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# Designing an Optimal Turnover-based Storage Rack for a 3D Compact AS/RS

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## Abstract

Compact, multi-deep (3D) automated storage and retrieval systems (AS/RS) are becoming increasingly popular for storing products. We study such a system where a storage and retrieval (S/R) machine takes care of movements in the horizontal and vertical directions of the rack, and an orthogonal conveying mechanism takes care of the depth movement. An important question is how to layout such systems under different storage policies to minimize the expected cycle time. We derive the expected single-command cycle time under the full-turnover-based storage policy and propose a model to determine the optimal rack dimensions by minimizing this cycle time. We simplify the model, and analytically determine optimal rack dimensions for any given rack capacity and ABC curve skewness. A significant cycle time reduction can be obtained compared with the random storage policy. We illustrate the findings of the study by applying them in a practical example.

*Keywords*: Order picking; Storage rack design; AS/RS; Travel time model; Warehousing; Turnover-based storage

## 1. Introduction

Space problems, the requirement to shorten customer response times, and the introduction of automated storage systems have led warehouse managers to change their storage solutions for unit-loads (pallets, containers or totes). Traditionally, such unit loads are stored in single-deep racks (with depths of only one unit-load per storage slot). To store or retrieve unit-loads, aisles are required between every two racks, wasting much floor space. Solutions exist in the form of multi-deep (3D) racks, also called ***compact***, or *very high-density* storage, saving much aisle space (e.g. Retrotech, 2006; e.g. 2006). In a 3D compact ***AS/RS*** (automated storage/retrieval system), every unit load can be automatically accessed individually, through a S/R (storage/retrieval) machine and a depth movement mechanism (De Koster et al., 2006). The full automation of tasks and the large storage capacity in a compact cubic space can make such systems more time efficient than traditional single-deep AS/RS. We have studied applications in dense container stacking at a container yard and the Distrivaart project in the Netherlands (Waals, 2005), where pallets are transported by barge shipping between several suppliers and several supermarket warehouses. This project has actually been implemented and has resulted in a fully automated storage system on a barge. Another example of a compact system can be found at Miele in Gütersloh (D), where a combination of cranes and shuttles store and retrieve individual palletized white goods (like washing machines and dish washers), and automatically sequence them for loading trains and trailers. More examples of realized compact storage projects can be found at the respective websites of companies like Siemens Dematic, Viscon Flexcom, Westfalia and Retrotech. As a result, compact AS/RSs have become increasingly popular (Hu et al., 2005; Van den Berg and Gademann, 2000) in many companies for storing and retrieving various products.

Several compact storage system technologies exist with different handling systems taking care of the horizontal, vertical and depth movements. Similar to De Koster et al. (2006), this paper studies compact systems with at least two variants of the depth movement of unit loads: one with gravity conveyors and the other with powered conveyors. Here we describe only the gravity conveyor system, but the powered-conveyor system works similarly.

The overall structure of the gravity conveyor system is sketched in Figure 1(a). It consists of a 3D storage rack, a depot (or I/O point), an S/R machine (or crane), and a conveying mechanism with gravity conveyors operating in pairs responsible for the depth movement in conjunction with elevating mechanisms at the back of the rack (see Figure 1(b) and (c)). The same system has been studied by De Koster et al. (2006). The unit loads enter and leave the system via the I/O point and are stored in the rack. The S/R machine can drive and lift simultaneously and takes care of the movements in the horizontal and vertical directions. It picks up unit loads from the I/O point to bring them to an inbound conveyor or retrieves them from an outbound conveyor to bring them to the I/O point. From Figure 1(b), it can be seen that there are many gravity conveyors working in pairs at every level of the rack.

**<Insert Figure 1 here>**

For each pair of gravity conveyors, a stop switch is needed at the front side of the rack to stop a unit load when it is needed for retrieval as sketched in Figure 1(c) (conveyor slopes are exaggerated). The two gravity conveyors have the same absolute slope angles (one is negative and the other is positive) and work in pairs. On the inbound conveyor, unit loads flow to the back end of the rack under gravity. At the back of the rack an inexpensive simple elevating mechanism lifts unit loads one by one from the down inbound conveyor to the upper outbound conveyor from which the unit load flows to the front end of the rack under gravity. In this way unit loads on the two conveyors can rotate continuously until the requested one is stopped at the S/R position at the front side of the rack. The lift drives the rotation of unit loads and, as it is slower than its two conveyors, it determines the effective rotation speed. In order to retrieve a unit load, the two neighboring gravity conveyors should have at least one empty slot. Due to gravity, slot "E" on the outbound conveyor will be empty when the stop switch is off, while at least one of the two slots, marked "E" will be empty when the stop switch is on. In order to retrieve a unit load "A", for example, turn off the stop switch and let the lifting mechanism work so that the unit loads can flow on the two conveyors continuously. When unit load "A" reaches the retrieval position, the stop switch is then turned on to make the unit load retrievable by the S/R machine. The depth movement mechanisms and the S/R machine also can be used to sequence unit loads according to their turnover; the S/R machine picks up a disordered (or new incoming) unit-load, and inserts it into a required relative position when this position flows to storage position "E", and becomes empty. The gravity conveyor system is innovative in its cheap construction: no motor-driven parts are used for the conveyors and the construction of the lifting mechanisms is simple as well. Because the inbound and outbound conveyors have only slight slopes, and are arranged in the same manners respectively, the total height of the rack is only increased a bit compared with the system that only uses conveyors without slopes.

Different from the above gravity conveyor system, a powered conveyor system does not use lifts and conveyors are mounted in the rack horizontally; the unit load movements in the rack are driven by power. Because the powered conveyors are horizontally mounted in the rack without slopes, the rack height of the rack is slightly shorter than that of the gravity conveyor system. The retrieval operation is identical to gravity conveyors, but for storage there are two differences in operation. First, the empty slots may be at any position on the two conveyors. Second, the storage time of a unit load may be longer than in the case of gravity conveyors because the powered conveyors may need time to rotate an empty slot to the storage position. In this paper, in order to obtain the expected cycle time for both variants of systems, we only consider retrieval travel time. This consideration is also motivated by the fact that retrieval time is more critical in operations.

Different storage policies are used to store unit loads in the rack, in particular random and dedicated storage policies (including full-turnover-based policies). Under the random storage policy, each unit load is equally likely to be stored in any of the storage positions in the rack. In reality, usually some items are requested more frequently than others, as described by an ABC curve. Full turnover-based storage policies store unit loads at a decreasing travel distance to the I/O point with increasing item turnover frequency, and thereby can shorten the S/R (storage and retrieval) machine travel time (Hausman et al., 1976; Park et al., 2003). In the studied 3D AS/RS, the full turnover-based storage policy is applied to sequentially assign items (on unit loads). During the storage and retrieval process, this storage sequence of the unit loads may be distorted, but it can be easily restored as the S/R machine and conveyor movement mechanisms can reshuffle unit loads according to their turnover frequencies when the S/R machine is idle.

Deriving the throughput or, alternatively, the single command cycle time, is one of the main issues in designing a compact system, and also in evaluating the system operations. It depends on material handling speeds and capabilities, but also on system dimensions, the crane dwell-point strategy, and storage and retrieval policies. While this problem has been tackled for 2D systems to a great extent (Hausman et al., 1976; Koh et al., 2002; Sarker and Babu, 1995), literature on 3D systems is far from abundant. We try to fill this gap by deriving the expected S/R machine travel time for single command cycle by using full turnover-based storage policies and the optimal rack dimensions.

