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The gauge theory of dislocations: static solutions of screw and edge dislocations

Markus Lazar*, Charalampos Anastassiadis,

Emmy Noether Research Group,
Department of Physics,
Darmstadt University of Technology,
Hochschulstr. 6,
D-64289 Darmstadt, Germany

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Abstract

We investigate the $T(3)$-gauge theory of static dislocations in continuous solids. We use the most general linear constitutive relations bilinear in the elastic distortion tensor and dislocation density tensor for the force and pseudomoment stresses of an isotropic solid. The constitutive relations contain six material parameters. In this theory both the force and pseudomoment stresses are asymmetric. The theory possesses four characteristic lengths ℓ_1, ℓ_2, ℓ_3 and ℓ_4 which are given explicitely. We first derive the three-dimensional Green tensor of the master equation for the force stresses in the translational gauge theory of dislocations. We then investigate the situation of generalized plane strain (anti-plane strain and plane strain). Using the stress function method, we find modified stress functions for screw and edge dislocations. The solution of the screw dislocation is given in terms of one independent length $\ell_1 = \ell_4$. For the problem of an edge dislocation, only two characteristic lengths ℓ_2 and ℓ_3 arise with one of them being the same $\ell_2 = \ell_1$ as for the screw dislocation. Thus, this theory possesses only two independent lengths for generalized plane strain. If the two lengths ℓ_2 and ℓ_3 of an edge dislocation are equal, we obtain an edge dislocation which is the gauge theoretical version of a modified Volterra edge dislocation. In the case of symmetric stresses we recover well known results obtained earlier.

Keywords: Gauge theory; dislocations; defects; incompatible elasticity, torsion, Green tensor.

*E-mail: lazar@fkp.tu-darmstadt.de (M. Lazar).

1 Introduction

1.1 Reviewing remarks

Gauge theories are very successful in high energy physics to understand and to explain fundamental interactions (see, e.g., [1, 2]). Important solutions in gauge theories are topological strings like the so-called Abrikosov-Nielson-Olesen string [3, 4]. It is also known that the theories of gravity and generalized gravity can be understood as gauge theories of the Poincaré group and the affine group, respectively [5, 6]. For example, the teleparallel formulation of Einstein's general relativity can be understood as a gauge theory of the four-dimensional translation group (see, e.g., Hehl [7]). Thus, the torsion tensor is not just a tensor, but rather a very special tensor that is related to the translation group [6]. In such theories mathematical solutions for cosmic strings and other topological defects in space-time are found [8]. But it is hard to observe them by experiment.

An application of the three-dimensional translational gauge theory is the gauge theoretical formulation of dislocations. The torsion tensor plays the role as the translational field strength. In such a theory dislocations appear quite naturally as topological defects (strings). Dislocations play an important role in solid state physics [9]. It is easy to investigate them experimentally. Therefore, the gauge theory of dislocations is the best example to investigate the translational gauge theory as a physical field theory. Pioneering work in the field of the translational gauge theory of defects in solids was made by Kadić and Edelen [10, 11], Edelen and Lagoudas [12]. It is well known that dislocations are the fundamental carrier of plasticity. The hope of the gauge theory of dislocations is to understand plasticity as a 'fundamental' interaction in solids. Materials display strong size effects when the characteristic length scales are of the order of micrometers down to a few nanometers. With the interest in miniaturization of devices, the length scales associated with nanodevices are very small and classical theories of elasticity and plasticity are not applicable. Classical elasticity and plasticity theories cannot explain the size dependence because their constitutive laws possess no internal material lengths. In metals, the physical origin of size effects can be related to dislocations. Gauge theory of dislocations may be used to explain size effects and to model mechanical properties of miniaturized devices. In the gauge theory of dislocations the dislocation density tensor which is the curl of the elastic distortion or of the negative plastic distortion multiplied by material parameters which define internal material lengths enter the constitutive relations. The gauge theory of dislocations shall be a step in the direction of a microscopic theory of plasticity that incorporates interaction of structural defects. The dislocation should play a similar role as a gauge boson like the photon as a gauge boson in Maxwell's theory. Nevertheless, the theory of Kadić and Edelen [11], Edelen and Lagoudas [12] has some lacks. They used very special constitutive relations. For example, the force stress tensor is symmetric. But in the continuum theory of dislocations it is known that the force stress tensor has to be asymmetric at least in the dislocation core region [13, 14] and that moment stress is the specific response to dislocations [15]. Such a moment stress cannot be calculated from elastic moduli only. A realistic constitutive law between the moment stress and the dislocation density tensor has to be used to built a physical dislocation theory.

Therefore, such a theory is not the most general isotropic dislocation gauge theory. Their theory is just able to give an acceptable solution of a screw dislocation after some ad hoc assumptions [16, 17]. Moreover, it is not possible to give a correct solution of an edge dislocation. Especially, the stress component σ_{zz} is incorrect and the condition of plane

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strain is not satisfied [18]. Such a dislocation gauge theory with symmetric force stresses possesses only one additional material parameter and two length scales can be defined to describe size-effects. In order to explain bending and torsion experiments one needs at least two length scales. One question that arises is how many characteristic length scales should appear in an improved gauge theory of dislocations? Recently, Sharma and Ganti [19] have combined Edelen's gauge model with a simple strain gradient elasticity. These authors applied it to a screw dislocation.

These drawbacks of the Edelen model have been a motivation for further investigations [18, 20]. The so-called Einstein choice has been investigated by Malyshev [18] and Lazar [21, 22]. Their solution of a screw dislocation looks to be acceptable for theories with symmetric force stresses. In the solution of an edge dislocation found by Malyshev [18], a strange far field emerges, which shows an oscillating behavior. In addition, the condition of plane strain is not fulfilled. Later, Lazar [23] tried to find a more realistic gauge theoretical solution for an edge dislocation. He used another constitutive relation and it was necessary to introduce additional gradients of the elastic rotation which is a little sophisticated. Thus, up to now, the case of an edge dislocation has not been treated satisfactory in the gauge theoretical framework. Quite recently, Malyshev [24] has used the Einsteinian $T(3)$-gauge approach to calculate the stress tensor of a screw dislocation in the second order. Some other proposals are given by Katanaev and Volovich [25], Popov and Kröner [26], Kröner [27]. We want to note that all these kinds of gauge theories of dislocations use symmetric force stresses and some special choices for the constitutive law of the moment (or hyper) stresses.

1.2 The subject of the paper

In this paper we want to investigate the translational gauge theory of dislocations with the most general linear isotropic constitutive relations quadratic in the physical state quantities for the force and pseudomoment stress tensors with six material coefficients. Such a theory was originally proposed by Lazar [21]. However, Lazar [21] used a simplified version for dislocations. Our purpose is to derive the general framework of such a theory and to apply it to find new analytical solutions of straight screw and edge dislocations in an infinite medium. The paper consists of five sections. Section 1 is the introduction. In section 2 we give the basics of the translational gauge theory of dislocations. We give the relation for positive definiteness of the elastic energy and the dislocation core energy. We give the relation between the pseudomoment stress tensor and the moment stress tensor. We derive the Euler-Lagrange equation in terms of the asymmetric stress tensor. This is the central equation for an isotropic material with dislocations. We set up the relation to other gauge theories of dislocations proposed earlier. We calculate the three-dimensional Green tensor for the central equation of the asymmetric force stresses. In this manner we are able to introduce four characteristic length scales of the gauge theory with asymmetric force stresses. Later we investigate dislocations in the framework of generalized plane strain. Sections 3 and 4 present the gauge theoretical solutions of straight screw and edge dislocations in an infinite medium with asymmetric force stresses. We will calculate the elastic distortion and dislocation density (torsion) tensors and the force and pseudomoment stresses of screw and edge dislocations. We prove that the force stresses of a screw dislocation and an edge dislocation have to be asymmetric instead of symmetric in order to guarantee nonnegative energy. Section 5 concludes the paper.

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Details of dislocations in asymmetric elasticity are given in the appendices A and B.

2 Gauge theory of dislocations

2.1 Foundations

In the translational gauge theory of dislocations the displacement vector u_i is defined up to a gauge transformation – a local translation $\tau_i(x)$

$$u_i \to u_i + \tau_i(x) \tag{2.1}$$

and the translational gauge field ϕ_{ij} transforming under the local translation in a suitable form

$$\phi_{ij} \to \phi_{ij} - \tau_{i,j}(x). \tag{2.2}$$

From the mathematical point of view, the gauge field ϕ_{ij} is the translational part of the generalized affine connection [20]. In the translational gauge theory, the elastic distortion is defined by the translational gauge-covariant derivative according to

$$\beta_{ij} = \nabla_j u_i := u_{i,j} + \phi_{ij}. \tag{2.3}$$

Thus, the elastic distortion is gauge independent and, therefore, a physical state quantity. The asymmetric tensor ϕ_{ij} may be identified as the negative plastic distortion tensor [20, 21]. The elastic strain energy is a function of the elastic distortion. It reads

$$W_{\rm el} = \frac{1}{2}\,\beta_{ij}\sigma_{ij} \tag{2.4}$$

with the specific response

$$\sigma_{ij} = \frac{\partial W_{\rm el}}{\partial \beta_{ij}}. \tag{2.5}$$

It is the analog of the Piola-Kirchhoff stress tensor. It is an asymmetric stress tensor. The most general isotropic constitutive relation has three material coefficients. Thus, the asymmetric force stress tensor has the form[1]

$$\sigma_{ij} = \lambda\,\delta_{ij}\beta_{kk} + 2\mu\,\beta_{(ij)} + 2\gamma\,\beta_{[ij]}. \tag{2.6}$$

Here μ and λ are the Lamé coefficients. The coefficient γ is an additional material parameter due to the skew-symmetric part of the elastic distortion (the elastic rotation). Thus, γ is the module of rotation (see also Kröner [28]). The skew-symmetric stress $\sigma_{[ij]}$ is caused by the (local) elastic distortion $\beta_{[ij]}$. If $\gamma = 0$ in Eq. (2.6), we obtain the Hooke law with $\sigma_{[ij]} = 0$. Thus, in this case the elastic rotation $\beta_{[ij]}$ is undetermined. The positive semi-definiteness of the elastic distortion energy $W_{\rm el} \geq 0$ requires the restriction

$$\mu \geq 0, \qquad \gamma \geq 0, \qquad 2\mu + 3\lambda \geq 0. \tag{2.7}$$

[1]We are using the notations $A_{(ij)} \equiv \frac{1}{2}(A_{ij} + A_{ji})$ and $A_{[ij]} \equiv \frac{1}{2}(A_{ij} - A_{ji})$.

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Another physical state quantity is the translational gauge field strength – the torsion tensor

$$T_{ijk} = \phi_{ik,j} - \phi_{ij,k}, \qquad T_{ijk} = -T_{ikj} \tag{2.8}$$

and

$$T_{ijk} = \beta_{ik,j} - \beta_{ij,k}. \tag{2.9}$$

It has the physical meaning of the dislocation density tensor

$$\alpha_{ij} = \frac{1}{2}\,\epsilon_{jkl} T_{ikl} = \epsilon_{jkl}\beta_{il,k}. \tag{2.10}$$

They fulfill the translational Bianchi identity

$$\epsilon_{jkl} T_{ijk,l} = 0, \qquad \alpha_{ij,j} = 0. \tag{2.11}$$

Thus, the physical state quantities are the elastic distortion, β_{ij}, and the dislocation density tensor T_{ijk}.

The dislocation core energy contains only the translational gauge field strength and is given by

$$W_{\rm disl} = \frac{1}{4}\, T_{ijk} H_{ijk} \tag{2.12}$$

with the translational gauge field momentum tensor

$$H_{ijk} = 2\frac{\partial W_{\rm disl}}{\partial T_{ijk}}, \qquad H_{ijk} = -H_{ikj}. \tag{2.13}$$

The translational gauge field momentum tensor H_{ijk} is a very special hyperstress tensor. It is the response quantity to the dislocation density tensor and has only nine independent components. It is convenient to perform an irreducible decomposition of the torsion, from which we can construct the most general linear and isotropic constitutive law between the dislocation density and the hyperstress in the following way [20, 21]

$$H_{ijk} = \sum_{I=1}^{3} a_I\, {}^{(I)}T_{ijk}. \tag{2.14}$$

Here a_1, a_2 and a_3 are nonnegative material coefficients due to the positive semi-definiteness if the dislocation core energy $W_{\rm disl} \geq 0$. They have the dimension of a force. The irreducible decomposition of the torsion under the rotation group $SO(3)$, with the numbers of independent components $9 = 5 \oplus 3 \oplus 1$, is given by

$$T_{ijk} = {}^{(1)}T_{ijk} + {}^{(2)}T_{ijk} + {}^{(3)}T_{ijk}. \tag{2.15}$$

The tensor, the trace and the axial tensor pieces are defined by (see also [6, 21])

$${}^{(1)}T_{ijk} = T_{ijk} - {}^{(2)}T_{ijk} - {}^{(3)}T_{ijk} \qquad \text{(tentor)} \tag{2.16}$$

$${}^{(2)}T_{ijk} := \frac{1}{2}\,(\delta_{ij} T_{llk} + \delta_{ik} T_{ljl}) \qquad \text{(trator)} \tag{2.17}$$

$${}^{(3)}T_{ijk} := \frac{1}{3}\,(T_{ijk} + T_{jki} + T_{kij}) \qquad \text{(axitor)}. \tag{2.18}$$

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The hyperstress tensor may be written as

$$H_{ijk} = c_1\, T_{ijk} + c_2\,\big(T_{jki} + T_{kij}\big) + c_3\,\big(\delta_{ij}T_{llk} + \delta_{ik}T_{ljl}\big) \tag{2.19}$$

with the abbreviations

$$c_1 := \frac{1}{3}\big(2a_1 + a_3\big), \qquad c_2 := \frac{1}{3}\big(a_3 - a_1\big), \qquad c_3 := \frac{1}{2}\big(a_2 - a_1\big). \tag{2.20}$$

The inverse relations are

$$a_1 = c_1 - c_2 \geq 0, \qquad a_2 = c_1 - c_2 + 2c_3 \geq 0, \qquad a_3 = 2c_2 + c_1 \geq 0. \tag{2.21}$$

Since β_{ij} and T_{ijk} are uncoupled from each other, the conditions of positive semi-definiteness, $W = W_{\text{el}} + W_{\text{disl}} \geq 0$, can be studied separately: $W_{\text{el}} \geq 0$ and $W_{\text{disl}} \geq 0$. Thus, the characteristic constants of the material have to satisfy the conditions (2.7) and (2.21).

With the relations $H_{im} = \frac{1}{2}\epsilon_{jkm}H_{ijk}$ and $T_{ijk} = \epsilon_{jkn}\alpha_{in}$, we obtain from Eq. (2.14) the irreducible decomposition

$$H_{ij} = a_1\Big(\underbrace{\alpha_{(ij)} - \frac{1}{3}\,\delta_{ij}\alpha_{kk}}_{\text{tentor: } {}^{(1)}\alpha_{ij}}\Big) + a_2 \underbrace{\alpha_{[ij]}}_{\text{trator: } {}^{(2)}\alpha_{ij}} + a_3 \underbrace{\frac{1}{3}\,\delta_{ij}\alpha_{kk}}_{\text{axitor: } {}^{(3)}\alpha_{ij}}. \tag{2.22}$$

Thus, the axitor represents a set of three perpendicular forests of screw dislocations of equal strength. The trator describes forests of edge dislocations: $\alpha_{xy} = -\alpha_{yx}$, $\alpha_{xz} = -\alpha_{zx}$ and $\alpha_{yz} = -\alpha_{zy}$. The tentor is the deviator of the dislocation density tensor and it represents forests of edge dislocations: $\alpha_{xy} = \alpha_{yx}$, $\alpha_{xz} = \alpha_{zx}$ and $\alpha_{yz} = \alpha_{zy}$ and screw dislocations: ${}^{(1)}\alpha_{xx} = \alpha_{xx} - \frac{1}{3}(\alpha_{xx} + \alpha_{yy} + \alpha_{zz})$, ${}^{(1)}\alpha_{yy} = \alpha_{yy} - \frac{1}{3}(\alpha_{xx} + \alpha_{yy} + \alpha_{zz})$, ${}^{(1)}\alpha_{zz} = \alpha_{zz} - \frac{1}{3}(\alpha_{xx} + \alpha_{yy} + \alpha_{zz})$ with ${}^{(1)}\alpha_{xx} + {}^{(1)}\alpha_{yy} + {}^{(1)}\alpha_{zz} = 0$. A similar constitutive relation was given earlier by Hehl and Kröner [13], Kröner [14], Katanaev and Volovich [25], Popov and Kröner [26]. Alternatively, the dislocation core energy (2.12) may be rewritten in the form

$$W_{\text{disl}} = \frac{1}{2}\,\alpha_{ij}H_{ij}. \tag{2.23}$$

We want to note that a similar hyperstress tensor like (2.22) in terms of the dislocation density tensor and with three material coefficients was derived by Eringen and Claus [29] in a special case of micromorphic elasticity. Therefore, the static gauge theory of dislocations is related to a micromorphic theory and not to a Cosserat theory if we identify the gauge field ϕ_{ij} with the micro-distortion tensor of micromorphic elasticity [30, 31]. In general, a hyperstress tensor possesses 27 independent components and is the response quantity to the gradient of the micro-distortion tensor. The isotropic constitutive relation of such hyperstress tensor has 11 material parameters (see, e.g., Mindlin [30]).

Since the tensor $H_{i[jk]}$ has only nine independent components like a couple stress tensor, the following question arises: In which way is the tensor H_{ijk} related to the couple stress tensor? The contortion tensor is defined in terms of the torsion tensor according to [32, 33, 34]

$$K_{ijk} = -\frac{1}{2}\big(T_{ijk} + T_{jki} - T_{kij}\big), \qquad K_{ijk} = -K_{jik}. \tag{2.24}$$

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A moment stress tensor possesses nine independent components and for an isotropic material the constitutive relations has just three material parameters. The contortion tensor contains nine independent rotational degrees of freedom. Obviously, K_{ijk} is antisymmetric in i, j, whereas T_{ijk} is so in j, k. With the identity $K_{lk} = \frac{1}{2}\epsilon_{ijl}K_{ijk}$, we recover the so-called Nye curvature tensor [35] (see also [36, 37, 38, 39])[2]

$$K_{ij} = \alpha_{ji} - \frac{1}{2}\,\delta_{ij}\alpha_{ll}, \qquad \alpha_{ij} = K_{ji} - \delta_{ij}K_{ll}. \tag{2.25}$$

The moment stress (or spin moment stress tensor) is the response of the crystal to the contortion

$$\tau_{ijk} = \frac{\partial W}{\partial K_{ijk}}, \qquad \tau_{ijk} = -\tau_{jik}. \tag{2.26}$$

Thus, contortion produces moment stress and vise versa. We can rewrite the tensor H_{ijk} in terms of the moment stress tensor τ_{ijk} as follows

$$H_{ijk} = -\tau_{ijk} + \tau_{jki} - \tau_{kij} \tag{2.27}$$

$$H_{[ij]k} = -\tau_{ijk}. \tag{2.28}$$

Therefore, the tensor H_{ijk} is given in terms of moment stresses similar like the torsion in terms of the contortion. Such a tensor $H_{i[jk]}$ is called pseudomoment stress tensor or sometimes spin energy potential tensor [41, 42, 5, 6]. In addition, we have

$$H_{ij} = \tau_{ji} - \frac{1}{2}\,\delta_{ij}\tau_{ll}, \qquad \tau_{ij} = H_{ji} - \delta_{ij}H_{ll}, \tag{2.29}$$

where $\tau_{ij} = \partial W/\partial K_{ij}$. We conclude, torsion produces pseudomoment stress and vise versa. Thus, pseudomoment stress is the specific response to dislocations. In the following we will use the expression pseudomoment stress tensor for H_{ijk}.

