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A time continuous analytic production model for service level, WIP, lead-time and utilization

H. JODLBAUER

Because of the fact that some logistical objectives are controversial a trade off between short actual lead times, low inventories, high utilization, high service levels and low backorders is necessary. To make good decisions in the design and in particular optimization of a production system or supply chain a deep understanding of the relationship between the logistical figures is essential. In this article relationships between these logistical figures are developed by an analytical approach for a time continuous multi item production model. Furthermore, the deviation of work in process is used to describe the fluctuation and disturbances of real systems.

Keywords: service level, WIP, lead time, utilization, throughput, backlog

1. Introduction

The relationship between logistical key figures like work in process (WIP), lead time, utilization, finished goods inventory (FGI) and service level is addressed by many authors.

One group of work is based on queuing theory represented for instance by Karmarkar (1987), Hopp/Spearman (1996) or Hopp/Root (2000). Karmarkar (1987) stated that the actual lead-times are highly dependent on actual work-loads and lot sizes. Hopp/Spearman (1996) showed that the lead time is an increasing function for the WIP. In addition they developed bounds describing the best and worst cases for the actual lead time. Spearman/Woodruff/Hopp (1990) define service level as fraction of jobs whose actual lead time is not greater than their planned lead time. In Hopp/Spearman (1996) a good overview and summary of the relationships are presented.

Another group based their work on empirical studies and their so called funnel model. Nyhuis/Wiendahl (1999) and his group developed the funnel model describing the relationship for WIP, utilization, lead time, FGI and

service level. In different studies it is shown that their model fits well, see for instance Wiendahl/Von Cieminski/ Wiendahl (2005) or Löd- ding/Windahl (2003).

Because of the complexity of real systems and a lot of disturbances and fluctuations in a practical environment some authors try to use computer simulation to study the relationship between the logistical key figures. Jodlbauer/Huber (2006) discussed for instance the service level as a func- tion of inventory in a sequential production line with divergent bill of ma- terial under several planning and control strategies like MRP, KANBAN, CONWIP or TOC.

For special situations there are time continuous models addressing some relationships between these logistical key figures. Moon/Choi (1998) pre- sented a further development of the (Q,r) model based on the ideas of Ouy- ang/Yeh/Wu (1996) with an additional decision variable lead time and al- lowing backorders and lost sales. Jodlbauer (2005) developed bounds and some additional relationships for the lead time as well as for the utilization with respect to the inventory for a time continuous model.

A good understanding of the relationship between the logistical figures and understanding their influence of the deviation of processing times, lead times and inventories on the performance measurements is crucial to find the right mix of logistical objectives. Ketokivi/Heikkilä (2003) proposed that strategic objectives derived from the market should be the basis for the necessary tradeoff between conflicting logistical goals like high utilization versus low WIP or high service level versus low FGI.

Erlebacher/Singh (1999) discussed the influence of variance for synchro- nous assembly lines. They introduced rules for determination of the set of stations which should receive variance reduction.

In this article a time continuous model for a multi item production system or supply chain with two stocks and one processing unit is developed. Based on differential equation for the inventories and for the backlog of the same structure and applying a time continuous generalization of Lit- tle's Law, see Jodlbauer/Stöcher 2006, relationships between average lead time, average utilization, average WIP, WIP deviation, average FGI, aver- age backlog and service level are developed and discussed.

The structure of the article is as follows. In section 2 a time continuous multi item model for one machine, one production buffer and a stock for the end items is introduced. It is shown that for every subsystem as well as for the backlog the same differential equation describes the situation well. In addition based on a time continuous generalization of Little's Law some basic equations are proven. In Section 3 the production subsystem consist- ing of the buffer and the machine is investigated and the relationship be- tween average WIP, average production lead time and utilization is devel-

oped. The stock for end items, especially the relationship between average finished goods inventory (FGI), average WIP and the end item queuing time is discussed in section 4. In section 5 the results of the production systems and of the end item stock are combined. In the last section we conclude and address some further research work.

2. The basic model

The objective of the model is to formulate analytically the relationship between the logistical figures lead-time, utilization, inventory, backlog and deviation of WIP for a production system. The basic model is based on a differential equation describing the relationship between the input flow, output flow and stored items. It is shown that for every subsystem for instance buffer, machine, finished goods inventory or the whole production system as well as for the backlog list the same differential equation governs the world. By using this differential equation, a generalization of Little's Law for a continuous setting and the statistical behavior of the WIP analytical relationships between the logistical figures are developed.

