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# Short note on the effect of lateral displacement on the surface stress distribution for cone and sphere contact

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## **Abstract**

In the present paper the effect of lateral displacement on the normal surface stress distribution in the case of conical and spherical contact problems is studied. After deriving the governing contact equations including the lateral displacement the new normal surface stress distributions are evaluated for a variety of amounts of lateral displacements and compared with the classical solutions of Hertz and Sneddon. It is shown, that Poisson's ratio is the main determinant of the lateral displacement. In addition the effect on the resulting combined approach of sample and indenter and the contact radius is evaluated as a function of the Poisson's ratio.

It is also shown, that for spherical indenters due to a rather surprising compensating effect of two approximations introduced by Hertz the classical Hertzian solution appears to be relatively immune against the effect of lateral displacement.

## **Introduction**

In 1995 Bolshakov and Pharr [1] demonstrated how the lateral displacement  $u(r)$  violates the boundary conditions in the case of a conical contact by applying the Sneddon approach [2]. In short their argumentation can be repeated as follows.

Assuming a rigid cone being pressed into an elastic half space  $z \geq 0$  of Young's modulus  $E$  and Poisson's ratio  $\nu$  the correct parametric form of the shape of the resulting contact zone beneath the indenter can be given as

$$\begin{pmatrix} \hat{r} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} r + u(r) \\ z + w(r) \end{pmatrix} = \begin{pmatrix} r + u(r) \\ h - \cot(\theta)(r + u(r)) \end{pmatrix}, \quad (1)$$

with  $\hat{r}$  and  $\hat{z}$  giving the surface positions of the deformed body and  $\theta$  denoting the cone indenter half angle. The resulting governing contact equation can be evaluated from (1) by setting  $z=0$  and one obtains [1] (for  $r \leq a$ , with  $a$  denoting the radius of contact)

$$w(r) + \cot(\theta) * u(r) = h - \cot(\theta) * r. \quad (2)$$

In the classical approach of Sneddon, equation (2) had been simplified by

$$w(r) = h - \cot(\theta) * r, \quad (3)$$

which however, can only be valid in those cases, where  $u(r)$  is small, i.e. where Poisson's ratio is close to 0.5 because the lateral displacement of  $u$  can be given in the form  $u(r) = (1-2\nu) * f(r)$ .

In the case of a rigid spherical indenter the parametric form of the shape of the contact zone must be given by

$$\begin{pmatrix} \hat{r} \\ \hat{z} \end{pmatrix} = \begin{pmatrix} r + u(r) \\ z + w(r) \end{pmatrix} = \begin{pmatrix} r + u(r) \\ h - \left[ R - \sqrt{R^2 - (r + u(r))^2} \right] \end{pmatrix}, \quad (4)$$

where  $R$  is the radius of the spherical indenter and which results in:

$$w(r) = h - \left[ R - \sqrt{R^2 - (r + u(r))^2} \right]. \quad (5)$$

The simplified Hertzian contact equations [3] can be derived from (5) by applying two approximations. At first the lateral displacement is neglected again and secondly the shape function of the sphere  $\left[ R - \sqrt{R^2 - r^2} \right]$  is approximated by the first term of its corresponding Taylor series resulting in

$$w(r) = h - \frac{r^2}{2R}. \quad (6)$$

## New Contact Equations

Applying now basic principles of linear elasticity and some tools of potential theory, equations (2) and (5) can be brought in the following form [4]:

For the conical indenter

$$H \int_s \frac{\sigma(r)r_0 dr_0 d\varphi_0}{\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)}} - \cot(\theta) * H\alpha \int_s \frac{\sigma(r)r_0 dr_0 d\varphi_0}{r * e^{-i\varphi} - r_0 * e^{-i\varphi_0}} = h - \cot(\theta) * r. \quad (7)$$

For the spherical indenter

$$H \int_s \frac{\sigma(r)r_0 dr_0 d\varphi_0}{\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\varphi - \varphi_0)}} = h - \left[ R - \sqrt{R^2 - \left( r - H\alpha \int_s \frac{\sigma(r)r_0 dr_0 d\varphi_0}{r * e^{-i\varphi} - r_0 * e^{-i\varphi_0}} \right)^2} \right], \quad (8)$$

where we have used the complex presentation of the lateral displacements with  $u^c = u + i * v$  ( $u$  displacement in x- and  $v$  in y-direction [not to be mixed up with the Poisson's ratio  $\nu$ ]). The constants  $H$  and  $\alpha$  are defined through

$$H = \frac{1 - \nu^2}{\pi E}; \quad \alpha = \frac{1 - 2\nu}{2(1 - \nu)}. \quad (9)$$

The integrals in (7) and (8) are surface integrals and the integration has to be performed over the contact surface. In order to solve these integral equations with respect to the normal stress distribution  $\sigma(r)$  a special numerical method is applied which is given in the appendix.