It is known that nontrivial traction boundary problems in the variational formulation of the gauge theory of defects can be formulated by means of a so-called null Lagrangian [12, 17, 43]. When the null Lagrangian is added to the Lagrangian of elasticity, it does not change the 'classical' Euler-Lagrange equation (force equilibrium in elasticity theory) because the associated Euler-Lagrange equation, $\sigma^0_{ij,j} = 0$, is identically satisfied. After minimal replacement, $u_{i,j} \to \beta_{ij}$, the null Lagrangian

$$W_{\rm bg} = \partial_j\big(\sigma^0_{ij}u_i\big) = \sigma^0_{ij,j}u_i + \sigma^0_{ij}u_{i,j} \longrightarrow \sigma^0_{ij}\beta_{ij}, \tag{2.30}$$

gives rise to a background stress tensor σ^0_{ij} which can be considered as the nucleation field in the gauge theory of defects [17].

The Euler-Lagrange equations of $W = W_{\rm disl} + W_{\rm el} - W_{\rm bg}$ are given by

$$\frac{\delta W}{\delta \beta_{ij}} \equiv \frac{\partial W}{\partial \beta_{ij}} - \partial_k \frac{\partial W}{\partial \beta_{ij,k}} = 0 \tag{2.31}$$

which can be expressed with Eqs. (2.5) and (2.13) as

$$H_{ijk,k} + \sigma_{ij} = \sigma^0_{ij} \qquad \text{(pseudomoment equilibrium)}. \tag{2.32}$$

[2]We want to mention that Nye [35] derived originally the Eq. (2.25) if the elastic strain is zero (see also discussion in Li *et al.* [40]). Here we have used the differential geometrical definition (2.24) of the contortion in order to derive (2.25). For non-zero elastic strain Eq. (2.25) was derived by deWit [38].

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Thus, the null Lagrangian gives a contribution to the pseudomoment equilibrium condition. Because the pseudomoment stress H_{ijk} is skewsymmetric in the indices j and k, the force equilibrium follows from Eq. (2.32) as a continuity equation

$$\sigma_{ij,j} = 0 \qquad \text{(force equilibrium)}. \tag{2.33}$$

We want to note that a similar equation like (2.32) has been earlier obtained by Trzęsowski and Sławianowski [44], Kröner [45] from a variational principle in nonlinear dislocation theory. We may rewrite Eq. (2.32) as follows

$$\epsilon_{kjn} H_{in,k} = \sigma_{ij} - \sigma^0_{ij} \tag{2.34}$$

in order to see that the pseudomoment stress tensor H_{in} is a kind of a stress function tensor of first order [46]. Eq. (2.32) may be decomposed into the symmetric and antisymmetric parts

$$H_{(ij)k,k} + \sigma_{(ij)} = \sigma^0_{(ij)} \tag{2.35}$$

$$H_{[ij]k,k} + \sigma_{[ij]} = \sigma^0_{[ij]} \qquad \text{(moment equilibrium)}. \tag{2.36}$$

Hence, Eq. (2.36) has an appearance close to Cosserat's moment of momentum equation and $H_{[ij]k}$ has the meaning of a moment stress tensor. The divergence of the moment stress tensor gives rise to asymmetric force stresses. The existence of moment stresses $H_{[ij]k}$ demands asymmetric force stresses $\sigma_{[ij]}$. However, due to Eq. (2.35) the equilibrium equations of dislocation theory deviate distinctly from those of the Cosserat theory (see also Kröner [45]). Using Eqs. (2.9) and (2.19), we can rewrite the moment equilibrium (2.32) in the following form

$$\begin{aligned} c_1(\beta_{ik,jk} - \beta_{ij,kk}) + c_2(\beta_{ji,kk} - \beta_{jk,ki} + \beta_{kj,ki} - \beta_{ki,kj}) \\ + c_3\big[\delta_{ij}(\beta_{lk,kl} - \beta_{ll,kk}) + \beta_{ll,ji} - \beta_{lj,li}\big] + \sigma_{ij} = \sigma^0_{ij}. \end{aligned} \tag{2.37}$$

With the inverse constitutive relation for β_{ij}

$$\beta_{ij} = \frac{\gamma+\mu}{4\mu\gamma}\,\sigma_{ij} + \frac{\gamma-\mu}{4\mu\gamma}\,\sigma_{ji} - \frac{\nu}{2\mu(1+\nu)}\delta_{ij}\,\sigma_{kk} \tag{2.38}$$

where the Poisson ration ν is expressed in terms of the Lamé coefficients

$$\nu = \frac{\lambda}{2\,(\lambda+\mu)}, \qquad \lambda = \frac{2\mu\nu}{1-2\nu} \tag{2.39}$$

we are able to rewrite completely the pseudomoment equilibrium condition (2.37) in terms of the force stress tensor according to

$$\begin{aligned} &\Big[(c_1 - c_2 + 2c_3)\,\frac{2\gamma\nu}{1+\nu} - 2c_3\gamma\Big](\delta_{ij}\,\sigma_{ll,kk} - \sigma_{ll,ij}) + 2c_3\gamma\,\delta_{ij}\,\sigma_{kl,kl} \\ &+ \big[c_1(\gamma+\mu) - c_2(\gamma-\mu)\big](\sigma_{ik,jk} - \sigma_{ij,kk}) + \big[c_1(\gamma-\mu) - c_2(\gamma+\mu)\big](\sigma_{ki,kj} - \sigma_{ji,kk}) \\ &+ \big[2c_2\mu - c_3(\gamma+\mu)\big]\,\sigma_{kj,ki} - \big[2c_2\mu + c_3(\gamma-\mu)\big]\,\sigma_{jk,ki} + 4\mu\gamma\,\sigma_{ij} = 4\mu\gamma\,\sigma^0_{ij}. \end{aligned} \tag{2.40}$$

If we use the force equilibrium condition $\sigma_{ij,j} = 0$, Eq. (2.40) simplifies to

$$\begin{aligned} &\Big[(c_1 - c_2 + 2c_3)\,\frac{2\gamma\nu}{1+\nu} - 2c_3\gamma\Big](\delta_{ij}\,\sigma_{ll,kk} - \sigma_{ll,ij}) - \big[c_1(\gamma+\mu) - c_2(\gamma-\mu)\big]\sigma_{ij,kk} \\ &+ \big[c_1(\gamma-\mu) - c_2(\gamma+\mu)\big](\sigma_{ki,kj} - \sigma_{ji,kk}) + \big[2c_2\mu - c_3(\gamma+\mu)\big]\,\sigma_{kj,ki} + 4\mu\gamma\,\sigma_{ij} = 4\mu\gamma\,\sigma^0_{ij}. \end{aligned} \tag{2.41}$$

Equation (2.41) is the master equation, being the central equation for the force stress tensor in this gauge theory of dislocations. In the next two sections, solutions for screw and edge dislocations will be given.

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2.2 Comparison to other gauge theories of dislocations

At this point we want to discuss the relation to other translational gauge theories of dislocations which differ in the form of the constitutive relations of the force stress and pseudomoment stress. The differences between the gauge theories of dislocations are mainly the choice of the constitutive relations of the force and pseudomoment stresses. Edelen [11, 12, 17] used a very simple constitutive relation for the pseudomoment stress tensor. The Edelen choice is $c_1 = 2s$, $c_2 = 0$ and $c_3 = 0$. Of course, this is not a general isotropic constitutive law. In addition he assumed symmetric force stresses. Also Valsakumar and Sahoo [16] used this choice for the constitutive relations. Malyshev [18] and Lazar [21, 22] discussed and used the so-called Einstein choice in three dimensions. It is given by: $a_2 = -a_1$ and $a_3 = -a_1/2$. Thus, $c_1 = a_1/2$, $c_2 = -a_1/2$ and $c_3 = -a_1$. In the Einstein choice the pseudomoment stress tensor is symmetric in the first two indices $H_{ijk} = H_{(ij)k}$ and is given in terms of 'gradients' of the elastic strain $\beta_{(ij)}$. In this choice, $H_{(ij)k,k}$ reduces to the Einstein tensor. Another choice $c_1 = a_1/2$, $c_2 = -a_1/2$ and $c_3 = \nu/(1-\nu)a_1$ was used by Lazar [23] to calculate symmetric force stresses of screw and edge dislocations. It can be seen in Eq. (2.21) that the Einstein-choice and also the latter choice violate the condition of nonnegative energy $W_{\text{disl}} \geq 0$. The material coefficients a_1, a_2 and a_3 have to be nonnegative. But in this paper we use the most general linear constitutive relations (2.6) and (2.19) of an isotropic material with six material parameters. In Table 1 we have listed the choice of the constitutive relations.

	Edelen [10]	Einstein choice [18, 21]	Lazar [23]	present paper
μ	μ	μ	μ	μ
ν	ν	ν	ν	ν
γ	0	0	0	γ
a_1	$2s$	a_1	a_1	a_1
a_2	$2s$	$-a_1$	$\frac{1+\nu}{1-\nu}a_1$	a_2
a_3	$2s$	$-\frac{a_1}{2}$	$-\frac{a_1}{2}$	a_3

Table 1: This table lists the material parameters a_I, μ, ν and γ for the translational gauge theory of dislocations.

2.3 The Green tensor

In this subsection, we want to derive the three-dimensional Green tensor of the master equation (2.41). First we set $\sigma^0_{ij} = L_{ij}\delta(x)\delta(y)\delta(z)$. The fundamental solution of Eq. (2.41) is defined by the equation

$$\begin{aligned}\Big[(c_1 - c_2 + 2c_3)\,\frac{2\gamma\nu}{1+\nu} - 2c_3\gamma\Big](\delta_{ij}\,\sigma_{ll,kk} - \sigma_{ll,ij}) - \big[c_1(\gamma+\mu) - c_2(\gamma-\mu)\big]\sigma_{ij,kk} \qquad (2.42)\\ + \big[c_1(\gamma-\mu) - c_2(\gamma+\mu)\big](\sigma_{ki,kj} - \sigma_{ji,kk}) + \big[2c_2\mu - c_3(\gamma+\mu)\big]\,\sigma_{kj,ki}\\ + 4\mu\gamma\,\sigma_{ij} = 4\mu\gamma\,L_{ij}\,\delta(x)\delta(y)\delta(z).\end{aligned}$$

We use the notation for the three-dimensional Fourier transform [47, 48]

$$\widetilde{f}(\boldsymbol{q}) \equiv \mathcal{F}_{(3)}\big[f(\boldsymbol{r})\big] = \int_{-\infty}^{\infty} f(\boldsymbol{r})\,\mathrm{e}^{+\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{r}, \qquad f(\boldsymbol{r}) \equiv \mathcal{F}^{-1}_{(3)}\big[\widetilde{f}(\boldsymbol{q})\big] = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} \widetilde{f}(\boldsymbol{q})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}}\mathrm{d}\boldsymbol{q}. \qquad (2.43)$$

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Applying the Fourier transform, it follows that

$$\Big[(c_1 - c_2 + 2c_3)\,\frac{2\gamma\nu}{1+\nu} - 2c_3\gamma\Big](\delta_{ij}\,q^2\widetilde{\sigma}_{ll} - q_iq_j\widetilde{\sigma}_{ll}) - \big[c_1(\gamma+\mu) - c_2(\gamma-\mu)\big]q^2\widetilde{\sigma}_{ij} \quad (2.44)$$
$$+ \big[c_1(\gamma-\mu) - c_2(\gamma+\mu)\big](q_kq_j\widetilde{\sigma}_{ki} - q^2\widetilde{\sigma}_{ji}) + \big[2c_2\mu - c_3(\gamma+\mu)\big]\,q_kq_i\widetilde{\sigma}_{kj}$$
$$- 4\mu\gamma\,\widetilde{\sigma}_{ij} = -4\mu\gamma\,L_{ij}.$$

The force equilibrium (2.33) reads in the $\boldsymbol{q}$-space: $\widetilde{\sigma}_{ij}q_j = 0$. Symmetrization, antisymmetrization, taking the trace and the inner product by $\boldsymbol{q}$ from left of Eq. (2.44) we deduce

$$[1+\ell_1^2q^2]\widetilde{\sigma}_{(ij)} - \frac{1}{2}\,(\ell_1^2-\ell_2^2)[\delta_{ij}q^2 - q_iq_j]\widetilde{\sigma}_{ll} - (\ell_1^2-\ell_3^2)q_k\widetilde{\sigma}_{k(i}q_{j)} = L_{(ij)} \quad (2.45)$$
$$[1+\ell_4^2q^2]\widetilde{\sigma}_{[ij]} + (\ell_4^2-\ell_3^2)q_k\widetilde{\sigma}_{k[i}q_{j]} = L_{[ij]} \quad (2.46)$$
$$[1+\ell_2^2q^2]\widetilde{\sigma}_{ll} = L_{ll} \quad (2.47)$$
$$[1+\ell_3^2q^2]q_i\widetilde{\sigma}_{ij} = q_iL_{ij} \quad (2.48)$$

where the characteristic length scales are defined by

$$\ell_1^2 = \frac{c_1-c_2}{2\mu} = \frac{a_1}{2\mu} \quad (2.49)$$
$$\ell_2^2 = \frac{(1-\nu)(c_1-c_2+2c_3)}{2\mu(1+\nu)} = \frac{(1-\nu)a_2}{2\mu(1+\nu)} \quad (2.50)$$
$$\ell_3^2 = \frac{(\mu+\gamma)(c_1-c_2+c_3)}{4\mu\gamma} = \frac{(\mu+\gamma)(a_1+a_2)}{8\mu\gamma} \quad (2.51)$$
$$\ell_4^2 = \frac{c_1+c_2}{2\gamma} = \frac{a_1+2a_3}{6\gamma}. \quad (2.52)$$

The lengths ℓ_1, ℓ_2 and ℓ_3 fulfill the following relation

$$\ell_3^2 = \frac{\mu+\gamma}{4\gamma}\Big(\ell_1^2 + \frac{1+\nu}{1-\nu}\,\ell_2^2\Big). \quad (2.53)$$

Now some remarks are in order. In the static gauge theory of dislocations we have found four characteristic length scales. These length scales may describe size effects produced by dislocations. They are important in the plastic zone namely the dislocation core. All four length scales depend on the six material parameters of an isotropic material. Thus, they describe the material property on which the influence of the pseudomoment stresses depends strongly. If the ratio of the smallest dimension of the body to the length scales is large, the effects of pseudomoment stresses are negligible. However, when this ratio is not large, pseudomoment stresses may produce effects of appreciable magnitude. This is the case in the dislocation core region. It is interesting to note that the internal length ℓ_1 depends on μ and has a similar form as the internal length in the couple stress theory [49, 50] and in gradient elasticity [30]. It can be seen that ℓ_2 depends on the Poisson number ν. This length ℓ_2 is the characteristic length of dilatation. Such a length appears also in gradient elasticity [30]. Because the characteristic lengths ℓ_3 and ℓ_4 depend on γ they look like the two characteristic lengths of micropolar elasticity namely the characteristic lengths for bending and for torsion (see e.g. [51]). Thus, it depends on the physical problem which length scale is of importance. It is clear that for $\gamma \to \infty$ just

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the bending length, which is the characteristic length in the theory of couple stresses, survives

$$\lim_{\gamma\to\infty}\ell_3^2=\frac{c_1-c_2+c_3}{4\mu}=\frac{a_1+a_2}{8\mu},\qquad \lim_{\gamma\to\infty}\ell_4^2=0. \tag{2.54}$$

On the other hand, if $\gamma=0$, then $\ell_3\to\infty$ and $\ell_4\to\infty$. For the Einstein choice $a_2=-a_1$ and $a_3=-a_1/2$ the lengths convert to

$$\ell_1^2=\frac{c_1}{\mu}=\frac{a_1}{2\mu},\qquad \ell_2^2=-\frac{(1-\nu)c_1}{\mu(1+\nu)}=-\frac{(1-\nu)a_1}{2\mu(1+\nu)},\qquad \ell_3^2=0,\qquad \ell_4^2=0. \tag{2.55}$$

Thus, only two lengths survive. These length scales ℓ_1^2 and ℓ_2^2 agree with $\mathcal{M}^{-2}$ and $-\mathcal{N}^{-2}$ used by Malyshev [18]. In addition, four length scales survive in the Edelen choice together with asymmetric force stresses

$$\ell_1^2=\frac{s}{\mu},\qquad \ell_2^2=\frac{(1-\nu)s}{\mu(1+\nu)},\qquad \ell_3^2=\frac{(\mu+\gamma)s}{2\mu\gamma},\qquad \ell_4^2=\frac{s}{\gamma}. \tag{2.56}$$

The length scales ℓ_1^2 and ℓ_2^2 coincide with M^{-2} and N^{-2} introduced by Kadić and Edelen [10], Edelen [52]. If $\gamma=0$, then $\ell_3\to\infty$ and $\ell_4\to\infty$.

From Eqs. (2.45)–(2.48) we obtain

$$\widetilde{\sigma}_{ll}=\frac{L_{ll}}{1+\ell_2^2q^2} \tag{2.57}$$

$$q_i\widetilde{\sigma}_{ij}=\frac{q_iL_{ij}}{1+\ell_3^2q^2} \tag{2.58}$$

and

$$\widetilde{\sigma}_{(ij)}=\frac{L_{(ij)}}{1+\ell_1^2q^2}+\frac{1}{2}\,(\ell_1^2-\ell_2^2)\frac{[\delta_{ij}q^2-q_jq_j]L_{ll}}{(1+\ell_1^2q^2)(1+\ell_2^2q^2)}+(\ell_1^2-\ell_3^2)\frac{q_kL_{k(i}q_{j)}}{(1+\ell_1^2q^2)(1+\ell_3^2q^2)} \tag{2.59}$$

$$\widetilde{\sigma}_{[ij]}=\frac{L_{[ij]}}{1+\ell_4^2q^2}-(\ell_4^2-\ell_3^2)\frac{q_kL_{k[i}q_{j]}}{(1+\ell_4^2q^2)(1+\ell_3^2q^2)}. \tag{2.60}$$

Using the inverse Fourier transformed function [47, 48]

$$\mathcal{F}_{(3)}^{-1}\Big[\frac{1}{q^2+\frac{1}{\ell^2}}\Big]=\frac{1}{4\pi r}\,\mathrm{e}^{-r/\ell},\qquad r=\sqrt{x^2+y^2+z^2} \tag{2.61}$$

we find

$$\sigma_{(ij)}=\frac{L_{(ij)}}{4\pi r\ell_1^2}\,\mathrm{e}^{-r/\ell_1}-\frac{L_{kk}}{8\pi}\,[\delta_{ij}\Delta-\partial_i\partial_j]\Big[\frac{1}{r}\big(\mathrm{e}^{-r/\ell_1}-\mathrm{e}^{-r/\ell_2}\big)\Big]-\frac{1}{4\pi}\,L_{k(i}\partial_{j)}\partial_k\Big[\frac{1}{r}\big(\mathrm{e}^{-r/\ell_1}-\mathrm{e}^{-r/\ell_3}\big)\Big] \tag{2.62}$$

$$\sigma_{[ij]}=\frac{L_{[ij]}}{4\pi r\ell_4^2}\,\mathrm{e}^{-r/\ell_4}+\frac{1}{4\pi}\,L_{k[i}\partial_{j]}\partial_k\Big[\frac{1}{r}\big(\mathrm{e}^{-r/\ell_4}-\mathrm{e}^{-r/\ell_3}\big)\Big]. \tag{2.63}$$

With $\sigma_{ij}=\sigma_{(ij)}+\sigma_{[ij]}$, the fundamental solution can be written in the following form

$$\sigma_{ij}=G_{ijkl}L_{kl} \tag{2.64}$$

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where G_{ijkl} is the Green tensor of Eq. (2.42). So, the Green tensor reads

$$G_{ijkl} = \frac{1}{8\pi}\Bigg\{(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\,\frac{\mathrm{e}^{-r/\ell_1}}{\ell_1^2 r} + (\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})\,\frac{\mathrm{e}^{-r/\ell_4}}{\ell_4^2 r} - (\delta_{ij}\Delta-\partial_i\partial_j)\delta_{kl}\Big[\frac{1}{r}\big(\mathrm{e}^{-r/\ell_1}-\mathrm{e}^{-r/\ell_2}\big)\Big] \\ -(\delta_{jl}\partial_i+\delta_{il}\partial_j)\partial_k\Big[\frac{1}{r}\big(\mathrm{e}^{-r/\ell_1}-\mathrm{e}^{-r/\ell_3}\big)\Big] - (\delta_{jl}\partial_i-\delta_{il}\partial_j)\partial_k\Big[\frac{1}{r}\big(\mathrm{e}^{-r/\ell_4}-\mathrm{e}^{-r/\ell_3}\big)\Big]\Bigg\}. \tag{2.65}$$

Using the convolution theorem, we obtain for the solution of Eq. (2.41)

$$\sigma_{ij} = G_{ijkl} * \sigma^0_{kl}. \tag{2.66}$$

With the help of this equation three-dimensional problems can be solved e.g. dislocation loops.