We are analyzing a multi-item production system S in a finite time horizon $[0, T]$ with a dynamic demand ${}_0w_{n+1}$. The production system S consists of one processing unit M_1 , one buffer B_1 just before the processing unit M_1 and a finished goods inventory B_2 . The input into buffer storage B_1 is defined by u_1 . The figure u_1 can be interpreted as the raw material input flow into the buffer storage B_1 . Procurement is not addressed in this article. The input into the processing unit M_1 is equal to the output of the buffer storage B_1 and is denoted by w_1 . The output of the processing unit is called u_2 and is equal to the input into the finished goods inventory B_2 . The output from the finished goods inventory is called w_2 . The unit of the vectors u_i and w_i ($j=1$ to 2) is number of items per time.

There is no restriction on the availability of the raw material, which is the input u_1 into the buffer stock B_1 . The processing unit M_1 has a product dependent processing time p_j needed for one item and a known available capacity. BL denotes the backlog list, which is the list of all open customer orders which past due dates. The left index 0 denotes planned values, for instance ${}_0w_2(t)$ are the planned customer orders with respect to the time t and $w_2(t)$ are the actual customer orders. The inventory in B_1 is denoted by $v_1(t)$, the finished goods inventory (FGI) is $v_2(t)$ and the Work In Process (WIP) is equal to the sum $v_1(t)$ plus the number of items within the processing unit M_1 . The unit of the inventory functions is number of items.

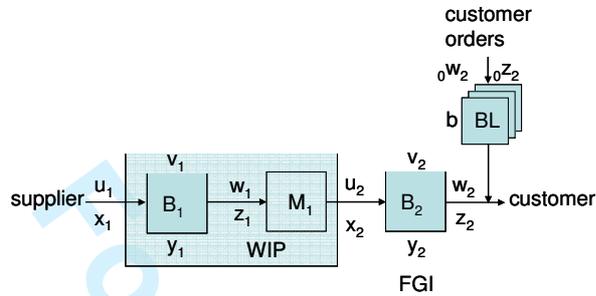


Fig. 1. Illustration of the system

Two views of the system are discussed. First, the number of items is observed. The j -th component of the vector functions $u_i(t)$ denotes the number of items entering into buffer B_i per time unit at time t of the j -th product type, $w_i(t)$ the leaving number of items per time-unit at time t and $v_i(t)$ the number of stored items. All three function values $u_i(t)$, $v_i(t)$ as well as $w_i(t)$ are vectors with the dimension number of different product types m .

The second view and more important one, is on the workload. The function $x_i(t)$ denotes the workload input into the system B_i . The value $x_i(t)$ is measured in standard processing time of the transformation processes M_i per time unit, thus the value $x_i(t)$ is dimensionless and has only one component. The workload input into the buffer is given by the scalar product of the two vectors input flow $u_i(t)$ and the standard processing time π needed for one item.

$$x_i(t) = \pi^T u_i(t), i = 1, 2 \tag{1}$$

whereby

$$\pi = \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix}$$

The expression $()^T$ denotes the transposed vector.

The workload output $z_i(t)$ from the buffer B_i is measured in standard processing time of the processing unit M_i per time unit.

$$z_i(t) = \pi^T w_i(t), i = 1, 2 \tag{2}$$

The value $z_2(t)$ is the sum of delivered products at time t to the customers measured in standard processing time per time.

It is assumed that only one product type is entering into a sub system at a time.

$$u_{ij}(t) \geq 0 \quad (3)$$

$$\exists j : u_{ij}(t) \geq 0 \wedge \forall k \neq j : u_{ik}(t) = 0$$

whereby

$u_{ij} \cdots j$ -th component of the vector u_i

and

$$w_{ij}(t) \geq 0$$

$$\exists j : w_{ij}(t) \geq 0 \wedge \forall k \neq j : w_{ik}(t) = 0$$

whereby

$w_{ij} \cdots j$ -th component of the vector w_i

As it is not possible to use more capacity than is available the following inequality holds true:

$$z_1(t) \leq 1 \quad \text{for } t \in [0, T] \quad (4)$$

$$x_2(t) \leq 1 \quad \text{for } t \in [0, T]$$

The functions $x_i(t)$ and $z_i(t)$ are dimensionless. We assume that the input and the output are piecewise continuous nonnegative functions. The jumping-points are called corners.

