_1,E_2)\omega_2^2};$$
$$T_{rev_{12}} := \frac{4\pi}{h\frac{\partial^2 F}{\partial X\partial Y}(E_1,E_2)\omega_1\omega_2}.$$

Clearly for all $j \in \{1,2,12\}$ we have $\lim_{h\to 0} T_{srev_j}/T_{rev_j} = 1$. The three semi-classical periods T_{srev_j} depend on h as well as their quotients. Since we will consider period quotients afterwards, is it preferably to study revival periods than semi-classical revival periods;for that we use indeed :

Proposition 4.5. *Let v a real number such that $v > -2\min\delta_i$. Then we have uniformly for $t \in [0, h^v]$:*

$$\sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}+\frac{(n-n_0)^2}{T_{srev_1}}+\frac{(m-m_0)^2}{T_{srev_2}}+\frac{(n-n_0)(m-m_0)}{T_{srev_{12}}}\right)}$$

$$= \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}+\frac{(n-n_0)^2}{T_{rev_1}}+\frac{(m-m_0)^2}{T_{rev_2}}+\frac{(n-n_0)(m-m_0)}{T_{rev_{12}}}\right)} + O\left(h^{v+\min(\delta_1,\delta_2)}\right).$$

Proof. The principle is the same as in the proof of proposition 3.5. With the partition $\mathbb{N}^2 = \Delta \amalg \Gamma$ and by triangular inequality we have

$$\left| \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \left[e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}+\frac{(n-n_0)^2}{T_{srev_1}}+\frac{(m-m_0)^2}{T_{srev_2}}+\frac{(n-n_0)(m-m_0)}{T_{srev_{12}}}\right)} \right.\right.$$

$$\left.\left. e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}+\frac{(n-n_0)^2}{T_{rev_1}}+\frac{(m-m_0)^2}{T_{rev_2}}+\frac{(n-n_0)(m-m_0)}{T_{rev12}}\right)} \right] \right|$$

$$\leq \sum_{n,m\in\Gamma} 2\,|a_{n,m}|^2$$

$$+2\sum_{n,m\in\Delta} |a_{n,m}|^2 \left[\left|2\pi t(n-n_0)^2\left(\frac{1}{T_{srev_1}}-\frac{1}{T_{rev_1}}\right)\right| + \left|2\pi t(m-m_0)^2\left(\frac{1}{T_{srev_2}}-\frac{1}{T_{rev_2}}\right)\right| \right.$$

$$\left. + \left|2\pi t(n-n_0)(m-m_0)\left(\frac{1}{T_{srev_{12}}}-\frac{1}{T_{rev_{12}}}\right)\right| ;$$

because $\left|e^{iX} - e^{iY}\right| \leq 2\,|X-Y|$.

Next we observe that

$$T_{srev_1} - T_{rev_1} = \frac{4\pi}{\omega_1^2 h}\left(\frac{\frac{\partial^2 F}{\partial X^2}(E_1,E_2) - \frac{\partial^2 F}{\partial X^2}(\tau_{n_0},\mu_{m_0})}{\frac{\partial^2 F}{\partial X^2}(\tau_{n_0},\mu_{m_0})\,\frac{\partial^2 F}{\partial X^2}(E_1,E_2)}\right).$$

First we have

$$\left|\frac{\partial^2 F}{\partial X^2}(E_1,E_2) - \frac{\partial^2 F}{\partial X^2}(\tau_{n_0},\mu_{m_0})\right|$$

$$\leq \sup_{(x,y)\in B((E_1,E_2),1)} \left\|\nabla\left(\frac{\partial^2 F}{\partial X^2}\right)(x,y)\right\|_{\mathbb{R}^2} \|(E_1,E_2) - (\tau_{n_0},\mu_{m_0})\|_{\mathbb{R}^2}$$

$$\leq M\sqrt{(E_1-\tau_{n_0})^2 + (E_2-\mu_{m_0})^2} \leq Mh\frac{\sqrt{2}}{2};$$

where $M > 0$ is a constant which does not depend on h.

On the other hand, since we suppose $\frac{\partial^2 F}{\partial X^2}(E_1,E_2) \neq 0$; there exists $\varepsilon_1 > 0$ and $r_1 > 0$ such that for all$(x,y) \in B((E_1,E_2),r_1)$ we get

$$\left|\frac{\partial^2 F}{\partial X^2}(x,y)\right| \geq \varepsilon_1;$$

and we have seen that there exists $h_1 > 0$ such that for all $h \in]0,h_1[$ we have

$$(\tau_{n_0},\mu_{m_0}) \in B((E_1,E_2),r_1);$$

therefore the application

$$h \mapsto \frac{1}{\frac{\partial^2 F}{\partial X^2}(E_1,E_2)\,\frac{\partial^2 F}{\partial X^2}(\tau_{n_0},\mu_{m_0})}$$

is bounded on the open set $]0,h_1[$, indeed for all $h \in]0,h_1[$ we have

$$\left| \frac{1}{\frac{\partial^2 F}{\partial X^2}(E_1,E_2)\,\frac{\partial^2 F}{\partial X^2}(\tau_{n_0},\mu_{m_0})} \right| \leq \frac{1}{\varepsilon_1^2} < +\infty;$$

and with $M' := 2\pi\frac{M\sqrt{2}}{\varepsilon_1^2}$ for all $h \in]0,h_1[$ we obtain $|T_{srev_1} - T_{rev_1}| \leq M'$.