3 Screw dislocation

We consider now a straight screw dislocation. We choose the dislocation line and the Burgers vector in z-direction: $b_x = b_y = 0$, $b_z = b$. First of all, a screw dislocation corresponds to the anti-plane strain problem. Thus, we only have the non-vanishing components of the elastic distortion tensor β_{zx} and β_{zy}. From the constitutive equation (2.6) we obtain the relations valid in anti-plane strain:

$$\sigma_{zx} = \frac{\mu-\gamma}{\mu+\gamma}\,\sigma_{xz}, \qquad \sigma_{zy} = \frac{\mu-\gamma}{\mu+\gamma}\,\sigma_{yz}. \tag{3.1}$$

In the case of a screw dislocation we want to solve Eq. (2.37) directly. From the force equilibrium condition (2.33), the following condition follows

$$\beta_{zx,x} + \beta_{zy,y} = 0. \tag{3.2}$$

It looks like a gauge condition, but it is nothing else than a consequence of the force equilibrium condition (2.33). Using Eq. (3.2), (2.37) simplifies to

$$\Big[1-\frac{c_1}{\mu+\gamma}\Delta\Big]\beta_{zx} = \beta^0_{zx}, \qquad \Big[1+\frac{c_2}{\mu-\gamma}\Delta\Big]\beta_{zx} = \beta^0_{zx}, \tag{3.3}$$

$$\Big[1-\frac{c_1}{\mu+\gamma}\Delta\Big]\beta_{zy} = \beta^0_{zy}, \qquad \Big[1+\frac{c_2}{\mu-\gamma}\Delta\Big]\beta_{zy} = \beta^0_{zy}. \tag{3.4}$$

Here Δ denotes the two-dimensional Laplacian. Because we have four equations for two components β_{zx} and β_{zy}, we obtain a relation between c_2 and c_1 as follows[3]

$$c_2 = -\frac{\mu-\gamma}{\mu+\gamma}\,c_1 \tag{3.5}$$

[3]If $\gamma = 0$, we obtain $c_2 = -c_1$. If $c_2 = 0$ like in the Edelen choice, we obtain $c_1 = 0$, $\ell_1^2 = 0$, $\beta_{zx} = \beta^0_{zx}$ and $\beta_{zy} = \beta^0_{zy}$. Thus, only the classical solution of a screw dislocation is allowed in the Edelen choice with symmetric force stresses. Edelen [17] used some ad hoc assumptions like $\beta^0_{xz} = \beta_{zx}$ to avoid this problems. But in the classical theory it must be: $\beta^0_{xz} = 0$. Also the approach of Valsakumar and Sahoo [16] possesses such lacks.

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or in the 'irreducible' parameters

$$a_3 = \frac{3\gamma - \mu}{2\mu}\, a_1. \tag{3.6}$$

Relation (3.5) is a consequence of (3.1). From the relation (3.6) together with the condition of non-negative dislocation core energy (2.21) we obtain the constraint between μ and γ:

$$\gamma \geq \frac{\mu}{3}. \tag{3.7}$$

It is important to note that in the case of $\gamma = 0$, $a_3 = -a_1/2$ violates the condition (2.21). Therefore, the condition of nonnegative dislocation core energy demands asymmetric force stresses. For the Edelen choice ($a_3 = a_1$) we obtain from (3.5) and (3.6): $\mu = \gamma$. Thus, it does not make sense to use symmetric force stresses in the Edelen model. Due to these reasons, the Edelen model demands asymmetric force stresses with $\mu = \gamma$ for a screw dislocation. Therefore, with $\gamma = \mu$ the correct solution of a screw dislocation of the Edelen model is contained in our general solution of the screw dislocation which we calculate in the following.

If we substitute (3.5) and (3.6) in (2.49) and (2.52), we obtain the characteristic length scale of the anti-plane strain problem

$$\ell_1^2 = \ell_4^2 = \frac{c_1}{\mu + \gamma} = \frac{a_1}{2\mu} \tag{3.8}$$

and the equations for the elastic distortion simplify to

$$\big[1 - \ell_1^2 \Delta\big]\beta_{zx} = \beta_{zx}^0, \qquad \big[1 - \ell_1^2 \Delta\big]\beta_{zy} = \beta_{zy}^0. \tag{3.9}$$

They are two-dimensional inhomogeneous Helmholtz equations. The inhomogeneous parts are given by Eq. (A.2). If we substitute (A.2) into Eq. (3.9), the elastic distortion is easily obtained as

$$\beta_{zx} = -\frac{b}{2\pi}\frac{y}{r^2}\Big[1 - \frac{r}{\ell_1} K_1\Big(\frac{r}{\ell_1}\Big)\Big], \qquad \beta_{zy} = \frac{b}{2\pi}\frac{x}{r^2}\Big[1 - \frac{r}{\ell_1} K_1\Big(\frac{r}{\ell_1}\Big)\Big] \tag{3.10}$$

where K_n is the modified Bessel function of the second kind and $n = 0, 1, \ldots$ denotes the order of this function. Substituting the elastic distortions (3.10) into (2.9), the torsion tensor or dislocation tensor is calculated as

$$T_{zxy} = \alpha_{zz} = \frac{b}{2\pi\ell_1^2}\, K_0\Big(\frac{r}{\ell_1}\Big). \tag{3.11}$$

The torsion (3.11) generates the following components of the pseudomoment stress tensor (2.19) according

$$H_{xx} = H_{xyz} = c_2\, T_{zxy} = -\frac{(\mu - \gamma)b}{2\pi}\, K_0\Big(\frac{r}{\ell_1}\Big) \tag{3.12}$$

$$H_{yy} = H_{yzx} = c_2\, T_{zxy} = -\frac{(\mu - \gamma)b}{2\pi}\, K_0\Big(\frac{r}{\ell_1}\Big) \tag{3.13}$$

$$H_{zz} = H_{zxy} = c_1\, T_{zxy} = \frac{(\mu + \gamma)b}{2\pi}\, K_0\Big(\frac{r}{\ell_1}\Big). \tag{3.14}$$

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Thus, this localized pseudomoment stress caused by a screw dislocation is of torsion-type. It is given in terms of only one length scale ℓ_1. The trace of the pseudomoment stresses is

$$H_{kk} = -\frac{(\mu - 3\gamma)b}{2\pi} K_0\Big(\frac{r}{\ell_1}\Big). \tag{3.15}$$

Now we want to show how we can solve the master equation (2.41) by means of a suitable stress function ansatz for the asymmetric force stresses of a screw dislocation. With the stress function ansatz we fulfill the force equilibrium condition (2.33). Then we substitute the stress function ansatz into the pseudomoment equilibrium condition (2.41). In the case of a straight screw dislocation Eq. (2.41) reduces to

$$\big[c_1(\gamma-\mu) - c_2(\gamma+\mu)\big](\sigma_{yz,yx} - \sigma_{xz,yy}) - \big[c_1(\gamma+\mu) - c_2(\gamma-\mu)\big]\Delta\sigma_{zx} + 4\mu\gamma\sigma_{zx} = 4\mu\gamma\sigma^0_{zx} \tag{3.16}$$

$$\big[c_1(\gamma-\mu) - c_2(\gamma+\mu)\big](\sigma_{xz,xy} - \sigma_{yz,xx}) - \big[c_1(\gamma+\mu) - c_2(\gamma-\mu)\big]\Delta\sigma_{zy} + 4\mu\gamma\sigma_{zy} = 4\mu\gamma\sigma^0_{zy} \tag{3.17}$$

$$\begin{aligned}\big[c_2(\gamma+\mu) - c_1(\gamma-\mu)\big]\Delta\sigma_{zx} + \big[c_2(\gamma-\mu) - c_1(\gamma+\mu)\big]\Delta\sigma_{xz} \\ + \big[2c_2\mu - c_3(\gamma+\mu)\big](\sigma_{xz,xx} + \sigma_{yz,xy}) + 4\mu\gamma\sigma_{xz} = 4\mu\gamma\sigma^0_{xz}\end{aligned} \tag{3.18}$$

$$\begin{aligned}\big[c_2(\gamma+\mu) - c_1(\gamma-\mu)\big]\Delta\sigma_{zy} + \big[c_2(\gamma-\mu) - c_1(\gamma+\mu)\big]\Delta\sigma_{yz} \\ + \big[2c_2\mu - c_3(\gamma+\mu)\big](\sigma_{xz,xy} + \sigma_{yz,yy}) + 4\mu\gamma\sigma_{yz} = 4\mu\gamma\sigma^0_{yz}.\end{aligned} \tag{3.19}$$

We introduce the following stress function ansatz, which is suitable for asymmetric force stresses in the anti-plane strain problem,

$$\sigma_{ij} = \begin{pmatrix} 0 & 0 & -\frac{\mu-\gamma}{\mu+\gamma}\,\partial_y F \\ 0 & 0 & \frac{\mu-\gamma}{\mu+\gamma}\,\partial_x F \\ -\partial_y F & \partial_x F & 0 \end{pmatrix} \tag{3.20}$$

where F is a modified Prandtl stress function. Using (3.5), (2.49) and (A.7), we have to solve the two-dimensional inhomogeneous Helmholtz equation

$$\big[1 - \ell_1^2\Delta\big]F = F^0. \tag{3.21}$$

Its solution reads

$$F = \frac{(\mu+\gamma)b}{2\pi}\Big[\ln r + K_0\Big(\frac{r}{\ell_1}\Big)\Big]. \tag{3.22}$$

With Eqs. (3.20) and (3.22) the force stress is calculated as

$$\sigma_{zx} = -\frac{(\mu+\gamma)b}{2\pi}\frac{y}{r^2}\Big[1 - \frac{r}{\ell_1}K_1\Big(\frac{r}{\ell_1}\Big)\Big], \qquad \sigma_{xz} = -\frac{(\mu-\gamma)b}{2\pi}\frac{y}{r^2}\Big[1 - \frac{r}{\ell_1}K_1\Big(\frac{r}{\ell_1}\Big)\Big], \tag{3.23}$$

$$\sigma_{zy} = \frac{(\mu+\gamma)b}{2\pi}\frac{x}{r^2}\Big[1 - \frac{r}{\ell_1}K_1\Big(\frac{r}{\ell_1}\Big)\Big], \qquad \sigma_{yz} = \frac{(\mu-\gamma)b}{2\pi}\frac{x}{r^2}\Big[1 - \frac{r}{\ell_1}K_1\Big(\frac{r}{\ell_1}\Big)\Big]. \tag{3.24}$$

In cylindrical coordinates the non-vanishing components of the force stress and elastic distortion tensors read

$$\sigma_{z\phi} = \frac{(\mu+\gamma)b}{2\pi}\frac{1}{r}\Big[1 - \frac{r}{\ell_1}K_1\Big(\frac{r}{\ell_1}\Big)\Big], \qquad \sigma_{\phi z} = \frac{(\mu-\gamma)b}{2\pi}\frac{1}{r}\Big[1 - \frac{r}{\ell_1}K_1\Big(\frac{r}{\ell_1}\Big)\Big] \tag{3.25}$$

$$\beta_{z\phi} = \frac{b}{2\pi}\frac{1}{r}\Big[1 - \frac{r}{\ell_1}K_1\Big(\frac{r}{\ell_1}\Big)\Big]. \tag{3.26}$$

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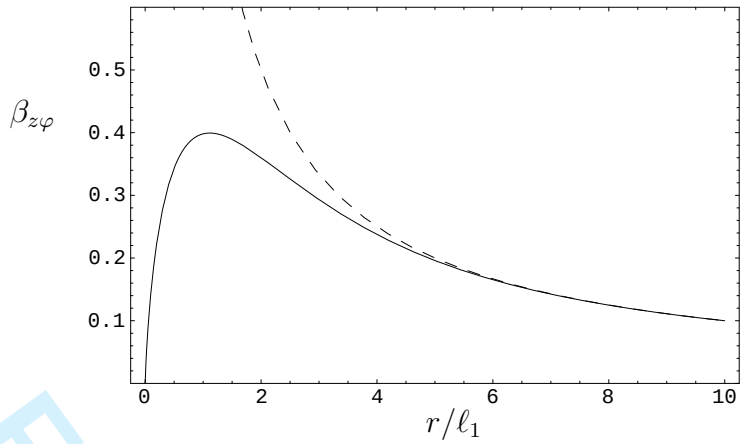

Figure 1: The component $\beta_{z\varphi}$ is given in units of $b/[2\pi\ell_1]$. The dashed curve represents the classical component.

The force stress and elastic distortion fields, calculated in the gauge theory, do not possess singularities. In fact, they have at $r \simeq 1.114\ell_1$ a maximum of: $\sigma_{z\phi} \simeq 0.399(\mu+\gamma)b/[2\pi\ell_1]$, $\sigma_{\phi z} \simeq 0.399(\mu-\gamma)b/[2\pi\ell_1]$, $\beta_{z\phi} \simeq 0.399b/[2\pi\ell_1]$ (see figure 1). We want to note that if $\gamma = 0$, we recover the force stress and pseudomoment stress, the elastic distortion and the dislocation tensor earlier calculated by Malyshev [18] and Lazar [21, 22] in the gauge theory of dislocations with the so-called Einstein choice. The modified Burgers vector is given by

$$b(r) = \oint_0^{2\pi} \beta_{z\phi}\, r\, \mathrm{d}\phi = \int_0^{2\pi}\int_0^r \alpha_{zz}(r')\, r'\, \mathrm{d}r'\, \mathrm{d}\phi = b\Big[1 - \frac{r}{\ell_1} K_1\Big(\frac{r}{\ell_1}\Big)\Big]. \tag{3.27}$$

This modified Burgers vector differs appreciably from the constant value b in the core region from $r = 0$ up to $r \simeq 6\ell_1$. Thus, it is suggestive to take $r_{\mathrm{c}} \simeq 6\ell_1$ as the dislocation core radius. Outside this core region, the Burgers vector reaches its constant value. Accordingly, the gauge theoretical solutions approaches the classical one outside the dislocation core region. Due to the cylindrical symmetry of a straight screw dislocation around the dislocation line the physical situation requires only one length scale ℓ_1.

The force between dislocations is the so-called Peach-Koehler force [53, 54, 55]. In the gauge theory of dislocations the gauge invariant formulation of the Peach-Koehler force has been derived by Lazar and Anastassiadis [56]. Kadić and Edelen [11] and Edelen and Lagoudas [12] just derived an expression which is not gauge invariant because they used the canonical Eshelby stress tensor instead of the gauge invariant one. It is given by the convolution of the stress and the dislocation density tensor

$$F_k = -\sigma_{li} * T_{lik} = \epsilon_{kij}\sigma_{li} * \alpha_{lj}. \tag{3.28}$$

It is obvious that (3.28) is similar in the form as the classical expression of the Peach-Koehler force. But, in (3.28) the stress and the dislocation density tensor are the gauge theoretical ones. For a screw dislocation it reads

$$F_x = \sigma_{zy} * \alpha_{zz}, \qquad F_y = -\sigma_{zx} * \alpha_{zz}. \tag{3.29}$$

By the convolution theorem, it is in the Fourier space

$$\widetilde{F}_x(\boldsymbol{q}) = \mathcal{F}_{(2)}\big[F_x\big] = \widetilde{\sigma}_{zy}(\boldsymbol{q})\widetilde{\alpha}_{zz}(\boldsymbol{q}), \qquad \widetilde{F}_y(\boldsymbol{q}) = \mathcal{F}_{(2)}\big[F_y\big] = -\widetilde{\sigma}_{zx}(\boldsymbol{q})\widetilde{\alpha}_{zz}(\boldsymbol{q}). \tag{3.30}$$

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We use the following notation for the two-dimensional Fourier transform [47, 48]

$$\widetilde{f}(\boldsymbol{q}) \equiv \mathcal{F}_{(2)}\big[f(\boldsymbol{r})\big] = \int_{-\infty}^{\infty} f(\boldsymbol{r})\, \mathrm{e}^{+\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \mathrm{d}\boldsymbol{r}, \qquad f(\boldsymbol{r}) \equiv \mathcal{F}_{(2)}^{-1}\big[\widetilde{f}(\boldsymbol{q})\big] = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \widetilde{f}(\boldsymbol{q})\, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{r}} \mathrm{d}\boldsymbol{q}. \tag{3.31}$$

The Fourier transformed functions are

$$\mathcal{F}_{(2)}\big[\ln r\big] = -\frac{2\pi}{q^2}, \qquad \mathcal{F}_{(2)}\Big[K_0\Big(\frac{r}{\ell_1}\Big)\Big] = \frac{2\pi}{q^2 + \frac{1}{\ell_1}} \tag{3.32}$$

and we obtain

$$\widetilde{\sigma}_{zy}(\boldsymbol{q}) = (\mu+\gamma) b\, \frac{\mathrm{i}q_x}{\ell_1^2 q^2 \big(q^2 + \frac{1}{\ell_1}\big)}, \qquad \widetilde{\sigma}_{zx}(\boldsymbol{q}) = -(\mu+\gamma) b\, \frac{\mathrm{i}q_y}{\ell_1^2 q^2 \big(q^2 + \frac{1}{\ell_1}\big)} \tag{3.33}$$

$$\widetilde{\alpha}_{zz}(\boldsymbol{q}) = b' \frac{1}{\ell_1^2 \big(q^2 + \frac{1}{\ell_1}\big)}. \tag{3.34}$$

The Fourier transformed Peach-Koehler force $\widetilde{F}_x$ is

$$\widetilde{F}_x(\boldsymbol{q}) = (\mu+\gamma) bb'\, \frac{\mathrm{i}q_x}{\ell_1^4 q^2 \big(q^2 + \frac{1}{\ell_1}\big)^2}. \tag{3.35}$$

Using the inverse Fourier transform, the Peach-Koehler force F_x is calculated as

$$\begin{aligned} F_x &= -(\mu+\gamma) bb'\, \partial_x\, \mathcal{F}_{(2)}^{-1}\Big[\frac{1}{q^2} - \frac{1}{q^2 + \frac{1}{\ell_1}} - \frac{1}{\ell_1^2 \big(q^2 + \frac{1}{\ell_1}\big)^2}\Big] \\ &= (\mu+\gamma) bb'\, \partial_x \Big[\ln r + K_0\Big(\frac{r}{\ell_1}\Big) + \frac{r}{2\ell_1} K_1\Big(\frac{r}{\ell_1}\Big)\Big] \\ &= \frac{(\mu+\gamma) bb'}{2\pi}\, \frac{x}{r^2} \Big[1 - \frac{r}{\ell_1} K_1\Big(\frac{r}{\ell_1}\Big) - \frac{r^2}{2\ell_1^2} K_0\Big(\frac{r}{\ell_1}\Big)\Big]. \end{aligned} \tag{3.36}$$

After an analogous computation we obtain for the component F_y

$$F_y = \frac{(\mu+\gamma) bb'}{2\pi}\, \frac{y}{r^2} \Big[1 - \frac{r}{\ell_1} K_1\Big(\frac{r}{\ell_1}\Big) - \frac{r^2}{2\ell_1^2} K_0\Big(\frac{r}{\ell_1}\Big)\Big]. \tag{3.37}$$

In polar coordinates, the non-vanishing component reads:

$$F_r = \frac{(\mu+\gamma) bb'}{2\pi}\, \frac{1}{r} \Big[1 - \frac{r}{\ell_1} K_1\Big(\frac{r}{\ell_1}\Big) - \frac{r^2}{2\ell_1^2} K_0\Big(\frac{r}{\ell_1}\Big)\Big]. \tag{3.38}$$

The Peach-Koehler force between two screw dislocations is a radial force. It has a maximum of: $F_r \simeq 0.2488(\mu+\gamma) bb'/[2\pi\ell_1]$ at $r \simeq 2.324\ell_1$ and is zero at $r = 0$ (see figure 2). It is interesting to note that the maximum of the Peach-Koehler force calculated in the framework of nonlocal elasticity is higher than our gauge theoretical result (see Eringen [57, 58], Lazar [59]). If one calculates the gradient of the elastic interaction energy between two screw dislocations, then also the result (3.38) follows. But, if one uses the 'total' interaction energy including the core contribution, one does not get the Peach-Koehler force. Because Valsakumar and Sahoo [60] used the 'total' interaction energy, they have not calculated the Peach-Koehler force in the gauge theory. Nevertheless, their result is in agreement with the nonlocal expression calculated by Eringen [57].