For a given rack storage capacity, we derive the single-command S/R machine travel time as a function of the rack dimensions and propose a corresponding model to optimize the dimensions by minimizing the travel time. Due to the complexity of the model, we first give several theorems to reduce the feasible area of the model variables without losing the optimal solution. We prove the optimal structure of the rack must be square-in-time (SIT) in the S/R machine's moving directions, and the conveyor's dimension should be the longest among all three dimensions. With these results, the proposed model simplifies to an equivalent non-linear convex programming model that can be solved numerically to obtain the optimal solution for ABC curve of any given steepness. Finally the results of the model are compared with those of the random storage policy in De Koster et al. (2006), and a practical example is given to illustrate the application of our research. Optimal racks using a full turnover-based storage policy can obtain significant reductions (up to 68%, depending on the steepness of the ABC curve) in the expected travel time compared with random storage.

## 2. Literature review

Designing an efficient AS/RS has interested many researchers for decades. Performance measures used include travel time per S/R operation cycle, number of S/R requests carried out per unit time, and average cost per S/R operation. Much literature focuses on the travel time per S/R operation cycle which depends on the shape of the storage rack (ratio between different dimensions: SIT (square in time), or non-SIT (NSIT)), unit load storage policies (random/dedicated S/R policies), the S/R crane's operation modes (single, dual, and multiple commands per cycle), dwell point policies (at the middle or corner of the rack), and the number of rack dimensions (2D or 3D racks). Because this paper discusses how to dimension the 3D rack by minimizing the expected travel time for a single command cycle under the full turnover-based storage policy, this section only reviews literature closely related to our research. We focus on travel time calculation with different rack shapes, on storage policies, and on 3D storage systems.

***Storage rack shape.*** Calculating the travel time based on different rack shapes with Chebyshev travel has received considerable interests since the study of Hausman et al. (1976). They calculate the one-way travel time for a single command cycle based on a SIT-rack system with different storage policies: random, turnover, and class-based storage. Bozer and White (1984) obtain the travel time for single and dual command cycles for NSIT rack systems under the random storage policy, and prove that with a constant AS/RS speed, the SIT rack is the optimal 2D-rack configuration. In practice other rack shapes exist, given the various cost components as well as height and length constraints. Based on Bozer and White's travel time model for NSIT racks, Eynan and Rosenblatt (1994) develop a procedure for dividing a rectangular rack into storage classes and calculate the travel time resulting from class-based storage. Recently, travel time as main performance criterion is used in Pan and Wang (1996), Park et al. (2003), Hu et al. (2005), Park et al. (2006), and Park (2006) for different types of NSIT racks systems. Only few papers (Bozer and White, 1984; Park et al., 2003) take the travel time as a function of the rack dimensions, and minimize the travel time by dimensioning the 2D rack.

***Storage policies.*** Under the random storage policy S/R requests are allocated randomly over the available storage locations in a rack. This method is considered widely in the literature, see for example Bozer and White (1984), Lee and Elsayed (2005), and De Koster et al. (2006). In many studies, like Hausman et al. (1976) and Lee and Elsayed (2005), it is used to benchmark improvements of other storage policies. The full turnover-based policy is first described by Heskett (1963; 1964) as the Cube-per-Order index (COI) rule without a proof of its optimality. Kallina and Lynn (1976) discuss the implementation of the COI rule in practice. The earlier mentioned work of Hausman et al. (1976) assumes a Pareto (or ABC)-demand curve and a basic EOQ (Economic Order Quantity)-based reordering policy, in the derivation of an expression for the expected single-command travel time for random and full turnover-based storage. Graves et al. (1977) extend this to an expression for the expected dual-command travel time under these storage policies. These analytical results under the full turnover-based storage policy are derived for SIT racks. The formulation by Hausman et al.

(1976) to calculate the one way travel time is a universal expression which can be used for NSIT racks and multi-deep racks as well, because in its derivation only EOQ assumptions and an ABC-demand function are used. It has been used by other researchers in different warehouse settings. For example, Koh et al. (2002) apply it to estimate the travel time for a warehousing system with a crane in combination with a carousel. Kim and Seidmann (1990) assume a product turnover distribution function different from that of Hausman et al. (1976) resulting in a different single-command travel time function. Their turnover distribution function has the advantage that it is analytically more tractable. However, it has the disadvantage that it is not concave as Figure 1 in their paper shows. For NSIT racks, Park et al. (2003) assume that the full turnover-based distribution function is given as $G(x) = x^s$ for $0 < x \le 1$ (the percentile of unit load stored) and $0 < s \le 1$ (a parameter representing the Pareto shapes) which is a simpler form than that of Hausman et al. (1976) who look at products rather than unit loads to model the Pareto curve. The same full turnover-based distribution function is also used by Park et al.(1999), Park (2006), and Park et al.(2006).

***3D storage systems.*** Park and Webster (1989b) propose a conceptual model that can help a warehouse planner in the design of certain 3D pallet-storage systems by minimizing the total storage system costs. The costs consist of land, building, handling equipment, storage-rack, labor, maintenance, and operating costs. Park and Webster (1989a) deal with a "cubic-in-time" layout, for minimizing the travel time of selected handling equipment. These two publications study conventional 2D (single deep) storage systems from a three-dimensional point of view by considering multiple 2D racks and aisles. The dimensions of the 3D storage systems are given. Their system work mechanism basically is the same to conventional 2D systems. Sari et al. (2005) study a 3D flow-rack AS/RS where the pallets are stored and retrieved at different rack sides by two cranes. In order to retrieve a particular pallet, the retrieval crane has to move all pallets in front of it and store these on a special restoring conveyer. They derive the travel time for the random storage policy with given lengths of the three rack dimensions. De Koster et al. (2006) extend the method of Bozer and White (1984) for 2D rack systems with rotating conveyors to three dimensions and find the optimal design of the 3D rack system by deriving the expected travel time of the S/R machine of random S/R requests under the random storage policy. They conclude that the optimal ratio of the three dimensions in vertical, horizontal and conveyor directions is $0.72:0.72:1$ for single-command systems.

No literature exists on travel time estimation and/or optimal system dimensioning for 3D AS/RS with the full turnover-based storage policy. In the following sections, we will step by step estimate the single-command travel time of the S/R machine after first introducing the problem assumptions.

## 3. Assumptions and general model

### *3.1 Assumptions*

The studied system is identical to that of De Koster et al. (2006), and sketched in Figure 1. We follow the assumptions of De Koster et al. (2006), (see also Ashayeri et al., 2002; Bozer and White, 1984, , 1990, , 1996; Foley et al., 2004; see also Hausman et al., 1976)

- The 3D rack is considered to have a continuous rectangular pick face, where the I/O point (or depot) is located at the lower left-hand corner of the rack (see Figure 1). The rack storage capacity is a given positive constant.
- The S/R machine is capable of simultaneously moving in vertical and horizontal direction at constant speeds. Thus, the travel time required to reach any location in the rack (or a storage conveyor pair in our case) is represented by the Chebyshev metric. When the crane is idle, it parks at the I/O point
- The conveyor can move unit loads in an orthogonal depth direction, independent of the S/R machine movement, at a constant speed.
- The S/R machine operates on a single-command basis (multiple stops in the aisle are not allowed).
- Following many papers on 2D rack AS/RSs (Bozer and White, 1984; Hausman et al., 1976), the prepostioning of unit loads when the system is idle is not considered.

- Each unit load holds only one item type. All storage locations and unit loads have the same size. Therefore all storage locations can be used for storing any unit load. The items are replenished according to the EOQ model.
- Following Hausman et al. (1976), we assume the pick-up/deposit (P/D) time for the crane to pick up or deposit a unit load can be ignored. This is justified if the P/D time is fairly small compared to the total crane travel time.
- We use a full turnover-based storage policy. That is, the storage position of each unit load is determined by its relative activity among all unit loads in the rack by sorting the unit loads from most to least active, starting from the I/O point. One item type can have multiple unit loads.

### 3.2 Notations and general model

For comparison convenience, we adopt the same notations from De Koster et al. (2006) when available. The length (*L*), the height (*H*) of the rack, and the length (2*P*) of every pair of conveyors form three orthogonal dimensions of the rack, in which the speed of the conveying mechanisms, and the S/R machine's speed in the horizontal and vertical direction are $s_c$, $s_h$, and $s_v$, respectively.

