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Colour symmetry of 25 colours in quasiperiodic patterns

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Abstract

Coloured periodic and quasiperiodic patterns with 25 colours and colour-symmetry were described for four-fold, six-fold, eight-fold, ten-fold, and twelve-fold symmetries. Eight-fold, twelve-fold and some four-fold patterns are related to coincidence site lattices (CSL). The procedures of delineation of coloured patterns with 25 colours must consider the property of a square number. Procedures for ten-fold patterns need special considerations and allow the description of superstructures.

Keywords: Quasicrystals, Colour Symmetry, Superstructure, Coincidence site lattice.

AMS Subject Classification:

1 Introduction

Colour symmetry was known before the discovery of quasicrystals in 1982 by D. Shechtman (published in 1984) [1]. In particular, black-and-white symmetry (combined with the symmetry operation of time inversion) was used to describe spin orientations ‘up’ and ‘down’ in magnetic materials. The black-and-white symmetry groups have been listed in the literature for periodic materials. The subject was dealt with in the textbook by Shubnikov and Koptsik [2] and other remarkable papers [3-7]. Coloured periodic patterns and colour groups can also be found in the standard textbook on tilings [8]. Shechtman’s discovery revived the interest in colour symmetry in the context of aperiodic structures.

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‘Colour’ in colour symmetry stands for any property, such as level or label, nucleus or numeral, atom or
adornment, and, of course, colour as well. ‘Symmetry’ means that a crystallographic symmetry operation on
some object may be accompanied by a global permutation of colours. The object may be a tiling, a crystal, a
quasicrystal, a model or something else. Therefore all colours should be equivalent, and their fractions
(relative frequencies) should be equal. Colour symmetry is strongly related to chemical and magnetic
superstructures and to coincidence site lattices.

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The number 25 has several properties which make the decoration of patterns attractive. There are several
rotational symmetries which allow a symmetric decoration with 25 colours. They are 4-, 5-, 6-, 8-, 10-, 12-,
15-, 20-, 24-, 25-, 40-, and 60-fold symmetry as described by M. Baake et al. [9-13], who derived the numbers
and its multiplicities by Dirichlet series and Dedekind zeta-functions. 25 is a square of a prime number, which
means that almost all decorations are based on the rank 2 as described by R. Lifshitz [14-19] in extensive
group theoretical descriptions including those of magnetic superstructures [17-19]. The rank describes the
dimensionality of the numbers indicating the colour. 25 is the square of the number 5, and thus is associated
with five-fold symmetry which is essential for icosahedral and decagonal quasicrystals. Some examples of
colour symmetry with rank 1 are described in the literature: 2 colours in ten-fold patterns [20], 11 colours in
ten-fold patterns [21], 17 colours in eight-fold patterns [22] and 13 colours in twelve-fold patterns [23]. An
example of rank 4 is described in [24] where the decoration with 16 colours of a ten-fold tiling is used to
model a superstructure of decagonal AlCoNi, and an example for rank 3 with 8 colours in seven-fold patterns
is discussed in [13].

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This paper is organized as follows: In Section 2 I briefly review some features of four-fold and six-fold
periodic patterns with 5 and 25 colours. This is followed (Section 3) by a discussion of 25 colours in eight-
and twelve-fold quasiperiodic patterns. The ten-(five-)fold symmetry requires a discussion in more detail.
Finally, some remarks on the decoration of various quasiperiodic four-fold grids and six-fold quasiperiodic
patterns will be made. Subsets are considered in Section 4, some general comments are added (Section 5).
The terms used in the present paper are defined or explained in the Appendix.

48 49 50 **2 Periodic patterns**

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The decoration of a square lattice with 5 colours is based on a well-known example for Pythagorean
Numbers [25]. The decoration is strongly related to a coincidence site lattice (CSL) with $\Sigma 5$. There exist two
enantiomorphic decorations. The procedure can be repeated for the derived sublattices to eventually end up
with 25 colours. The different combinations result in three decorations [9]. Two enantiomorphic colourings

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3 correspond to CSL decorations of the $\Sigma 25$ type; these two are the examples which are not based on rank 2.
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5 The third one is really based on rank 2; for instance, the first digit may count the horizontal progression (0
6 through 4) and the second one the vertical progression (0 through 4 as well). Decoration of vertices with 5 or
7 25 colours uses the operation mod 5, for 25 it is used for both digits. Only the CSL decoration of the square
8 lattice with $\Sigma 25$ is based on mod 25. Translation is based on adding numbers, rotation is based on
9 multiplication of the additive number by (well-defined) numbers; in case of rank 2 (and higher) it is based on
10 multiplication of the additive vector by matrices. However, there exists a variety of possibilities for the
11 numbers and matrices if a special colour symmetry is considered..

12
13 Figure 1 represents simultaneously numbers 0 through 24 and the corresponding colours. The numbers form
14 a so-called Devil's Magic Square. That means that in every 5×5 square of the tiling the sum of columns, rows
15 and diagonals is identical; it is 60 in the present case.
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[Insert Figure 1 about here]

The six-fold periodic symmetry allows a single decoration with 25 colours which is not related to any CSL and which is not very different from the decoration of the square lattice.

3 Quasiperiodic patterns

The multiplicity for eight-fold and twelve-fold patterns is 2 for both, while the rank of decoration is 2. Vertices and tiles can be decorated as well. The decoration of tiles is based on the average of the numbers of adjacent vertices using the mod 5 operation for both digits. Both decorations, of vertices and of tiles, are related to CSL and have two enantiomorphic solutions. As in any other CSL related coloured (two-dimensional) pattern, there are no colour mirror symmetries. First empirical investigations on CSLs of quasicrystals were published in [26-28]. The CSLs were put on a systematic basis [29, 30], more publications [31-37] followed. Figure 2 is a decoration of an eight-fold quasiperiodic pattern with a decoration of 'one of twentyfive' corresponding to a CSL lattice $\Sigma 25$. Decoration is shown for vertices and tiles as well. The corresponding illustration for a twelve-fold quasiperiodic pattern is given in Fig. 3.