An additional interpretation of the function $z_i(t)$ is the utilization of the machine M_i . Workload input flow into the processing unit equals one means full utilization of the system.

The stored workload within the system B_i is modeled by the inventory function $y_i(t)$.

The inventory functions are assumed to be piecewise smooth (continuous differentiable between corners) and nonnegative.

$$y_i(t) \geq 0 \quad \text{for } t \in [0, T] \wedge i = 1, 2 \quad (5)$$

The unit of the inventory function y_i is the time unit.

6

H. Jodlbauer

The inventory at time t is the inventory at the beginning plus the cumulated input minus the cumulated output within the time period $[0, t]$

$$v_i(t) = v_i(0) + \int_0^t u_i(\tau) d\tau - \int_0^t w_i(\tau) d\tau \in \square_m \quad (6)$$

for $t \in [0, T] \wedge i = 1, 2$

The standard processing time π^T multiplied by Equation (6) yields in

$$y_i(t) = y_i(0) + \int_0^t x_i(\tau) d\tau - \int_0^t z_i(\tau) d\tau \quad (7)$$

for $t \in [0, T] \wedge i = 1, 2$

By differentiation of (7) the formula

$$y'_i(t) = x_i(t) - z_i(t) \quad (8)$$

for $t \in [0, T]$, t not a corner $\wedge i = 1, 2$

is obtained. The right side of (8) is often called the net-put process, see for instance Sigman (1993). Equation (9) says that the change in inventory is equal to the difference between input and output.

In the next paragraph the lead-time or queuing time in the stock $l_i(t)$ is introduced. The lead-time defines the time a item spends within the buffer B_i . Consequently the lead-time is only defined for a time during which there is an input into the buffer.

By the implicit equation

$$y_i(t) = \int_t^{t+\Delta_i(t)} z_i(\tau) d\tau \quad (9)$$

$t \in [0, T], i = 1, 2$

the value $\Delta_i(t)$ is determined. The value $\Delta_i(t)$ is not uniquely defined by (9). The minimum possible positive value of $\Delta_i(t)$, which fulfills (9) has the following interpretation: $\Delta_i(t)$ can be interpreted as the range that is the time it takes to empty the inventory by applying the output without any in-

put. Furthermore, $\Delta_i(t)$ is the FIFO lead-time, because the entering item leaves the production system just before emptying the inventory if the first in first out rule is applied.

Summarizing, we define the lead-time $l_i(t)$ as follows:

$$l_i(t) := \min \left\{ \Delta_i(t) \geq 0 \left| y_i(t) = \int_t^{t+\Delta_i(t)} z_i(\tau) d\tau \right. \right\} \quad (10)$$

whereby $t \in \{ \tau \in [0, T] \mid x_i(\tau) > 0 \}, i = 1, 2$

Thus, the lead time $l_i(t)$ is the time period it takes for a item which enters stock at time t to leave stock. The function $l_i(t)$ is piecewise continuous. The lead time of the processing unit M_I has to be equal to the processing time. That yields in the following formula for the production lead-time of a item entering the subsystem (B_I, M_I) at time t .

$$l_{(B_I, M_I)}(t) = l_1(t) + \pi_{j(u_1(t))} \quad (11)$$

$t \in [0, T]$

whereby

$$\pi_{j(u_1(t))} = \pi^T \begin{pmatrix} \text{sgn}(u_{i1}(t)) \\ \vdots \\ \text{sgn}(u_{im}(t)) \end{pmatrix}$$

The index function $j(\cdot)$ references the standard processing time of the loaded product type at time t .

Assuming constant processing times and no material loss the output from the machine M_I at time $t + \pi_{j(w_1(t))}$ has to be equal to the input with respect to the time t .

$$u_2(t + \pi_{j(w_1(t))}) = w_1(t) \quad (12)$$

$$x_2(t + \pi_{j(w_1(t))}) = z_1(t)$$

In the next section, backlog (stock outs of the finished goods inventory) is modeled. For that reason we have to distinguish between the planned values, which are noted with the left index 0, and the actual values.

8

H. Jodlbauer

Backlog b is the cumulated difference between planned demand ${}_0z_2$ and actual output z_2 .

$$b(t) = b(0) + \int_0^t {}_0z_2(\tau) d\tau - \int_0^t z_2(\tau) d\tau \quad (13)$$

$$b'(t) = {}_0z_2(t) - z_2(t) \quad (14)$$

$$b(t) \geq 0 \quad (15)$$

Non-negativity for the backlog means that there is no actual delivery without an order. Analogous to the lead time the delivery delay l_b is defined by

$$l_b(t) := \min \left\{ \Delta_b(t) \geq 0 \mid b(t) = \int_t^{t+\Delta_b(t)} z_2(\tau) d\tau \right\} \quad (16)$$

The delivery delay $l_b(t)$ illustrates how long a customer order with delivery date t is late.

