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Elastic stress field beneath an arbitrary axisymmetric punch

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Abstract.

The nanoindentation test is commonly used for the local determination of mechanical properties (hardness, Elastic modulus, ...) and also to study the initial stages of plasticity (dislocation nucleation, dislocation interaction mechanisms) at nanometer scale. In that case, the determination of the elastic stress field beneath the indenter is of primary interest. The purpose of this paper is to give an analytical expression for the elastic stress and strain fields beneath an axisymmetric punch. Most solutions, in the literature, are given for simple indenter shapes, such as flat, conical or spherical indenters. Complete

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solution for arbitrary indenter profile described by a power law, with exponent integer or not, is proposed here. The stress is given as the real part of complex analytical expressions.

## 1. Introduction

Over the last two decades, the nanoindentation test has become a common technique for the determination of local mechanical properties such as hardness and elastic modulus [1-4]. The principle consists in applying a low load with an indenter on the surface while the progressive penetration depth is monitored. The main experimental parameter for hardness and elastic modulus determination is the contact area between the indenter and the surface under load. Several models have been proposed to determine this crucial parameter through the analysis of the unloading curve shape. One of the first attempts has been given by Doerner and Nix [6] who considered that the contact area was not modified during an elastic unloading, as for a flat punch. In a more realistic approach, , Oliver and Pharr [2] proposed to determine the initial value of the contact area by taking into account its variation during elastic unloading. For this purpose, the principle of the equivalent indenter, assumed to be parabolic was used. This principle, initially introduced by Solomon [7], suggests that the elastic unloading of an indenter on a plastically deformed surface is equivalent to that of a “modified” indenter giving the same displacement when pressed against a plane. However, the shape of the equivalent indenter which depends on the plastic behaviour of the material, is more complex than a paraboloid and

should be determined for each unloading. One method consists in describing the equivalent indenter shape by a polynomial [4,8-9] in cylindrical coordinates  $(\rho, z)$ :

$$z(\rho) = \sum_n C_n \rho^n \quad (1)$$

In this case, for a rigid indenter the load  $P$  and the displacement  $h$  are functions of the contact radius  $a$  [10-11] :

$$h(a) = \sqrt{\pi} \sum_n \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)} C_n a^n \quad (2)$$

$$P(a) = 2E_r \sqrt{\pi} \sum_n \frac{n}{n+1} \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)} C_n a^n \quad (3)$$

where  $\Gamma$  is the gamma function and  $E_r$  is the reduced modulus defined by

$$E_r = \frac{E}{1-\nu^2} \quad (4)$$

with  $E$  and  $\nu$  the Young's modulus and the Poisson's ratio of the sample respectively. It has been shown elsewhere [4], that with the stiffness:

$$S = \frac{dP}{dh} \quad (5)$$

the penetration-stiffness relation can be fitted by a polynomial:

$$h(S) = \frac{\sqrt{\pi}}{(2E_r)^n} \sum_n \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)} C_n S^n \quad (6)$$

allowing the determination of the  $C_n$  coefficients and the  $n$  exponents.

Unloading curves can often be fitted by simple power law

$$P(h) = \alpha(h - h_0)^m \quad (7)$$

where  $h_0$  is the residual penetration depth. In this case, the shape function of the equivalent indenter reduces to a single term with a non integer exponent  $n$

$$z(r) = C r^n \quad (8)$$

The corresponding load-displacement relationship being:

$$P(h) = \frac{2E_r}{(\sqrt{\pi}C)^{\frac{1}{n}+1}} \frac{n}{n+1} \frac{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} h^{\frac{1}{n}+1} \quad (9)$$