[Insert figures 2 and 3 about here]

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The multiplicity for ten-fold patterns is 1. The colouring is not related to a CSL and mirror symmetries for colours do exist. Problems arise from the fact that vertices and tiles cannot be coloured simultaneously with 5 or 25 colours. It is empirically found that patterns with a single translation class can be decorated with 5 and therefore also with 25 colours at vertices. The decoration of the Penrose pentagon pattern (PPP) with 5 colours at vertices was demonstrated in 1987, published with numbers [38] and as a coloured figure afterwards [14, 15]. The sublattice of each one of the five colours forms a tiling [38-40] which is now called the Tübingen triangle tiling (TTT). As the TTT is formed by a single translation class, it can be decorated with 5 colours as well. This procedure can be performed again and again. Table 1 shows this hierarchy; the first column is dedicated to the parent tiling and its subsets of 5, 5^2 , 5^3 , ... colours. Their acceptance domain in internal space is a decagon. The second and the third columns refer to tilings with different acceptance domains, these can be derived from the decagons by stellated extensions [41].

‘[Insert Table 1 about here]’

There are other decagonal patterns formed by a single translation class, such as e.g. Kepler’s Aa patterns [42, 23], and all patterns formed by the zero-level of ‘ $\Gamma = \frac{1}{2}$ ’ patterns, including the so-called anti-Penrose pattern described by Kléman and Pavlovitch [43], the t_A and t_B patterns by Ingalls [44], and the ‘three-level’ binary tiling formed by rhombi [39].

The centres of decagons of Ingalls’ t_C [44] pattern form the zero-level of the anti-Penrose pattern. Therefore, its decagons can be decorated with 25 colours and with full colour symmetry (Fig. 4). Colouring of tiles in ten-fold patterns with 5 and 25 colours is not a simple procedure. The Penrose rhombus pattern is derivable from PPP; the PPP can be decorated at vertices as mentioned above. The fat rhombi are decorated by a vertex of the PPP, its colour can be taken for the decoration of the fat rhombus, the thin rhombi may be decorated according to the colour symmetry of the surroundings. Such coloured patterns with 5 colours were published in the literature [45, 23].

The centres of rhombi of the mentioned three-level tiling are at the zero-level and can therefore be decorated with 5 and 25 colours. The tiles may be coloured with the colour of the centre (see Fig. 5). The centres of all rhombi or of fat rhombi form novel tilings in the same manner as other special points of the Penrose rhombus pattern form the PPP. The novel tilings have a single translation class and are composed of different tiles including pentagons, ten-fold stars and large rhombi; the centres of the ten-fold stars form the TTT.

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3 A third type of special points within the rhombi is determined by crossing points of Ammann bar pentagrids.
4 Several five-colour symmetries were determined for different rhombus patterns from Ammann bar pentagrids
5 coloured with 5 colours, but no analogous solution for 25 colours has been found.
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10 '[Insert figure 4 and figure 5 as a coloured figure about here]'

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14 Four-fold (τ -related) Fibonacci-based and (σ -related) Octonacci-based (see e.g. [46-48]) grids and their
15 colourings have been discussed in the literature and at several meetings. Posters dealing with colour-
16 symmetry patterns were presented but not published. Hence, slightly different ideas concerning the definition
17 of colour symmetry exist. Any quasiperiodic sequence – including those with an integer scaling factor or
18 integer ratio of two lengths – can be used to construct a grid with four-fold symmetry. Several examples were
19 tested for the number 25 in the present work. It turned out that for all quasiperiodic patterns decorations
20 according to CSL Σ_{25} (with multiplicity 2) do exist. However, an 'unrotated' decoration with 25 colours as in
21 the periodic case was only found for Fibonacci-based grids and for those with an integer ratio of lengths; this
22 was especially found for the 'Copper Mean' with the ratio 2.
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30 There are also six-fold quasiperiodic patterns. A six-fold Fibonacci related tiling was constructed based
31 upon dualisation of Ammann bar tria-grids, 4 different LI (local isomorphism or local indistinguishability)
32 classes resulted [49]. A different method based on substitution of two equilateral triangles and two
33 enantiomorphic parallelograms was presented later [50]. In any case, the six-fold Fibonacci related patterns
34 can be coloured with 25 colours with multiplicity 1. As in the periodic case (Section 2), no CSL related
35 decoration exists for 25 colours. Other six-fold quasiperiodic grids and their duals decorated with 25 colours
36 are a challenge.
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44 **4 Subsets with five colours**

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46 All decorations with 25 colours based upon rank 2 can be subdivided into 5 subsets of 5 colours each. In all
47 cases there are 6 systems of formation. In the simple case of a periodic square lattice, these subsets are rows,
48 columns, 2 diagonals and two enantiomorphic sets related to a CSL with Σ_5 . This is based on the fact that the
49 field of 5×5 numbers can be separated by the additive vectors into 5 sets of 5 numbers each. Single sets are
50 formed by the additive vectors (0,1); (1,0); (1,1); (1,4); (1,2); and (2,1). All other pairs of numbers will result
51 in one of these sets due to the mod 5 argument.
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3 The correlation of these sets with with coloured patterns was checked. A similar result as for periodic
4 patters was found for all quasiperiodic four-fold lattices which are not related to $\Sigma 25$. In case of six-fold
5 periodic and Fibonacci related patterns these are 3 + 3 lines in different orientations. For ten-fold patterns
6 based on a single translation class 5 lines according to symmetry and one superstructure [38] are found. Only
7 lines and no superstructures were found for eight-fold, twelve-fold and $\Sigma 25$ related four-fold patterns.
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15 The present synopsis summarizes some key points for coloured periodic and quasiperiodic tilings in
16 Euclidean 2D geometry. There is no general problem to extend colour symmetry to higher dimensions;
17 visualisation may be more difficult. Extension to spherical geometry is almost trivial, whereas extension to
18 hyperbolic geometry opens a new field. Some colour symmetries are known from M. Escher's coloured wood
19 engravings 'Circle limit I – IV'; in recent papers [51, 52] special hyperbolic tilings and its colourings are
20 discussed. A symmetric decoration with 25 colours in hyperbolic space should be possible.
21 In all cases, superstructures of one or more colours are related to chemically or magnetically ordered (quasi-)
22 crystals. Sublattices formed by colour symmetry bear the opportunity to build up novel tilings.
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26 The problems described for 25 (and 5) colours are similar for 49 (and 7) colours in eight-, twelve- and
27 fourteen-(seven-)fold patterns. Features of 9 colours in eight-fold patterns are similar to those of 25 colours in
28 eight-fold patterns; however, the problem of 9 colours in twelve-fold patterns is similar to that of 25 (5)
29 colours in ten-fold patterns. Vertices can be decorated with 9 colours provided there is a single translation
30 class.
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34 The permutation of colours caused by a symmetry operation is assigned to so-called orbits [51]. Colours
35 permute cyclically or do not change within an orbit. A twelve-fold rotation separates 25 colours into 3 orbits,
36 one orbit with a single colour and two orbits with 12 colours each. In case of eight-fold rotation, 25 colours
37 are distributed on 4 orbits, one with a single colour and three with 8 colours. The separation at a ten-fold
38 centre is more complex, there are five orbits, one with a single colour, two with 2 colours and two with 10
39 colours. If there is no orbit with a single colour, the rotation centre can not be coloured, this was found for
40 five-fold centres and 25 colours; there are five orbits with 5 colours each. The distribution of 25 colours on
41 the orbits and the sequence of colours within an orbit follows the rules of forming subsets (Section 4) and
42 depends on the rotation centre.
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46 A question concerning the relation of colouring and projection may arise. In order to investigate this issue,
47 the coloured vertices should be lifted to superspace. Lifting is not a simple process; results are often not
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3 unique; it depends on the number of dimensions of the selected superspace. Even for a well-defined
4 superspace different lattices can result. The colouring of vertices is generally connected with a formation of
5 superlattices in superspace. In most cases the symmetry of superlattices is reduced with respect to the
6 symmetry of the superspace lattice. However, the symmetry along the projection direction survives. For
7 instance., each one of five colours may form a separated level in superspace. Colouring Ammann bar
8 pentagrids is based on the projection of a coloured square lattice with five colours of the $\Sigma 5$ type.
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14 15 16 **Acknowledgment**