The structure for the equations for the inventory (6), (7) and (8), for the backlog (13) to (15) as well as for their corresponding lead-times, see (10) and (16) is the same. In Jodlbauer/Stöcher (2006) the following basic theorem is formulated and proven:

Theorem 1

For a system

$$y'(t) = x(t) - z(t) \quad (17)$$

and the function $l(t)$ defined by the implicit equation

$$y(t) = \int_t^{t+l(t)} z(\tau) d\tau \quad (18)$$

the following identity for the average values holds true

H. Jodlbauer

9

$$X - Z = \frac{y(T) - y(0)}{T} \quad (19)$$

$$L = \frac{Y}{X} + \frac{B - A}{TX} = \frac{Y}{X} \left(1 + \frac{B - A}{T} \right)$$

whereby

$$X = \frac{1}{T} \int_0^T x(t) dt, \quad Z = \frac{1}{T} \int_0^T z(t) dt, \quad Y = \frac{1}{T} \int_0^T y(t) dt$$

$$L = \frac{\int_0^T x(t) l(t) dt}{\int_0^T x(t) dt}, \quad A = \int_0^{l(0)} \left(y(0) - \int_0^t z(\tau) d\tau \right) dt$$

$$B = \int_T^{T+l(T)} \left(y(0) + \int_0^T x(\tau) d\tau - \int_0^t z(\tau) d\tau \right) dt$$

For a system with periodic boundary condition $B-A$ as well as $y(t)-y(0)$ equals to zero, so (19) simplifies to $L = \frac{Y}{X}$ and $X = Z$. For bounded inventory and lead time and a long time period $[0, T]$ compared to the maximum inventory or lead time this simplification is approximately true – that lead us to the next corollary, for details see Jodlbauer/Stöcher (2006). In addition in Jodlbauer/Stöcher (2006) it is shown that the equation (19) holds true for every applied priority rule. Because of this the propositions of this article are valid for all dispatching rules not only for the FIFO used for the definition of the lead time $l(t)$.

Corollary 1

For a system with equation (17), definition (18) and bounded functions $y(t)$ and $l(t)$

$$\max_{t \in [0, T]} y(t) = y_{\max} \ll T \quad (20)$$

$$\max_{t \in [0, T]} l(t) = l_{\max} \ll T \quad (21)$$

10

H. Jodlbauer

and non negative functions $x(t)$, $y(t)$, $z(t)$ and $l(t)$ the following identities hold approximately true:

$$X = Z \quad (22)$$

$$L = \frac{Y}{X} = \frac{Y}{Z}$$

Applying equation (18) and assuming a bounded function z the function y is bounded if the function l is bounded. Theorem 1 and Corollary 1 can be interpreted as Little's Law, see Little (1961), for a time continuous system. Based on Corollary 1 general relationships between the inventory, backlog, utilization, lead time and delivery delay are yielded. Before summarizing all these equations a bounded system S and the average values are defined.

Defintion 1

A production system S consisting of a processing unit, a buffer just before the processing unit and a finished goods inventory which fulfills (1)-(8), (10), (13)-(16) is called bounded if and only if all lead times are bounded and their bounds are very small in comparison to the time T .

$$l_i(t) \leq l_{i,\max} \ll T, \text{ for all } t \in [0, T] \wedge i = 1, 2 \quad (23)$$

$$l_b(t) \leq l_{b,\max} \ll T, \text{ for all } t \in [0, T]$$

Definition 2

Time averages

$$\begin{aligned}
 X_i &:= \frac{1}{T} \int_0^T x_i(t) dt && \text{Average workload input into the buffer } B_i \\
 Z_i &:= \frac{1}{T} \int_0^T z_i(t) dt && \text{Average workload output from the buffer } B_i \\
 Y_i &:= \frac{1}{T} \int_0^T y_i(t) dt && \text{Average inventory of buffer } B_i \\
 Y_{M_1} &&& \text{Average stored workload in } M_1 \\
 Y_{(B_1, M_1)} &= Y_1 + Y_{M_1} && \text{WIP of the sub system } (B_1, M_1) \\
 Y &= Y_{(B_1, M_1)} + Y_2 && \text{total average inventory (WIP + FGI)} \\
 {}_0Z_1 &:= \frac{1}{T} \int_0^T z_1(t) dt && \text{Average planned workload input into } M_1 \\
 {}_0Z_2 &:= \frac{1}{T} \int_0^T z_2(t) dt && \text{Average planned demand measured in time} \\
 B &:= \frac{1}{T} \int_0^T b(t) dt && \text{Average backlog measured in time}
 \end{aligned}
 \tag{24}$$