The relation between the unloading curve exponent  $m$  and the indenter shape exponent  $n$  is thus given by:

$$n = \frac{1}{m-1} \quad (10)$$

This method is commonly used to estimate the true penetration depth and thus the contact area under load  $A_c$ . The hardness  $H$  and the reduced elastic modulus  $E_r$  are then easily determined through the equations

$$H = \frac{P}{A_c} \quad (11)$$

$$S = 2E_r \frac{\sqrt{A_c}}{\pi} \quad (12)$$

Beside the determination of elasto-plastic characteristics of materials, nanoindentation is also of fundamental interest for studying elementary deformation mechanisms and incipient plasticity, involving small material volumes and few dislocations[12-21]. The precise knowledge of the stress field generated in the material is required to understand the individual behaviour of these defects [20]. The surface stress has been previously calculated in the

case of an elastic contact between a flat surface and an axisymmetric indenter [22-23], or a more complex indenter described by a power function with a non integer exponent  $n$  [4,5,24]. This latter case is of great interest since it can be applied to any unloading curve, even for plastically deformed material, through the concept of the equivalent indenter [7].

In the following, analytical solutions will be proposed for the calculation of the complete stress field in the case of arbitrary shaped axisymmetric indenters.

**2. Theoretical results**

Close form expressions of the stress field, either on the surface or in the volume, beneath simple indenters, including flat punches, cones or spheres have been previously published [25-30], but very few results are available for more complex profiles, except a recent work of Schwarzer [31-32] proposing a partial solution in the case of axisymmetric punches described by polynomials. However this theory, named the “extended Hertzian theory”, is limited to even power of  $r$  which cannot be applied to real equivalent indenters with non integer exponent. For more complex shapes, the only results, to our best knowledge, have been obtained by finite element methods (FEM) [33-34].

In this work, Sneddon’s [22,35] results, will be used to determine the complete stress fields in the quite general case of power laws with exponents integer or not.

Sneddon [36] showed that the normal stress and displacements can be derived from the  $f(t)$  function, solution of the dual integral equation:

$$u_z(r) = -\frac{(\lambda + 2\mu)a}{\mu} \int_0^\infty f(t) J_0(rt) dt = f(r) = h - ca^n r^n \quad \text{for } r \leq 1 \quad (13)$$

$$\sigma_z(r) = 2(\lambda + \mu) \int_0^\infty t f(t) J_0(rt) dt = 0 \quad \text{for } r \geq 1 \quad (14)$$

With

$$r = \frac{\rho}{a} \quad (15)$$

And  $\lambda$  and  $\mu$  the Lamé's coefficient. To find a  $f(t)$  function satisfying the preceding equations, Sneddon proposed to introduce the function  $g(u)$  obeying:

$$f(t) = \int_0^1 g(u) \cos(ut) du \quad (16)$$

With this form of function  $g(u)$  we find that:

$$\int_0^\infty t f(t) J_0(rt) dt = \int_0^\infty J_0(rt) dt \int_0^1 t g(u) \cos(ut) du \quad (17)$$

and integrating by parts:

$$\int_0^\infty t f(t) J_0(rt) dt = g(1) \int_0^\infty J_0(rt) \sin(t) dt - \int_0^\infty J_0(rt) dt \int_0^1 \frac{\partial g(u)}{\partial u} \sin(ut) du \quad (r > 1) \quad (18)$$

Interchanging the order of integration in the double integral and knowing that:

$$\int_0^\infty J_0(rt) \sin(ut) dt = 0 \quad (19)$$

we see that equation (14) is satisfied and we must have  $g(1)=0$  since:

$$\sigma_z = 0 \quad \text{for } r=1$$

Equation (13) can be written:

$$-\frac{(\lambda + 2\mu)a}{\mu} \int_0^\infty J_0(rt) dt \int_0^1 g(u) \cos(ut) du = h - ca^n r^n \quad (20)$$



If we interchange the order of integration and make use of:

$$\int_0^{\infty} J_0(r t) \cos(u t) dt = \frac{1}{\sqrt{r^2 - u^2}} \quad (r > u) \quad (21)$$

$$\int_0^{\infty} J_0(r t) \cos(u t) dt = 0 \quad (0 < r < u) \quad (22)$$

we see that  $g(u)$  is a solution of the integral equation:

$$-\frac{(\lambda + 2\mu)a}{\mu} \int_0^r \frac{g(u) du}{\sqrt{r^2 - u^2}} = h - Ca^n r^n \quad (23)$$