17 I am grateful for the collaboration during the last years with several colleagues on both colour symmetries and
18 coincidence site lattices. Michael Baake, Uwe Grimm and Ron Lifshitz assisted in the preparation of the
19 present manuscript. Further advice of two unknown Referees and Guest Editor Shelomo Ben-Abraham is
20 appreciated.
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27 28 **Appendix**

29 In the following a glossary of a few used terms is listed.
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33 *Coincidence site lattice (CSL)* [26-37]. The common lattice points of two lattices of the same type after a
34 rigid rotation about a common lattice point. The fraction of common points of periodic lattices is given by
35 $1/\Sigma$. $\Sigma 5$ indicates that 1/5 of lattice points coincide. The Σ index is used to characterize coincidence grain
36 boundaries. In case of quasiperiodic lattices, the fraction of common points is additionally influenced by the
37 common area of the two acceptance domains in internal space. Several colour symmetries are characterized
38 by a relationship to CSL, the fraction of each colour is $1/\Sigma$. The CSL related colourings occur as
39 enantiomorph pairs and do not have a colour mirror symmetry, whereas the non-CSL type colourings have.
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48 *Colour Group* [15, 51]. A symmetry group of an uncoloured pattern is subdivided into several subgroups:
49 (i) operations leaving all colours unchanged; (ii) operations permuting systematically at least some colours
50 depending on the involved orbits; (iii) operations not being symmetry operations of the coloured pattern, these
51 can be mirror operations as found for several CLS type colourings.
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Colour Symmetry Operation [15, 51]. A symmetry operation which permutes systematically at least some colours, some others may be unchanged.

Enantiomorphic Pairs. If the mirror symmetry of the uncoloured crystal is broken by colouring (c.f. CSL), the pattern appears in right-handed and left-handed versions.

Multiplicity [9]. Number of different coloured versions of a pattern for a fixed number of colours.

PPP. Penrose pentagon pattern [8, 38].

Orbits of Colours [51]. Orbits are formed by symmetry operations and the number of different orbits is determined by the number of colours. Colours permute cyclically or do not change within a single orbit.

Pythagorean Numbers [25]. A set of natural numbers which fulfil the condition $a^2 + b^2 + c^2 \dots = d^2$, depending on the number of dimensions.

Rank. According to R. Lifshitz [15] it is the smallest number of permutations that can generate the colour permutation group. For the present purpose, it is used for the dimensionality of the numbers indicating the colour [13, 24].

Sigma Index Σ [25-37]. $1/\Sigma$ indicates the fraction (or relative frequency) of common lattice points of a CSL. $\Sigma=5$ is usually written as $\Sigma 5$.

Single Translation Class. The acceptance domain occupies only a single level in internal space.

Sublattice, Superstructure. A subset of lattice points (vertices) forms a novel lattice different from the parent lattice. All sublattices of a parent lattice form a superstructure. A single colour of a coloured lattice forms a sublattice.

TTT. Tübingen triangle tiling [38-49].

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Figure Captions

Fig. 1 Periodic square tiling simultaneously decorated with numbers 0 through 24 and corresponding colours. The numbers form a so-called Devil's Magic Square, in every 5×5 square of the tiling the sum of columns, rows and diagonals is 60.

Fig. 2 Eight-fold quasiperiodic pattern with a colour decoration 'one of twentyfive' at vertices and tiles. This corresponds to a coincidence lattice with $\Sigma 25$.

Fig. 3 Twelve-fold quasiperiodic pattern with a colour decoration 'one of twentyfive' to a CLS with $\Sigma 25$.

Fig. 4 Ingalls' ten-fold t_c pattern [44] decorated at decagons with 25 colours.

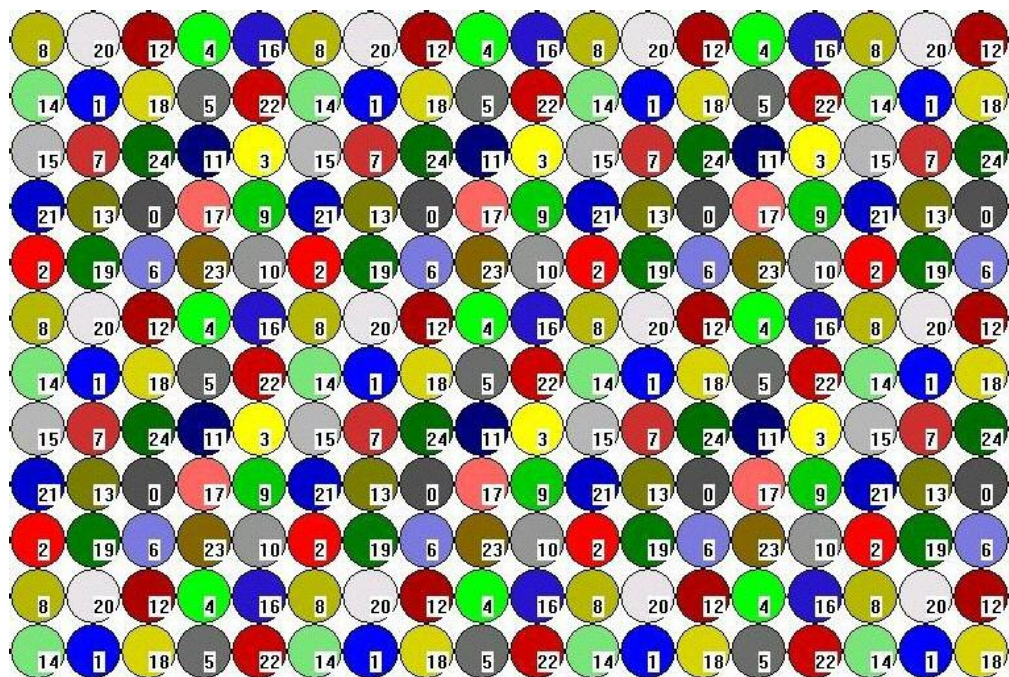
Fig. 5 A ten-fold quasiperiodic 'three-level-tiling' decorated with 25 colours.