To standardize the system, we define the following quantities.

$t_c = \dfrac{2*P}{s_c}$: length (in time) of the conveyor.

$t_h = \dfrac{L}{s_h}$: length (in time) of the rack.

$t_v = \dfrac{H}{s_v}$: height (in time) of the rack.

$T = \max\{t_h, t_v, t_c\}$.

$b = \min\left\{\dfrac{t_h}{T}, \dfrac{t_v}{T}, \dfrac{t_c}{T}\right\}$. Note that $0 < b \le 1$ and $b = 1$ if $t_h = t_v = t_c$.

$a$ is the remaining element (besides $b$ and 1) of the set $\left\{\frac{t_h}{T}, \frac{t_v}{T}, \frac{t_c}{T}\right\}$, thus $0 < b \le a \le 1$.

For determining the optimal dimensions of the rack, we suppose that $H * L * P$ is a constant. As a result $t_h t_v t_c = V$ (the storage capacity in cubic time) is also a positive constant. Set $H * L * P = V'$ (volume in cubic meter units), i.e., $(t_v s_v)(t_h s_h)(0.5 t_c s_c) = V'$. The relationship between $V'$ and $V$ can be expressed as:

$$V = \frac{2V'}{s_h s_v s_c}. \qquad (1)$$

Assume that a S/R request location is represented by $(x, y, z)$, on the movement directions of the S/R machine or conveyor: the longest dimension refers to the $z$ direction, the shortest dimension refers to the $y$ dimension and the left medium dimension refers to the $x$ direction. The S/R machine's retrieval time consists of the following components:

- Time needed for the S/R machine to go from the depot to an S/R position (as shown in Figure 1(c)) to pick-up an available unit load. The unit load is made available to the S/R position by the conveyor movement mechanism. Because the movements in three dimensions are independent, this time, denoted by $W$, is the maximum of the following three quantities:
  - time needed for the S/R machine to travel horizontally from the depot to the S/R position,
  - time needed for the S/R machine to travel vertically from the depot to the S/R position,
  - time needed for the conveyor to circulate the load from the current position to the S/R position.
- Time needed for the S/R machine to return to the depot from the S/R point, $U$.

Hence, the corresponding expected S/R machine travel time, called expected single-command cycle (*ESC*), can be expressed as follows:

$$ESC = E(W) + E(U). \qquad (2)$$

In order to derive $ESC$ under the full turnover-based policy, we recall Hausman et al.

(1976). In their paper, in order to calculate the turnover of each item in a storage space, they model the well-known ABC curve as

$$G(i) = i^{\delta}, \quad 0 < \delta \le 1, \tag{3}$$

where $i$ is the percentage of inventoried items, $0 < i \le 1$, $\delta$ is the skewness of the ABC curve, and $G(i)$ is the cumulative percentage of demand in full unit loads. Under the full turnover-based storage policy, for a fraction or percentage $i$ of the items, they derive the expected one-way travel time for the crane traveling from the I/O point to a random P/D position of a request to pick-up or store a unit load (denoted by $T_T^{'}$ in their paper) as:

$$T_T^{'} = \frac{\int_{j=0}^{1} \lambda(j) y(j) dj}{\int_{j=0}^{1} \lambda(j) dj}, \tag{4}$$

in which $\lambda(j)$ is the turnover of the $j$th unit load in the rack. Assuming that an EOQ order policy is implemented, Hausman et al. (1976) obtain:

$$\lambda(j) = (\frac{2\delta}{K})^{1/2} j^{(\delta-1)/(\delta+1)}, \quad 0 < j \le 1, \tag{5}$$

where $K$ is the ratio of order cost to holding cost which is assumed to be identical for all items. $y(j)$ is the ranked one-way time to travel from the I/O point to location $j$ and $0 < y(j) \le 1$, where by definition the $j$th percentile of the locations is closer to the I/O point than the location under consideration. For 2D SIT racks, $y(j)$ equals $j^{1/2}$ (Hausman et al., 1976). However, in a multi-deep system, a third dimension has to be added to retrieve and move a load to the S/R position from inside of the rack as shown in Figure 1(c). This additional dimension makes the derivation of $y(j)$ more complex than for a single-deep 2D storage system. The derivation process of $y(j)$ can be found in Appendix A, from which we have

$$y(j) = \begin{cases} (abj)^{1/3} & 0 < j \le b^2/a \\ (aj)^{1/2} & b^2/a < j \le a \\ j & a < j \le 1 \end{cases}. \tag{6}$$

Substituting (6) into (4), and multiplying the result with $T$ results in

$$E(W) = T \times \frac{\int_{j=0}^{b^2/a} \lambda(j)\sqrt[3]{abj}dj + \int_{j=b^2/a}^{a} \lambda(j)\sqrt{aj}dj + \int_{j=a}^{1} \lambda(j)jdj}{\int_{j=0}^{1} \lambda(j)dj}$$

$$\Rightarrow E(W) = T\left(\frac{sb^{2s+1}}{(2s+1)(3s+1)a^s} + \frac{sa^{s+1}}{(s+1)(2s+1)} + \frac{s}{s+1}\right), \quad (7)$$

where $s = 2\delta/(1+\delta)$.

$E(U)$ can be obtained in a similar fashion, by neglecting the depth movement.

Without loss of generality, we suppose $t_h \geq t_v$. Set $\beta = t_v / t_h$ which is the rack shape in the crane's moving directions. If we standardize $t_h = 1$, then similar to the above procedure for obtaining (6), we have

$$y(j) = \begin{cases} (\beta j)^{1/2} & 0 < j \leq \beta \\ j & \beta < j \leq 1 \end{cases}. \quad (8)$$

Substituting (8) into (4) and multiplying the result with $T = t_h$, $E(U)$ is obtained as

$$E(U) = \frac{\int_{j=0}^{\beta} \lambda(j)\sqrt{\beta j}dj + \int_{j=\beta}^{1} \lambda(j)jdj}{\int_{j=0}^{1} \lambda(j)dj} t_h = \frac{s(2s+1+\beta^{s+1})}{(s+1)(2s+1)} t_h, \quad (9)$$

From (2), (7) and (9), the mathematical model to dimension the optimal storage rack system then can be determined by the following general model (denoted as GM):

**Model GM:**

$$\begin{aligned} &Minimize \quad ESC(a,b,T) = E(U) + E(W) \\ &= \frac{s(2s+1+\beta^{s+1})}{(s+1)(2s+1)} t_h + \left(\frac{sb^{2s+1}}{(2s+1)(3s+1)a^s} + \frac{sa^{s+1}}{(s+1)(2s+1)} + \frac{s}{s+1}\right)T \\ &subject\ to \quad abT^3 = V \\ &\beta = \begin{cases} b/a & if\ t_c = T \\ b & if\ t_c = aT \\ a & if\ t_c = bT \end{cases} \\ &t_h = \begin{cases} aT & if\ t_c = T \\ T & if\ t_c = aT \\ T & if\ t_c = bT \end{cases} \end{aligned} \quad (10)$$

where $T > 0$ and $0 < b \le a \le 1$.

When the optimal values of variables a, b, $T$ of model GM can be determined, the expected travel time is minimized for a given rack capacity $V$. In order to find these optimal solutions, we distinguish the following three cases:

- The conveyor's length is the longest dimension (denoted by **CL**), or $t_c = T$ and $t_v : t_h : t_c \equiv b : a : 1$;
- The conveyor's length is the medium dimension (denoted by **CM**), or $t_c = aT$ and $t_v : t_h : t_c \equiv b : 1 : a$;
- The conveyor's length is the shortest dimension (denoted by **CS**), or $t_c = bT$ and $t_v : t_h : t_c \equiv a : 1 : b$.

## 4. Model properties and equivalent model

Solving Model GM directly based on the three cases CL, CM, and CS is difficult. Therefore, we propose several theorems to simplify it. Theorems 1 and 2 show that the cases CS and CM can be neglected respectively. Theorem 3 shows the optimal rack shape is SIT. These theorems lead to a much easier nonlinear convex programming model equivalent to model GM.