Next, since

$$\left| \frac{1}{T_{srev_1} T_{rev_1}} \right| \leq h^2 \frac{\left| \frac{\partial^2 F}{\partial X^2}(E_1,E_2)\,\frac{\partial^2 F}{\partial X^2}(\tau_{n_0},\mu_{m_0}) \right|}{16\pi^2} \leq Kh^2$$

where $K := \frac{1}{16\pi^2}\sup_{(x,y)\in B((E_1,E_2),1)} \left|\frac{\partial^2 F}{\partial X^2}(x,y)\right|^2$, there exists a constant $C_1 > 0$ (which does not depend on h) such that for all $h \in]0,h_1[$ we have $|1/T_{sren_1} - 1/T_{ren_1}| \leq C_1 h^2$ (eg. take $C_1 := KM$). In a similary way : there exists $C_2, C_{12} > 0$ such that for all $h \in]0,h_2[$ we get $|1/T_{sren_2} - 1/T_{ren_2}| \leq C_2 h^2$ and for all $h \in]0,h_{12}[$ we get also $|1/T_{sren_{12}} - 1/T_{ren_{12}}| \leq C_{12} h^2$.

Next, for all $t \geq 0$, for all pair $(n,m) \in \Delta$ and for all $h < \min(h_1,h_2,h_{12})$ we have

$$\left| t\,(n-n_0)^2 \left(\frac{1}{T_{srev_1}} - \frac{1}{T_{rev_1}} \right) \right| \leq C_1 |t| h^{2\delta_1};$$

$$\left| t\,(n-n_0)\,(m-m_0) \left(\frac{1}{T_{srev_{12}}} - \frac{1}{T_{rev_{12}}} \right) \right| \leq C_{12} |t| h^{\delta_1+\delta_2};$$

$$\left| t\,(m-m_0)^2 \left(\frac{1}{T_{srev_2}} - \frac{1}{T_{rev_2}} \right) \right| \leq C_2 |t| h^{2\delta_2};$$

hence for all $t \in [0,h^v]$ where v is a real number such that $v > -2\min\delta_i$ we obtain

$$\left| t\,(n-n_0)^2 \left(\frac{1}{T_{srev_1}} - \frac{1}{T_{rev_1}} \right) \right| + \left| t\,(m-m_0)^2 \left(\frac{1}{T_{srev_2}} - \frac{1}{T_{rev_2}} \right) \right|$$
$$+ \left| t\,(n-n_0)\,(m-m_0) \left(\frac{1}{T_{srev_{12}}} - \frac{1}{T_{rev_{12}}} \right) \right| \leq M h^{v+2\min\delta_i}$$

where $M := 3\max(C_1,C_2,C_{12})$.

So we proove that : there exists a constant $M > 0$ such that for all $h < \min(h_1,h_2,h_{12})$ and for all $v > -2\min\delta_i$ we get

$$\left| \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \left[e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}} + \frac{m-m_0}{T_{scl_2}} + \frac{(n-n_0)^2}{T_{srev_1}} + \frac{(m-m_0)^2}{T_{srev_2}} + \frac{(n-n_0)(m-m_0)}{T_{srev_{12}}} \right)} \right.\right.$$
$$\left.\left. e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}} + \frac{m-m_0}{T_{scl_2}} + \frac{(n-n_0)^2}{T_{rev_1}} + \frac{(m-m_0)^2}{T_{rev_2}} + \frac{(n-n_0)(m-m_0)}{T_{rev_{12}}} \right)} \right] \right|$$
$$\leq 2 \sum_{n,m\in\Gamma} |a_{n,m}|^2 + M h^{v+2\min\delta_i} \sum_{n,m\in\Delta} |a_{n,m}|^2 .$$

It follows from the lemma 2.5 (the lemma 2.5 says $\sum_{n,m\in\Gamma} |a_{n,m}|^2 = O(h^\infty)$) and from

$$\sum_{n,m\in\Delta} |a_{n,m}|^2 \leq \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 = 1 + O(h^\infty)$$

that

$$\left| \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \left[e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}} + \frac{m-m_0}{T_{scl_2}} + \frac{(n-n_0)^2}{T_{srev_1}} + \frac{(m-m_0)^2}{T_{srev_2}} + \frac{(n-n_0)(m-m_0)}{T_{srev_{12}}} \right)} \right.\right.$$

$$e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}+\frac{(n-n_0)^2}{T_{rev_1}}+\frac{(m-m_0)^2}{T_{rev_2}}+\frac{(n-n_0)(m-m_0)}{T_{rev_{12}}}\right)}\Bigg]\Bigg|$$