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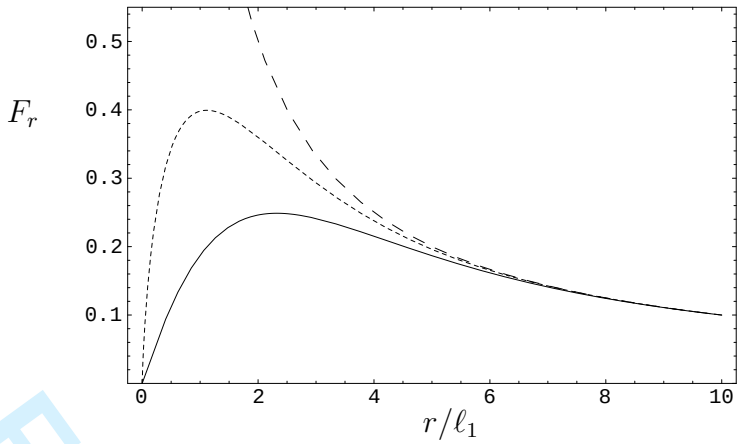

Figure 2: The component F_r is given in units of $\mu bb'/[2\pi\ell_1]$ ($\gamma = 0$). The small dashed curve represents the nonlocal result and the dashed curve represents the classical component.

4 Edge dislocation

In this section we consider a straight edge dislocation. We choose the dislocation line in the z-direction and the Burgers vector reads $b_x = b$, $b_y = 0$, $b_z = 0$. The problem of an edge dislocation corresponds to plane strain.

4.1 The general case

Let us turn to the edge dislocation in the framework of the gauge theory of dislocations. In this case the master equation (2.37) reduces to

$$\begin{aligned}
&\big[(c_1 - c_2 + 2c_3)\,2\gamma\nu - 2c_3\gamma(1+\nu)\big](\sigma_{xx,yy} + \sigma_{yy,yy}) - 2\gamma(c_1 - c_2)\Delta\sigma_{xx} \qquad (4.1)\\
&\qquad + \big[(c_1 - c_2)(\gamma - \mu) - c_3(\gamma + \mu)\big](\sigma_{xx,xx} + \sigma_{yx,yx}) + 4\mu\gamma\,\sigma_{xx} = 4\mu\gamma\,\sigma^0_{xx}\\
&\big[(c_1 - c_2 + 2c_3)\,2\gamma\nu - 2c_3\gamma(1+\nu)\big](\sigma_{xx,xx} + \sigma_{yy,xx}) - 2\gamma(c_1 - c_2)\Delta\sigma_{yy} \qquad (4.2)\\
&\qquad + \big[(c_1 - c_2)(\gamma - \mu) - c_3(\gamma + \mu)\big](\sigma_{xy,xy} + \sigma_{yy,yy}) + 4\mu\gamma\,\sigma_{yy} = 4\mu\gamma\,\sigma^0_{yy}\\
&\big[c_1(\gamma - \mu) - c_2(\gamma + \mu)\big](\sigma_{xx,xy} - \sigma_{yx,xx}) + \big[2c_2\mu - c_3(\gamma + \mu)\big](\sigma_{xy,xx} + \sigma_{yy,yx})\\
&\qquad - \big[(c_1 - c_2 + 2c_3)\,2\gamma\nu - 2c_3\gamma(1+\nu)\big](\sigma_{xx,xy} + \sigma_{yy,xy}) \qquad (4.3)\\
&\qquad\qquad - \big[c_1(\gamma + \mu) - c_2(\gamma - \mu)\big]\Delta\sigma_{xy} + 4\mu\gamma\,\sigma_{xy} = 4\mu\gamma\,\sigma^0_{xy}\\
&\big[c_1(\gamma - \mu) - c_2(\gamma + \mu)\big](\sigma_{yy,yx} - \sigma_{yx,yy}) + \big[2c_2\mu - c_3(\gamma + \mu)\big](\sigma_{xx,xy} + \sigma_{yx,yy})\\
&\qquad - \big[(c_1 - c_2 + 2c_3)\,2\gamma\nu - 2c_3\gamma(1+\nu)\big](\sigma_{xx,xy} + \sigma_{yy,xy}) \qquad (4.4)\\
&\qquad\qquad - \big[c_1(\gamma + \mu) - c_2(\gamma - \mu)\big]\Delta\sigma_{yx} + 4\mu\gamma\,\sigma_{yx} = 4\mu\gamma\,\sigma^0_{yx}\\
&\Big[1 - \frac{(1-\nu)c_3}{2\mu\nu}\Delta\Big]\sigma_{zz} = \sigma^0_{zz}. \qquad (4.5)
\end{aligned}$$

We used the plane strain condition $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ because we require plane strain for an edge dislocation in the gauge theory.

For the asymmetric force stress we use the stress function ansatz of Mindlin-type with

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the two modified stress functions f and Ψ:

$$\sigma_{ij} = \begin{pmatrix} \partial_{yy}^2 f - \partial_{xy}^2 \Psi & -\partial_{xy}^2 f + \partial_{xx}^2 \Psi & 0 \\ -\partial_{xy}^2 f - \partial_{yy}^2 \Psi & \partial_{xx}^2 f + \partial_{xy}^2 \Psi & 0 \\ 0 & 0 & \nu \Delta f \end{pmatrix}. \tag{4.6}$$

This stress function ansatz (4.6) has been also used by Lazar and Maugin [61] in gradient micropolar elasticity. If we substitute (4.6) into the coupled system of equations (4.1)–(4.5), we obtain the decoupled equations

$$\frac{\partial^2}{\partial y^2}\left[\left(1 - \frac{(c_1 - c_2 + c_3)(1-\nu)}{2\mu}\Delta\right) f - f^0\right] - \frac{\partial^2}{\partial x \partial y}\left[\left(1 - \frac{(c_1 - c_2 + c_3)(\mu+\gamma)}{4\mu\gamma}\Delta\right)\Psi - \Psi^0\right] = 0 \tag{4.7}$$

$$\frac{\partial^2}{\partial x^2}\left[\left(1 - \frac{(c_1 - c_2 + c_3)(1-\nu)}{2\mu}\Delta\right) f - f^0\right] + \frac{\partial^2}{\partial x \partial y}\left[\left(1 - \frac{(c_1 - c_2 + c_3)(\mu+\gamma)}{4\mu\gamma}\Delta\right)\Psi - \Psi^0\right] = 0 \tag{4.8}$$

$$\frac{\partial^2}{\partial x \partial y}\left[\left(1 - \frac{(c_1 - c_2 + c_3)(1-\nu)}{2\mu}\Delta\right) f - f^0\right] - \frac{\partial^2}{\partial x^2}\left[\left(1 - \frac{(c_1 - c_2 + c_3)(\mu+\gamma)}{4\mu\gamma}\Delta\right)\Psi - \Psi^0\right] = 0 \tag{4.9}$$

$$\frac{\partial^2}{\partial x \partial y}\left[\left(1 - \frac{(c_1 - c_2 + c_3)(1-\nu)}{2\mu}\Delta\right) f - f^0\right] + \frac{\partial^2}{\partial y^2}\left[\left(1 - \frac{(c_1 - c_2 + c_3)(\mu+\gamma)}{4\mu\gamma}\Delta\right)\Psi - \Psi^0\right] = 0 \tag{4.10}$$

$$\Delta\left[\left(1 - \frac{c_3(1-\nu)}{2\mu\nu}\Delta\right) f - f^0\right] = 0. \tag{4.11}$$

The addition of Eqs. (4.7) and (4.8) gives

$$\Delta\left[\left(1 - \frac{(c_1 - c_2 + c_3)(1-\nu)}{2\mu}\Delta\right) f - f^0\right] = 0. \tag{4.12}$$

For the moment equilibrium condition (2.36), we obtain

$$\Delta\left[\left(1 - \frac{(c_1 - c_2 + c_3)(\mu+\gamma)}{4\mu\gamma}\Delta\right)\Psi - \Psi^0\right] = 0. \tag{4.13}$$

If we compare (4.11) with (4.12), we obtain the following constraint between c_1, c_2 and c_3:[4]

$$c_3 = \frac{\nu}{1-\nu}(c_1 - c_2) \tag{4.14}$$

[4]This relation is not fulfilled in the Einstein choice and $c_1 - c_2 + c_3 = 0$. In the Edelen choice $c_3 = 0$, $c_2 = 0$, we obtain from (4.14) $c_1 = 0$, $\ell_2 = 0$ and $\ell_3 = 0$. Thus, only $f = f^0$ and $\Psi = \Psi^0$ are allowed with the Edelen choice if the plane strain condition is fulfilled. Due to these reasons, Edelen [52], Kadić and Edelen [11], Malyshev [18] used a stress function ansatz not fulfilling the plane strain condition.

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or in 'irreducible' parameters

$$a_2 = \frac{1+\nu}{1-\nu}\, a_1. \tag{4.15}$$

This constraint is nothing but a consequence of the relation $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ valid in the plane strain problem. We want to note that this constraint (4.15) has been earlier derived by Lazar [23]. With $-1 \leq \nu \leq 1/2$, we get from (4.15) the relation

$$0 \leq a_2 \leq 3a_1. \tag{4.16}$$

If we substitute (4.14) and (4.15) into (2.50) and (2.51), we obtain the two characteristic length scales of the plane strain problem

$$\ell_2^2 = \frac{c_1 - c_2}{2\mu} = \frac{1}{2\mu}\, a_1 \tag{4.17}$$

$$\ell_3^2 = \frac{(c_1 - c_2)(\mu+\gamma)}{4\mu(1-\nu)\gamma} = \frac{\mu+\gamma}{4\mu(1-\nu)\gamma}\, a_1. \tag{4.18}$$

The relation between these internal lengths is

$$\ell_3^2 = \frac{\mu+\gamma}{2(1-\nu)\gamma}\, \ell_2^2. \tag{4.19}$$

Therefore, we obtain from Eqs. (4.7)–(4.11) two inhomogeneous Helmholtz equations which we have to solve

$$[1 - \ell_2^2\, \Delta] f = f^0 \tag{4.20}$$

$$[1 - \ell_3^2\, \Delta] \Psi = \Psi^0 \tag{4.21}$$

where the inhomogeneous parts are given by (B.14) and (B.15). The solutions of Eqs. (4.20) and (4.21) are the following modified 'Airy' stress functions

$$f = -\frac{\mu b}{4\pi(1-\nu)}\, \partial_y \Big\{ r^2 \ln r + 4\, \ell_2^2 \Big[\ln r + K_0\Big(\frac{r}{\ell_2}\Big) \Big] \Big\} \tag{4.22}$$

$$\Psi = \frac{\mu\gamma\, b}{2\pi(\mu+\gamma)}\, \partial_x \Big\{ r^2 \ln r + 4\, \ell_3^2 \Big[\ln r + K_0\Big(\frac{r}{\ell_3}\Big) \Big] \Big\}. \tag{4.23}$$

If we substitute the stress functions (4.22) and (4.23) into the stress function ansatz (4.6),

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the following components of the force stress tensor follow

$$\sigma_{xx} = -\frac{y}{r^4}\Bigg\{A\Big[(y^2+3x^2)+\frac{4\,\ell_2^2}{r^2}(y^2-3x^2)-2y^2\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)-2(y^2-3x^2)K_2\Big(\frac{r}{\ell_2}\Big)\Big] - B\Big[(x^2-y^2)-\frac{4\,\ell_3^2}{r^2}(3x^2-y^2)+2x^2\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big)-2(y^2-3x^2)K_2\Big(\frac{r}{\ell_3}\Big)\Big]\Bigg\} \quad (4.24)$$

$$\sigma_{yy} = -\frac{y}{r^4}\Bigg\{A\Big[(y^2-x^2)-\frac{4\,\ell_2^2}{r^2}(y^2-3x^2)-2x^2\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)+2(y^2-3x^2)K_2\Big(\frac{r}{\ell_2}\Big)\Big] + B\Big[(x^2-y^2)-\frac{4\,\ell_3^2}{r^2}(3x^2-y^2)+2x^2\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big)+2(3x^2-y^2)K_2\Big(\frac{r}{\ell_3}\Big)\Big]\Bigg\} \quad (4.25)$$

$$\sigma_{xy} = \frac{x}{r^4}\Bigg\{A\Big[(x^2-y^2)-\frac{4\,\ell_2^2}{r^2}(x^2-3y^2)-2y^2\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)+2(x^2-3y^2)K_2\Big(\frac{r}{\ell_2}\Big)\Big] + B\Big[(x^2+3y^2)+\frac{4\,\ell_3^2}{r^2}(x^2-3y^2)-2x^2\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big)-2(x^2-3y^2)K_2\Big(\frac{r}{\ell_3}\Big)\Big]\Bigg\} \quad (4.26)$$

$$\sigma_{yx} = \frac{x}{r^4}\Bigg\{A\Big[(x^2-y^2)-\frac{4\,\ell_2^2}{r^2}(x^2-3y^2)-2y^2\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)+2(x^2-3y^2)K_2\Big(\frac{r}{\ell_2}\Big)\Big] - B\Big[(x^2-y^2)-\frac{4\,\ell_3^2}{r^2}(x^2-3y^2)-2y^2\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big)+2(x^2-3y^2)K_2\Big(\frac{r}{\ell_3}\Big)\Big]\Bigg\} \quad (4.27)$$

$$\sigma_{zz} = -2\nu\,A\,\frac{y}{r^2}\Big[1-\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big] \quad (4.28)$$

with

$$A := \frac{\mu b}{2\pi(1-\nu)}, \qquad B := \frac{\mu\gamma b}{\pi(\mu+\gamma)}. \quad (4.29)$$

The trace of the stress tensor is

$$\sigma_{kk} = -2(1+\nu)\,A\,\frac{y}{r^2}\Big[1-\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big]. \quad (4.30)$$

The skew-symmetric part of the force stress tensor reads

$$\sigma_{[xy]} = \frac{\mu\gamma b}{\pi(\mu+\gamma)}\,\frac{x}{r^2}\Big[1-\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big)\Big]. \quad (4.31)$$

Now we want to discuss some details of the core modification of the force stress fields (4.24)–(4.28). The spatial distributions of the force stresses near the dislocation line are presented in figure 3. The force stress fields have no artificial singularities at the core and the maximum stress occurs at a short distance away from the dislocation line (see figures 4 and 5). They are zero at $r = 0$. It can be seen that the stresses have the following extreme values: $|\sigma_{xx}(0,y)| \simeq 0.546A/\ell_2 + 0.260B/\ell_3$ at $|y| \simeq (0.996\ell_2 + 1.494\ell_3)/2$, $|\sigma_{yy}(0,y)| \simeq 0.260A/\ell_2 - 0.260B/\ell_3$ at $|y| \simeq (1.494\ell_2 + 1.494\ell_3)/2$, $|\sigma_{xy}(x,0)| \simeq 0.260A/\ell_2 + 0.546B/\ell_3$ at $|x| \simeq (1.494\ell_2 + 0.996\ell_3)/2$, $|\sigma_{yx}(x,0)| \simeq 0.260A/\ell_2 - 0.260B/\ell_3$ at $|x| \simeq (1.494\ell_2 + 1.494\ell_3)/2$, and $|\sigma_{zz}(0,y)| \simeq 0.399A$ at $|y| \simeq 1.114\ell_2$. Thus, the characteristic internal lengths ℓ_2 and ℓ_3 determine the position and the magnitude of the stress maxima. For $\gamma > 0$ the stresses σ_{xx} and σ_{xy} are bigger than in the case $\gamma = 0$ and σ_{yy} and σ_{yx} are smaller than in the case $\gamma = 0$ (see figure 5).

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Using Eq. (2.38), we find for the elastic distortion

$$\beta_{xx} = -\frac{y}{r^4}\Big\{\frac{A}{2\mu}\Big[(1-2\nu)r^2 + 2x^2 + \frac{4\,\ell_2^2}{r^2}(y^2-3x^2) - 2(y^2-\nu\,r^2)\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big) - 2(y^2-3x^2)K_2\Big(\frac{r}{\ell_2}\Big)\Big] - \frac{B}{2\mu}\Big[(x^2-y^2) - \frac{4\,\ell_3^2}{r^2}(3x^2-y^2) + 2x^2\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big) - 2(y^2-3x^2)K_2\Big(\frac{r}{\ell_3}\Big)\Big]\Big\} \tag{4.32}$$

$$\beta_{yy} = -\frac{y}{r^4}\Big\{\frac{A}{2\mu}\Big[(1-2\nu)r^2 - 2x^2 - \frac{4\,\ell_2^2}{r^2}(y^2-3x^2) - 2(x^2-\nu\,r^2)\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big) + 2(y^2-3x^2)K_2\Big(\frac{r}{\ell_2}\Big)\Big] + \frac{B}{2\mu}\Big[(x^2-y^2) - \frac{4\,\ell_3^2}{r^2}(3x^2-y^2) + 2x^2\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big) + 2(3x^2-y^2)K_2\Big(\frac{r}{\ell_3}\Big)\Big]\Big\} \tag{4.33}$$

$$\beta_{xy} = \frac{x}{r^4}\Big\{\frac{A}{2\mu}\Big[(x^2-y^2) - \frac{4\,\ell_2^2}{r^2}(x^2-3y^2) - 2y^2\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big) + 2(x^2-3y^2)K_2\Big(\frac{r}{\ell_2}\Big)\Big] + \frac{B}{2\mu}\Big[2y^2 + \frac{4\,\ell_3^2}{r^2}(x^2-3y^2) - (x^2-y^2)\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big) - 2(x^2-3y^2)K_2\Big(\frac{r}{\ell_3}\Big)\Big] + \frac{B}{2\gamma}\,r^2\Big[1 - \frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big)\Big]\Big\} \tag{4.34}$$

$$\beta_{yx} = \frac{x}{r^4}\Big\{\frac{A}{2\mu}\Big[(x^2-y^2) - \frac{4\,\ell_2^2}{r^2}(x^2-3y^2) - 2y^2\frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big) + 2(x^2-3y^2)K_2\Big(\frac{r}{\ell_2}\Big)\Big] + \frac{B}{2\mu}\Big[2y^2 + \frac{4\,\ell_3^2}{r^2}(x^2-3y^2) - (x^2-y^2)\frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big) - 2(x^2-3y^2)K_2\Big(\frac{r}{\ell_3}\Big)\Big] - \frac{B}{2\gamma}\,r^2\Big[1 - \frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big)\Big]\Big\}. \tag{4.35}$$

The dilatation reads

$$\beta_{kk} = -\frac{(1-2\nu)A}{\mu}\,\frac{y}{r^2}\Big[1 - \frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big]. \tag{4.36}$$

In addition the elastic rotation is

$$\beta_{[xy]} = \frac{\mu b}{2\pi(\mu+\gamma)}\,\frac{x}{r^2}\Big[1 - \frac{r}{\ell_3}K_1\Big(\frac{r}{\ell_3}\Big)\Big]. \tag{4.37}$$

In Eqs. (4.36) and (4.37) it can be seen that ℓ_2 and ℓ_3 are the characteristic lengths for the elastic dilatation and elastic rotation, respectively.