Table Caption

Table 1. Hierarchy of ten-fold tilings with a single translation class and its subsets of $5, 5^2, 5^3, \dots$ colours. The second and third columns are designated to tilings of stellated extensions of acceptance domains. The dotted lines at the top and at the bottom indicate that the table can be extended in both directions to an infinitely large number of classes which have, however, a decreasing significance.

Table 1. Hierachy of 5^n colour sublattices

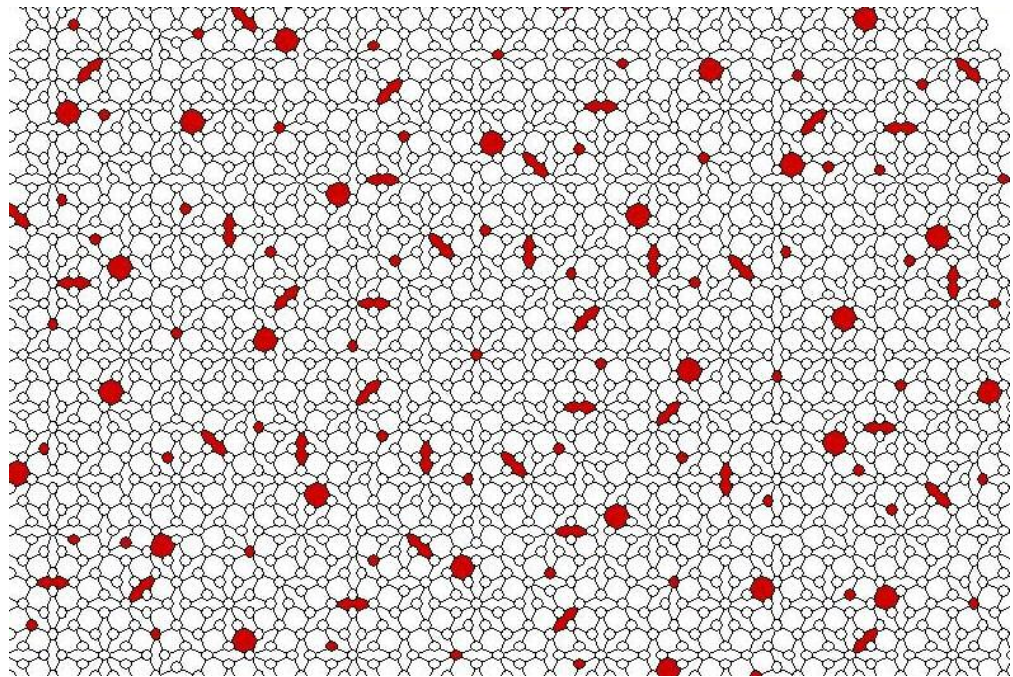
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Undetermined pattern	1st stellation of undet. pat.	2nd stellation of undet. pat.
Penrose Pentagon Pattern (PPP) = 5-colour sublattice of undetermined pattern.	1st stellation of PPP	2nd stellation of PPP
TTT, 5-colour sublattice of PPP	1st stellation of TTT	2nd stellation of TTT
25-colour sublattice of PPP	1st stellation of 25 c.s. PPP	2nd stellation of 25 c.s. PPP
= 5-colour sublattice of TTT		
125-colour sublattice of PPP	1st stellation of 25c.s. TTT	2nd stellation of 25 c.s. TTT
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Acceptance domains are decagons	Acceptance domains are stars	Acceptance domains are pointed stars