We can try for  $g(u)$ :

$$g(u) = A + B u^n \quad (24)$$

and equation (20) may be written :

$$\int_0^r \frac{g(u) du}{\sqrt{r^2 - u^2}} = A \frac{\pi}{2} + B \frac{\sqrt{\pi} \Gamma(\frac{n+1}{2})}{2 \Gamma(\frac{n}{2} + 1)} r^n \quad (25)$$

We finally get:

$$A = -\frac{2}{\pi} \frac{\mu}{(\lambda + 2\mu)a} h \quad (26)$$

$$B = \frac{\mu}{(\lambda + 2\mu)a} \frac{2 \Gamma(\frac{n}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{n+1}{2})} Ca^n \quad (27)$$

Since  $g(1)=0$  we have:

$$h = \sqrt{\pi} \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n+1}{2})} Ca^n \quad (28)$$

and:

$$g(u) = -\frac{2\mu}{(\lambda + 2\mu)a} \frac{\Gamma(\frac{n}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{n+1}{2})} Ca^n (1 - u^n) \quad (29)$$

$$f(t) = -\frac{2\mu}{(\lambda + 2\mu)a} \frac{\Gamma(\frac{n}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{n+1}{2})} C a^n \int_0^1 (1-u^n) \cos(tu) du \quad (30)$$

Let  $k$  be :

$$k = -\frac{2\mu}{(\lambda + 2\mu)a} \frac{\Gamma(\frac{n}{2} + 1)}{\sqrt{\pi} \Gamma(\frac{n+1}{2})} C a^n \quad (31)$$

We see that  $f(t)$  is the real part of the complex quantity:

$$f^*(t) = k \int_0^1 (1-u^n) e^{-iut} du \quad (32)$$

To compute the stress field we have to evaluate the following complex integrals:

$$I^* = \int_0^\infty e^{-zt} f^*(t) J_0(rt) dt \quad (33)$$

$$K^* = \int_0^\infty e^{-zt} t f^*(t) J_1(rt) dt \quad (34)$$

and the components of the stresses are the real parts of the complex stresses:

$$\sigma_z^* = 2k(\lambda + \mu) \left( -\frac{\partial I^*}{\partial z} + z \frac{\partial^2 I^*}{\partial z^2} \right) \quad (35)$$

$$\tau_r^* = 2k(\lambda + \mu) z \frac{\partial^2 K^*}{\partial z^2} \quad (36)$$

$$\sigma_\theta^* = -2k\lambda \frac{\partial I^*}{\partial z} + \frac{2\mu}{r} k \left( K^* + \frac{\mu + \lambda}{\mu} z \frac{\partial K^*}{\partial z} \right) \quad (37)$$

$$\sigma_r^* + \sigma_\theta^* = -2k(\lambda + \mu) \left( \frac{2\lambda + \mu}{\lambda + \mu} \frac{\partial I^*}{\partial z} + z \frac{\partial^2 I^*}{\partial z^2} \right) \quad (38)$$

These components of the stress are easily derived from the  $I^*$  and  $K^*$  integrals

and their derivatives relative to  $z$ :

$$I^* = \int_0^1 (1-u^n) du \int_0^\infty e^{-(z+iu)t} J_0(rt) dt \quad (39)$$

$$K^* = \int_0^1 (1-u^n) du \int_0^\infty e^{-(z+iu)t} J_1(rt) dt \quad (40)$$

If we note :

$$p_0^* = \int_0^\infty e^{-(z+iu)t} J_0(rt) dt = \frac{1}{\sqrt{r^2 + (iu + z)^2}} \quad (41)$$

$$p_1^* = \int_0^\infty e^{-(z+iu)t} J_1(rt) dt = \frac{1}{r} - \frac{iu + z}{r\sqrt{r^2 + (iu + z)^2}} \quad (42)$$

we can write:

$$I^* = I_0^* - I_n^* \quad (43)$$

$$K^* = K_0^* - K_n^* \quad (44)$$

with:

$$I_0^* = \int_0^1 p_0^* du \quad (45)$$

$$K_0^* = \int_0^1 p_1^* du \quad (46)$$

$$I_n^* = \int_0^1 p_0^* u^n du \quad (47)$$

$$K_n^* = \int_0^1 p_1^* u^n du \quad (48)$$

Finally the integrals and their derivatives take the form:

$$I_0^* = i \ln \frac{z + \sqrt{r^2 + z^2}}{i + z + \sqrt{r^2 + (i + z)^2}} \quad (49)$$

$$\frac{\partial I_0^*}{\partial z} = \frac{i}{\sqrt{r^2 + z^2}} - \frac{i}{\sqrt{r^2 + (i + z)^2}} \quad (50)$$

$$\frac{\partial^2 I_0^*}{\partial z^2} = \frac{iz - 1}{[r^2 + (i + z)^2]^{\frac{3}{2}}} - \frac{iz}{(r^2 + z^2)^{\frac{3}{2}}} \quad (51)$$

$$K_0^* = \frac{1 - i\sqrt{r^2 + z^2} + i\sqrt{r^2 + (i + z)^2}}{r} \quad (52)$$

$$\frac{\partial K_0^*}{\partial z} = \frac{iz-1}{r[r^2+(i+z)^2]^{\frac{3}{2}}} - \frac{iz}{r(r^2+z^2)^{\frac{3}{2}}} \quad (53)$$

$$\frac{\partial^2 K_0^*}{\partial z^2} = \frac{ir(i+z)}{[r^2+(i+z)^2]^{\frac{3}{2}}} - \frac{irz}{(r^2+z^2)^{\frac{3}{2}}} \quad (54)$$

Let us note, with  $F_1(a,b,c,d,z_1,z_2)$  the Appell hypergeometric function:

$$F_{n+1,\frac{1}{2}} = F_1\left(1+n, \frac{1}{2}, \frac{1}{2}, 2+n, \frac{1}{r+iz}, \frac{1}{-r+iz}\right) \quad (55)$$

$$F_{n+2,\frac{1}{2}} = F_1\left(2+n, \frac{1}{2}, \frac{1}{2}, 3+n, \frac{1}{r+iz}, \frac{1}{-r+iz}\right) \quad (56)$$

$$F_{n+1,\frac{3}{2}} = F_1\left(1+n, \frac{3}{2}, \frac{3}{2}, 2+n, \frac{1}{r+iz}, \frac{1}{-r+iz}\right) \quad (57)$$

$$F_{n+2,\frac{3}{2}} = F_1\left(2+n, \frac{3}{2}, \frac{3}{2}, 3+n, \frac{1}{r+iz}, \frac{1}{-r+iz}\right) \quad (58)$$

$$F_{n+1,\frac{5}{2}} = F_1\left(1+n, \frac{5}{2}, \frac{5}{2}, 2+n, \frac{1}{r+iz}, \frac{1}{-r+iz}\right) \quad (59)$$

$$F_{n+2,\frac{5}{2}} = F_1\left(2+n, \frac{5}{2}, \frac{5}{2}, 3+n, \frac{1}{r+iz}, \frac{1}{-r+iz}\right) \quad (60)$$

$$F_{n+3,\frac{5}{2}} = F_1\left(3+n, \frac{5}{2}, \frac{5}{2}, 4+n, \frac{1}{r+iz}, \frac{1}{-r+iz}\right) \quad (61)$$