We first reformulate Model GM for the three cases: CL, CM, and CS respectively.

For the case CL, $t_c = T$ and the corresponding model can be presented as:

$$\begin{aligned} &\textit{Minimize}\ \ ESC_{CL}(a,b) = \frac{V^{1/3} a^{-s} s}{(1+s)(1+2s)(1+3s)(ab)^{1/3}} (a^s + a^{1+s} + a^{1+2s} + b^{1+s} \\ &\qquad + b^{1+2s} + 5a^s s + 5a^{1+s} s + 3a^{1+2s} s + 3b^{1+s} s + b^{1+2s} s + 6a^s s^2 + 6a^{1+s} s^2) \quad (11) \\ &\textit{subject to}\quad 0 < b \le a \le 1. \end{aligned}$$

For the case CM, $t_c = aT$ and the corresponding model turns out to be:

$$\begin{aligned} &\textit{Minimize}\ \ ESC_{CM}(a,b)=\frac{V^{1/3}a^{-s}s}{(1+s)(1+2s)(1+3s)(ab)^{1/3}}(2a^{s}+a^{1+2s}+a^{s}b^{1+s}\\ &\qquad +b^{1+2s}+10a^{s}s+3a^{1+2s}s+3a^{s}b^{1+s}s+b^{1+2s}s+12a^{s}s^{2}) \qquad (12)\\ &\textit{subject to}\quad 0<b\le a\le 1. \end{aligned}$$

For the case CS, $t_c = bT$ and the corresponding model can be presented as:

$$\begin{aligned} &\textit{Minimize}\ \ ESC_{CS}(a,b)=\frac{V^{1/3}a^{-s}s}{(1+s)(1+2s)(1+3s)(ab)^{1/3}}(2a^{s}+2a^{1+2s}+b^{1+2s}\\ &\qquad +10a^{s}s+6a^{1+2s}s+b^{1+2s}s+12a^{s}s^{2}) \qquad (13)\\ &\textit{subject to}\quad 0<b\le a\le 1. \end{aligned}$$

We denote $(a_l, b_l)$, $(a_m, b_m)$, and $(a_s, b_s)$ as the optimal variable values of Models (11), (12), and (13) where the minimal objective function values are denoted by $ESC^*_{CL}(a_l, b_l)$, $ESC^*_{CM}(a_m, b_m)$, and $ESC^*_{CS}(a_s, b_s)$, respectively.

The optimal variable value of $(a,b)$, denoted by $(a^*, b^*)$, of Model GM satisfies

$$(a^*,b^*)=\arg\min_{a,b}\{ESC^*_{CS}(a_s,b_s),\ ESC^*_{CM}(a_m,b_m),\ ESC^*_{CL}(a_l,b_l)\}. \qquad (14)$$

The minimum objective value of Model GM is $ESC^*(a^*, b^*, T^*)$ where $T^* = V^{1/3}/(a^*b^*)^{1/3}$.

### 4.1 *Simplifying Model GM*

**Theorem 1**. The minimal objective function value $ESC^*_{CM}(a_m, b_m)$ of model (12) is

(I) equal to the minimal objective function value $ESC^*_{CS}(a_s, b_s)$ of model (13) if $a_s = b_s = a_m = b_m$.

(II) less than the minimal objective function value $ESC^*_{CS}(a_s, b_s)$ otherwise.

**Proof**. See Appendix B.

Case I, or $a_s = b_s$ in Theorem 1, represents the situation where the optimal rack's $y$-dimension ($b_s$) equals the $x$-dimension ($a_s$) in the CS case. This rack configuration is included in the CM case if $a_m = b_m$ (note that $(a,b)$ represent different dimensions in the two cases). We conclude from Theorem 1:

"*The case CS in Model GM can be neglected for calculating the optimal solution of Model GM.*"

Theorem 2 is similar to Theorem 1. We state it here without proof (this follows the same lines as the proof of Theorem 1). It shows the case CM can be neglected in calculating the optimal solution of Model GM.

**Theorem 2**. The minimal objective function value $ESC^*_{CL}(a_l, b_l)$ of model (12) is

(I) equal to the minimal objective function value $ESC^*_{CM}(a_m, b_m)$ of model (13) if $a_l = b_l = b_m = a_m$=1 (i.e. cubic-in-time).

(II) less than the minimal objective function value $ESC^*_{CM}(a_m, b_m)$ otherwise.

From Theorem 1 and Theorem 2, we conclude:

"*All optimal solutions of Model GM exist in the case CL (i.e., Model (11)). The model (11) is an equivalent to Model GM.*"

It is obvious that finding the optimal expected travel time of Model (11) is easier than Model GM. However it is still too complicated to prove whether its objective function is convex or concave. We therefore decide to analyze the problem further.

With Theorem 3 we prove the optimal 3D rack must be SIT.

**Theorem 3**. For the 3D rack, the expected travel time with the full turnover-based storage policy will be minimized **only when** the rack is SIT and the conveyor's length is the longest.

**Proof**. See Appendix C.

### *4.2 The equivalent model of Model GM and its solution*

From Theorem 3, we conclude the optimal solution of Model GM has the following properties: $T = t_c$, $a = b\left(\text{thus } \beta = 1\right)$, $t_h = t_v = at_c$, and $a^2 t_c{}^3 = V$. Therefore a model equivalent to model GM is the following constrained-optimization problem:

$$\begin{aligned} &\textit{Minimize} \quad ESC(a) = \frac{V^{1/3}s}{(1+s)(1+2s)(1+3s)(a)^{2/3}}(1+2a+2a^{1+s}+5s \\ &\qquad\qquad +8as+4a^{1+s}s+6s^2+6as^2) \qquad (15) \\ &\textit{subject to} \quad D = \left\{a \middle| 0 < a \le 1\right\}. \end{aligned}$$

Since $\dfrac{d^2 ESC(a)}{da^2} = \dfrac{2sV^{1/3}}{9a^{8/3}(1+s)(1+2s)}(5-2a-2a^{1+s}+10s-2as-a^{1+s}s+6a^{1+s}s^2)$ >0 and constraint $D$ is linear, the problem is a strict convex non-linear programming problem.

At this point, if the critical point $a^*$ of equation $\dfrac{dESC(a)}{da}$=0 is in $D$, we have found the minimum objective function value $ESC(a^*)$, where

$$\frac{dESC(a)}{da} = \frac{2sV^{1/3}}{3a^{5/3}(1+s)(1+2s)}(-1+a+a^{1+s}-2s+as+2a^{1+s}s). \qquad (16)$$

Because $\lim\limits_{a\to 0}\dfrac{dESC(a)}{da} = -\infty$, $\dfrac{dESC(a)}{da}\Big|_{a=1} = 2V^{1/3}s/(3(1+2s))$ >0, and $\dfrac{dESC(a)}{da}$ is continuous, the unique critical point $a^*$ of equation $\dfrac{dESC(a)}{da}$=0 must be in $D$.

Equation $\dfrac{dESC(a)}{da}$=0 can be solved numerically for any given $s$. The optimal solution of Model (15) is given by the optimal decision variable $a = a^*$ and the optimal objective function value $ESC(a^*)$. $a^*$ is a function of $s$ according to (16).

Because $a^2 t_c{}^3 = V$, we have $t_c{}^* = V^{1/3}/a^{*2/3}$. Again, because $t_h = t_v = at_c$, we have $t_v{}^* = t_h{}^* = (a^* V)^{1/3}$.

From the above analysis, we conclude the following for Model GM:

(a) Given a 3D rack with a total storage capacity $V$, the expected travel time of the S/R machine will be minimized if $t_v^* = t_h^* = (a^* V)^{1/3}$ and $t_c^* = V^{1/3}/a^{*2/3}$ (i.e.