Using the elastic distortion (4.32)–(4.35) in terms of the stress functions f and Ψ, we obtain for the torsion (2.9) or the dislocation density of a straight edge dislocation according

$$\alpha_{xz} = T_{xxy} = -\frac{1-\nu}{2\mu}\,\partial_y\Delta f + \frac{\mu+\gamma}{4\mu\gamma}\,\partial_x\Delta\Psi \tag{4.38}$$

$$\alpha_{yz} = T_{yxy} = \frac{1-\nu}{2\mu}\,\partial_x\Delta f + \frac{\mu+\gamma}{4\mu\gamma}\,\partial_y\Delta\Psi. \tag{4.39}$$

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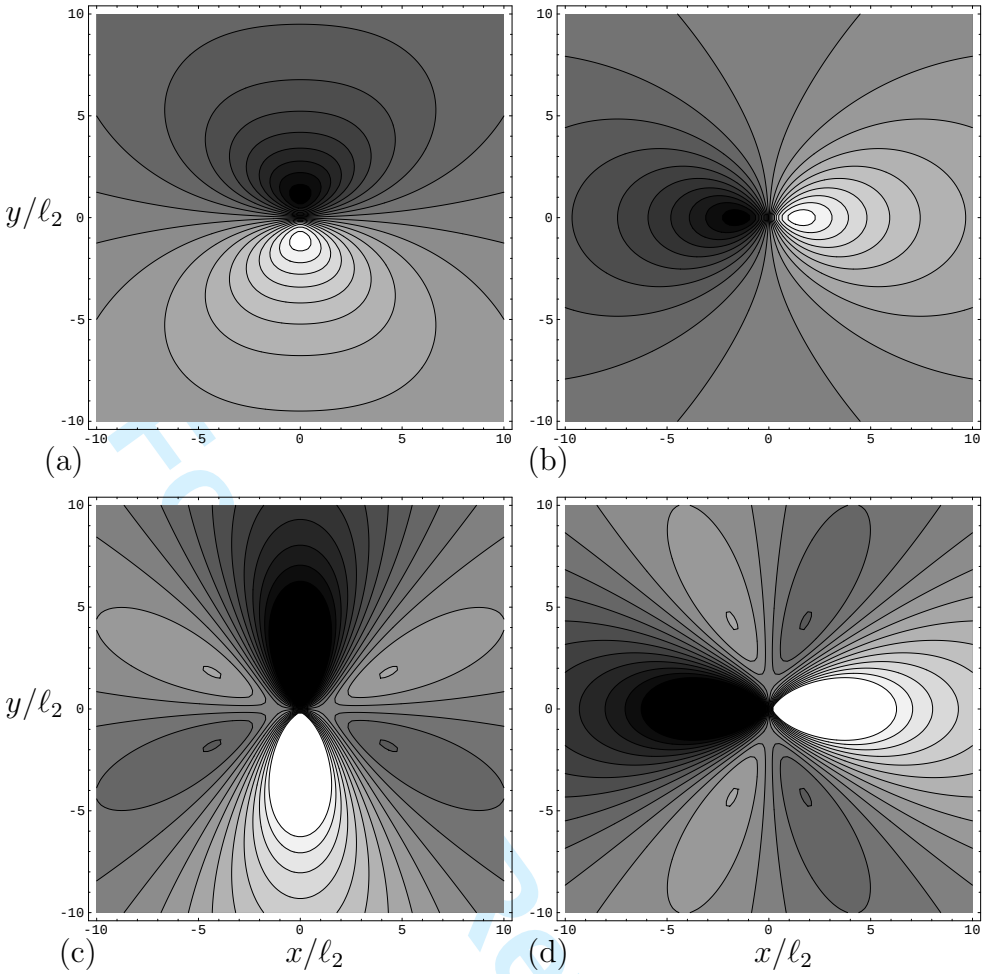

Figure 3: Force stress contours of an edge dislocation near the dislocation line: (a) σ_{xx}, (b) σ_{xy}, (c) σ_{yy}, (d) σ_{yx} with $\nu = 0.3$ and $\gamma = \mu/2$.

Simple differentiation gives the non-vanishing expressions

$$\alpha_{xz} = \frac{b}{4\pi}\left\{\frac{1}{\ell_2^2}K_0\Big(\frac{r}{\ell_2}\Big) + \frac{1}{\ell_3^2}K_0\Big(\frac{r}{\ell_3}\Big) - \frac{x^2-y^2}{r^2}\Big[\frac{1}{\ell_2^2}K_2\Big(\frac{r}{\ell_2}\Big) - \frac{1}{\ell_3^2}K_2\Big(\frac{r}{\ell_3}\Big)\Big]\right\} \tag{4.40}$$

$$\alpha_{yz} = -\frac{b}{2\pi}\frac{xy}{r^2}\Big[\frac{1}{\ell_2^2}K_2\Big(\frac{r}{\ell_2}\Big) - \frac{1}{\ell_3^2}K_2\Big(\frac{r}{\ell_3}\Big)\Big]. \tag{4.41}$$

So far, it is surprising that the component (4.41), which is usually the dislocation density of an edge dislocation with Burgers vector b_y, is non-zero. The components (4.40) and (4.41) are necessary to fulfill the pseudomomentum equilibrium condition (2.32). Also we would like to note that these non-vanishing components of the torsion tensor do not have cylindrical symmetry due to the K_2-terms (see figure 6). Since an edge dislocation is lacking cylindrical symmetry around the dislocation line two length scales, ℓ_2 and ℓ_3, are needed for a proper model. Fig. 6a shows this asymmetry in the dislocation core region of an edge dislocation.

With Eqs. (2.19) and (2.22) the localized pseudomoment stresses of bending type are

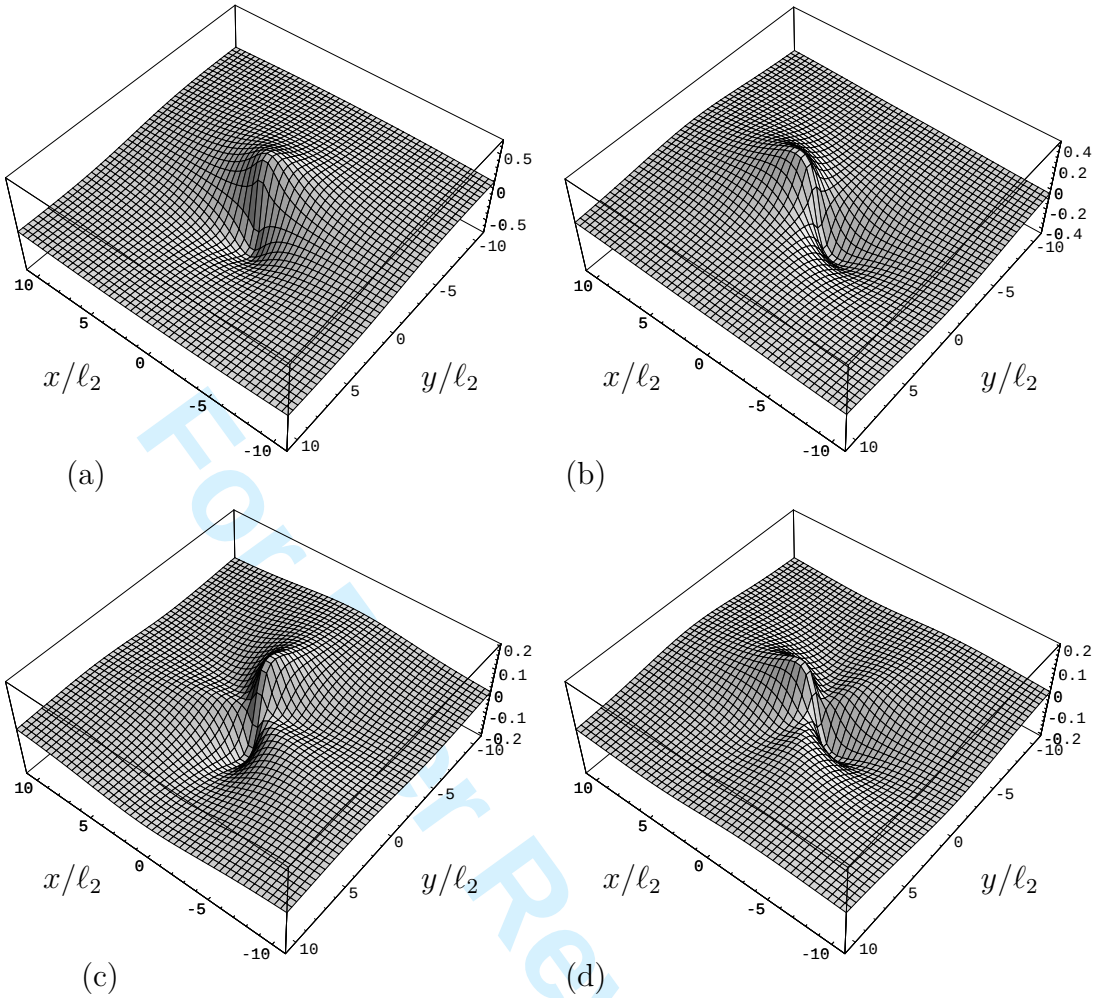

Figure 4: Force stress of an edge dislocation: (a) σ_{xx}, (b) σ_{xy}, (c) σ_{yy}, (d) σ_{yx} are given in units of A with $\nu = 0.3$ and $\gamma = \mu/2$.

given by

$$H_{xz} = H_{xxy} = \frac{1}{1-\nu}(c_1 - c_2)\alpha_{xz} \tag{4.42}$$
$$= A\Big[K_0\Big(\frac{r}{\ell_2}\Big) - \frac{x^2-y^2}{r^2}K_2\Big(\frac{r}{\ell_2}\Big)\Big] + B\Big[K_0\Big(\frac{r}{\ell_3}\Big) + \frac{x^2-y^2}{r^2}K_2\Big(\frac{r}{\ell_3}\Big)\Big]\Big\}$$
$$H_{zx} = H_{zxy} = -\frac{\nu}{1-\nu}(c_1 - c_2)\alpha_{xz} \tag{4.43}$$
$$= -\nu A\Big[K_0\Big(\frac{r}{\ell_2}\Big) - \frac{x^2-y^2}{r^2}K_2\Big(\frac{r}{\ell_2}\Big)\Big] - \nu B\Big[K_0\Big(\frac{r}{\ell_3}\Big) + \frac{x^2-y^2}{r^2}K_2\Big(\frac{r}{\ell_3}\Big)\Big]\Big\}$$
$$H_{yz} = H_{yxy} = \frac{1}{1-\nu}(c_1 - c_2)\alpha_{yz} = -2\,\frac{xy}{r^2}\Big[A\,K_2\Big(\frac{r}{\ell_2}\Big) - BK_2\Big(\frac{r}{\ell_3}\Big)\Big] \tag{4.44}$$
$$H_{zy} = H_{zzx} = -\frac{\nu}{1-\nu}(c_1 - c_2)\alpha_{yz} = 2\nu\,\frac{xy}{r^2}\Big[A\,K_2\Big(\frac{r}{\ell_2}\Big) - BK_2\Big(\frac{r}{\ell_3}\Big)\Big]. \tag{4.45}$$

Therefore, plane strain bending is given in terms of the two length scales ℓ_2 and ℓ_3. If we use the components (4.40) and (4.41) of the dislocation density, the Burgers vector is

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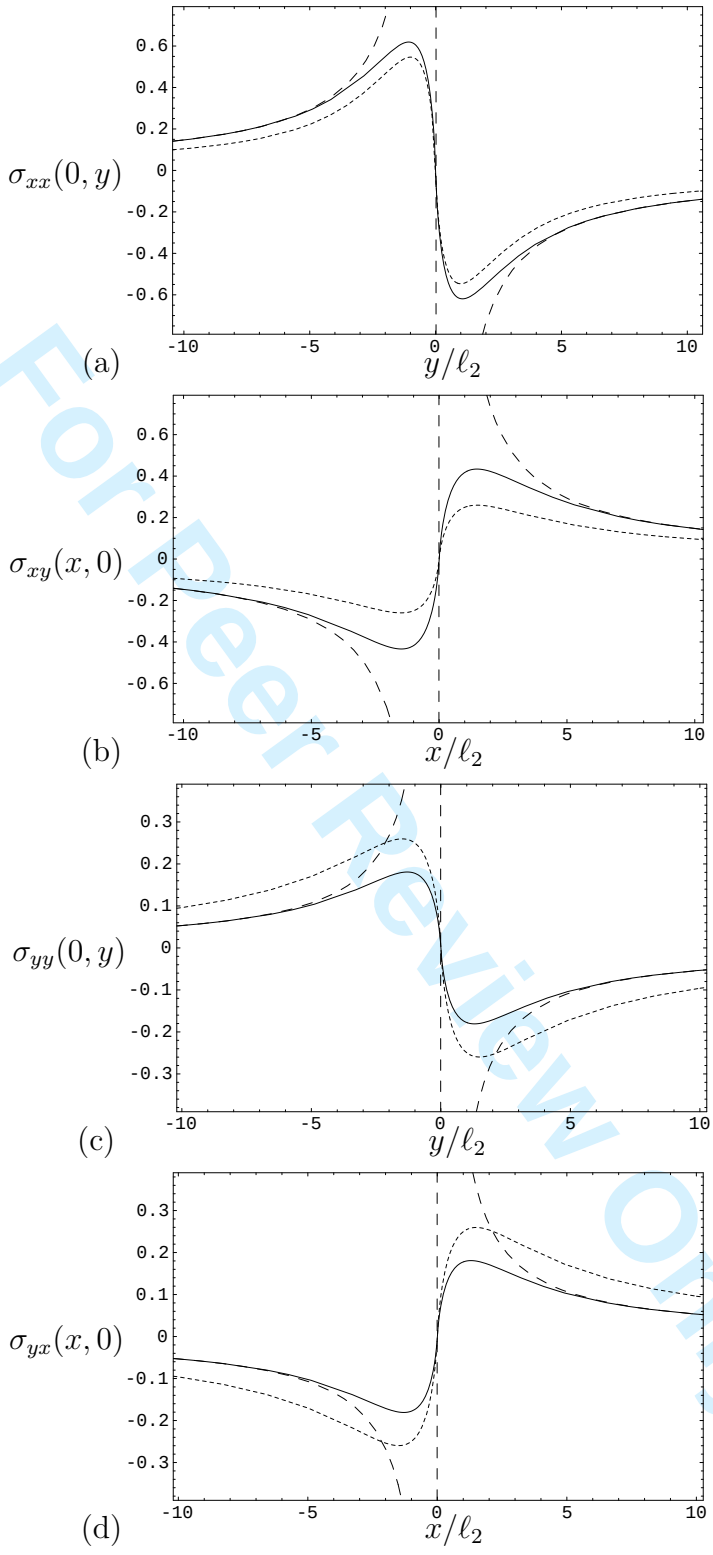

Figure 5: The force stresses components near the dislocation line: (a) $\sigma_{xx}(0,y)$, (b) $\sigma_{xy}(x,0)$, (c) $\sigma_{yy}(0,y)$, (d) $\sigma_{yx}(x,0)$ are given in units of A with $\nu = 0.3$ and $\gamma = \mu/2$. The dashed curves represent the stresses in asymmetric elasticity and the small dashed curves the symmetric force stresses ($\gamma = 0$).

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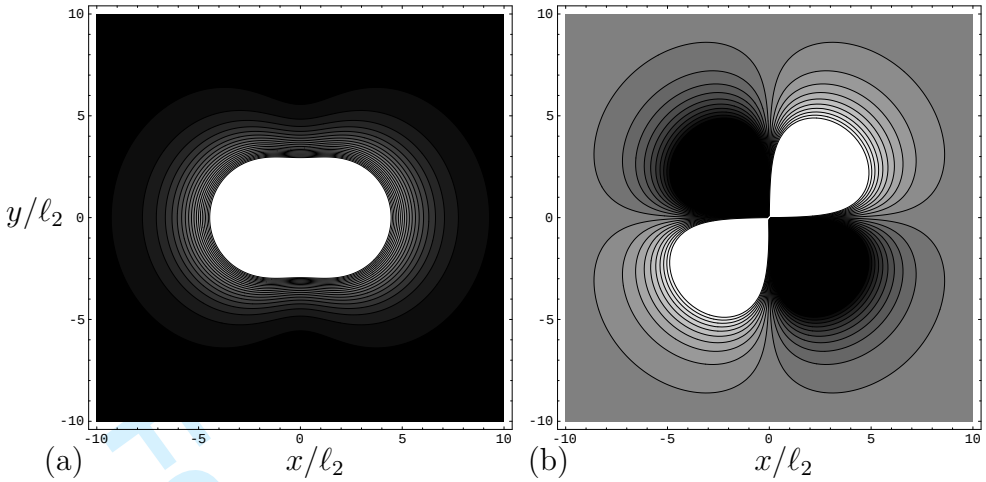

Figure 6: Torsion contours of an edge dislocation near the dislocation line: (a) α_{xz}, (b) α_{yz} with $\nu = 0.3$ and $\gamma = \mu/2$ (in units of $b/[4\pi]$).

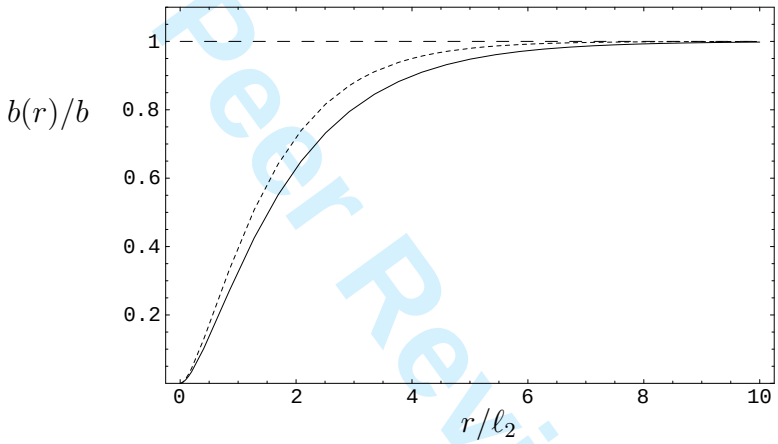

Figure 7: The modified Burgers vector of an edge dislocation $b(r)/b$ for $\ell_2 = \ell_3$ (small dashed) and for $\nu = 0.3$ and $\gamma = \mu/2$ (solid).

calculated as

$$b(r) = \oint (\beta_{xx}\,\mathrm{d}x + \beta_{xy}\,\mathrm{d}y) = \int_0^{2\pi}\int_0^r \alpha_{xz}(r',\phi')\,r'\,\mathrm{d}r'\,\mathrm{d}\phi'$$
$$= b\Big\{1 - \frac{1}{2}\Big[\frac{r}{\ell_2}\,K_1\Big(\frac{r}{\ell_2}\Big) + \frac{r}{\ell_3}\,K_1\Big(\frac{r}{\ell_3}\Big)\Big]\Big\} \tag{4.46}$$
$$0 = \oint (\beta_{yx}\,\mathrm{d}x + \beta_{yy}\,\mathrm{d}y) = \int_0^{2\pi}\int_0^r \alpha_{yz}(r',\phi')\,r'\,\mathrm{d}r'\,\mathrm{d}\phi'. \tag{4.47}$$

Thus, it can be seen that the torsion (4.41) does not contribute to the Burgers vector. Only the K_0-terms in (4.40) give a contribution to the Burgers vector (4.46). The plot is given in figure 7.