The real part of the  $F_1$  function can be easily calculated with mathematical packages. Then, we can write:

$$I_n^* = \frac{F_{n+1,\frac{1}{2}}}{(1+n)\sqrt{r^2+z^2}} \quad (62)$$

$$\frac{\partial I_n^*}{\partial z} = \frac{1}{(r^2+z^2)^{\frac{3}{2}}} \left( \frac{z F_{n+1,\frac{3}{2}}}{n+1} + \frac{F_{n+2,\frac{3}{2}}}{n+2} \right) \quad (63)$$

$$\frac{\partial^2 I_n^*}{\partial z^2} = \frac{1}{(n+1)(r^2+z^2)^{\frac{5}{2}}} [2(r^2+z^2) F_{n+1,\frac{3}{2}} - 3r^2 F_{n+1,\frac{5}{2}}] \quad (64)$$

$$K_n^* = \frac{1}{(n+1)r} - \frac{1}{r\sqrt{r^2+z^2}} \left( \frac{z}{n+1} F_{n+1, \frac{1}{2}} + \frac{i}{n+2} F_{n+2, \frac{1}{2}} \right) \quad (65)$$

$$\frac{\partial K_n^*}{\partial z} = -\frac{r}{(n+1)(r^2+z^2)^{\frac{3}{2}}} F_{n+1, \frac{3}{2}} \quad (66)$$

$$\frac{\partial^2 K_n^*}{\partial z^2} = \frac{3r}{(r^2+z^2)^{\frac{5}{2}}} \left( \frac{z}{n+1} F_{n+1, \frac{5}{2}} + \frac{i}{n+2} F_{n+2, \frac{5}{2}} \right) \quad (67)$$

The formulae are undetermined when  $n$  is an integer, but in that case we can use a recurrence relationship:

$$I_n^* = -\frac{\sqrt{r^2+(i+z)^2}}{n} + \frac{n-1}{n}(r^2+z^2)I_{n-2}^* + \frac{2n-1}{n}izI_{n-1}^* \quad (68)$$

$$K_n^* = \frac{1}{(n+1)r} [1+i\sqrt{r^2+(i+z)^2} - ni(r^2+z^2)I_{n-1}^* + nziI_n^*] \quad (69)$$

with:

$$I_0^* = i \ln \frac{z + \sqrt{r^2+z^2}}{i+z+\sqrt{r^2+(i+z)^2}} \quad (70)$$

$$I_1^* = \sqrt{r^2+z^2} - \sqrt{r^2+(i+z)^2} - z \ln \frac{z + \sqrt{r^2+z^2}}{i+z+\sqrt{r^2+(i+z)^2}} \quad (71)$$

$$K_0^* = \frac{1-i\sqrt{r^2+z^2} + i\sqrt{r^2+(i+z)^2}}{r} \quad (72)$$

Again, following equations (35-38), the stress components can be derived from the real parts and their first and second derivatives relative to  $z$  of the expressions (43) and (44).

When the exponent  $n$  is an integer, the  $I$  and  $K$  integrals, real parts of the  $I^*$  and  $K^*$  complex ones, can be calculated knowing that:

$$\sqrt{r^2+(i+z)^2} = a + ib \quad (73)$$

$$\text{Log}[i + z + \sqrt{r^2 + (i + z)^2}] = \ln \sqrt{(z + a)^2 + (1 + b)^2} + i \text{ArcCos}\left(\frac{z + a}{\sqrt{(z + a)^2 + (1 + b)^2}}\right) \quad (74)$$

with:

$$a = \frac{\sqrt{\sqrt{r^2 + z^2 - 1)^2 + 4z^2} + (r^2 + z^2 - 1)}{\sqrt{2}} \quad (75)$$

$$b = \frac{\sqrt{\sqrt{r^2 + z^2 - 1)^2 + 4z^2} - (r^2 + z^2 - 1)}{\sqrt{2}} \quad (76)$$

For  $n=0$  (flat punch) equations (68) and (69) fail but we have [22]:

$$f(t) = \frac{\sin t}{t} \quad (77)$$

and the I and K integrals are the imaginary parts of:

$$I^* = \int_0^\infty e^{(-z+i)t} J_0(rt) dt = \frac{1}{\sqrt{r^2 + (i + z)^2}} \quad (78)$$

$$K^* = \int_0^\infty e^{(-z+i)t} J_1(rt) dt = \frac{\sqrt{r^2 + (i + z)^2} - (i + z)}{r} \quad (79)$$

For example, the normal stress is the imaginary part of:

$$\sigma_z^* = 2k(\lambda + \mu) \left( \frac{1}{\sqrt{r^2 + (i + z)^2}} + \frac{z(i + z)}{[r^2 + (i + z)^2]^{\frac{3}{2}}} \right) \quad (80)$$

When  $n$  is different from zero we can use the (68) to (69) equations.