$$= O\left(h^{v+2\min\delta_i}\right).$$

□

4.2.2. *Comparison between revivals periods an the time scale* $[0,h^{\alpha}]$. Recall here that the parameters $\left(\delta_i^{'},\delta_i\right)\in]\frac{1}{2},1[^2$ with $\delta_i^{'}>\delta_i$; recall also that the real coefficients α,β would require $\alpha>1-2\min\delta_i$ and $\beta>1-3\min\delta_i$. Next we observe that $1-3\min\delta_i-(1-2\min\delta_i)=-\min\delta_i<0$ and $-2\min\delta_i-(1-3\min\delta_i)=\min\delta_i-1<0$; hence for h small enough we obtain

$$h^{1-2\min\delta_i}<h^{1-3\min\delta_i}<h^{-2\min\delta_i}.$$

So we can make a "good choice" for parameters α, β and v : indeed we can choose α, β and v such that : $v\leq\beta<-1<\alpha<0$, therefore (for h small enough) we have :

$$\left[0,T_{cl_i}\right]\subset[0,h^{\alpha}]\subset[0,T_{rev_i}]\subset\left[0,h^{\beta}\right]\subseteq[0,h^{v}].$$

4.2.3. *The quadradic approximation* $\mathbf{a_2}$. So, the quadradic approximation of the autocorrelation function on the time scale $\left[0,h^{\beta}\right]$ is :

Definition 4.6. The quadradic approximation of the autocorrelation function is

$$\mathbf{a_2}:t\mapsto\sum_{n,m\in\mathbb{N}^2}\left|a_{n,m}\right|^2e^{-2i\pi t\left(\frac{n-n_0}{T_{scl_1}}+\frac{m-m_0}{T_{scl_2}}+\frac{(n-n_0)^2}{T_{rev_1}}+\frac{(m-m_0)^2}{T_{rev_2}}+\frac{(n-n_0)(m-m_0)}{T_{rev_{12}}}\right)}.$$

4.3. **Revival theorems.**

4.3.1. *Preliminaries.*

Resonance hypothesis.

Definition 4.7. We say that the revival periods T_{rev_1}, T_{rev_2}, $T_{rev_{12}}$ are in resonance if an only if there exists $\left(\frac{p_1}{q_1},\frac{p_2}{q_2},\frac{p_{12}}{q_{12}}\right)\in\mathbb{Q}^3$ such that

$$\frac{p_1}{q_1}T_{rev_1}=\frac{p_2}{q_2}T_{rev_2}=\frac{p_{12}}{q_{12}}T_{rev_{12}}.$$

Notation 4.8. In this case, we introduce the notation $T_{frac}:=\frac{p_1}{q_1}T_{rev_1}=\frac{p_2}{q_2}T_{rev_2}=\frac{p_{12}}{q_{12}}T_{rev_{12}}$. And for all $j\in\{1,2\}$ let us also consider the numbers $r_j:=p_{12}q_j$, $s_j:=q_{12}p_j$, and clearly for all $j\in\{1,2\}$ we have $T_{rev_j}=\frac{r_j}{s_j}T_{rev_{12}}$.

Preliminaries. To make progress in our study we need to introduce a new function ψ_{cl} with two artificial variables t_1,t_2.

Definition 4.9. Let us define the pseudo-classical function ψ_{cl} :

$$\psi_{cl}\left(t_1,t_2\right):=\sum_{n,m\in\mathbb{N}^2}\left|a_{n,m}\right|^2e^{-2i\pi t_1\frac{n-n_0}{T_{scl_1}}-2i\pi t_2\frac{m-m_0}{T_{scl_2}}}.$$

So we get the obvious following property; first the function ψ_{cl} is doubly-periodic :

(i) for all pair $t_1,t_2\geq0$ we have $\psi_{cl}\left(t_1+T_{scl_1},t_2\right)=\psi_{cl}\left(t_1,t_2\right)$;

(ii) and for all pair $t_1,t_2\geq0$ we have also $\psi_{cl}\left(t_1,t_2+T_{scl_2}\right)=\psi_{cl}\left(t_1,t_2\right)$.

This function have no immediate physical significance, but if the time t_1 and t_2 are equal :

(iii) for all $t\geq0$ we have $\psi_{cl}(t,t)=\mathbf{a_1}(t)$.

Some lemmas.

Notation 4.10. Let us consider the sequence $(\theta_{n,m})_{n,m} = (\theta_{n,m}(p_1, q_1, p_2, q_2, p_{12}, q_{12}, h))_{n,m}$ with $(n, m) \in \mathbb{Z}^2$ defined by :

$$\theta_{n,m} := e^{-2i\pi\left(\frac{p_1}{q_1}(n-n_0)^2+\frac{p_{12}}{q_{12}}(n-n_0)(m-m_0)+\frac{p_2}{q_2}(m-m_0)^2\right)}.$$

The periodicity of this sequence is caracterised by the following easy proposition.