4.2 The case $\ell_2 = \ell_3$

Now we turn to a special case of the gauge theoretical edge dislocation we have found. We obtained two internal lengths ℓ_2 and ℓ_3. If we require that the dislocation density (4.41)

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has to be zero, the internal lengths must fulfill $\ell_2 = \ell_3$. Therefore, we may reduce them to only one internal length. If the internal lengths ℓ_2 and ℓ_3 are equal, we find

$$\ell_2 = \ell_3 : \qquad \Rightarrow \qquad \gamma = \frac{\lambda}{2\nu} = \frac{\mu}{1-2\nu}, \qquad A = B. \tag{4.48}$$

Thus, in this case the parameter γ is expressed in terms of μ and ν. Eventually, the non-vanishing components of the stress tensor are

$$\sigma_{xx} = -\frac{\mu b}{\pi(1-\nu)}\frac{y}{r^2}\Big[1 - \frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big] \tag{4.49}$$

$$\sigma_{xy} = \frac{\mu b}{\pi(1-\nu)}\frac{x}{r^2}\Big[1 - \frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big] \tag{4.50}$$

$$\sigma_{zz} = -\frac{\mu\nu b}{\pi(1-\nu)}\frac{y}{r^2}\Big[1 - \frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big]. \tag{4.51}$$

Thus, $\sigma_{yx} = 0$ and $\sigma_{yy} = 0$. The non-vanishing components of the elastic distortion are

$$\beta_{xx} = -\frac{b}{2\pi}\frac{y}{r^2}\Big[1 - \frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big] \tag{4.52}$$

$$\beta_{xy} = \frac{b}{2\pi}\frac{x}{r^2}\Big[1 - \frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big] \tag{4.53}$$

$$\beta_{yx} = \frac{\nu b}{2\pi(1-\nu)}\frac{x}{r^2}\Big[1 - \frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big] \tag{4.54}$$

$$\beta_{yy} = \frac{\nu b}{2\pi(1-\nu)}\frac{y}{r^2}\Big[1 - \frac{r}{\ell_2}K_1\Big(\frac{r}{\ell_2}\Big)\Big]. \tag{4.55}$$

Because the components β_{yx} and β_{yy} are compatible, they can be expressed in terms of a displacement field u_y according to

$$\beta_{yx} = u_{y,x}, \qquad \beta_{yy} = u_{y,y}, \qquad u_y = \frac{\nu b}{2\pi(1-\nu)}\Big[\ln r + K_0\Big(\frac{r}{\ell_2}\Big)\Big]. \tag{4.56}$$

Let us compare this type of edge dislocation with the edge dislocation originally introduced by Volterra [62] (see also [66]). The displacement field of a Volterra edge dislocation reads $u_x = b/(2\pi)\arctan y/x$ and $u_y = b/(2\pi)\ln r$. But the corresponding force stress of this displacement field calculated in the theory of (symmetric) elasticity does not fulfill the force equilibrium condition because it produces line forces at the dislocation line and the dilatation and the hydrostatic pressure are zero. These drawbacks are not acceptable from the physical point of view. In the limit $\ell_2 \to 0$, our displacement field u_y has a pre-factor $\nu/(1-\nu)$ instead of 1. But the modified pre-factor is necessary to satisfy the force equilibrium condition with asymmetric force stresses and to give the correct dilatation. Thus, we have found the gauge theoretical version of an edge dislocation of Volterra type which is able to remove the original drawbacks of the edge dislocation of Volterra type. The limiting value $\nu = 1/2$ defines incompressibility and we obtain the pre-factor 1. In this limit λ and γ tend to infinity. Thus, in the limit of incompressibility the Volterra dislocation is also valid.

The non-vanishing component of the dislocation density tensor is given by

$$\alpha_{xz} = \frac{b}{2\pi}\frac{1}{\ell_2^2}K_0\Big(\frac{r}{\ell_2}\Big). \tag{4.57}$$

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Thus, this dislocation density has cylindrical symmetry and has the same form as a screw dislocation (3.11). The localized pseudomoment stress reads

$$H_{xz} = \frac{\mu b}{\pi(1-\nu)}\, K_0\Big(\frac{r}{\ell_2}\Big), \qquad H_{zx} = -\frac{\nu\mu b}{\pi(1-\nu)}\, K_0\Big(\frac{r}{\ell_2}\Big). \tag{4.58}$$

It is the pseudomoment stress of an edge dislocation with cylindrical symmetry (see also [23]). The corresponding Burgers vector is

$$b(r) = b\Big[1 - \frac{r}{\ell_2}\, K_1\Big(\frac{r}{\ell_2}\Big)\Big]. \tag{4.59}$$

4.3 The case $\gamma = 0$: symmetric force stresses

Now we want to revert the case of symmetric force stresses. If we set $\gamma = 0$ (symmetric force stress), we obtain from Eqs. (3.5), (2.49), (4.14), (2.50) and (2.51):

$$a_3 = -\frac{a_1}{2}, \qquad a_2 = \frac{1+\nu}{1-\nu}\, a_1, \qquad \ell_2^2 = \ell_1^2 = \frac{a_1}{2\mu}, \qquad \ell_3^2 \to \infty, \qquad B = 0. \tag{4.60}$$

It is obvious that a_3 in (4.60) violates the condition of positive dislocation core energy (2.21). Thus, the condition of nonnegative dislocation core energy requires asymmetric force stresses. The condition (4.60) has only one independent internal length and it is the choice introduced by Lazar [23]. With $B = 0$ we recover from Eqs. (4.24)–(4.28) the force stress earlier calculated by [23] and [63, 64] in the framework of a gauge theory with symmetric force stresses and strain gradient elasticity, respectively. Also the distortions (4.32)–(4.35) become symmetric with $B = 0$ and the elastic rotation (4.37) becomes zero.

4.4 The case $\gamma \to \infty$

If we set $\gamma \to \infty$, then we obtain from Eqs. (3.5), (2.49), (4.14), (2.50) and (2.51):

$$a_3 \to \infty, \qquad a_2 = \frac{1+\nu}{1-\nu}\, a_1, \qquad \ell_2^2 = \ell_1^2 = \frac{a_1}{2\mu}, \qquad \ell_3^2 = \frac{a_1}{4\mu(1-\nu)}, \qquad B = \frac{\mu b}{\pi}. \tag{4.61}$$

This case does not violate the positive definiteness of the dislocation core energy (2.21). So we obtain the relation between ℓ_2 and ℓ_3

$$\ell_3^2 = \frac{1}{2(1-\nu)}\, \ell_2^2, \qquad \frac{\ell_2}{2} \le \ell_3 \le \ell_2. \tag{4.62}$$

The skewsymmetric stress (4.31) converts to

$$\sigma_{[xy]} = \frac{\mu b}{\pi}\, \frac{x}{r^2}\Big[1 - \frac{r}{\ell_3} K_1\Big(\frac{r}{\ell_3}\Big)\Big] \tag{4.63}$$

and the skewsymmetric distortion (4.37) vanishes $\beta_{[xy]} = 0$. All other quantities can be easily calculated with (4.61). Because the elastic rotation is zero in the present case the corresponding energy converts to

$$\lim_{\gamma\to\infty} W(\boldsymbol{\beta}, \operatorname{curl}\boldsymbol{\beta}) = W(\boldsymbol{e}, \operatorname{curl}\boldsymbol{e}) = W(\boldsymbol{e}, \operatorname{curl}\boldsymbol{e}^{\mathrm{P}}), \qquad \boldsymbol{e} = \operatorname{sym}\boldsymbol{\beta} \tag{4.64}$$

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where $\boldsymbol{e}$ and $\boldsymbol{e}^{\mathrm{P}}$ denote the elastic and plastic strains. Such a theory looks like a theory of strain gradient plasticity (see, e.g., Gurtin and Anand [65].) In such a theory the trace of the dislocation density tensor is zero

$$\alpha_{jj} = -\epsilon_{jkl} e_{jk,l} = \epsilon_{jkl} e^{\mathrm{P}}_{jk,l} = 0. \tag{4.65}$$

Thus, it is a theory with two constraints

$$\begin{aligned} \gamma \to \infty : \qquad & \beta_{[ij]} = 0 \\ a_3 \to \infty : \qquad & \alpha_{jj} = 0. \end{aligned} \tag{4.66}$$

5 Conclusion

In this paper, we have investigated the (static) translational gauge theory of dislocations. We have used the most general linear and isotropic constitutive relations between the force stress and elastic distortion tensors and the pseudomoment stress and dislocation density (torsion) tensors. Thus, the linear theory possesses six material coefficients and the force and pseudomoment stress tensors are asymmetric. We have found four characteristic length scales for the gauge theory of dislocation given in terms of the six material coefficients to account for size effects of small-scale problems. We have derived the conditions of positive energy for the six material coefficients. We observed that the so-called Einstein choice of the three coefficients of the pseudomoment stress tensor violates the positive definiteness of the dislocation core energy. This fact demonstrates that the three-dimensional Einsteinian gauge approach is not a realistic model for dislocations. We have observed that in the gauge theory of dislocations moment stress is the response to contortion and pseudomoment stress is the response to pseudomoment stress. Using Fourier transform, we have calculated the three-dimensional Green tensor of our so-called master equation of the force stress tensor. In the case of generalized plane strain the four characteristic lengths reduce to only two lengths and the number of independent material coefficients simplifies to four. Later, we have solved the anti-plane strain problem of a screw dislocation and the plane strain problem of an edge dislocation in the framework of gauge theory. In this turn, we have found new solutions for all physical state quantities of a screw dislocation and an edge dislocation.

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A Screw dislocation in asymmetric elasticity

In asymmetric elasticity the force stress tensor is asymmetric and fulfills the force equilibrium condition

$$\sigma^0_{ij,j} = 0, \qquad \sigma^0_{ij} \neq \sigma^0_{ji}. \tag{A.1}$$

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The elastic distortion of a Volterra screw dislocation is given by [66]

$$\beta^0_{zx} = -\frac{b}{2\pi}\frac{y}{r^2}, \qquad \beta^0_{zy} = \frac{b}{2\pi}\frac{x}{r^2}, \qquad r^2 = x^2 + y^2. \tag{A.2}$$

Thus, the present problem is of anti-plane strain type. The non-vanishing component of the dislocation density of a screw dislocation reads

$$\alpha^0_{zz} = \beta^0_{zy,x} - \beta^0_{zx,y}. \tag{A.3}$$

It is the incompatibility condition. In terms of force stresses it reads

$$\alpha^0_{zz} = \frac{1}{\mu+\gamma}(\sigma^0_{zy,x} - \sigma^0_{zx,y}). \tag{A.4}$$

For a straight screw dislocation it has the form

$$\alpha^0_{zz} = b\,\delta(x)\delta(y). \tag{A.5}$$

In terms of the Prandtl stress function F^0, the incompatibility condition is given by

$$\Delta F^0 = (\mu+\gamma) b\,\delta(x)\delta(y). \tag{A.6}$$

The Prandtl stress function is nothing but the Green function of the two-dimensional Laplace equation. It reads [39]

$$F^0 = \frac{(\mu+\gamma)b}{2\pi}\,\ln r. \tag{A.7}$$

B Edge dislocation in asymmetric elasticity

In the case of straight edge dislocations the equations of incompatibility take the form

$$\alpha^0_{xz} = \beta^0_{xy,x} - \beta^0_{xx,y} \tag{B.1}$$
$$\alpha^0_{yz} = \beta^0_{yy,x} - \beta^0_{yx,y}. \tag{B.2}$$

We introduce the following combinations [51]

$$A_1 := \alpha^0_{yz,x} - \alpha^0_{xz,y} = \beta^0_{yy,xx} + \beta^0_{xx,yy} - \beta^0_{xy,xy} - \beta^0_{yx,xy} \tag{B.3}$$
$$A_2 := -\alpha^0_{xz,x} - \alpha^0_{yz,y} = \beta^0_{yx,yy} - \beta^0_{xy,xx} + \beta^0_{xx,xy} - \beta^0_{yy,xy}. \tag{B.4}$$

Expressing the elastic distortions in terms of force stresses and using $\sigma^0_{ij,j} = 0$ and $\sigma^0_{zz} = \nu(\sigma^0_{xx} + \sigma^0_{yy})$, we obtain

$$A_1 = \frac{1-\nu}{2\mu}\,\Delta(\sigma^0_{xx} + \sigma^0_{yy}) \tag{B.5}$$
$$A_2 = \frac{1}{2\mu}(\sigma^0_{xx,xy} - \sigma^0_{yy,xy}) + \frac{\gamma-\mu}{4\mu\gamma}(\sigma^0_{xy,yy} - \sigma^0_{yx,xx}) + \frac{\gamma+\mu}{4\mu\gamma}(\sigma^0_{yx,yy} - \sigma^0_{xy,xx}). \tag{B.6}$$

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Because we deal with asymmetric force stresses we use the stress function ansatz given by Mindlin for couple-stress theory [50]

$$\sigma_{ij}^0 = \begin{pmatrix} \partial_{yy}^2 f^0 - \partial_{xy}^2 \Psi^0 & -\partial_{xy}^2 f^0 + \partial_{xx}^2 \Psi^0 & 0 \\ -\partial_{xy}^2 f^0 - \partial_{yy}^2 \Psi^0 & \partial_{xx}^2 f^0 + \partial_{xy}^2 \Psi^0 & 0 \\ 0 & 0 & \nu \Delta f^0 \end{pmatrix} \tag{B.7}$$

where f^0 and Ψ^0 are stress functions of second order. The stress function ansatz (B.7) is the generalization of the stress function ansatz with the Airy stress function f^0 for symmetric stresses. If Ψ^0 is zero, (B.7) reduces to the usual expression for the stresses in terms of the Airy stress function f^0. Equations (B.5) and (B.6) are reduced to the following inhomogeneous bi-harmonic equations

$$\Delta\Delta\, f^0 = \frac{2\mu}{1-\nu} A_1 \tag{B.8}$$

$$\Delta\Delta\, \Psi^0 = -\frac{4\mu\gamma}{\mu+\gamma} A_2. \tag{B.9}$$

Because we want to consider a straight edge dislocation with the Burgers vector $b = b_x$, the dislocation density tensor has the form

$$\alpha_{yz}^0 = 0, \qquad \alpha_{xz}^0 = b\,\delta(x)\delta(y). \tag{B.10}$$

So we obtain

$$\Delta\Delta\, f^0 = -\frac{2\mu b}{1-\nu}\, \partial_y[\delta(x)\delta(y)] \tag{B.11}$$

$$\Delta\Delta\, \Psi^0 = \frac{4\mu\gamma b}{\mu+\gamma}\, \partial_x[\delta(x)\delta(y)]. \tag{B.12}$$

Since the two-dimensional Green function of the bi-harmonic equation is

$$\Delta\Delta\, G = \delta(x)\delta(y), \qquad G = \frac{1}{8\pi}\, r^2 \ln r \tag{B.13}$$

the solutions of (B.11) and (B.12) are the following Airy stress functions [39]

$$f^0 = -\frac{\mu b}{4\pi(1-\nu)}\, \partial_y(r^2 \ln r) \tag{B.14}$$

$$\Psi^0 = \frac{\mu\gamma\, b}{2\pi(\mu+\gamma)}\, \partial_x(r^2 \ln r). \tag{B.15}$$

(B.14) is the well-known Airy stress function for an edge dislocation with Burgers vector b_x and (B.15) looks like an Airy stress function for an edge dislocation with Burgers vector b_y with a different pre-factor.

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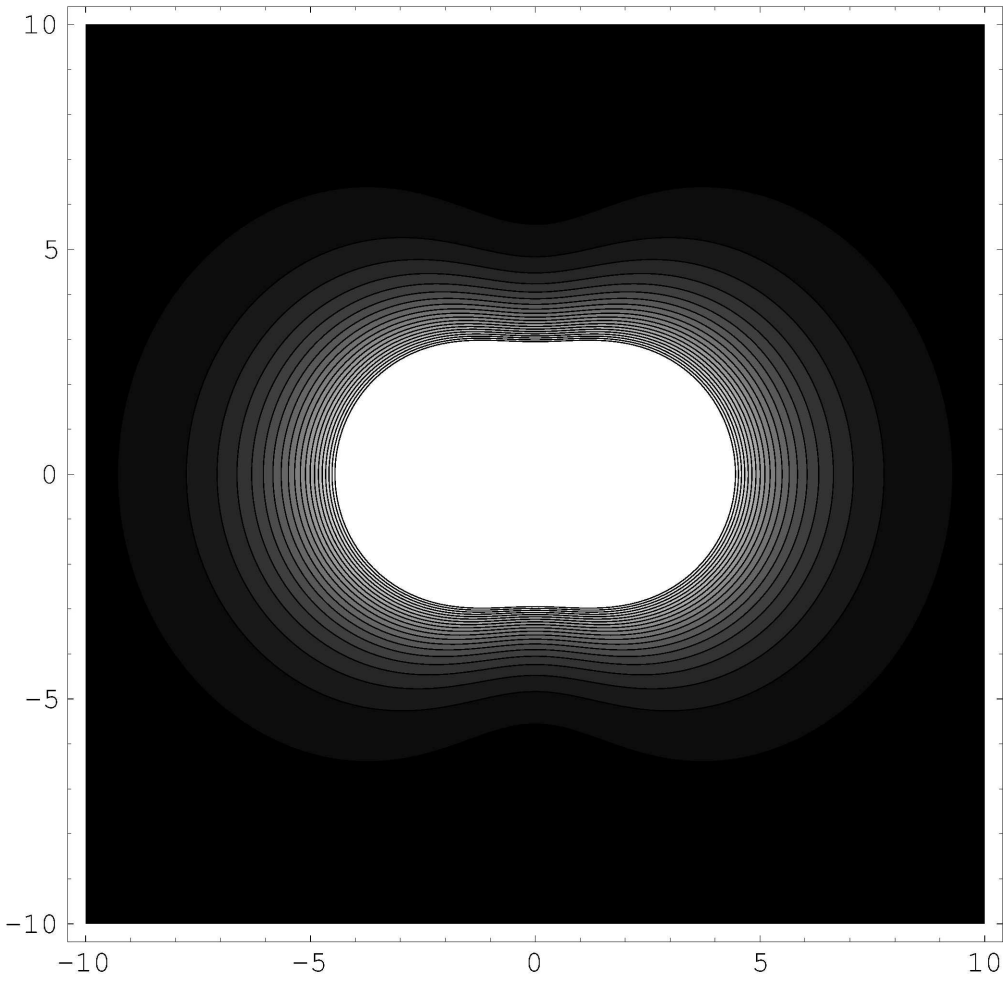