Input weighted Averages

12

H. Jodlbauer

$$L_i := \frac{\int_0^T l_i(t) x_i(t) dt}{\int_0^T x_i(t) dt} \quad \text{Average queuing time in buffer } B_i \quad (25)$$

$$P := \frac{\int_0^T \pi_{j(w(u_1(t)))}(t) z_1(t) dt}{\int_0^T z_1(t) dt} \quad \text{Average processing time in machine } M_1$$

$$L_{(B_1, M_1)} = L_i + P \quad \text{Average production lead time of the sub system } (B_1, M_1)$$

$$L = L_{(B_1, M_1)} + L_2 \quad \text{total average lead time}$$

$$L_b := \frac{\frac{1}{T} \int_0^T l_b(t) z_2(t) dt}{\frac{1}{T} \int_0^T z_2(t) dt} \quad \text{Average delivery delay}$$

Some parameters in the above definition have special names.

Z_1	Average utilization of machine M_1
X_2	Average throughput of machine M_1
Y_2	Average Finished Goods Inventory (FGI)
$Y_{(B_1, M_1)}$	Average Work In Process (WIP) of the sub system (B_1, M_1)
L_1	Average queuing time before processing
L_2	Average queuing time of the end items

Theorem 2

For a bounded production system S consisting of a processing unit, a buffer just before the processing unit and a finished goods inventory which fulfills (1)-(8), (10), (13)-(16) the following identities hold approximately true:

Inventory and backlog are bounded

H. Jodlbauer

13

$$y_i(t) \leq k_i l_{i,\max}, i = 1, 2 \quad (26)$$

$$b(t) \leq k_2 l_{b,\max}$$

whereby

$$k_1 = 1, k_2 = \max_t(z_2(t))$$

Average input is equal to average output for every sub system

$${}_0Z_2 = Z_2, {}_0W_2 = W_2 \quad (27)$$

$$X_i = Z_i, U_i = W_i \text{ for } i = 1, 2$$

$$X_2 = Z_1, U_2 = W_1$$

Average lead time is equal to fraction average inventory over average input

$$L_i = \frac{Y_i}{X_i}, \text{ for } i = 1, 2 \quad (28)$$

$$P = \frac{Y_{M_1}}{Z_1}$$

$$L_{(B_1, M_1)} = \frac{Y_{(B_1, M_1)}}{X_1}$$

$$L = \frac{Y}{X_1}$$

Average delivery delay is equal to fraction average backlog over average demand

$$L_b = \frac{B}{{}_0Z_2} \quad (29)$$

Proof.

$$y_i(t) \stackrel{(9),(10)}{\leq} \int_t^{t+l_i(t)} z_i(\tau) d\tau \stackrel{\substack{\exists k_i < \infty \\ i=1: k_1=1, \text{see (4)} \\ i=2: k_2 = \max_t(z_2(t))}}{\leq} k_i l_i(t) \stackrel{S \text{ bounded}}{\leq} k_i l_{i,\max} \quad (30)$$

14 *H. Jodlbauer*

$$b(t) \stackrel{(16)}{\equiv} \int_t^{t+l_b(t)} z_2(\tau) d\tau \stackrel{k_2 = \max(z_2(t))}{\leq} k_2 l_b(t) \stackrel{S \text{ bounded}}{\leq} k_2 l_{b,\max} \tag{31}$$

Formula (26) is proven.

$${}_0Z_2 - Z_2 \stackrel{(24)}{\equiv} \frac{1}{T} \int_0^T z_2(t) - z_2(t) dt \stackrel{(14)}{\equiv} \frac{1}{T} (b(T) - b(0)) \stackrel{(23),(26)}{\approx} 0 \tag{32}$$

For all the other sub systems as well for the vector valued functions the proof is analogously yield. For the sub system M_I the following additional consideration is necessary. For the sub system M_I the analogous equations to the differential equation (8) and the implicit integral equation (9) for the processing time make sense. The input for M_I is defined by z_I and the output by x_2 . It is important to take into account that the processing time function has to be considered as piecewise constant and that the first time derivative of the processing time function is zero between the corners. That is why results of Theorem 1 and Corollary 1 are applicable to subsystem M_I .