## Results

For the conical indenter the effect of lateral displacement on both the surface stress distribution and the resulting total approach  $h$  is increasing with decreasing half angle of the cone (table 1). As to be expected we obtain no deviation from the classical solution in the case  $\nu=0.5$ . The deviation behaves monotonic (increasing) with decreasing  $\nu$  (table 1 and fig. 1). Concerning the penetration depth  $h$  for smaller  $\nu$  the differences between correct and classical solutions are quite significant even in the less pronounced case of a  $70.3^\circ$ -cone, which is often used as an equivalent of symmetry of revolution for a Berkovich indenter. Even more

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2  
3 pronounced are the differences of the resulting contact radii (fig. 2). As this is an important  
4 parameter in analysing nanoindentation data the deviations to the classical Sneddon solution  
5 for a variety of Poisson's ratios are given in Table 2. In fig. 2 one can clearly see, that the new  
6 approach does in fact provide the required linear displacement beneath the indenter of the  
7 combined (parametric) presentation of normal  $w(r)$  and lateral  $u(r)$  displacement, while the  
8 normal displacement alone does not (dashed line in fig. 2).

9  
10 In the case of spherical indenters the picture changes completely because here the influence of  
11 the lateral displacement is relatively small if the contact radius is small compared to the radius  
12 of the indenter. In addition for  $\nu < 0.5$  two effects are responsible for the differences to the  
13 classical Hertzian case, namely the lateral displacement of course and the error made by the  
14 parabolic approximation of the shape of the sphere. Fortunately both effects compensate each  
15 other almost completely for  $\nu = 0$ , so that we find very Hertzian like stress distributions even  
16 for rather big ratios of contact radius  $a$  to indenter radius  $R$  and small Poisson's ratios (fig. 3  
17 and table 1). We see (fig. 3) that, even for ratios  $a/R = 0.5$  the deviation of the Hertzian and the  
18 correct penetration is well below 2% (exactly  $100\% * (h_{\text{Hertz}} - h) / h_{\text{Hertz}} = 1.64\%$  at the  
19 maximum found at  $\nu = 0.386$ ).

## 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39 40 41 42 **Conclusions**

43  
44 It was found out, that it is important to take the lateral displacement during normal indentation  
45 into account for conical indenters (not only for small half angles). The influence of the lateral  
46 displacement effect increases with decreasing Poisson's ratio (from 0.5 to 0) and decreasing  
47 half angle. The equivalent cone for a Berkovich indenter with a half angle of  $70.3^\circ$  shows  
48 already a significant deviation from the classical Sneddon solution if only elastic  
49 deformations are considered. Cube corner indenters with smaller half angles will produce  
50 bigger deviations.  
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In the case of spherical indenters it was found that the classical Hertzian solution provides an “almost perfect” approximation for smaller Poisson’s ratios ( $\nu < 0.3$ ) because the two “Hertzian approximation errors”, namely the parabolic approximation of the shape of the sphere (given due to  $\left[ R - \sqrt{R^2 - r^2} \right] \approx \frac{r^2}{2R}$ ) and the neglected effect of the lateral displacement, almost cancel each other out, so that the Hertzian approach appears to be valid for ratios of contact radius  $a$  to indenter radius  $R$  up to  $a/R=0.5$  with very good accuracy. As the Hertzian field is a rather simple and well known mathematical tool in analysing indentation data, the latter result might be considered as the most important part of this short note.