Proposition 4.11. *For all $p_1, q_1, p_2, q_2, p_{12}, q_{12} \in \mathbb{Z}$, the sequence $(\theta_{n,m})_{n,m}$ verify*

$$\begin{cases} \theta_{n+\ell_1,m} = \theta_{n,m} \\ \\ \theta_{n,m+\ell_2} = \theta_{n,m} \end{cases}$$

if and only if the integers ℓ_1 and ℓ_2 satisfy the following equations :

$$\forall (n, m) \in \mathbb{Z}^2,\ \frac{\ell_1^2 p_1}{q_1} + \frac{2p_1\ell_1}{q_1}n + \frac{p_1 r_1 \ell_1}{s_1 q_1}m \equiv 0\ [1]$$

$$\forall (n, m) \in \mathbb{Z}^2,\ \frac{\ell_2^2 p_2}{q_2} + \frac{2p_2\ell_2}{q_2}m + \frac{p_2 r_2 \ell_2}{s_2 q_2}n \equiv 0\ [1].$$

Example 4.12. An obvious solution is $\ell_1 = q_1 s_1$ and $\ell_2 = q_2 s_2$.

For two periods $\ell_1, \ell_2 \in \mathbb{Z}^2$ let us consider the set of sequences ℓ_1, ℓ_2-periodic with his natural scalar product.

Definition 4.13. For a fixed pair $\ell_1, \ell_2 \in (\mathbb{Z}^*)^2$ we define $\mathfrak{S}_{\ell_1,\ell_2}(\mathbb{Z})$ the set of sequences ℓ_1, ℓ_2-periodic in the following sense :

$$\mathfrak{S}_{\ell_1,\ell_2}(\mathbb{Z}) := \left\{ u_{n,m} \in \mathbb{C}^{\mathbb{Z}^2};\ \forall n, m \in \mathbb{Z}^2,\ u_{n+\ell_1,m} = u_{n,m}\ \text{ and }\ u_{n,m+\ell_2} = u_{n,m} \right\}.$$

So we have the elementary :

Proposition 4.14. *The application*

$$\langle\ ,\ \rangle_{\mathfrak{S}_{\ell_1,\ell_2}} : \begin{cases} \mathfrak{S}_{\ell_1,\ell_2}(\mathbb{Z})^2 \to \mathbb{C} \\ \\ (u, v) \mapsto \langle u, v\rangle_{\mathfrak{S}_{\ell_1,\ell_2}} := \frac{1}{|\ell_1\ell_2|}\sum_{n=0}^{|\ell_1|-1}\sum_{m=0}^{|\ell_2|-1} u_{n,m}\overline{v_{n,m}} \end{cases}.$$

is a Hermitean product on the space $\mathfrak{S}_{\ell_1,\ell_2}(\mathbb{Z})$.

We have also the obvious following remark :

Proposition 4.15. *Let us consider $\phi^{k,p}_{n,m} := e^{-\frac{2i\pi kn}{\ell_1}} e^{-\frac{2i\pi pm}{\ell_2}}$ where $(k, p) \in \mathbb{Z}^2$; then the family $\left\{ \left(\phi^{k,p}_{n,m}\right)_{n,m\in\mathbb{Z}^2} \right\}_{k=0\ldots\ell_1-1, p=0\ldots\ell_2-1}$ is an orthonormal basis of the space vector $\mathfrak{S}_{\ell_1,\ell_2}(\mathbb{Z})$.*

4.3.2. *The main theorem.* In the following theorem we show that the function $t \mapsto \mathbf{a_2}(t)$ near the period T_{frac} can be written as a finite sum of ψ_{cl} with arguments shifted. Indeed we have :

Theorem 4.16. Suppose resonance hypothesis holds; then there exists a family of $\ell_1 + \ell_2$ complex numbers (depends on h) : $\left(c_{k_1,k_2}\right)_{k_1\in\{0\ldots\ell_1-1\}, k_2\in\{0\ldots\ell_2-1\}}$ where the integers $\ell_1, \ell_2 \in \mathbb{Z}^2$ are solutions of equations from proposition 4.11; such that

$$\mathbf{a_2}\left(t + T_{frac}\right) = \sum_{k_1=0}^{\ell_1-1}\sum_{k_2=0}^{\ell_2-1} c_{k_1,k_2}\psi_{cl}\left(t + T_{frac} + \frac{k_1}{\ell_1}T_{scl_1}, t + T_{frac} + \frac{k_2}{\ell_2}T_{scl_2}\right)$$

$$+O\left(h^{\alpha+2\min\delta_i-1}\right).$$

holds for all $t\in[0,h^\alpha]$. The numbers c_{k_1,k_2} are called fractionnals coefficients; and for all $k_1\in\{0...\ell_1-1\}$, $k_2\in\{0...\ell_2-1\}$

$$c_{k_1,k_2}=e^{-\frac{2i\pi k_1 n_0}{\ell_1}}e^{-\frac{2i\pi k_2 m_0}{\ell_2}}b_{k_1,k_2}$$

with $b_{k_1,k_2}=b_{k_1,k_2}(h)=\left\langle\sigma_h,\phi^{k_1,k_2}\right\rangle_{\mathfrak{S}_{\ell_1,\ell_2}}$.