The formulas $L_i = \frac{Y_i}{X_i}$, $P = \frac{Y_{M_1}}{Z_1}$ follow directly from Corollary 1.

$$L_{(B_1, M_1)} \stackrel{(25)}{\equiv} L_1 + P \stackrel{L_i = \frac{Y_i}{X_i}, P = \frac{Y_{M_1}}{Z_1}}{\equiv} \frac{Y_1}{X_1} + \frac{Y_{M_1}}{Z_1} \stackrel{(24),(27)}{\equiv} \frac{Y_{(B_1, M_1)}}{X_1} \tag{33}$$

$$L \stackrel{(25)}{\equiv} L_{(B_1, M_1)} + L_2 \stackrel{L_{(B_1, M_1)} = \frac{Y_{(B_1, M_1)}}{X_1}, L_i = \frac{Y_i}{X_i}}{\equiv} \frac{Y_{(B_1, M_1)}}{X_1} + \frac{Y_2}{X_2} \stackrel{(24),(27)}{\equiv} \frac{Y}{X_1}$$

Finally formulas (28) and (29) follow directly from Corollary 1. □

Formula (27) is a consequence of the postulation S is a bounded system. Wight (1970) stated nearly forty years ago: The input into a shop has to be equal to the output. The equations in (28) and (29) are a generalization of Little's Law, see Little (1961).

3. Discussion of the sub system (B_1, M_1) and the WIP

In this paragraph the sub systems (B_1, M_1) are investigated in more detail. The objective is to determine the production lead time as well the utilization with respect to the WIP. More precisely the next theorem states the relationship between the average production lead time and the WIP without using the average input or average output. Instead of the average input or output the statistical distribution of the inventory is taken into account. Other authors discussed the importance of the deviation of the actual lead time, see for instance Hopp/Spearman/Woodruff (1990), Spearman/Zhang (1999) or Hopp/Roof (2000), for reducing the inventories or defining an optimal planned lead time. In this article the variation of WIP is used to model the fluctuations of a real system.

Theorem 3

For a production system with inventory equation (7) or (8), lead time definition (9) and (10) and bounded lead time according to (23) the following identities hold approximately true:

Average production lead time with respect to average WIP and WIP deviation

$$L_{(B_1, M_1)} = \frac{Y_{(B_1, M_1)}}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{Y_{(B_1, M_1)}}^2)}(y) dy} \quad (34)$$

Average utilization with respect to average WIP and WIP deviation

$$Z_1 = 1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{Y_{(B_1, M_1)}}^2)}(y) dy \quad (35)$$

Whereby

$$F_{(Y_{(B_1, M_1)}, \sigma_{Y_{(B_1, M_1)}}^2)}(y) \quad \text{statistical distribution function} \quad (36)$$

of the WIP

$$\sigma_{Y_{(B_1, M_1)}}^2 \quad \text{variance of the WIP}$$

Proof

16 *H. Jodlbauer*

First of all we prove that the expectation value $E(z_t)$ of the input z_t equals to $1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{Y_{(B_1, M_1)}^2})}(y) dy$. The expectation values are approximated by the average values.

$$\begin{aligned}
 E(z_1) &= Z_1 = \frac{1}{T} \int_0^T z_1(t) dt = & (37) \\
 &= 1 - \frac{1}{T} \int_0^T 1 - z_1(t) dt \\
 &= 1 - \frac{1}{T} \int_{\{t \in [0, T] / z_1(t) < 1 \wedge y_1(t) = 0\}} 1 - z_1(t) dt \\
 &= 1 - \frac{1}{T} \int_{\{t \in [0, T] / y_{(B_1, M_1)}(t) < P \wedge y_1(t) = 0\}} 1 - z_1(t) dt \\
 &= 1 - \int_0^P \left(1 - \frac{y}{P}\right) f_{y_{(B_1, M_1)}}(y) dy \\
 &= 1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{Y_{(B_1, M_1)}^2})}(y) dy
 \end{aligned}$$