$t_v^* : t_h^* : t_c^* = a^* : a^* : 1$) and the optimal expected travel time for the single command cycle is $ESC(a^*)$ where $ESC(a)$ is the objective function of Model (15) and $a^*$ is the solution of the equation $\frac{dESC(a)}{da}$ =0.

(b) The optimal ratio of the three dimensions $t_v^* : t_h^* : t_c^*$ is a function of the skewness parameter $s$, but independent of the rack capacity $V$.

Like in any AS/RS design, physical restrictions may limit the dimensions' choices, for example in case of implementation of an AS/RS in a low building. However, as compact systems of large sizes exist, in above we follow common literature (e.g. Bozer and White, 1984; Hausman et al., 1976; Park et al., 2003) and consider unconstrained optimization. We can easily solve a problem constrained in one or more of the dimensions if we can solve the unconstrained problem (Model GM). For example, to design the compact system with a limitation on the height, we need to add an upper bound constraint for the vertical dimension into Model GM to have a new model (denoted by Model GM'). For solving Model GM', we can first check whether the added constraint effects the optimal solution of Model GM. If it does not, the optimal solution of Model GM is the optimal solution of Model GM'. Otherwise, we can convert Model GM' into 2 two-dimensional sub-problems: one is to set the vertical dimension equal to its upper bound and the other is to set $t_h = t_v$ (refer to Theorem 3), and solve these. After that an optimal solution of Model GM' can be obtained by comparing the optimal solutions of the two sub-problems.

## 5. Comparing the results with those of De Koster et al. (2006)

De Koster et al. (2006) consider randomized storage policies in which any point within the rack is equally likely to be selected for storage or retrieval. Their problem corresponds to $\delta = 1$ or $s = 1$ in our paper. Let $s = 1$, the objective function of Model GM turns into:

$$ESC(a,b,T) = \left(\frac{\beta^2}{6} + \frac{1}{2}\right)t_h + T\left(\frac{b^3}{12a} + \frac{a^2}{6} + \frac{1}{2}\right), \tag{17}$$

which is the same as that of De Koster et al. (2006). The optimal solution for a random storage policy can be found from De Koster et al. (2006) as: $t_h^* = t_v^* = 0.90V^{1/3}$, $T = t_c^* = 1.24V^{1/3}$ and $ESC^* = 1.38\,V^{1/3}$.

In our paper, using the conclusion in Section 4.2, we can find the optimal solution and its expected travel time for the 3D AS/RS rack system for every skewness parameter value corresponding to a particular ABC curve. Using Equation (3) to represent an ABC curve, the notation $i/G(i)$ denotes that a fraction $i$ of the inventoried items represents a fraction $G(i)$ of the total demand. For a given $i/G(i)$ combination we can obtain $\delta$ from Equation (3), by solving $\delta = \ln G(i)/\ln i$. According to the relationship between $s$ and $\delta$, $s$ can be further determined by $s = \delta/(2-\delta)$.

Table 1 tabulates values of the optimal solutions for different $i/G(i)$ combinations and their corresponding $s$ or $\delta$ values for a given $V$. In this table, the time values for $t_h^*$, $t_v^*, t_c^*$ and $ESC^*$ are expressed in the quantities $V^{1/3}$. In Figure 2, the expected travel time of our full turnover-based storage policy is compared with that of the random storage policy in De Koster et al. (2006), for various ABC curves, and shows the corresponding expected travel time improvement. In this Figure, Series "$ESC_{FT}$", "$ESC_{RAN}$" and "Time saved" represent the optimal $ESC^*$ value of this paper, the optimal $ESC^*$ value of De Koster et al. (2006), and the percentage improvement $(ESC_{FT} - ESC_{RAN})/ESC_{RAN} \times 100\%$, respectively.

**<Insert Table 1 here>**

**<Insert Figure 2 here>**

From Table 1 and Figure 2, it can be seen that

(1) When $0 < \delta < 1$ (all cases except 20%/20%), reductions in the expected travel time are obtainable from the turnover-based storage policy compared with the random storage policy in the 3D rack system. The reduction percentage depends on the steepness of the ABC curve. For a 20%/90% ABC curve with $\delta = 0.07$, the improvement is significant and the percentage of the travel time saved is 67.68%.

(2) When $\delta = 1$ (20%/20%), our result is the same as that of De Koster et al. (2006) with

the random storage policy. The problem in their paper is a special case of that in this paper with $\delta = 1$.

(3) For the turnover-based storage policy, the smaller the skewness parameter $\delta$ is in the ABC curve, the more sensitive the expected travel time is. For example, when $\delta$ decreases from 1 to 0.75 (20%/20% to 20%/30%), the relative ESC decreases by $\frac{1.38V^{1/3} - 1.31V^{1/3}}{1 - 0.75} = 0.26V^{1/3}$, however when $\delta$ decreases from 0.14 to 0.07 (20%/80% to 20%/90%), the relative ESC decrease is $\frac{0.72V^{1/3} - 0.45V^{1/3}}{0.14 - 0.07} = 3.74V^{1/3}$, which is much bigger than $0.26V^{1/3}$.

## 6. An example

As an illustrating example, assume that we have to design a 3D compact system with data in Table 2, based on those in De Koster et al. (2006). The layout of the system refers to Figure 1. The problem is to 1) find the near optimal dimensions of the system so for two given ABC curves in Table 2, and 2) compare the results with those of random storage policy.

**<Insert Table 2 here>**

The rack should have sufficient capacity to store 1000 pallets, which means that the rack should have at least $V' = 1.2 \times 1.2 \times 2 \times 1000 = 2880$ ($m^3$). Or $V = 2V'/(s_h s_v s_c) = 3600$ ($\text{seconds}^3$) according to equation (1).

Recalling the conclusion in Subsection 4.2, we obtain the optimal solutions for a continuous rack system: (1) for the 20%/20% ABC curve, $t_c^* = 1.24\sqrt[3]{V} = 19.07$ (seconds), $t_h^* = t_v^* = 0.72t_c^* = 13.74$ (seconds) and the optimal travel time $ESC^* = 1.38\sqrt[3]{V} = 21.18$ (seconds); (2) for the 20%/90% ABC curve, $t_c^* = 1.50\sqrt[3]{V} = 22.93$ (seconds), $t_h^* = t_v^* = 0.55t_c^* = 12.53$ (seconds) and the optimal travel time $ESC^* = 0.45\sqrt[3]{V} = 6.84$ (seconds).

However, in a real-world setting, AS/RS systems are discrete and the rack dimensions must be integral multiples of the pallet dimensions. The rack horizontal dimension must be an even multiple of the pallet dimension since it consists of multiples of two conveyors working in pairs. Therefore, we choose ‘practical optimal’ dimensions such that they are as close as possible to the corresponding optimal dimensions found while the system storage capacity is at least 1000 pallets. We obtain the following practical near optimal dimensions and expected travel time for both ABC curves: (1) for the 20%/20% ABC curve, $\overline{t_h}^* = 14.4$ seconds (30 pallets), $\overline{t_v}^*$ =12.5 seconds (5 pallets), $\overline{t_c}^*$ =21 seconds (7 pallets), the approximate optimal travel time $\overline{ESC}^*$ =21.53 seconds, with a real rack capacity of 1050 pallets; (2) for the 20%/90% ABC curve, $\overline{t_h}^* = 12.48$ seconds (26 pallets), $\overline{t_v}^*$ =12.50 seconds (5 pallets), $\overline{t_c}^*$ =24 seconds (8 pallets), the near optimal travel time $\overline{ESC}^*$ =6.93 seconds, and the real rack capacity is 1040 pallets. From the above results we find that the deviation of the near optimal solutions from the optimal solutions is fairly small: the deviation percentages (i.e. $(\overline{ESC}^* - ESC^*)/ESC^* \times 100\%$) are 0.16% and 0.13% respectively. Note that the resulting rack dimensions do not differ much in $ESC$. This phenomenon makes it is possible to find robust layouts good for various ABC-curves. If for a 20%/90% ABC curve, the random storage policy is implemented, the travel time will equal that of a 20%/20% ABC curve, and be tripled (21.53/6.93≈3).