## Appendix

As the author was not successful in finding a solution due to direct integration of the equations (7) and (8) he introduced an approach of the form

$$w_0 = \frac{3H}{4} \sum_{i=1}^N \frac{\lambda_i}{a_i^3} \left\{ (2a_i^2 - r^2) \sin^{-1} \frac{a_i}{l_{02i}} + (l_{02i}^2 - a_i^2)^{1/2} \left( \frac{3l_{01i}^2 - 2a_i^2}{a_i} \right) \right\}$$

$$u_0^c = u + i * v = -\frac{3H\alpha}{2} \sum_{i=1}^N \frac{\lambda_i r e^{i\varphi}}{a_i^3} \left\{ (a_i^2 - l_{01i}^2)^{1/2} \left( 1 - \frac{l_{01i}^2 + 2a_i^2}{3r^2} \right) + \frac{2a^3}{3r^2} \right\}$$

$$l_{01i} = \frac{1}{2} \left( \sqrt{(r+a_i)^2} - \sqrt{(r-a_i)^2} \right).$$

$$l_{02i} = \frac{1}{2} \left( \sqrt{(r+a_i)^2} + \sqrt{(r-a_i)^2} \right).$$

where the constants  $\lambda_i$  have to be fitted to the integral contact equations. The reader should note that this approach just is a superposition of Hertzian loads with different contact radii  $a_i$  (figure 4). Now an approximated solution to the equations (7) and (8) can be found. Together with the normalisation condition one obtains

$$\sigma_{zz0}(r, \varphi) = \sigma_{zz0}(r) = \frac{3p}{2\pi} \sum_{i=1}^N \frac{\lambda_i}{a_i^3} \sqrt{a_i^2 - r^2}; \quad \sum_{i=1}^N \lambda_i = 1.$$

Choosing a sufficiently high number of loads  $N$  one can approximate any displacement (indenter shape) of symmetry of revolution. The approach has the advantage, that the evaluation of the complete elastic field for the whole body would be only a question of summing up a series of Hertzian fields. In addition the solution can easily be extended to layered materials [5]. The interested reader can find more information and other applications of the method in [6]. Instead of classical Hertzian loads also any other load solution could be applied. So, in the case of the cone the classical Sneddon approach [2] had been used.

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online at <http://authors.elsevier.com/sd/article/S0257897205007772>

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## Tables

$\nu$	Cone: $100\% \cdot (h_{\text{Sneddon}} - h) / h_{\text{Sneddon}}$		Sphere: $100\% \cdot (h_{\text{Hertz}} - h) / h_{\text{Hertz}}$	
	$\theta=45^\circ$	$\theta=70.3^\circ$	$a/R=0.01$	$a/R=0.25$
0.0	14.2	6.0	0.073	0.043
0.1	13.0	5.4	0.062	0.259
0.2	11.4	4.6	0.058	0.476
0.3	9.04	3.5	0.036	0.681
0.4	5.58	2.1	0.022	0.764
0.5	0.0	0.0	0.001	0.428

Table 1: Deviation between the absolute normal displacement  $h$  of the classical solutions and the new results including the effect of lateral displacement for sphere and cone for two examples of cone half angles and ratios of contact radius to Sphere radius

$\nu$	$a_{\text{new}} / a_{\text{Sneddon}}$		$hc_{\text{new}} / h_{\text{new}}$	
	$\theta=45^\circ$	$\theta=70.3^\circ$	$\theta=45^\circ$	$\theta=70.3^\circ$
0.0	0.750	0.910	0.43384	0.389801
0.1	0.778	0.920	0.426696	0.386898
0.2	0.812	0.933	0.417431	0.383253
0.3	0.857	0.949	0.405101	0.378544
0.4	0.917	0.970	0.387998	0.372239
0.5	1.0	1.0	0.36338	0.36338

Table 2: Resulting contact radius  $a_{\text{new}}$  in units of the contact radius of the corresponding classical Sneddon solution  $a_{\text{Sneddon}}$  and ratio between of the elastic displacement above the contact area  $hc$  and the absolute normal displacement  $h$  of the new approach (in the classical Sneddon solution:  $hc/h = 1 - 2/\pi = 0.36338$ )

## Figure Captions

Fig. 1: Normal surface stress distribution for conical indenter with half angle of  $45^\circ$  half angle normalized by ratio of loading force  $p$  to square of contact radius  $a$ . It is given for three different Poisson's ratios  $\nu$  with  $\nu=0.5$  (solid line = classical solution),  $\nu=0.3$  (short dashes) and  $\nu=0.1$  (long dashes)