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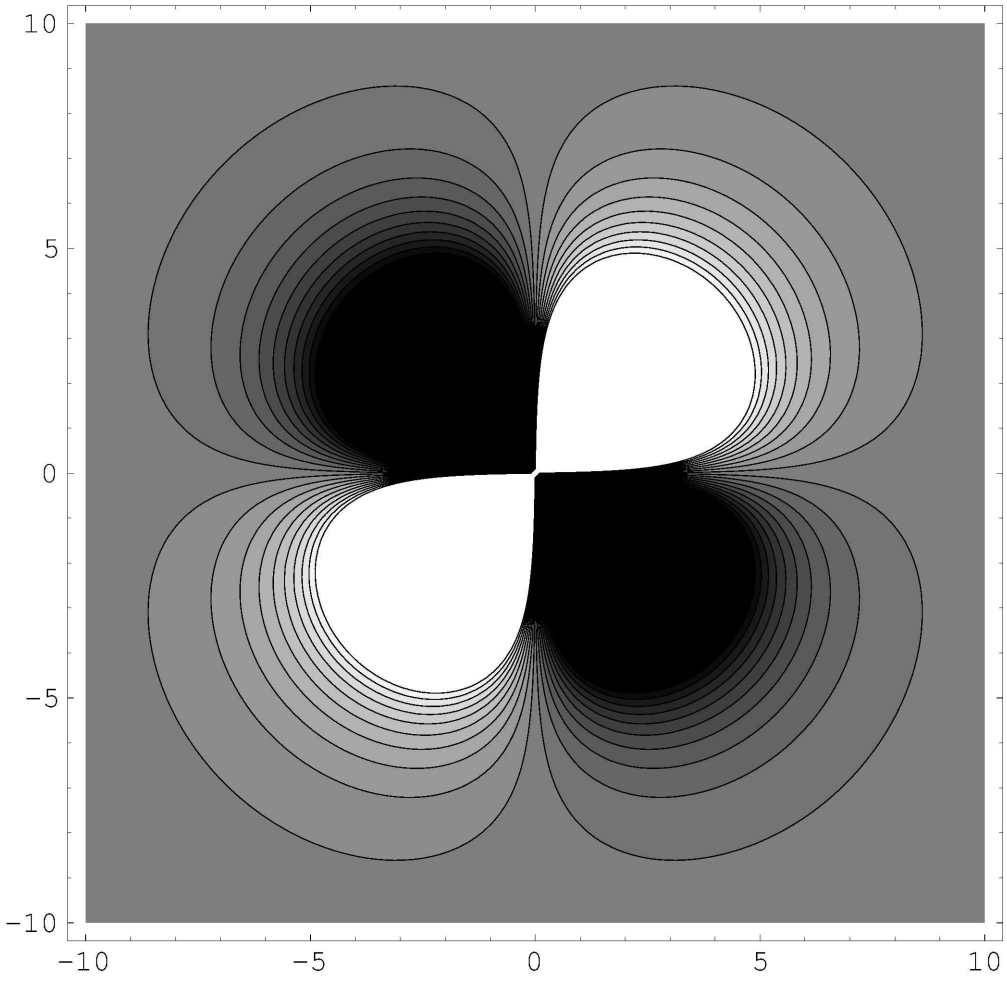

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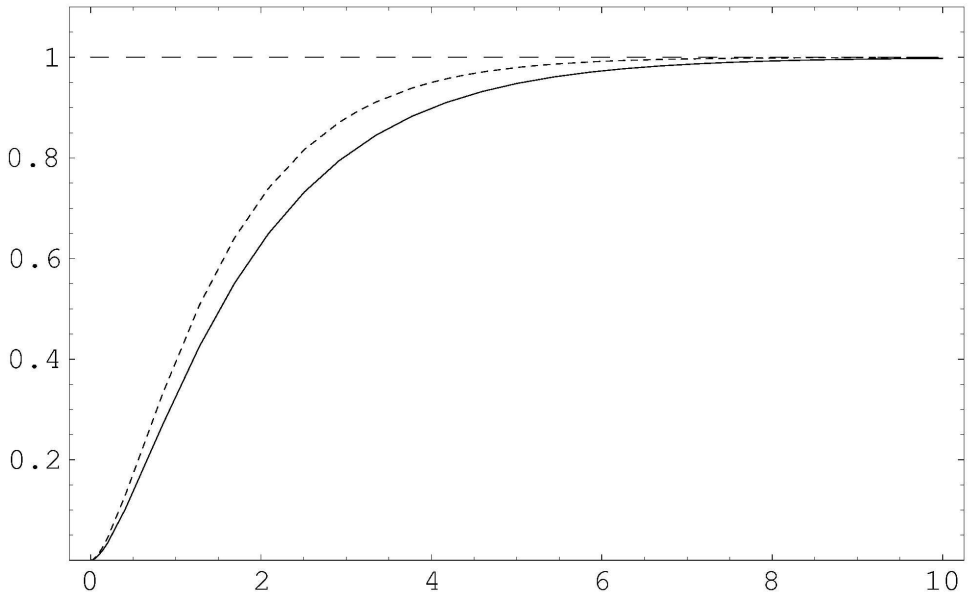

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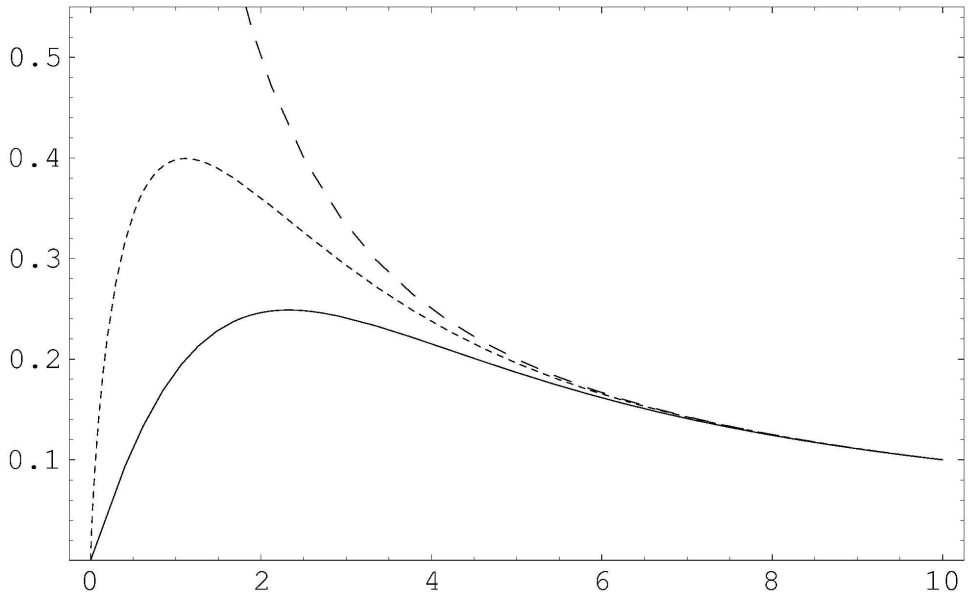
eview Only



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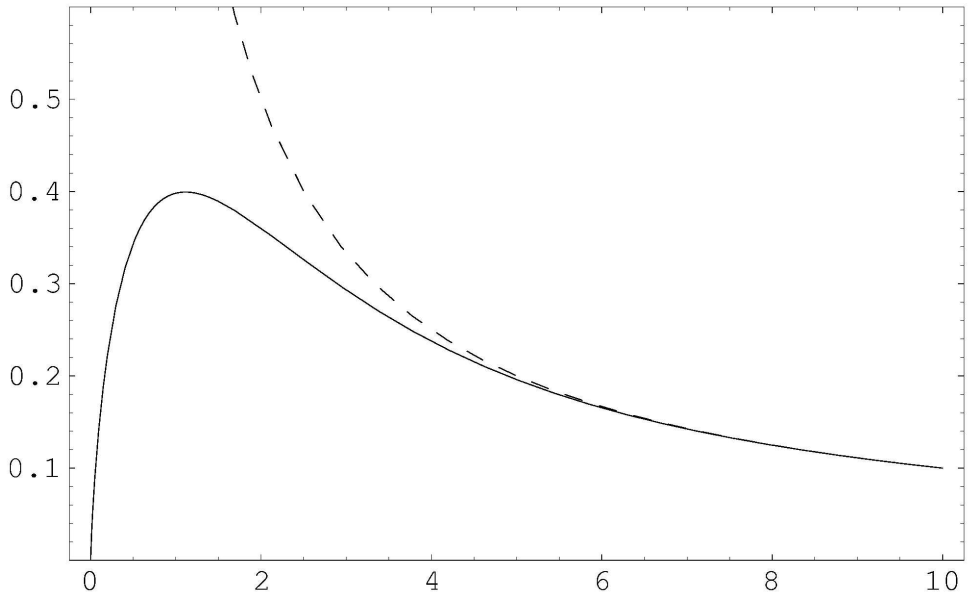

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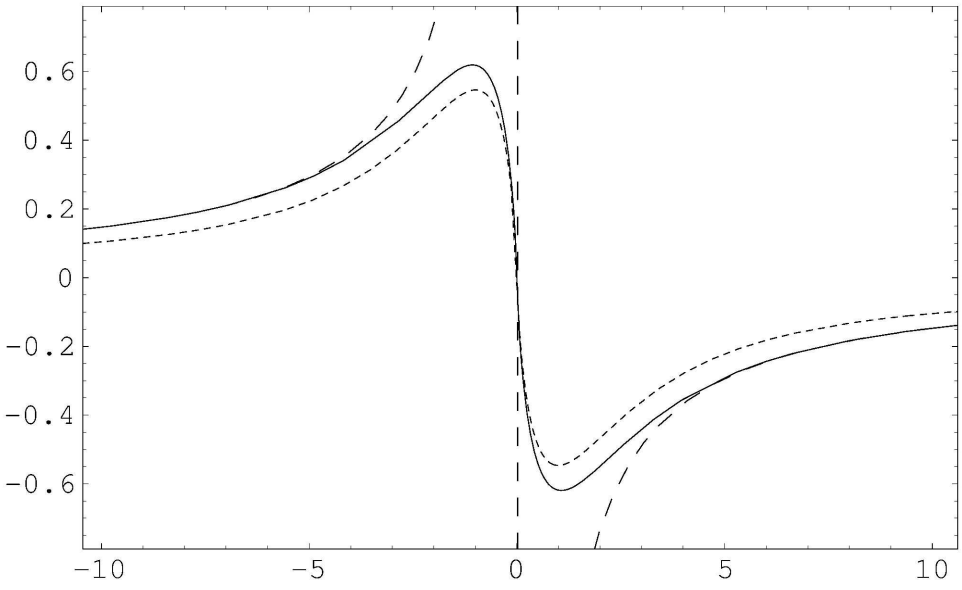

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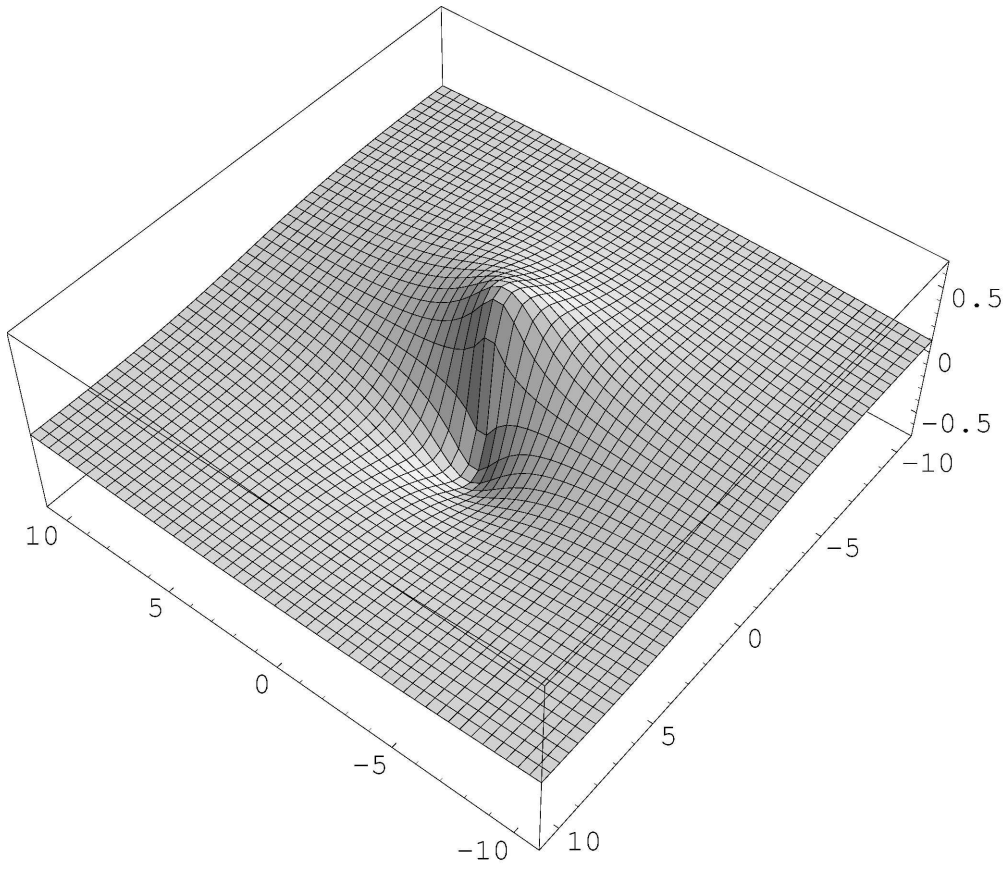
For Review Only



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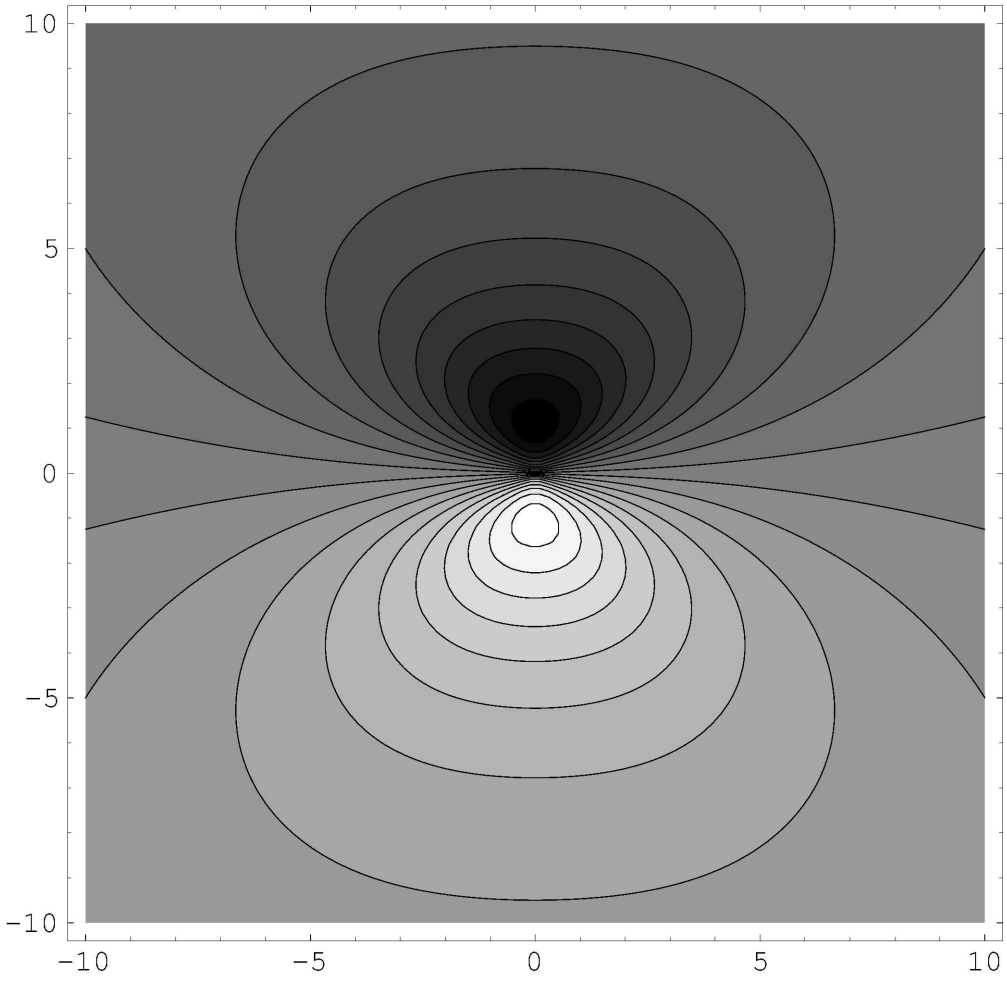
For Peer Review Only



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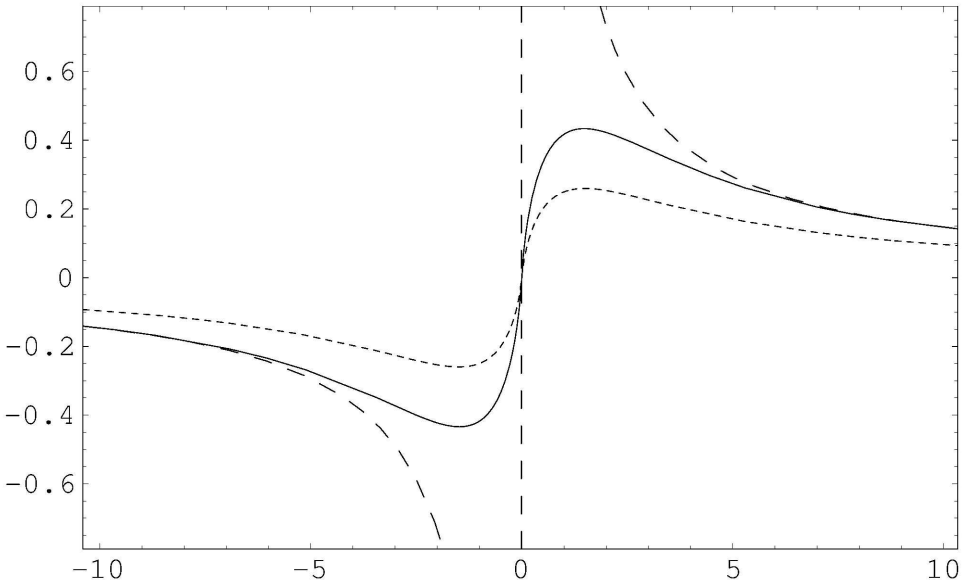

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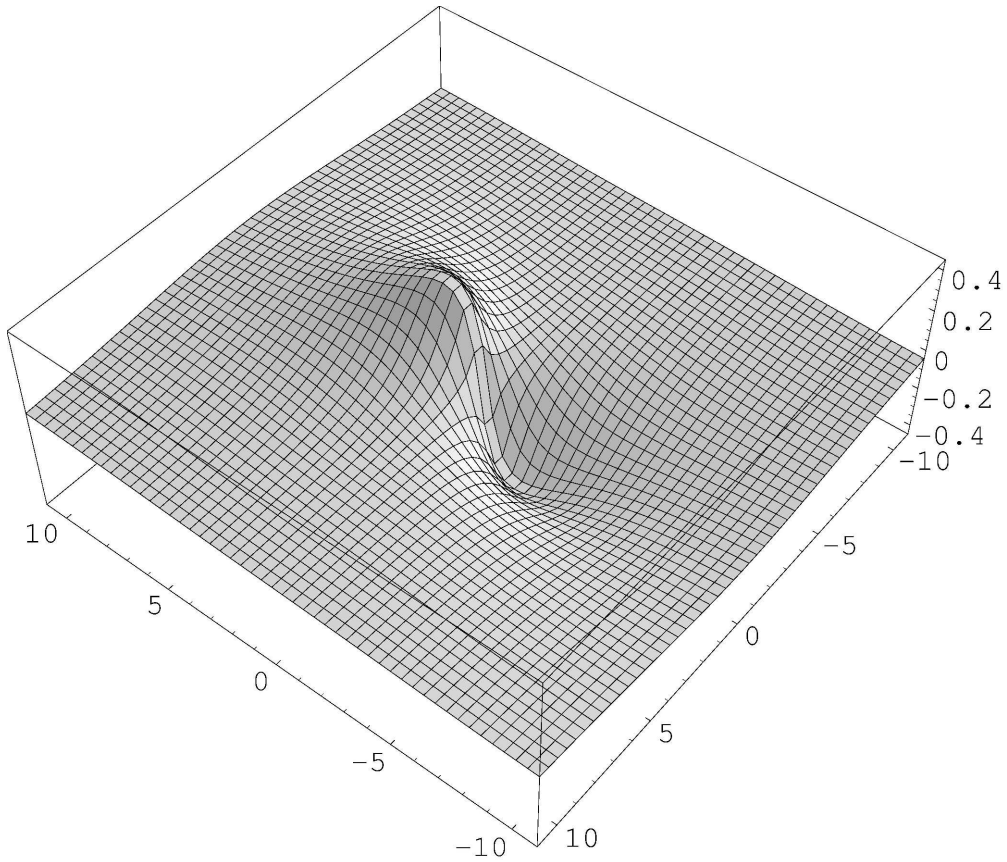