## 7. Conclusion

In this paper we have derived the travel time for compact 3D AS/RSs with a full turnover-based storage policy, discussed its optimal dimension design, and compared the results with those in De Koster et al. (2006) who studied the system with random storage policy. From the results of the present paper, we find that

(1) The optimal ratio between the three dimensions $t_c^* : t_h^* : t_v^*$ varies with the skewness parameter $\delta$ of the ABC curve. For a decreasing $\delta$, or increasing turnover frequency for a given percentage of the inventoried items in the rack inventory, the optimal ratio $t_c^* / t_h^*$ or $t_c^* / t_v^*$ will increase. The problem with the random storage policy discussed by De Koster et

al. (2006) is a special case of our problem with skewness parameter $\delta = 1$.

(2) For the 3-dimemsional rack system, the expected travel time will be minimized only when the rack is SIT in horizontal and vertical directions, which is similar to the results of Bozer and White (1984) and De Koster et al. (2006), but not cubic in time for any $s \in (0,1]$.

(3) The full turnover-based storage policy is a good assignment rule for improving the performance of the expected travel time of S/R machine for single command cycle. The more skewed (smaller $\delta$) the ABC curve is, the more expected time is saved compared to the random storage policy. For example, for $\delta = 0.07$ (a 20%/90% ABC curve), the saved time is 67.68%.

(4) From Section 6, it can be seen that the optimal results for our discussed continuous 3D AS/RS are helpful to find a near optimal solution for practical examples.

(5) Our model and results potentially can be applicable to those 3D storage systems that have similar work mechanisms.

Moreover, the derived ESC can be applied to evaluate the operational performance of an existing 3D AS/RS, and may be extended further in several directions, albeit the analysis may become cumbersome. It is interesting to study the class-based storage assignment. Although class-based storage is not optimal, it is easier to implement in practice while it still can improve travel time substantially, compared to random storage. Second, the impact of dwell point strategies of the S/R machine can be studied. Third, multiple commands for a single cycle are particularly interesting to consider since the conveyors might then preposition loads. The analysis however will become very cumbersome then. But numerical results might still be feasible. Finally, the time needed for pickup/drop-off a unit load may also be considered in a 3D rack system although it is commonly omitted by researchers in 2D systems.



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## Appendix A. Calculation of Equation (6)

Because $W = \max\{t_h, t_v, t_c\}$ and $0 < b \le a \le 1$, the calculation of $y(j)$ should be classified into three cases (see Figure 3).

**<Insert Figure 3 here>**

Case 1: Let $j \le b^3/(ab) = b^2/a$; or $0 < W \le b$. Consider a location $(x, y, z)$ in the $j$th fractile or percentile of the distance distribution (region A in Figure 3). By definition *j%* of the locations are closer to the I/O point than the location under consideration. These *j%* locations must be arranged in a cube in time, since the total time taken by the crane and conveyor to move from the I/O point to the P/D position of any point $(x, y, z)$ is max(*x,y,*z). Since the dimension of the total warehouse is *ab,* the volume of this cube is $j$ by the total volume $ab$, or $abj$. Therefore, for $j \le b^2/a$, the travel time from the depot to the location *j*th percentile is

$$y(j) = (abj)^{1/3}. \tag{18}$$

Case 2: Any point $(x, y, z)$ located in region B (Figure 3) satisfies the location percentile $j \ge b^3/(ab) = b^2/a$ and $j \le a^2b/(ab) = a$; or $b \le W \le a$ where $W = \max(x, z)$. The racked locations of the $j$th percentile must be arranged in a rectangular block in time with $\sqrt{aj} \times b \times \sqrt{aj}$ in the horizontal, vertical and depth dimensions (so that the total volume is $abj$ and $\sqrt{aj} \ge b$). Then, for $b^2/a \le j \le a$, the travel time from the depot to *j*th percentile location is

$$y(j) = \sqrt{aj}. \tag{19}$$

Case 3: Any point $(x, y, z)$ located in the region C (Figure 3) satisfies $a \le j \le 1$ or $a \le W \le 1$ where $W = z$. The locations of $j$th percentile must be arranged in a rectangular block in time with $a \times b \times j$ in the horizontal, vertical and depth dimensions (so that the total volume is $abj$). Then for $a \le j \le 1$, the travel time from the depot to the location *j*th percentile is

$$y(j) = j. \tag{20}$$

Considering (18), (19) and (20), Equation (6) is obtained.

## Appendix B. Proof of Theorem 1

Because the optimal solution $(a,b)$ of model (13) is $(a_s, b_s)$, its objective function value is

$$ESC^*_{CS}(a_s,b_s) = \frac{V^{1/3} a_s^{-s} s}{(1+s)(1+2s)(1+3s)(a_s b_s)^{1/3}} (2a_s^{\ s} + 2a_s^{\ 1+2s} + b_s^{\ 1+2s} + 10a_s^{\ s} s + 6a_s^{\ 1+2s} s + b_s^{\ 1+2s} s + 12a_s^{\ s} s^2). \tag{21}$$

The constraint is the same for Models (12) and (13), so $(a_s, b_s)$ is a feasible solution of model (12), and its corresponding objective function value of model (12) is

$$ESC_{CM}(a_s,b_s) = \frac{V^{1/3} a_s^{-s} s}{(1+s)(1+2s)(1+3s)(a_s b_s)^{1/3}} (2a_s^{\ s} + a_s^{\ 1+2s} + a_s^{\ s} b_s^{\ 1+s} + b_s^{\ 1+2s} + 10a_s^{\ s} s + 3a_s^{\ 1+2s} s + 3a_s^{\ s} b_s^{\ 1+s} s + b_s^{\ 1+2s} s + 12a_s^{\ s} s^2). \tag{22}$$

From Equations (21) and (22), we have

$$ESC^*_{CS}(a_s,b_s) - ESC_{CM}(a_s,b_s)$$

$$= \frac{V^{1/3} a_s^{-s} s}{(1+s)(1+2s)(1+3s)(a_s b_s)^{1/3}} (a_s^{\ 1+2s} - a_s^{\ s} b_s^{\ 1+s} + 3a_s^{\ 1+2s} s - 3a_s^{\ s} b_s^{\ 1+s} s). \tag{23}$$

Because $a, b, s$, and $V > 0$, we have

$$\frac{V^{1/3} a_s^{-s} s}{(1+s)(1+2s)(1+3s)(a_s b_s)^{1/3}} > 0. \tag{24}$$

When $b_s = a_s$ for $a$ and $b$ in Models (12) and (13), $a_s^{\ 1+2s} - a_s^{\ s} b_s^{\ 1+s} = 0$ and $3a_s^{\ 1+2s} s - 3a_s^{\ s} b_s^{\ 1+s} s = 3a_s^{\ s} s (a_s^{\ 1+s} - b_s^{\ 1+s}) = 0$. Considering Equations (24) and (23), we have $ESC^*_{CS}(a_s,b_s) - ESC_{CM}(a_s,b_s) = 0$. If $(a_s,b_s)$ equals $(a_m,b_m)$ and $b_s = a_s$, then $(a_s,b_s)$ is also the optimal solution of Model (12) and $ESC^*_{CM}(a_m,b_m) = ESC_{CM}(a_s,b_s)$. In this case $ESC^*_{CM}(a_m,b_m) = ESC^*_{CS}(a_s,b_s)$ holds. (I) is proven.