Proof. Let us denote the integers $\tilde{n}:=n-n_0$, $\tilde{m}:=m-m_0$ and consider the function $\varepsilon(t)$:

$$\begin{aligned}\varepsilon(t):=&\left|\mathbf{a_2}\left(t+T_{frac}\right)-\sum_{k_1=0}^{\ell_1-1}\sum_{k_2=0}^{\ell_2-1}c_{k_1,k_2}\psi_{cl}\left(t+T_{frac}+\frac{k_1}{\ell_1}T_{scl_1},t+T_{frac}+\frac{k_2}{\ell_2}T_{scl_2}\right)\right|\\
=&\left|\sum_{n,m\in\mathbb{N}^2}|a_{n,m}|^2\,e^{-2i\pi t\frac{\tilde{n}}{T_{scl_1}}}e^{-2i\pi T_{frac}\frac{\tilde{n}}{T_{scl_1}}}e^{-2i\pi t\frac{\tilde{m}}{T_{scl_2}}}e^{-2i\pi T_{frac}\frac{\tilde{m}}{T_{scl_2}}}\right.\\
&e^{-2i\pi t\frac{\tilde{n}^2}{T_{rev_1}}}e^{-2i\pi\frac{p_1\tilde{n}^2}{q_1}}e^{-2i\pi t\frac{\tilde{m}^2}{T_{rev_2}}}e^{-2i\pi\frac{p_2\tilde{m}^2}{q_2}}e^{-2i\pi t\frac{\tilde{n}\tilde{m}}{T_{rev_{12}}}}e^{-2i\pi\frac{p_{12}\tilde{n}\tilde{m}}{q_{12}}}\\
&\left.-\sum_{k_1=0}^{\ell_1-1}\sum_{k_2=0}^{\ell_2-1}c_{k_1,k_2}\psi_{cl}\left(t+T_{frac}+\frac{k_1}{\ell_1}T_{cl_1},t+T_{frac}+\frac{k_2}{\ell_2}T_{cl_2}\right)\right|\\
=&\left|\sum_{n,m\in\mathbb{N}^2}|a_{n,m}|^2\,\theta_{n,m}e^{-2i\pi t\frac{\tilde{n}}{T_{scl_1}}}e^{-2i\pi T_{frac}\frac{\tilde{n}}{T_{scl_1}}}e^{-2i\pi t\frac{\tilde{m}}{T_{scl_2}}}e^{-2i\pi T_{frac}\frac{\tilde{m}}{T_{scl_2}}}\right.\\
&e^{-2i\pi t\frac{\tilde{n}^2}{T_{rev_1}}}e^{-2i\pi t\frac{\tilde{m}^2}{T_{rev_2}}}e^{-2i\pi t\frac{\tilde{n}\tilde{m}}{T_{rev_{12}}}}\\
&\left.-\sum_{k_1=0}^{\ell_1-1}\sum_{k_2=0}^{\ell_2-1}c_{k_1,k_2}\psi_{cl}\left(t+T_{frac}+\frac{k_1}{\ell_1}T_{scl_1},t+T_{frac}+\frac{k_2}{\ell_2}T_{scl_2}\right)\right|.\end{aligned}$$

Since the sequence $(\theta_{n,m})_{n,m}\in\mathfrak{S}_{\ell_1,\ell_2}(\mathbb{Z})$ with $\ell_1=q_1s_1$ and $\ell_2=q_2s_2$ there exists a unique decomposition of the sequence $(\theta_{n,m})_{n,m}$ on the basis $\left\{\left(\phi^{k,p}_{n,m}\right)_{n,m\in\mathbb{Z}^2}\right\}_{k=0...\ell_1-1,p=0...\ell_2-1}$; indeed we have :

$$\theta_{n,m}=\sum_{k_1=0}^{\ell_1-1}\sum_{k_2=0}^{\ell_2-1}b_{k_1,k_2}\phi^{k_1,k_2}_{n,m}.$$

where $b_{k_1,k_2}=\left\langle\theta,\phi^{k_1,k_2}\right\rangle_{\mathfrak{S}_{\ell_1,\ell_2}}$. Therefore we get

$$\begin{aligned}\varepsilon(t)=&\left|\sum_{n,m\in\mathbb{N}^2}\sum_{k_1=0}^{\ell_1-1}\sum_{k_2=0}^{\ell_2-1}|a_{n,m}|^2\,b_{k_1,k_2}e^{-2i\pi t\frac{\tilde{n}}{T_{scl_1}}}e^{-2i\pi T_{frac}\frac{\tilde{n}}{T_{scl_1}}}e^{-2i\pi t\frac{\tilde{m}}{T_{scl_2}}}e^{-2i\pi T_{frac}\frac{\tilde{m}}{T_{scl_2}}}\right.\\
&e^{-2i\pi t\frac{\tilde{n}^2}{T_{rev_1}}}e^{-2i\pi t\frac{\tilde{m}^2}{T_{rev_2}}}e^{-2i\pi t\frac{\tilde{n}\tilde{m}}{T_{rev_{12}}}}\phi^{k_1,k_2}_{n,m}\\
&\left.-\sum_{k_1=0}^{\ell_1-1}\sum_{k_2=0}^{\ell_2-1}c_{k_1,k_2}\psi_{cl}\left(t+T_{frac}+\frac{k_1}{\ell_1}T_{scl_1},t+T_{frac}+\frac{k_2}{\ell_2}T_{scl_2}\right)\right|\end{aligned}$$