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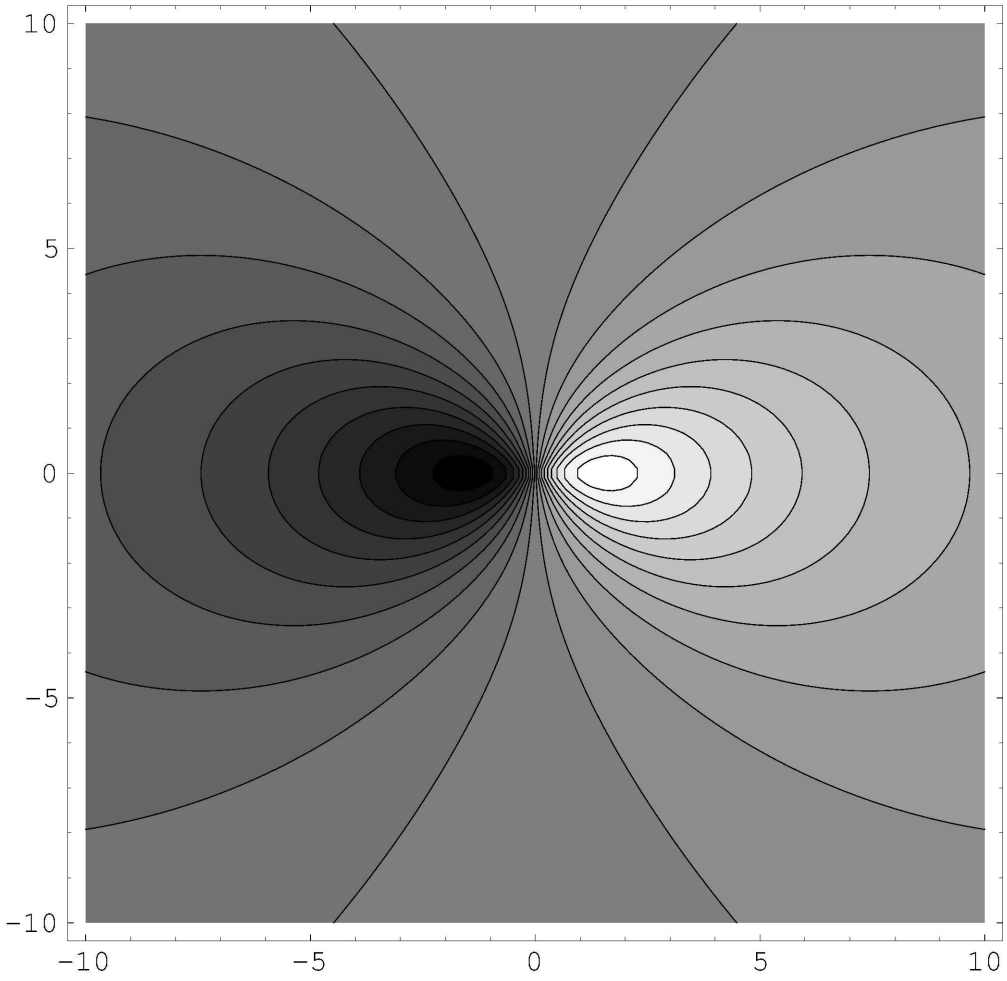

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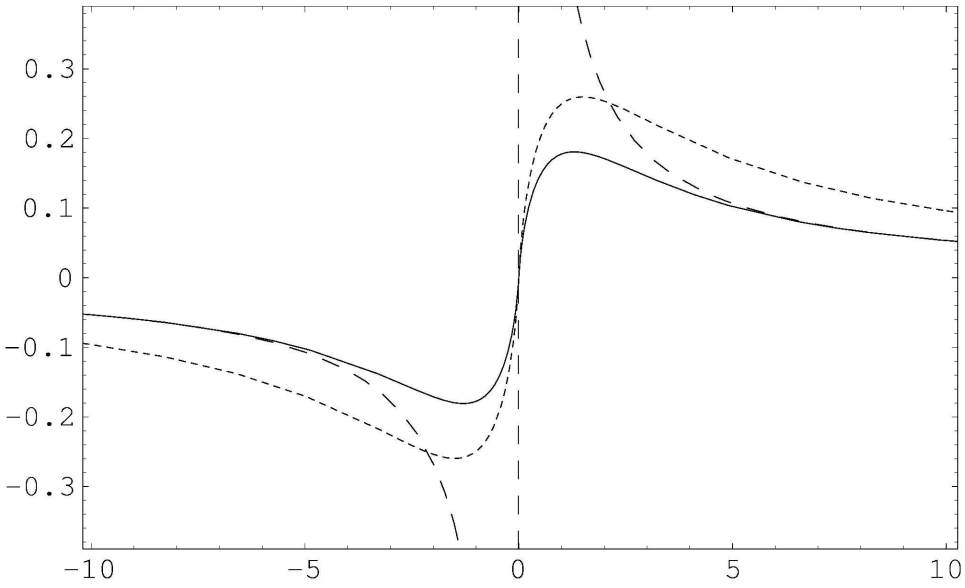

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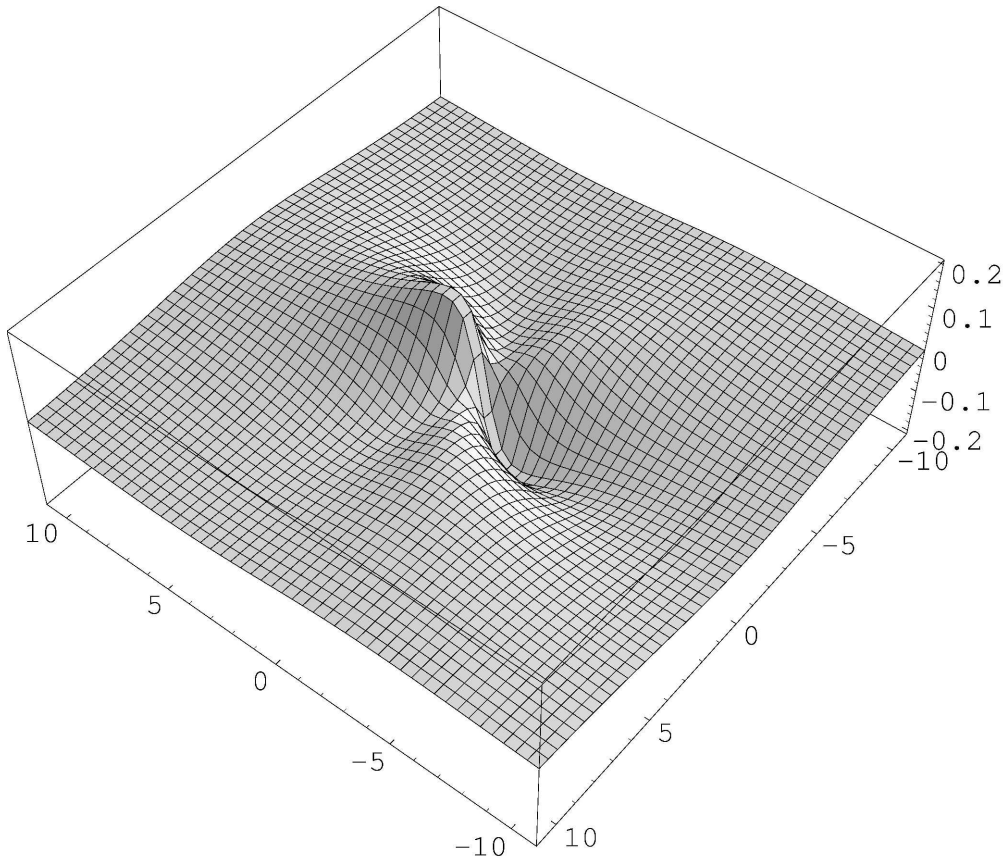

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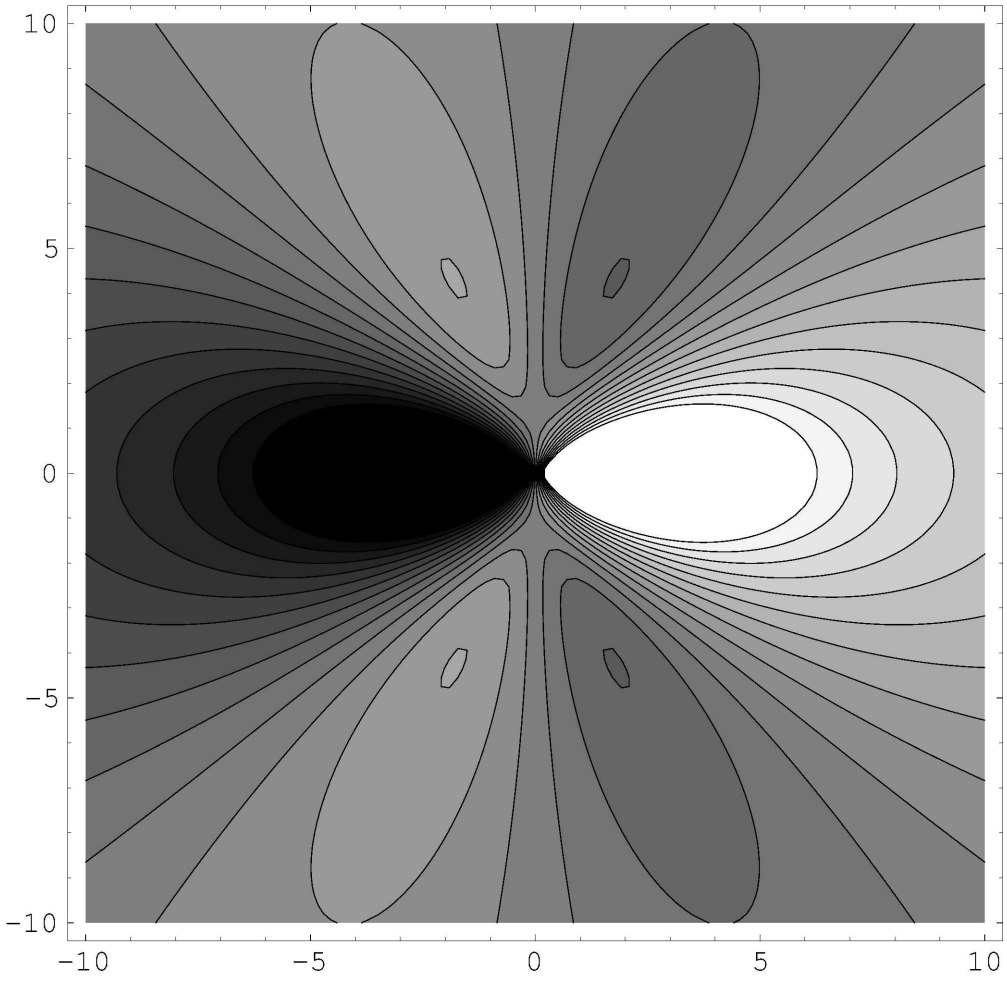

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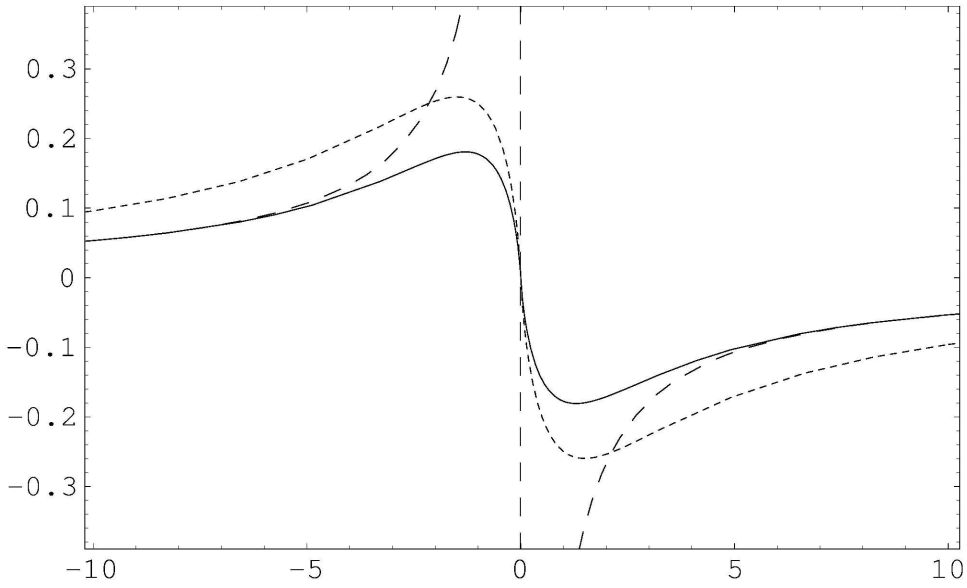

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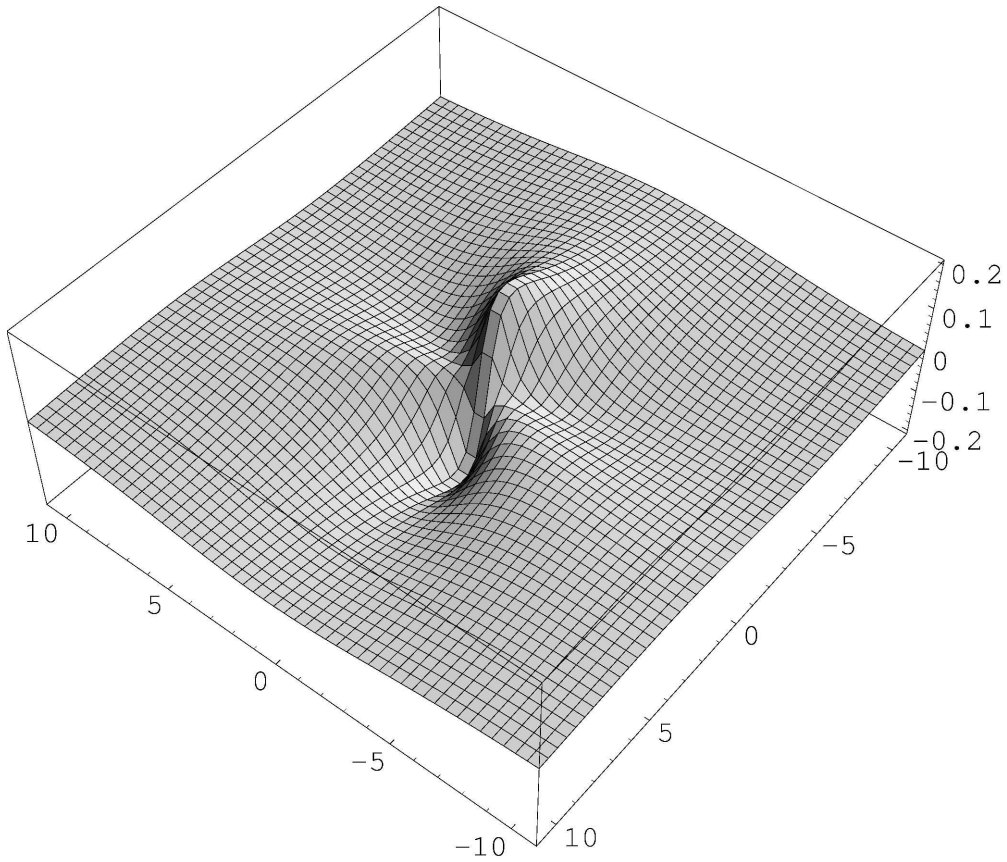

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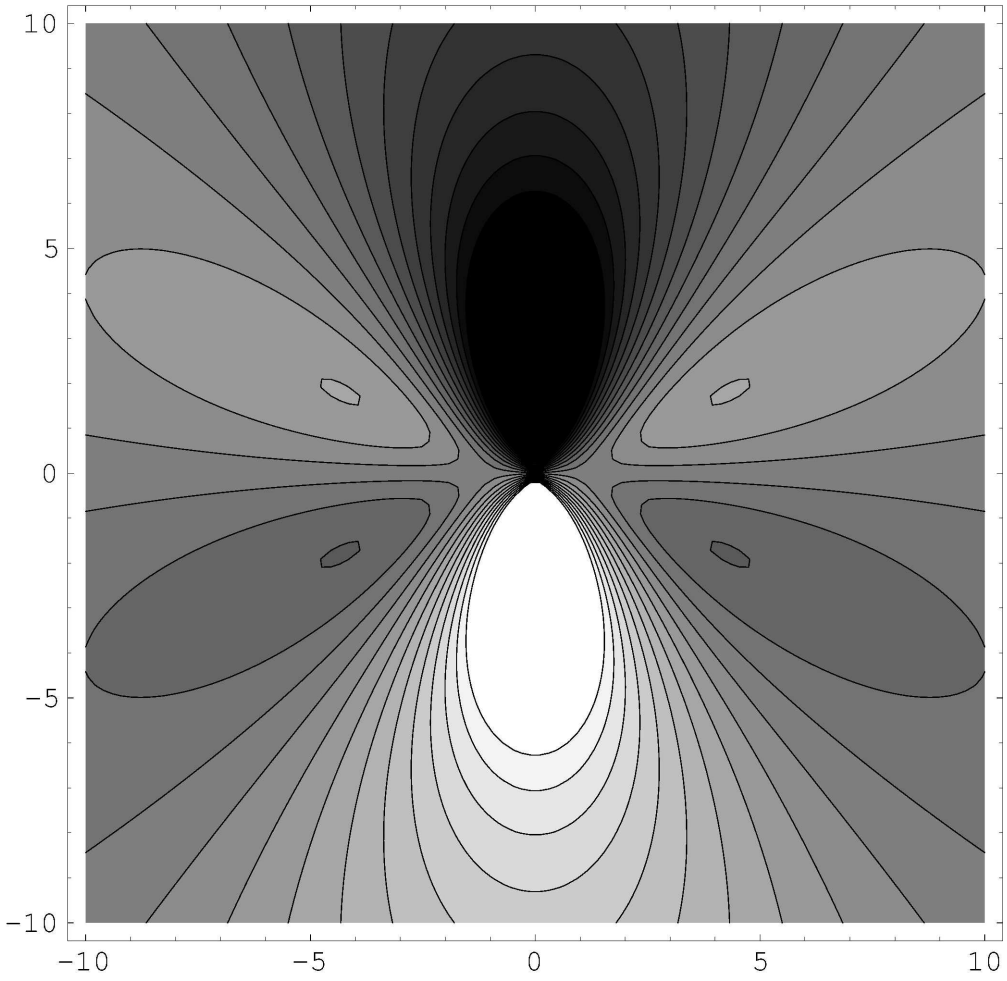

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