Otherwise, $b_s < a_s$ , in Equation (23), $a_s^{1+2s} - a_s^{\ s} b_s^{\ 1+s} = a_s^{\ s}(a_s^{\ 1+s} - b_s^{\ 1+s}) > 0$ and $3a_s^{\ 1+2s} s - 3a_s^{\ s} b_s^{\ 1+s} s \ = 3a_s^{\ s} s(a_s^{\ 1+s} - b_s^{\ 1+s}) > 0$. Considering Equations (24) and (23), we have $ESC_{CS}^*(a_s, b_s) - ESC_{CM}(a_s, b_s) > 0$. Because $(a_m, b_m)$ is the optimal solution of Model (12), for the minimized model, $ESC_{CM}^*(a_m, b_m) \le ESC_{CM}(a_s, b_s)$ . Thus $ESC_{CS}^*(a_s, b_s) - ESC_{CM}^*(a_m, b_m) > 0$. (II) is proven.

**Appendix C. Proof of Theorem 3**

From the above Theorems 1-2, we know the optimal result of Model GM exists in the case CL, where the conveyor's length is the longest. Thus, if we can prove the 3D rack must be SIT for $ESC_{CL}$ to be minimized, then Theorem 3 is proven. For convenience, we use an equivalent version of the model, different from Model (11) only in form, for the case CL as follows:

$$\begin{aligned}
&Minimize\ \ ESC_{CL}(a,b,t_c) = \frac{a^{-s} s t_c}{(1+s)(1+2s)(1+3s)}(a^s + a^{1+s} + a^{1+2s} + b^{1+s} \\
&\qquad + b^{1+2s} + 5a^s s + 5a^{1+s} s + 3a^{1+2s} s + 3b^{1+s} s + b^{1+2s} s + 6a^s s^2 + 6a^{1+s} s^2) \\
&subject\ to \quad abt_c^3 = V \qquad\qquad (25)\\
&\qquad 0 < b \le a \le 1 \\
&\qquad t_c > 0.
\end{aligned}$$

We use reduction to absurdity to prove that the optimal 3D rack is SIT when $ESC_{CL}$ is minimized.

Suppose the optimal rack were not SIT for CL. Let $(\overline{a}, \overline{b}, \overline{t}_c)$ and $\overline{ESC}_{CL}$ denote the optimal solution and objective function value of Model (25). Then we have $\overline{a} > \overline{b}$, and $\overline{ESC}_{CL} \le ESC_{CL}$.

Because $abt_c^3 = V$, we have $\bar{a}\bar{b} = V/\bar{t}_c^3$. Let $\bar{a}\bar{b} = V/\bar{t}_c^3 = k$ ($k$ is a positive constant).

We can design a new solution: $a = b = k^{1/2}$, and $t_c = \bar{t}_c = \sqrt[3]{V/(\bar{a}\bar{b})} = \sqrt[3]{V/k}$ that provides the 3D SIT rack. Then we obtain

$$\begin{aligned}\overline{ESC}_{CL}(\bar{a},\bar{b},\bar{t}_c) - ESC_{CL}(k^{1/2},k^{1/2},\bar{t}_c) = \frac{\bar{a}^{-(1+3s)} s V^{1/3}}{(1+s)(1+2s)(1+3s)k^{1/3}}(\bar{a}^{2+3s}\\ + \bar{a}^{2+4s} - 2\bar{a}^{1+3s}\sqrt{k} - 2\bar{a}^{1+3s}\sqrt{k^{1+s}} + \bar{a}^{s}k^{1+s} + k^{1+2s} + 5\bar{a}^{2+3s}s + 3\bar{a}^{2+4s}s\\ - 8\bar{a}^{1+3s}s\sqrt{k} - 4\bar{a}^{1+3s}s\sqrt{k^{1+s}} + 3\bar{a}^{s}k^{1+s}s + k^{1+2s}s + 6\bar{a}^{2+3s}s^2 - 6\bar{a}^{1+3s}s^2\sqrt{k}).\end{aligned} \tag{26}$$

Because $\bar{a}, s, k$ and $V > 0$, we have $\frac{\bar{a}^{-(1+3s)} s V^{1/3}}{(1+s)(1+2s)(1+3s)k^{1/3}}$>0. Define

$$\begin{aligned}f(x) = x^{2+3s} + x^{2+4s} - 2x^{1+3s}\sqrt{k} - 2x^{1+3s}\sqrt{k^{1+s}} + x^{s}k^{1+s} + k^{1+2s} + 5x^{2+3s}s + 3x^{2+4s}s\\ - 8x^{1+3s}s\sqrt{k} - 4x^{1+3s}s\sqrt{k^{1+s}} + 3x^{s}k^{1+s}s + k^{1+2s}s + 6x^{2+3s}s^2 - 6x^{1+3s}s^2\sqrt{k}.\end{aligned} \tag{27}$$

Then to check $\overline{ESC}_{CL}(\bar{a},\bar{b},\bar{t}_c) - ESC_{CL}(k^{1/2},k^{1/2},\bar{t}_c)$<0 or not is equivalent to check $f(\bar{a})$<0 where $\bar{a} \neq k^{1/2}$.

With $x, s$ and $k > 0$, we have

$$\frac{\partial^2 f(x)}{\partial x^2} = \frac{x^{-3(1+s)}s}{1+2s}(2x^{s}k^{1+s} + 2k^{1+2s} + x^{2(1+2s)}s + 4x^{s}k^{1+s}s + 3k^{1+2s}s) > 0 \tag{28}$$

Equation (28) shows that $f(x)$ is a strictly convex function of $x$. Therefore at most one critical point exists and satisfies

$$\frac{\partial f(x)}{\partial x} = \frac{x^{-3(1+s)}s}{1+2s}(x^{2+3s} + x^{2+4s} - x^{s}k^{1+s} - k^{1+2s} + 2x^{2+3s}s + x^{2+4s}s - 2x^{s}k^{1+s}s - k^{1+2s}s) = 0. \tag{29}$$

If the point exists, the corresponding value of $f(x)$ must be the overall minimal point. Set $x = \sqrt{k}$, we find that $\frac{\partial f(x)}{\partial x}\Big|_{x=\sqrt{k}}$ =0 and $\sqrt{k}$ is the critical point. That is, $\min f(x) = f(\sqrt{k}) = 0$. Then we have $f(x) > 0$ for all $x \neq \sqrt{k}$.

Because the optimal rack were not SIT for CL (i.e. $\bar{a} > \bar{b}$ ), and $\bar{a}\bar{b} = k > 0$, we have $\bar{a} > \sqrt{k} > \bar{b}$ (here $\bar{a} \neq \sqrt{k}$), and then $f(\sqrt{k}) < f(\bar{a})$. Then $f(\bar{a}) > 0$, which implies that

$\overline{ESC}_{CL}(\overline{a},\overline{b},\overline{t}_c) - ESC_{CL}(k^{1/2},k^{1/2},\overline{t}_c) > 0$, contradicting that $(\overline{a},\overline{b},\overline{t}_c)$ is the optimal solution of Model (25). Hence, we have completed the proof of Theorem 3.