Fig. 2: Parametric plot of the combined displacements in normal and lateral direction on the surface near the contact zone presenting the shape of the surface in the deformed state (thick solid line, with cone half angle  $\theta=45^\circ$  and  $\nu=0.1$ ). For comparison the normal displacement is also shown alone clearly demonstrating, that this does not mirror the required linear behaviour beneath the indenter. The vertical lines are marking the radii of contact at  $a_{\text{new}}=0.778$  and  $a_{\text{Sneddon}}=1$  (in units of  $a_{\text{Sneddon}}$ , c.f. table 2). The thin solid lines symbolise the surface of a corresponding cone clearly showing, that the combined normal and lateral displacement of the new approach perfectly follows the conical shape, while the normal displacement alone does not.

Fig. 3: Normal surface stress distribution for a spherical indenter with ratio of contact to indenter radius of 0.5 for a Poisson's ratios of  $\nu=0.386$  (solid line). For comparison also the classical Hertzian stress distribution is shown (short dashes)

Fig. 4: Illustration of the mathematical method of superposition of Hertzian load distributions with different contact radii  $a_i$  (and in principle also different loads).

Figures

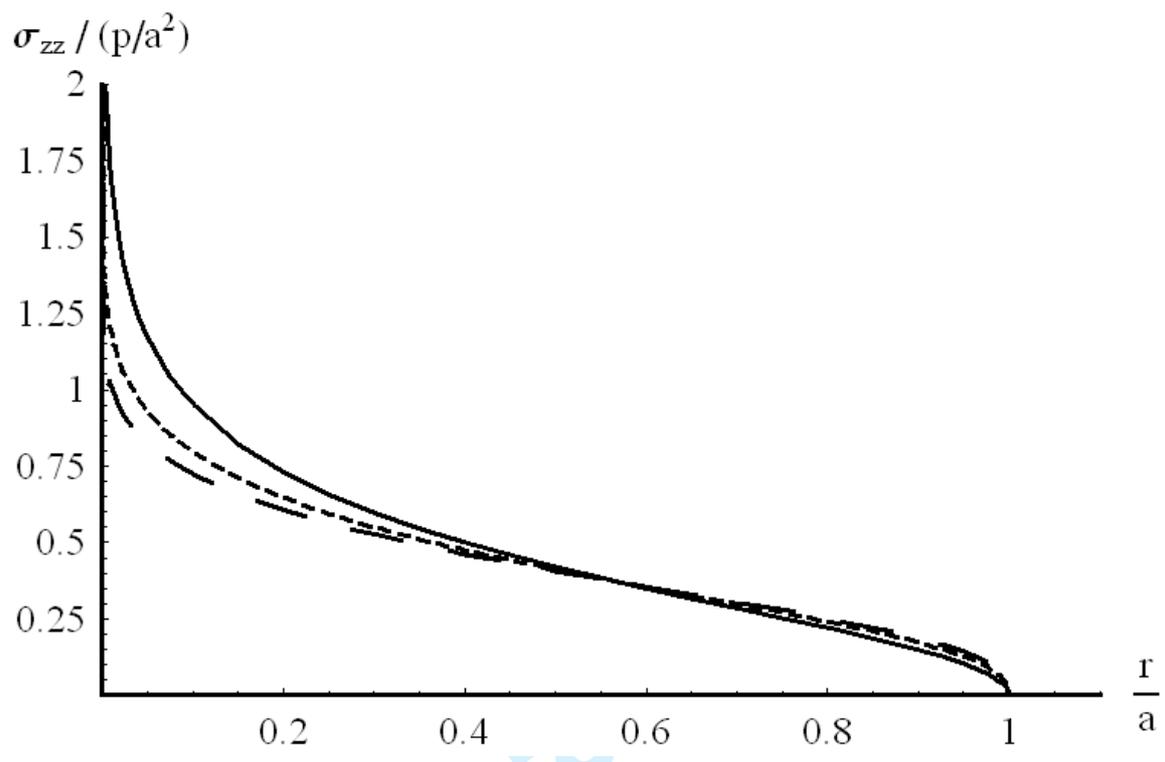


Fig. 1

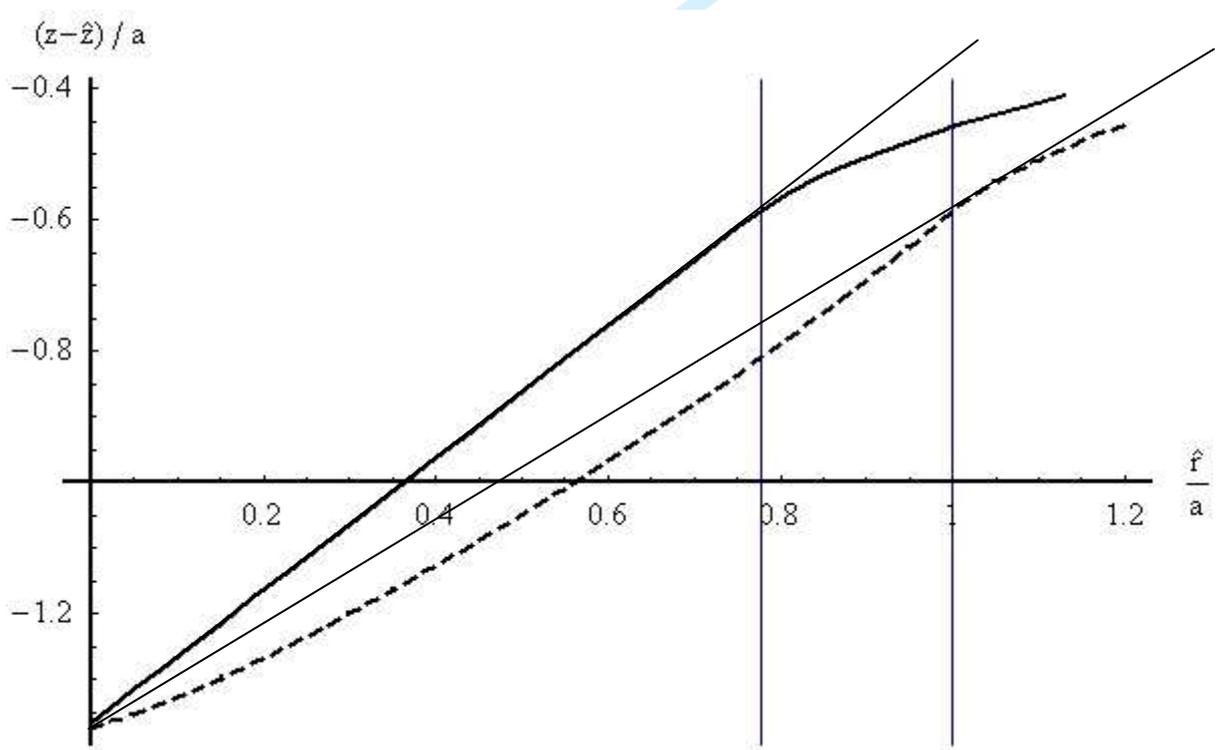


Fig. 2

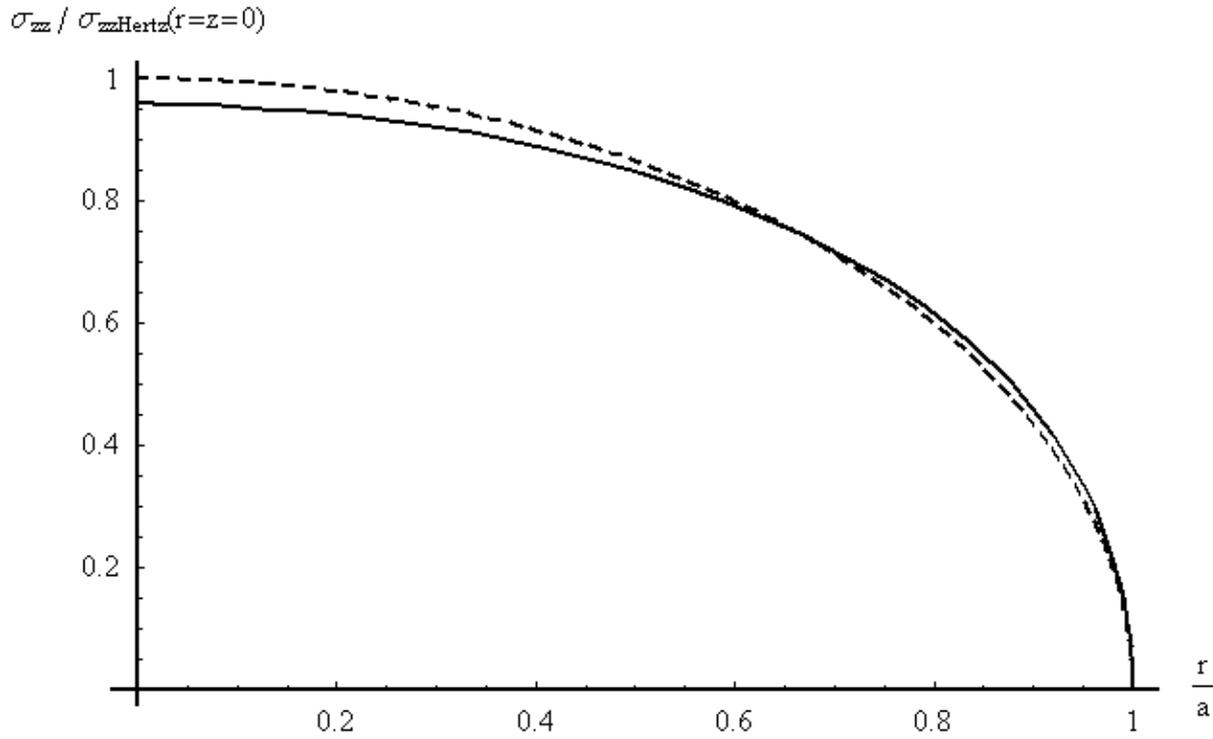


Fig. 3

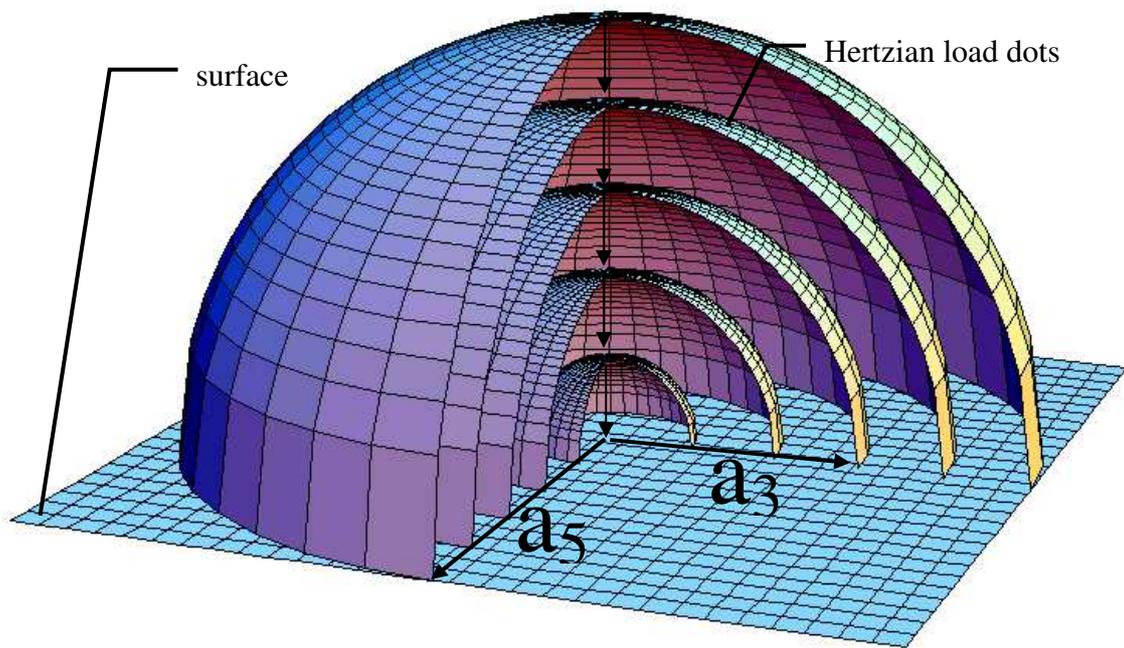


Fig. 4