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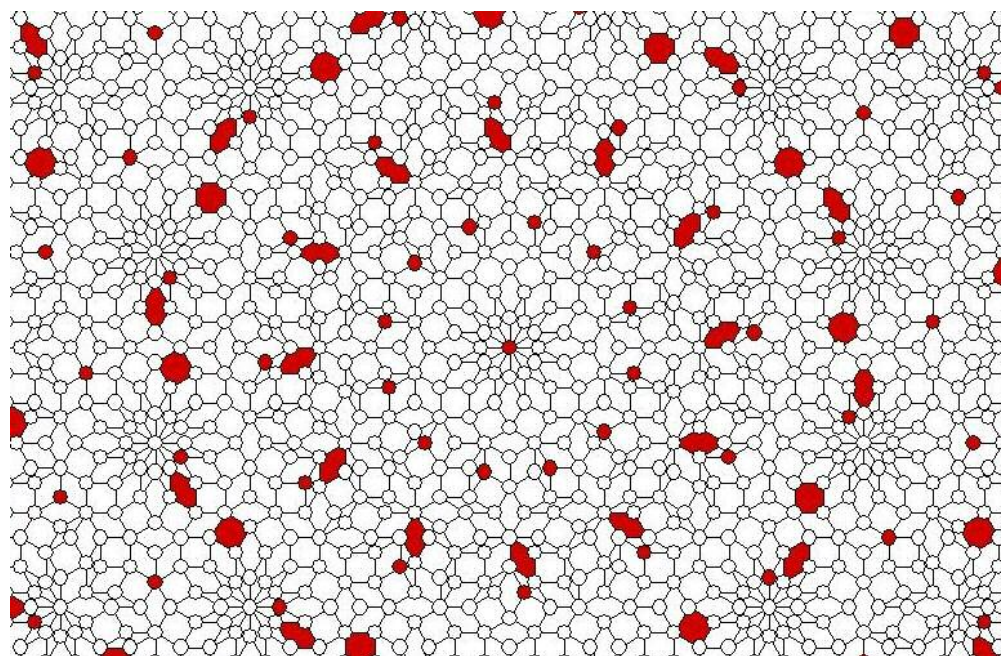
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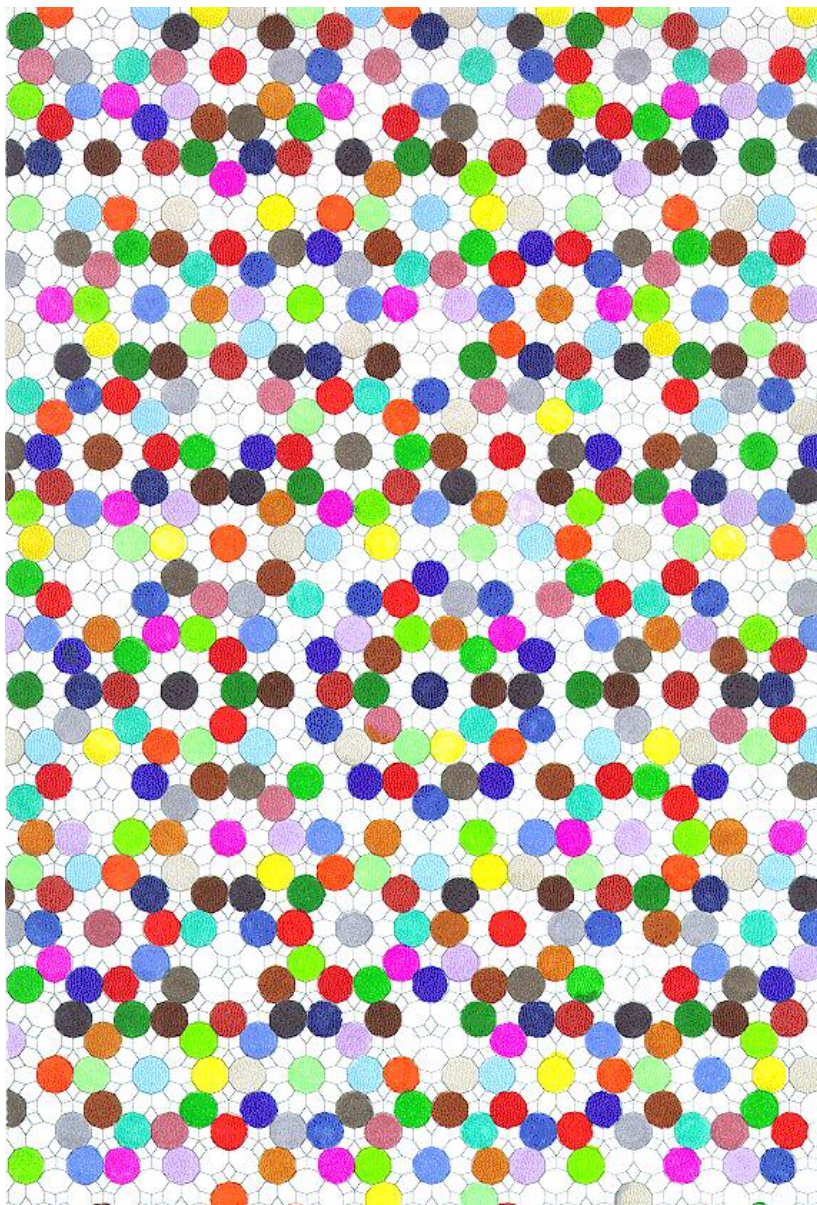
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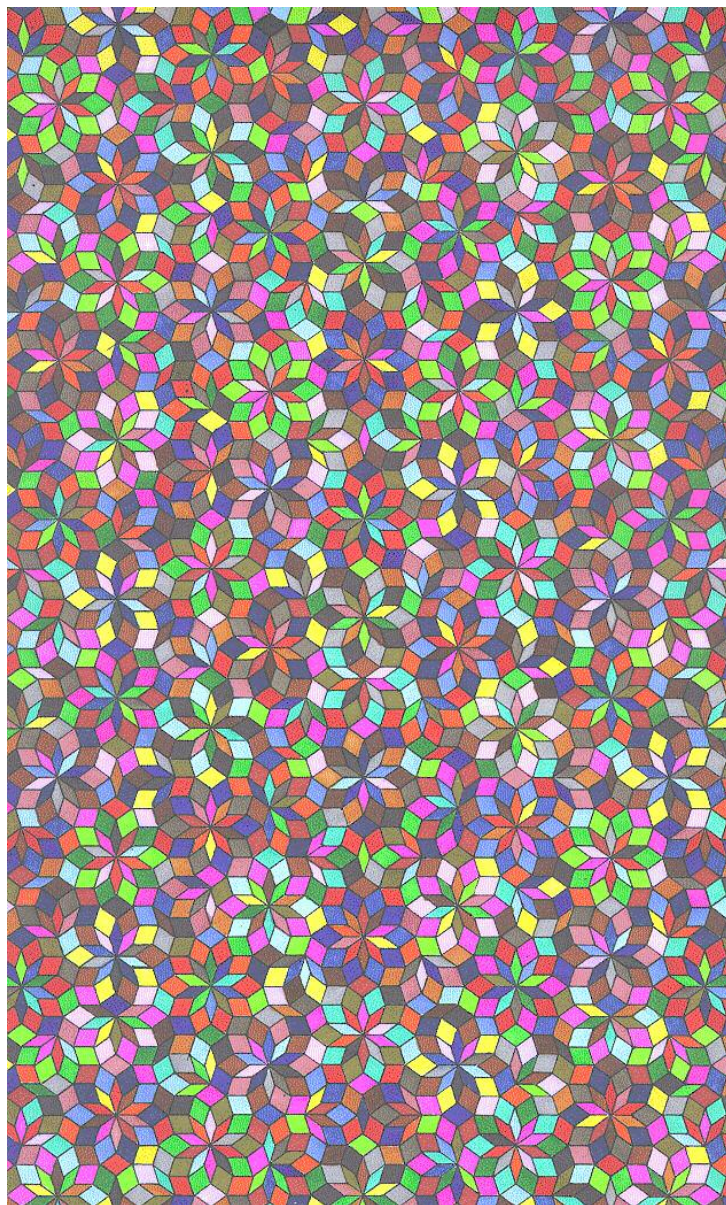
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Catchline (head of first page only) *Philosophical Magazine*, Vol. X, No. X, Month 2008, xxx–xxx

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Running heads (verso) *R. Lück*
(recto) *25 colours in quasiperiodic patterns*

Article Type (e.g. Review Article, Research Note, Brief Communication – if known)

Colour symmetry of 25 colours in quasiperiodic patterns

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Abstract

Coloured periodic and quasiperiodic patterns with 25 colours and colour-symmetry were described for four-fold, six-fold, eight-fold, ten-fold, and twelve-fold symmetries. Eight-fold, twelve-fold and some four-fold patterns are related to coincidence site lattices (CSL). The procedures of delineation of coloured patterns with 25 colours must consider the property of a square number. Procedures for ten-fold patterns need special considerations and allow the description of superstructures.