$\begin{aligned} & \stackrel{\equiv}{=} z_1(t)=1 \text{ for } t \notin \{t \in [0, T] / z_1(t) < 1 \wedge y_1(t) = 0\} \\ & \text{(not an empty inventory causes full utilization)} \\ & \stackrel{\equiv}{=} \{t \in [0, T] / z_1(t) < 1 \wedge y_1(t) = 0\} = \\ & = \{t \in [0, T] / y_{M_1}(t) < P \wedge y_1(t) = 0\} = \\ & = \{t \in [0, T] / y_{(B_1, M_1)}(t) < P \wedge y_1(t) = 0\} \\ & \stackrel{\equiv}{=} E(g(h)) = \int_{-\infty}^{\infty} f_h(h) g(h) dh \\ & (28) \\ & f_{(B_1, M_1)} \text{ density function} \\ & \stackrel{\equiv}{=} \text{integration by parts} \end{aligned}$

The proof continues by:

$$L_{(B_1, M_1)} \stackrel{\equiv}{=} \frac{Y_{(B_1, M_1)}}{(28), (37) \left(1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{Y_{(B_1, M_1)}^2)}(y) dy\right)} \quad (38)$$

Formula (35) says that a smaller deviation of the WIP, higher average WIP or a shorter average processing time cause a higher utilization. For more interesting characteristics of the average processing time P see Jodlbauer

(2004). The lead time of the sub system (B_I, M_I) is decreased by reducing the deviation of WIP, the average WIP or the average processing time. The next figures illustrate the result of theorem 3.

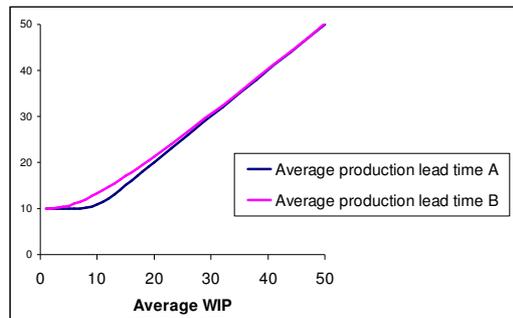


Fig. 2. Illustration of relationship production lead time and WIP for the sub system (B_I, M_I)

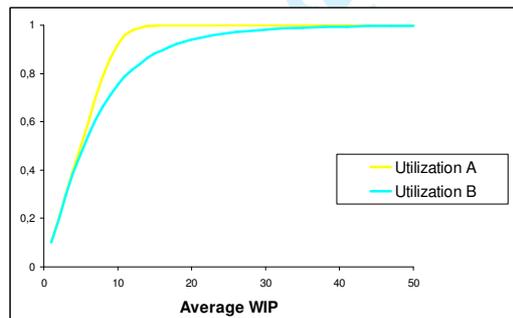


Fig. 3. Illustration of relationship utilization and WIP for the sub system (B_I, M_I)

The unit of the average WIP as well as of the average production lead time is the time unit. Utilization is a dimensionless number between zero and one. Case A is generated by an average standard processing time 10 and a coefficient of variation for the WIP of 20%. This means that the deviation of the WIP is equal to 20% of the average WIP. For case B all parameters are the same with the exception of the coefficient of variation of 70%. In both cases the lognormal distribution for the WIP is assumed. Both figures clearly demonstrate that for a higher WIP deviation, the utilization is lower and the lead time longer. Consequently a low deviation of the WIP should be aimed at. In addition the figures demonstrate that the utilization as well as the average production lead time is an increasing function with respect to the average WIP.

Figure 4 addresses the influence of the average processing time on the utilization and average lead time.

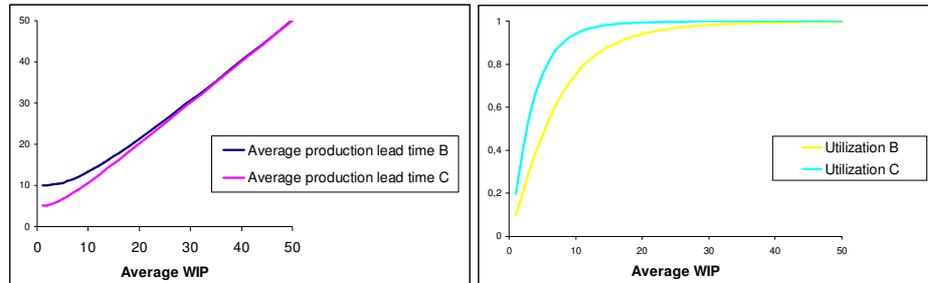


Fig. 4. Illustration of the influence of the average processing time on utilization and average lead time for the sub system (B_1, M_1)

The only difference between Case B and C is that in Case C an average processing time of 5 instead of 10 is chosen. Figure 4 illustrates that a reduced average processing time decreases the average lead time and increases utilization. The influence of the average processing time on utilization and average lead time is higher for lower average WIP. The two measurements reducing the deviation of the WIP and decreasing the average processing time improve the logistical system in the sense that with lower average WIP a higher utilization and a shorter average production lead time is yielded.