$$\begin{aligned}=&\left|\sum_{n,m\in\mathbb{N}^2}\sum_{k_1=0}^{\ell_1-1}\sum_{k_2=0}^{\ell_2-1}|a_{n,m}|^2\,b_{k_1,k_2}e^{-2i\pi t\frac{\tilde{n}}{T_{scl_1}}}e^{-2i\pi T_{frac}\frac{\tilde{n}}{T_{scl_1}}}e^{-2i\pi t\frac{\tilde{m}}{T_{scl_2}}}e^{-2i\pi T_{frac}\frac{\tilde{m}}{T_{scl_2}}}\right.\\
&\left.e^{-2i\pi t\frac{\tilde{n}^2}{T_{rev_1}}}e^{-2i\pi t\frac{\tilde{m}^2}{T_{rev_2}}}e^{-2i\pi t\frac{\tilde{n}\tilde{m}}{T_{rev_{12}}}}\phi^{k_1,k_2}_{n,m}\right|\end{aligned}$$

$$- \sum_{k_1=0}^{\ell_1-1} \sum_{k_2=0}^{\ell_2-1} c_{k_1,k_2} \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \, e^{-2i\pi t \frac{\tilde{n}}{T_{scl_1}}} e^{-2i\pi T_{frac} \frac{\tilde{n}}{T_{scl_1}}} e^{-2i\pi k_1 \frac{\tilde{n}}{\ell_1}}$$

$$e^{-2i\pi t \frac{\tilde{m}}{T_{scl_2}}} e^{-2i\pi T_{frac} \frac{\tilde{m}}{T_{scl_2}}} e^{-2i\pi k_2 \frac{\tilde{m}}{\ell_2}} \Bigg|$$

$$= \Bigg| \sum_{n,m\in\mathbb{N}^2} \sum_{k_1=0}^{\ell_1-1} \sum_{k_2=0}^{\ell_2-1} |a_{n,m}|^2 \, b_{k_1,k_2} e^{-2i\pi t \frac{\tilde{n}}{T_{scl_1}}} e^{-2i\pi T_{frac} \frac{\tilde{n}}{T_{scl_1}}} e^{-2i\pi t \frac{\tilde{m}}{T_{scl_2}}} e^{-2i\pi T_{frac} \frac{\tilde{m}}{T_{scl_2}}}$$

$$e^{-2i\pi t \frac{\tilde{n}^2}{T_{rev_1}}} e^{-2i\pi t \frac{\tilde{m}^2}{T_{rev_2}}} e^{-2i\pi t \frac{\tilde{n}\tilde{m}}{T_{rev_{12}}}} e^{-\frac{2i\pi k_1 n}{\ell_1}} e^{-\frac{2i\pi k_2 m}{\ell_2}}$$

$$- \sum_{k_1=0}^{\ell_1-1} \sum_{k_2=0}^{\ell_2-1} c_{k_1,k_2} \sum_{n,m\in\mathbb{N}^2} |a_{n,m}|^2 \, e^{-2i\pi t \frac{\tilde{n}}{T_{scl_1}}} e^{-2i\pi T_{frac} \frac{\tilde{n}}{T_{scl_1}}} e^{-2i\pi k_1 \frac{\tilde{n}}{\ell_1}}$$

$$e^{-2i\pi t \frac{\tilde{m}}{T_{scl_2}}} e^{-2i\pi T_{frac} \frac{\tilde{m}}{T_{scl_2}}} e^{-2i\pi k_2 \frac{\tilde{m}}{\ell_2}} \Bigg|.$$

And since $c_{k_1,k_2} e^{-2i\pi k_1 \frac{\tilde{n}}{\ell_1}} e^{-2i\pi k_2 \frac{\tilde{m}}{\ell_2}} = b_{k_1,k_2} e^{-\frac{2i\pi k_1 n}{\ell_1}} e^{-\frac{2i\pi k_2 m}{\ell_2}}$ we deduce that

$$\varepsilon(t) =$$

$$\Bigg| \sum_{n,m\in\mathbb{N}^2} \sum_{k_1=0}^{\ell_1-1} \sum_{k_2=0}^{\ell_2-1} |a_{n,m}|^2 \, b_{k_1,k_2} e^{-2i\pi t \frac{\tilde{n}}{T_{scl_1}}} e^{-2i\pi T_{frac} \frac{\tilde{n}}{T_{scl_1}}} e^{-2i\pi t \frac{\tilde{m}}{T_{scl_2}}} e^{-2i\pi T_{frac} \frac{\tilde{m}}{T_{scl_2}}} e^{-\frac{2i\pi k_1 n}{\ell_1}} e^{-\frac{2i\pi k_2 m}{\ell_2}}$$

$$\left(-1 + e^{-2i\pi t \frac{\tilde{n}^2}{T_{rev_1}}} e^{-2i\pi t \frac{\tilde{m}^2}{T_{rev_2}}} e^{-2i\pi t \frac{\tilde{n}\tilde{m}}{T_{rev_{12}}}} \right) \Bigg|.$$