Keywords: Quasicrystals, Colour Symmetry, Superstructure, Coincidence site lattice.

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1 Introduction

Colour symmetry was known before the discovery of quasicrystals in 1982 by D. Shechtman (published in 1984) [1]. In particular, black-and-white symmetry (combined with the symmetry operation of time inversion) was used to describe spin orientations ‘up’ and ‘down’ in magnetic materials. The black-and-white symmetry groups have been listed in the literature for periodic materials. The subject was dealt within the textbook by Shubnikov and Koptsik [2] and other remarkable papers [3-7]. Coloured periodic patterns and colour groups can also be found in the standard textbook on tilings [8]. Shechtman’s discovery revived the interest in colour symmetry in the context of aperiodic structures.

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‘Colour’ in colour symmetry stands for any property, such as level or label, nucleus or numeral, atom or adornment, and, of course, colour as well. ‘Symmetry’ means that a crystallographic symmetry operation on some object may be accompanied by a global permutation of colours. The object may be a tiling, a crystal, a quasicrystal, a model or something else. Therefore all colours should be equivalent, and their fractions (relative frequencies) should be equal. Colour symmetry is strongly related to chemical and magnetic superstructures and to coincidence site lattices.

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The number 25 has several properties which make the decoration of patterns attractive. There are several rotational symmetries which allow a symmetric decoration with 25 colours. They are 4-, 5-, 6-, 8-, 10-, 12-, 15-, 20-, 24-, 25-, 40-, and 60-fold symmetry as described by M. Baake et al. [9-13], who derived the numbers and its multiplicities by Dirichlet series and Dedekind zeta-functions. 25 is a square of a prime number, which means that almost all decorations are based on the rank 2 as described by R. Lifshitz [14-19] in extensive group theoretical descriptions including those of magnetic superstructures [17-19]. The rank describes the dimensionality of the numbers indicating the colour. 25 is the square of the number 5, and thus is associated with five-fold symmetry which is essential for icosahedral and decagonal quasicrystals. Some examples of colour symmetry with rank 1 are described in the literature: 2 colours in ten-fold patterns [20], 11 colours in ten-fold patterns [21], 17 colours in eight-fold patterns [22] and 13 colours in twelve-fold patterns [23]. An example of rank 4 is described in [24] where the decoration with 16 colours of a ten-fold tiling is used to model a superstructure of decagonal AlCoNi, and an example for rank 3 with 8 colours in seven-fold patterns is discussed in [13].

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This paper is organized as follows: In Section 2 I briefly review some features of four-fold and six-fold periodic patterns with 5 and 25 colours. This is followed (Section 3) by a discussion of 25 colours in eight- and twelve-fold quasiperiodic patterns. The ten-(five-)fold symmetry requires a discussion in more detail. Finally, some remarks on the decoration of various quasiperiodic four-fold grids and six-fold quasiperiodic patterns will be made. Subsets are considered in Section 4, some general comments are added (Section 5). The terms used in the present paper are defined or explained in the Appendix.

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2 Periodic patterns

The decoration of a square lattice with 5 colours is based on a well-known example for Pythagorean Numbers [25]. The decoration is strongly related to a coincidence site lattice (CSL) with $\Sigma 5$. There exist two enantiomorphic decorations. The procedure can be repeated for the derived sublattices to eventually end up with 25 colours. The different combinations result in three decorations [9]. Two enantiomorphic colourings

correspond to CSL decorations of the $\Sigma 25$ type; these two are the examples which are not based on rank 2.

The third one is really based on rank 2; for instance, the first digit may count the horizontal progression (0 through 4) and the second one the vertical progression (0 through 4 as well). Decoration of vertices with 5 or 25 colours uses the operation mod 5, for 25 it is used for both digits. Only the CSL decoration of the square lattice with $\Sigma 25$ is based on mod 25. Translation is based on adding numbers, rotation is based on multiplication of the additive number by (well-defined) numbers; in case of rank 2 (and higher) it is based on multiplication of the additive vector by matrices. However, there exists a variety of possibilities for the numbers and matrices if a special colour symmetry is considered.

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Figure 1 represents simultaneously numbers 0 through 24 and the corresponding colours. The numbers form a so-called Devil's Magic Square. That means, that in every 5×5 square of the tiling the sum of columns, rows and diagonals is identical; it is 60 in the present case.

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'[Insert Figure 1 about here]'

The six-fold periodic symmetry allows a single decoration with 25 colours which is not related to any CSL and which is not very different from the decoration of the square lattice.

3 Quasiperiodic patterns

The multiplicity for eight-fold and twelve-fold patterns is 2 for both, while the rank of decoration is 2. Vertices and tiles can be decorated as well. The decoration of tiles is based on the average of the numbers of adjacent vertices using the mod 5 operation for both digits. Both decorations, of vertices and of tiles, are related to CSL and have two enantiomorphic solutions. As in any other CSL related coloured (two-dimensional) pattern, there are no colour mirror symmetries. First empirical investigations on CSLs of quasicrystals were published in [26-28]. The CSLs were put on a systematic basis [29, 30], more publications [31-37] followed. Figure 2 is a decoration of an eight-fold quasiperiodic pattern with a decoration of 'one of twentyfive' corresponding to a CSL lattice $\Sigma 25$. Decoration is shown for vertices and tiles as well. The corresponding illustration for a twelve-fold quasiperiodic pattern is given in Fig. 3.