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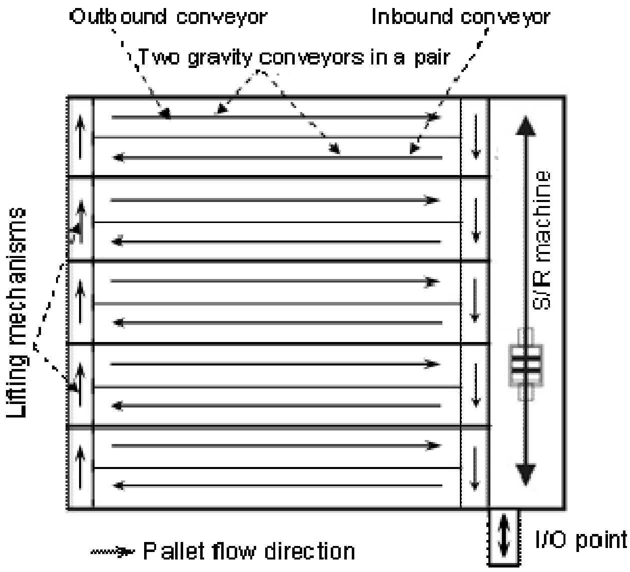
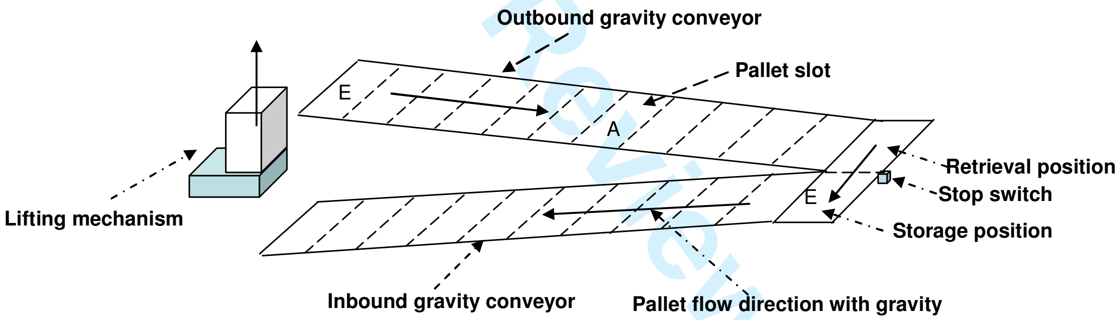
## List of the Tables and Figures

**(a)** Overall sketch




**(b)** Top-view sketch for unit-load flow directions




**(c)** Side-top view sketch for the operation of a pair of gravity conveyors

**Figure 1:** A compact AS/RS with gravity conveyors in pairs (De Koster et al., 2006)

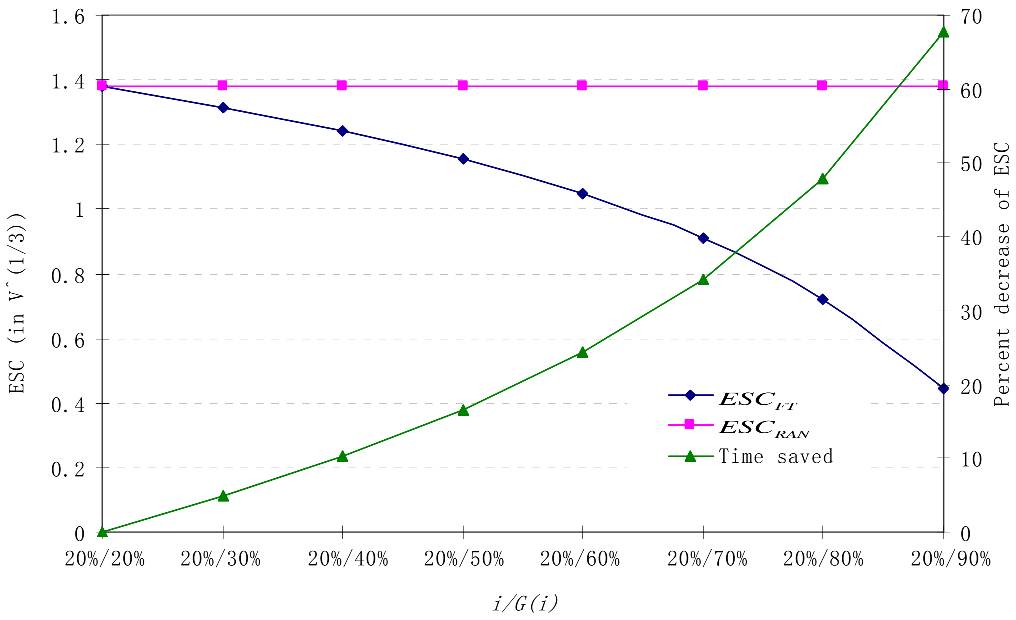



Figure **2**: Reduction of ESC from the full turnover-based storage policy compared with that of the random storage for various $i/G(i)$

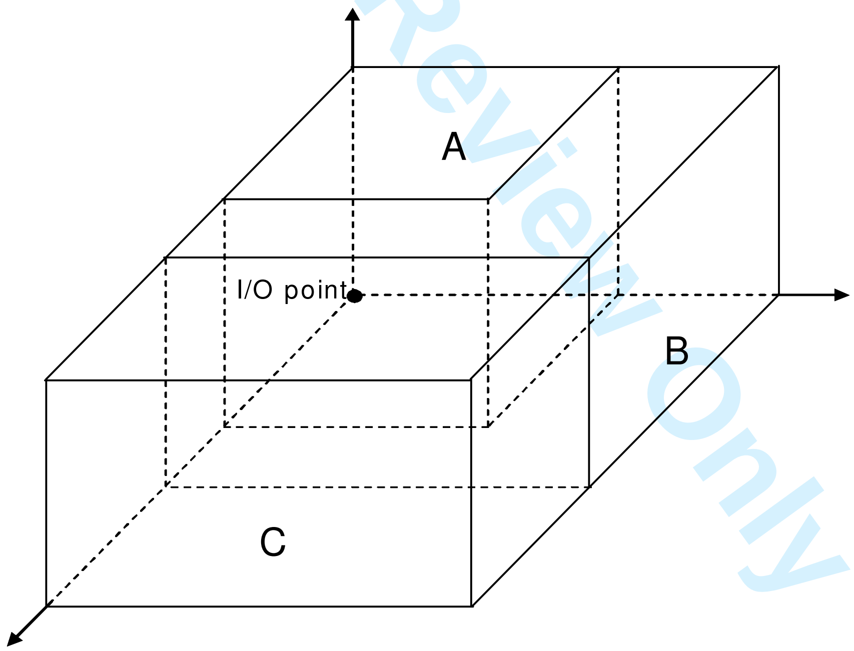



**Figure 3:** Three storage regions in the 3D rack

**Table 1:** The optimal solutions for different skewness parameters(ABC curves)

| $\delta$ | $s$ | ABC Curve | $a^*$ | $b^*$ | $t_h^*$ | $t_v^*$ | $t_c^*$ | $ESC^*$ |
|---|---|---|---|---|---|---|---|---|
| 1.00 | 1.00 | 20%/20% | 0.72 | 0.72 | 0.90 | 0.90 | 1.24 | 1.38 |
| 0.75 | 0.86 | 20%/30% | 0.70 | 0.70 | 0.89 | 0.89 | 1.27 | 1.31 |
| 0.57 | 0.73 | 20%/40% | 0.68 | 0.68 | 0.88 | 0.88 | 1.29 | 1.24 |
| 0.43 | 0.60 | 20%/50% | 0.66 | 0.66 | 0.87 | 0.87 | 1.31 | 1.15 |
| 0.32 | 0.48 | 20%/60% | 0.64 | 0.64 | 0.86 | 0.86 | 1.35 | 1.05 |
| 0.22 | 0.36 | 20%/70% | 0.61 | 0.61 | 0.85 | 0.85 | 1.38 | 0.91 |
| 0.14 | 0.24 | 20%/80% | 0.58 | 0.58 | 0.84 | 0.84 | 1.43 | 0.72 |
| 0.07 | 0.12 | 20%/90% | 0.55 | 0.55 | 0.82 | 0.82 | 1.50 | 0.45 |

**Table 2:** System parameters

| | | |
|---|---|---|
| Total system capacity in pallets | | 1000 pallets |
| Storage policy | | Full turnover-based storage |
| Pallet size in meter (width x length x height) | Net | 1 x 1 x 1.5 |
| | Gross* | 1.2 x 1.2 x 2 |
| S/R machine | Operating policy | Single-command cycle |
| | Vertical speed ( $s_v$ ) | 0.8 (*meter per second*) |
| | Horizontal speed ( $s_h$ ) | 2.5 (*meter per second*) |
| Conveyor speed ( $s_c$ ) | | 0.8 (*meter per second*) |
| Cases of ABC curve considered | | 20%/20% and 20%/90% |

*The gross pallet size includes not only the net pallet size, but also space for uprights, pallet beams, sloping conveyors, and free lift height between the top of the load and the bottom of the next higher beam