To finish, we use the partition $\mathbb{N}^2 = \Delta \amalg \Gamma$ and we just consider indices in the set Δ for the sum. Hence there exists constants $C_1, C_2, C_{12} > 0$ which does not depend on h, such that

$$\left| t \frac{(n-n_0)^2}{T_{rev_1}} \right| \leq C_1 h^{\alpha+2\delta_1-1}; \quad \left| t \frac{(m-m_0)^2}{T_{rev_2}} \right| \leq C_2 h^{\alpha+2\delta_2-1};$$

$$\left| t \frac{(n-n_0)(m-m_0)}{T_{rev_{12}}} \right| \leq C_{12} h^{\alpha+\delta_1+\delta_2-1}$$

holds for all pair $(n,m) \in \Delta$ and for all $t \in [0, h^\alpha]$.

As a consequence for all $t \in [0, h^\alpha]$ and for all pair $(n,m) \in \Delta$ we get :

$$e^{-2i\pi t \frac{\tilde{n}^2}{T_{rev_1}}} e^{-2i\pi t \frac{\tilde{m}^2}{T_{rev_2}}} e^{-2i\pi t \frac{\tilde{n}\tilde{m}}{T_{rev_{12}}}} - 1 = O\left(h^{\alpha+2\min\delta_i-1} \right).$$

□

If we take $t = 0$ we obtain :

Corollary 4.17. *Under the same hypothesis we have*

$$\mathbf{a_2}\left(T_{frac}\right) = \sum_{k_1=0}^{\ell_1-1} \sum_{k_2=0}^{\ell_2-1} c_{k_1,k_2} \psi_{cl}\left(T_{frac} + \frac{k_1}{\ell_1} T_{scl_1}, T_{frac} + \frac{k_2}{\ell_2} T_{scl_2} \right).$$

4.4. **Explicit values of modulus for revival coefficients.** Our final aim is to compute the modulus of revival coefficients. The idea is to split the sum $\left|c_{k_1,k_2}\right|$ in two simple parts. These parts look like that Gauss sums, but in fact with a little difference. Here we propose a simple way to compute this sums and we don't use sophisticated theory. Start with a notation and a remark :

Notation 4.18. For $\ell \geq 1$ and for integers p et q such that $p \wedge q = 1$, let us consider for all integer $k \in \{0...\ell - 1\}$ the following sum

$$d_k(\ell, p, q) = \frac{1}{\ell} \sum_{n=0}^{\ell-1} e^{-2i\pi \frac{p}{q}(n-n_0)^2} e^{\frac{2i\pi kn}{\ell}}.$$

Therefore for all $k \in \{0...\ell - 1\}$

$$|d_k(\ell, p, q)| = \frac{1}{\ell} \left| \sum_{m \in \mathbb{Z}/\ell\mathbb{Z}} e^{-2i\pi \frac{p}{q} m^2} e^{\frac{2i\pi km}{\ell}} \right|.$$

Theorem 4.19. Suppose resonance hypothesis holds and suppose also $\frac{p_{12}}{q_{12}} \in \mathbb{Z}$; then we obtain :

$$\left|b_{k_1,k_2}\right|^2 = \left|d_{k_1}\left(\ell_1, p_1 s_1, \ell_1\right)\right|^2 \left|d_{k_2}\left(\ell_2, p_2 s_2, \ell_2\right)\right|^2.$$

Proof. For all $k_1 \in \{0...\ell_1 - 1\}$, $k_2 \in \{0...\ell_2 - 1\}$ we have

$$\left|b_{k_1,k_2}\right| = \frac{1}{\ell_1 \ell_2} \left| \sum_{(n,m) \in \mathbb{Z}/\ell_1\mathbb{Z} \times \mathbb{Z}/\ell_2\mathbb{Z}} e^{-2i\pi\left(\frac{p_1}{q_1} n^2 + \frac{p_{12}}{q_{12}} nm + \frac{p_2}{q_2} m^2\right)} e^{\frac{2i\pi k_1 n}{\ell_1}} e^{\frac{2i\pi k_2 m}{\ell_2}} \right|;$$

$$= \frac{1}{\ell_1 \ell_2} \left| \sum_{m=0}^{\ell_2-1} e^{\frac{2i\pi k_2 m}{\ell_2}} e^{-2i\pi \frac{p_2}{q_2} m^2} \sum_{n=0}^{\ell_1-1} e^{-2i\pi \frac{p_1}{q_1} n^2} e^{\frac{2i\pi n}{\ell_1}\left(k_1 - \frac{p_{12}}{q_{12}} \ell_1 m\right)} \right|$$

and since $\ell_1 = q_1 s_1 = q_1 q_{12} p_1$ we obtain

$$\left|b_{k_1,k_2}\right| = \frac{1}{\ell_1 \ell_2} \left| \sum_{m=0}^{\ell_2-1} e^{\frac{2i\pi k_2 m}{\ell_2}} e^{-2i\pi \frac{p_2}{q_2} m^2} \sum_{n=0}^{\ell_1-1} e^{-2i\pi \frac{p_1}{q_1} n^2} e^{\frac{2i\pi n}{\ell_1}(k_1 - p_{12} q_1 s_1 m)} \right|.$$