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2 The multiplicity for ten-fold patterns is 1. The colouring is not related to a CSL and mirror symmetries for
3 colours do exist. Problems arise from the fact that vertices and tiles cannot be coloured simultaneously with 5
4 or 25 colours. It is empirically found that patterns with a single translation class can be decorated with 5 and
5 therefore also with 25 colours at vertices. The decoration of the Penrose pentagon pattern (PPP) with 5
6 colours at vertices was demonstrated in 1987, published with numbers [38] and as a coloured figure
7 afterwards [14, 15]. The sublattice of each one of the five colours forms a tiling [38-40] which is now **called**
8 afterwards [14, 15]. The sublattice of each one of the five colours forms a tiling [38-40] which is now **called**
9 **the** Tübingen triangle tiling (TTT). As the TTT is formed by a single translation class, it can be decorated
10 with 5 colours as well. This procedure can be performed again and again. Table 1 shows this hierarchy; the
11 first column is dedicated to the parent tiling and its subsets of 5, 5², 5³, ... colours. Their acceptance domain in
12 internal space is a decagon. The second and the third columns **refer** to tilings with different acceptance
13 domains, these can be derived from the decagons by stellated extensions [41].
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19 '[Insert Table 1 about here]'

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22 There are other decagonal patterns formed by a single translation class, **such** as e.g. Kepler's Aa patterns
23 [42, 23], and all patterns formed by the zero-level of ' $\Gamma = \frac{1}{2}$ ' patterns, including the so-called anti-Penrose
24 pattern described by Kléman and Pavlovitch [43], the t_A and t_B patterns by Ingalls [44], and the 'three-level'
25 binary tiling formed by rhombi [39].
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28 The centres of decagons of Ingalls' t_C [44] pattern form the zero-level of the anti-Penrose pattern.
29 Therefore, its decagons can be decorated with 25 colours and with full colour symmetry (Fig. 4). Colouring
30 of tiles in ten-fold patterns with 5 and 25 colours is not a simple procedure. The Penrose **rhombus** pattern is
31 derivable from PPP; the PPP can be decorated at vertices as mentioned above. The fat rhombi are decorated
32 by a vertex of the PPP, its colour can be taken for the decoration of the fat rhombus, the **thin** rhombi may be
33 decorated according to the colour symmetry of the surroundings. **Such** coloured patterns with 5 colours were
34 published in the literature [45, 23].
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38 The centres of rhombi of the mentioned three-level tiling are at the zero-level and can therefore be decorated
39 with 5 and 25 colours. The tiles may be coloured with the colour of the centre (see Fig. 5). The centres of all
40 rhombi or of fat rhombi form novel tilings in the same manner as other special points of the Penrose **rhombus**
41 pattern form the PPP. The novel tilings have a single translation class and are composed of different tiles
42 including pentagons, ten-fold stars and large rhombi; the centres of the ten-fold stars form the TTT.
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2 A third type of special points within the rhombi is determined by crossing points of Ammann bar pentagrids.
3 Several five-colour symmetries were determined for different rhombus patterns from Ammann bar pentagrids
4 coloured with 5 colours, but no analogous solution for 25 colours has been found.

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8 '[Insert figure 4 and figure 5 as a coloured figure about here]'

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10 Four-fold (τ -related) Fibonacci-based and (σ -related) Octonacci-based (see e.g. [46-48]) grids and their
11 colourings have been discussed in the literature and at several meetings. Posters dealing with colour-
12 symmetry patterns were presented but not published. Hence, slightly different ideas concerning the definition
13 of colour symmetry exist. Any quasiperiodic sequence – including those with an integer scaling factor or
14 integer ratio of two lengths – can be used to construct a grid with four-fold symmetry. Several examples were
15 tested for the number 25 in the present work. It turned out that for all quasiperiodic patterns decorations
16 according to CSL $\Sigma 25$ (with multiplicity 2) do exist. However, an 'unrotated' decoration with 25 colours as in
17 the periodic case was only found for Fibonacci-based grids and for those with an integer ratio of lengths; this
18 was especially found for the 'Copper Mean' with the ratio 2.

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20 There are also six-fold quasiperiodic patterns. A six-fold Fibonacci related tiling was constructed based
21 upon dualisation of Ammann bar tria-grids, 4 different LI (local isomorphism or local indistinguishability)
22 classes resulted [49]. A different method based on substitution of two equilateral triangles and two
23 enantiomorphic parallelograms was presented later [50]. In any case, the six-fold Fibonacci related patterns
24 can be coloured with 25 colours with multiplicity 1. As in the periodic case (Section 2), no CSL related
25 decoration exists for 25 colours. Other six-fold quasiperiodic grids and their duals decorated with 25 colours
26 are a challenge.

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34 4 Subsets with five colours

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36 All decorations with 25 colours based upon rank 2 can be subdivided into 5 subsets of 5 colours each. In all
37 cases there are 6 systems of formation. In the simple case of a periodic square lattice, these subsets are rows,
38 columns, 2 diagonals and two enantiomorphic sets related to a CSL with $\Sigma 5$. This is based on the fact that the
39 field of 5×5 numbers can be separated by the additive vectors into 5 sets of 5 numbers each. Single sets are
40 formed by the additive vectors (0,1); (1,0); (1,1); (1,4); (1,2); and (2,1). All other pairs of numbers will result
41 in one of these sets due to the mod 5 argument.

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2 The correlation of these sets with with coloured patterns was checked. A similar result as for periodic
3 patters was found for all quasiperiodic four-fold lattices which are not related to $\Sigma 25$. In case of six-fold
4 periodic and Fibonacci related patterns these are 3 + 3 lines in different orientations. For ten-fold patterns
5 based on a single translation class 5 lines according to symmetry and one superstructure [38] are found. Only
6 lines and no superstructures were found for eight-fold, twelve-fold and $\Sigma 25$ related four-fold patterns.
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9 **5 Concluding remarks**

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11 The present synopsis summarizes some key points for coloured periodic and quasiperiodic tilings in
12 Euclidean 2D geometry. There is no general problem to extend colour symmetry to higher dimensions;
13 visualisation may be more difficult. Extension to spherical geometry is almost trivial, whereas extension to
14 hyperbolic geometry opens a new field. Some colour symmetries are known from M. Escher's coloured wood
15 engravings 'Circle limit I – IV'; in recent papers [51, 52] special hyperbolic tilings and its colourings are
16 discussed. A symmetric decoration with 25 colours in hyperbolic space should be possible.
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19 In all cases, superstructures of one or more colours are related to chemically or magnetically ordered (quasi-)
20 crystals. Sublattices formed by colour symmetry bear the opportunity to build up novel tilings.
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22 The problems described for 25 (and 5) colours are similar for 49 (and 7) colours in eight-, twelve- and
23 fourteen-(seven)-fold patterns. Features of 9 colours in eight-fold patterns are similar to those of 25 colours in
24 eight-fold patterns; however, the problem of 9 colours in twelve-fold patterns is similar to that of 25 (5)
25 colours in ten-fold patterns. Vertices can be decorated with 9 colours provided there is a single translation
26 class.
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30 The permutation of colours caused by a symmetry operation is assigned to so-called orbits [51]. Colours
31 permute cyclically or do not change within an orbit. A twelve-fold rotation separates 25 colours into 3 orbits,
32 one orbit with a single colour and two orbits with 12 colours each. In case of eight-fold rotation, 25 colours
33 are distributed on 4 orbits, one with a single colour and three with 8 colours. The separation at a ten-fold
34 centre is more complex, there are five orbits, one with a single colour, two with 2 colours and two with 10
35 colours. If there is no orbit with a single colour, the rotation centre can not be coloured, this was found for
36 five-fold centres and 25 colours; there are five orbits with 5 colours each. The distribution of 25 colours on
37 the orbits and the sequence of colours within an orbit follows the rules of forming subsets (Section 4) and
38 depends on the rotation centre.
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43 A question concerning the relation of colouring and projection may arise. In order to investigate this issue,
44 the coloured vertices should be lifted to superspace. Lifting is not a simple process; results are often not
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unique; it depends on the number of dimensions of the selected superspace. Even for a well-defined superspace different lattices can result. The colouring of vertices is generally connected with a formation of superlattices in superspace. In most cases the symmetry of superlattices is reduced with respect to the symmetry of the superspace lattice. However, the symmetry along the projection direction survives. For instance, each one of five colours may form a separated level in superspace. Colouring Ammann bar pentagrids is based on the projection of a coloured square lattice with five colours of the $\Sigma 5$ type.