For example, for  $n=1$  (cone):

$$\sigma_z^* = -k(\lambda + \mu) \left[ \frac{z}{\sqrt{r^2 + z^2}} - \frac{z}{\sqrt{r^2 + (i + z)^2}} + \ln \frac{i + z + \sqrt{r^2 + (i + z)^2}}{z + \sqrt{r^2 + z^2}} \right] \quad (81)$$

For  $n=2$  (paraboloid) :

$$\sigma_z^* = -k(\lambda + \mu) \left[ 2i \sqrt{r^2 + (i + z)^2} - \frac{2z}{\sqrt{r^2 + (i + z)^2}} \right] \quad (82)$$

For  $n=3$ : (83)

$$\sigma_z^* = k(\lambda + \mu) \left[ \frac{3z}{2} \sqrt{r^2 + z^2} - \frac{3(i+z)}{2} \sqrt{r^2 + (i+z)^2} - \frac{3z}{\sqrt{r^2 + (i+z)^2}} + \frac{3(r^2 + 2z^2)}{2} \ln \frac{i+z + \sqrt{r^2 + (i+z)^2}}{z + \sqrt{r^2 + z^2}} \right]$$

Values of the I and K integrals, for n=1 to 5, and listed in appendix.

Using equations (61) to (64), the normal stresses on the surface (z=0) and below the surface (z=a/2), are plotted in Figures 1 and 2 for non integer values of n: 1.5, 2.5 and 3.5.

### 3. Conclusion

In this paper, analytical expressions have been proposed for the determination of the complete stress field generated by an arbitrary axisymmetric indenter described by an exponent n, integer or not. This calculus is not restricted to elastic indentation, but can also be applied through the concept of the equivalent indenter to elasto-plastic contact. The elastic stress field can thus be readily obtained once the exponent m of the elastic unloading curve has been determined. One of the main interests of this study is to give the ability to determine the forces acting on individual dislocations involved in the imprint formation. As an example, this result can be applied in discrete dislocation dynamic (DDD) study of the indentation test. This result, combined with recent insights in the description of the complete microstructure generated during a nanoindentation test, represent a significant step towards the comprehension of the early plastic deformation mechanisms in materials.

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## Appendix

$n=0$ :

$$I_0^* = -i \ln \frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

$$K_0^* = \frac{1}{r} - \frac{i\sqrt{r^2+z^2}}{r} + \frac{i\sqrt{r^2+(i+z)^2}}{r}$$

$n=1$ :

$$I_1^* = \sqrt{r^2+z^2} - \sqrt{r^2+(i+z)^2} + z \ln \frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

$$K_1^* = \frac{1}{2r} + \frac{z}{2r} \sqrt{r^2+z^2} + \frac{i+z}{2r} \sqrt{r^2+(i+z)^2} - \frac{1}{2r} \ln \frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

$n=2$ :

$$I_2^* = \frac{3}{2} i z \sqrt{r^2+z^2} - \left( \frac{1}{2} + \frac{3 i z}{2} \right) \sqrt{r^2+(i+z)^2} - \left( \frac{i r^2}{2} - i z^2 \right) \ln \frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

$$K_2^* = \left( -\frac{2 i r}{3 r} + \frac{i z^2}{3 r} \right) \sqrt{r^2+z^2} + \left( \frac{i}{3 r} + \frac{2 i r}{3} - \frac{z}{3 r} + \frac{i z^2}{3 r} \right) \sqrt{r^2+(i+z)^2} - i r z \ln \frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

n=3:

$$I_3^* = \left(\frac{2r^2}{3} - \frac{11z^2}{6}\right)\sqrt{r^2+z^2} - \left(\frac{1}{3} + \frac{2r^2}{3} - \frac{11z^2}{6} + \frac{5iz}{6}\right)\sqrt{r^2+(i+z)^2} + \left(\frac{3r^2z}{2} - z^3\right)\ln\frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

$$K_3^* = \left(\frac{13rz}{8} - \frac{z}{4r}\right)\sqrt{r^2+z^2} + \left(\frac{i}{4r} + \frac{3ir}{8} - \frac{z}{4r} - \frac{13rz}{8} - \frac{iz^2}{4r} + \frac{z^3}{4r}\right)\sqrt{r^2+(i+z)^2} - \left(\frac{3r^3}{8} - \frac{3rz^2}{2}\right)\ln\frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

n=4:

$$I_4^* = \left(\frac{55ir^2z}{24} + \frac{25iz^2}{12}\right)\sqrt{r^2+z^2} - \left(\frac{1}{4} + \frac{3r^2}{8} + \frac{7iz}{12r} + \frac{55ir^2}{24} - \frac{13z^2}{12} - \frac{25iz^3}{12}\right)\sqrt{r^2+(i+z)^2} - \left(\frac{3ir^4}{8} - 3ir^2z^2 + iz^4\right)\ln\frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

$$K_4^* = \left(-\frac{8ir^3}{15} + \frac{83irz^2}{30} - \frac{iz^4}{5r}\right)\sqrt{r^2+z^2} + \left(\frac{i}{5r} + \frac{4ir}{15} + \frac{8ir^3}{15} - \frac{z}{5r} - \frac{29rz}{30} - \frac{iz^2}{5r} - \frac{83irz^2}{30} + \frac{z^3}{5r} + \frac{iz^4}{5r}\right)\sqrt{r^2+(i+z)^2} + \left(-\frac{3ir^3z}{2} + 2irz^3\right)\ln\frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

n=5:

$$I_5^* = \left(\frac{8r^4}{15} - \frac{607r^2z^2}{120} + \frac{137z^4}{60}\right)\sqrt{r^2+z^2} - \left(\frac{1}{5} + \frac{4r^2}{15} + \frac{8r}{15} - \frac{47z^2}{60} + \frac{137z^4}{60} - \frac{607r^2z^2}{120} + \frac{9iz}{20} + \frac{161ir^2z}{120} - \frac{77iz^3}{60}\right)\sqrt{r^2+(i+z)^2} + \left(\frac{15r^4z}{8} - 5r^2z^3 + z^5\right)\ln\frac{i+z+\sqrt{r^2+(i+z)^2}}{z+\sqrt{r^2+z^2}}$$

$$\begin{aligned} K_s^* = & \left( \frac{113r^3z}{48} - \frac{97rz^3}{24} + \frac{z^5}{6r} \right) \sqrt{r^2 + z^2} \\ & + \left( \frac{1}{6r} + \frac{5ir}{24} + \frac{5ir^3}{16} - \frac{z}{6r} - \frac{17rz}{24} - \frac{113r^3z}{48} - \frac{iz^2}{6r} - \frac{41irz^2}{24} + \frac{z^3}{6r} + \frac{97rz^3}{24} + \frac{iz^4}{6r} - \frac{z^5}{6r} \right) \sqrt{r^2 + (i+z)^2} \\ & - \left( \frac{5r^5}{16} - \frac{15r^3z^2}{4} + \frac{5rz^4}{2} \right) \ln \frac{i+z+\sqrt{r^2 + (i+z)^2}}{z+\sqrt{r^2 + z^2}} \end{aligned}$$

## Figures captions

Figure1:

Normal stress on the surface,  $z=0$ , for indenters described by power laws with exponents  $n$ : 1.5, 2.5 and 3.5

Figure2:

Normal stress in volume,  $z=a/2$ , for  $n$ : 1.5, 2.5, and 3.5.