For $j \in \{1, 2\}$, let us consider χ_j the following characters :

$$\chi_j : \begin{cases} \mathbb{Z}/\ell_j\mathbb{Z} \hookrightarrow \mathbb{C}^* \\ \\ a \mapsto e^{-2i\pi \frac{a}{\ell_j}}; \end{cases}$$

as a consequence we get

$$\left|b_{k_1,k_2}\right|^2 = \frac{1}{\ell_1^2 \ell_2^2} \sum_{(x,y) \in \mathbb{Z}/\ell_1\mathbb{Z} \times \mathbb{Z}/\ell_2\mathbb{Z}} \chi_1\left(p_1 s_1 x^2 - x(k_1 - p_{12} q_1 p_1 y\right) \chi_2\left(p_2 s_2 y^2 - k_2 y\right)$$

$$\sum_{(z,t) \in \mathbb{Z}/\ell_1\mathbb{Z} \times \mathbb{Z}/\ell_2\mathbb{Z}} \chi_1\left(-p_1 s_1 z^2 + z(k_1 - p_{12} q_1 p_1 t\right) \chi_2\left(-p_2 s_2 t^2 + k_2 t\right)$$

$$= \frac{1}{\ell_1^2 \ell_2^2} \sum_{((x,z),(y,t)) \in (\mathbb{Z}/\ell_1\mathbb{Z})^2 \times (\mathbb{Z}/\ell_2\mathbb{Z})^2} \left(\chi_1\left(p_1 s_1\left(x^2 - z^2\right) - k_1(x - z) + p_{12} q_1 p_1 (xy - zt)\right)\right.$$

$$\left.\chi_2\left(p_2 s_2\left(y^2 - t^2\right) - k_2(y - t)\right)\right).$$

Now, because $\frac{p_{12}}{q_{12}} \in \mathbb{Z}$ then $\ell_1 = q_1 q_{12} p_1 | q_1 p_1 p_{12}$ hence for all x, y, z, t we have $p_{12} q_1 p_1 (xy - zt) \in \ell_1\mathbb{Z}$. Therefore for all x, y, z, t we have $\chi_1\left(p_{12} q_1 p_1 (xy - zt)\right) = 1$. Hence

$$\left|b_{k_1,k_2}\right|^2 =$$

$$\frac{1}{\ell_1^2\ell_2^2}\sum_{((x,z),(y,t))\in(\mathbb{Z}/\ell_1\mathbb{Z})^2\times(\mathbb{Z}/\ell_2\mathbb{Z})^2}\chi_1\left(p_1s_1\left(x^2-z^2\right)-k_1(x-z)\right)\chi_2\left(p_2s_2\left(y^2-t^2\right)-k_2(y-t)\right)$$

$$=\frac{1}{\ell_1^2\ell_2^2}\sum_{(x,z)\in(\mathbb{Z}/\ell_1\mathbb{Z})^2}\chi_1\left(p_1s_1\left(x^2-z^2\right)-k_1(x-z)\right)\sum_{(y,t)\in(\mathbb{Z}/\ell_2\mathbb{Z})^2}\chi_2\left(p_2s_2\left(y^2-t^2\right)-k_2(y-t)\right)$$

$$=\frac{1}{\ell_1^2}\left|\sum_{x\in\mathbb{Z}/\ell_1\mathbb{Z}}e^{-2i\pi\frac{p_1s_1}{\ell_1}x^2}e^{\frac{2i\pi k_1}{\ell_1}x}\right|^2\frac{1}{\ell_2^2}\left|\sum_{y\in\mathbb{Z}/\ell_2\mathbb{Z}}e^{-2i\pi\frac{p_2s_2}{\ell_2}y^2}e^{\frac{2i\pi k_2}{\ell_2}y}\right|^2.$$

□

To finish we can compute

$$\left|d_{k_1}\left(\ell_1,p_1s_1,\ell_1\right)\right|^2\left|d_{k_2}\left(\ell_2,p_2s_2,\ell_2\right)\right|^2.$$

with the following results (see for example **[Lab2]**) :

Proposition 4.20. *For all pair p,q with $p\wedge q=1$ and q odd, then for all $k\in\{0...q-1\}$ we get :*

$$|d_k(q,p,q)|^2=\frac{1}{q}.$$

And

Proposition 4.21. *For all pair p,q with $p\wedge q=1$ and q even, then for all $k\in\{0...q-1\}$ we get :*

$$if\ \frac{q}{2}\ is\ even\ then\ \ |d_k(q,p,q)|^2=\begin{cases}\frac{2}{q}\ \ if\ k\ is\ even\\ \\ 0\ \ else;\end{cases}$$

$$if\ \frac{q}{2}\ is\ odd\ then\ \ |d_k(q,p,q)|^2=\begin{cases}0\ \ if\ k\ is\ pair\\ \\ \frac{2}{q}\ else.\end{cases}$$

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