Acknowledgment

I am grateful for the collaboration during the last years with several colleagues on both colour symmetries and coincidence site lattices. Michael Baake, Uwe Grimm and Ron Lifshitz assisted in the preparation of the present manuscript. Further advise of two unknown Referees and the Guest Editor is appreciated.

Appendix

In the following a glossary of a few used terms is listed.

Coincidence site lattice (CSL) [26-37]. The common lattice points of two lattices of the same type after a rigid rotation about a common lattice point. The fraction of common points of periodic lattices is given by $1/\Sigma$. $\Sigma 5$ indicates that 1/5 of lattice points coincide. The Σ index is used to characterize coincidence grain boundaries. In case of quasiperiodic lattices, the fraction of common points is additionally influenced by the common area of the two acceptance domains in internal space. Several colour symmetries are characterized by a relationship to CSL, the fraction of each colour is $1/\Sigma$. The CSL related colourings occur as enantiomorphic pairs and do not have a colour mirror symmetry, whereas the non-CSL type colourings have.

Colour Group [15, 51]. A symmetry group of an uncoloured pattern is subdivided into several subgroups: (i) operations leaving all colours unchanged; (ii) operations permuting systematically at least some colours depending on the involved orbits; (iii) operations not being symmetry operations of the coloured pattern, these can be mirror operations as found for several CLS type colourings.

Colour Symmetry Operation [15, 51]. A symmetry operation which permutes systematically at least some colours, some others may be unchanged.

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Enantiomorphic Pairs. If the mirror symmetry of the uncoloured crystal is broken by colouring (c.f. CSL), the pattern appears in right-handed and left-handed versions.

Multiplicity [9]. Number of different coloured versions of a pattern for a fixed number of colours.

PPP. Penrose pentagon pattern [8, 38].

Orbits of Colours [51]. Orbits are formed by symmetry operations and the number of different orbits is determined by the number of colours. Colours permute cyclically or do not change within a single orbit.

Pythagorean Numbers [25]. A set of natural numbers which fulfil the condition $a^2 + b^2 + c^2 \dots = d^2$, depending on the number of dimensions.

Rank. According to R. Lifshitz [15] it is the smallest number of permutations that can generate the colour permutation group. For the present purpose, it is used for the dimensionality of the numbers indicating the colour [13, 24].

Sigma Index Σ [25-37]. $1/\Sigma$ indicates the fraction (or relative frequency) of common lattice points of a CSL. $\Sigma=5$ is usually written as $\Sigma 5$.

Single Translation Class. The acceptance domain occupies only a single level in internal space.

Sublattice, Superstructure. A subset of lattice points (vertices) forms a novel lattice different from the parent lattice. All sublattices of a parent lattice form a superstructure. A single colour of a coloured lattice forms a sublattice.

TTT. Tübingen triangle tiling [38-49].

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Figure Captions

Fig. 1 Periodic square tiling simultaneously decorated with numbers 0 through 24 and corresponding colours. The numbers form a so-called Devil's Magic Square, in every 5×5 square of the tiling the sum of columns, rows and diagonals is 60.

Fig. 2 Eight-fold quasiperiodic pattern with a colour decoration 'one of twentyfive' at vertices and tiles. This corresponds to a coincidence lattice with $\Sigma 25$.

Fig. 3 Twelve-fold quasiperiodic pattern with a colour decoration 'one of twentyfive' to a CLS with $\Sigma 25$.

Fig. 4 Ingalls' ten-fold t_c pattern [44] decorated at decagons with 25 colours.

Fig. 5 A ten-fold quasiperiodic 'three-level-tiling' decorated with 25 colours.

Table Caption

Table 1. Hierarcy of ten-fold tilings with a single translation class and its subsets of 5, 5², 5³...colours. The second and third columns are designated to tilings of stellated extensions of acceptance domains. The dotted lines at the top and at the bottom indicate that the table can be extended in both directions to an infinitely large number of classes which have, however, a decreasing significance.

Table 1. Hierarcy of 5ⁿ colour sublattices

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Undetermined pattern	1st stellation of undet. pat.	2nd stellation of undet. pat.
Penrose Pentagon Pattern (PPP) = 5-colour sublattice of undetermined pattern.	1st stellation of PPP	2nd stellation of PPP
TTT, 5-colour sublattice of PPP	1st stellation of TTT	2nd stellation of TTT
25-colour sublattice of PPP = 5-colour sublattice of TTT	1st stellation of 25 c.s. PPP	2nd stellation of 25 c.s. PPP
125-colour sublattice of PPP = 25-colour sublattice of TTT	1st stellation of 25c.s. TTT	2nd stellation of 25 c.s. TTT
.....
Acceptance domains are decagons	Acceptance domains are stars	Acceptance domains are pointed stars

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- Deleted: Fig. 4 Ingalls' ten-fold t_c pattern [44] decorated at decagons with 25 colours.¶
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Table 1. Hierarchy of ten-fold tilings with a single translation class and its subsets of 5, 5², 5³,...colours. The second and third columns are designated to tiling of stellated extension of acceptance domains. The dotted lines at the top and at the bottom indicate that the table can be extended in both directions to an infinitely large number of classes which have, however, a decreasing significance.

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Hierarchy of 5n colour sublattices

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