The gained relationships between WIP, lead time and utilization by the proposed time continuous approach are confirmed by the works of many other authors based on queuing theory, experimental studies or simulation, see for instance Hopp/Spearman (1996), Lödning/Yu/Wiendahl (2003) or Jodlbauer/Huber (2006). In Jodlbauer (2005) the relationship between WIP, utilization and lead time was discussed without taking the deviation of the WIP into account.

4. Discussion of the sub system B_2 and the finished goods inventory (FGI)

In this section the finished goods stock is investigated. The goal is to describe the queuing time in the end item stock with respect to the WIP and the finished goods inventory. More precisely the next theorem states the relationship between the average queuing time in the end item stock and the average finished good inventory and the statistical behavior of the WIP.

Theorem 4

For a production system with inventory equation (7) or (8), lead time definition (9) and (10) and bounded lead time according to (23) the following identities hold approximately true:

Average end item queuing time with respect to average finished inventory, average WIP and WIP deviation

$$L_2 = \frac{Y_2}{1 - \frac{1}{P} \int_0^P F_{(\gamma_{(B_1, M_1)}, \sigma_{\gamma_{(B_1, M_1)}^2})}(y) dy} \tag{39}$$

Proof

$$L_2 \stackrel{(28)}{=} \frac{Y_2}{X_1} \stackrel{(37)}{=} \frac{Y_2}{1 - \frac{1}{P} \int_0^P F_{(\gamma_{(B_1, M_1)}, \sigma_{\gamma_{(B_1, M_1)}^2})}(y) dy} \tag{40}$$

□

The next two figures illustrate the result of theorem 4.

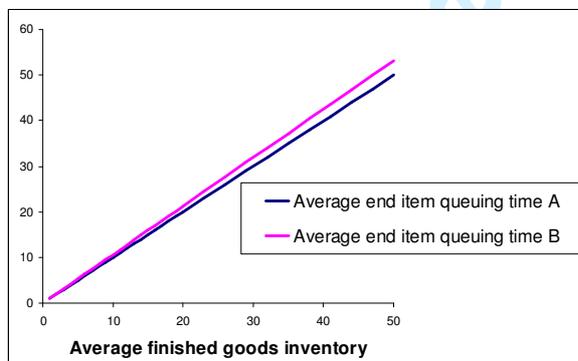


Fig. 5. Illustration of the relationship queuing time in the end item stock and average finished goods inventory (FGI) for constant average WIP

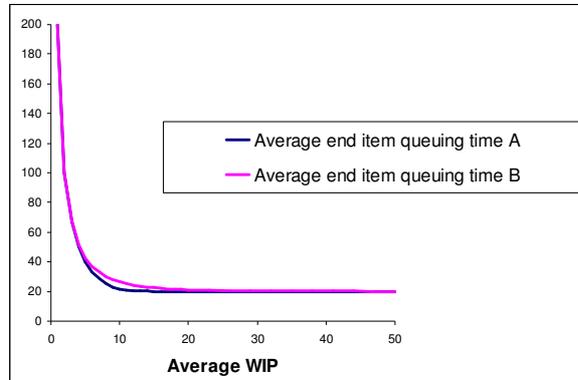


Fig. 6. Illustration of the relationship queuing time in the end item stock and average WIP for constant average FGI

The unit of the average WIP, average finished goods inventory as well as of the average end item queuing time is the time unit. Case A and case B are defined in the same manner as for figure 3 or figure 4. For figure Fig. 5 a constant WIP equal to 20 and for figure Fig. 6 a constant FGI equal to 20 is assumed.

Figure 5 demonstrates that the average end item queuing time is a linear function with respect to the average finished goods inventory. The slope depends on the utilization – a higher utilization (that means greater average WIP, smaller WIP deviation, or less average processing time) causes a smaller slope. Once again a small WIP deviation is advantageous.

Figure Fig. 6 shows that the average queuing time of end items is a decreasing function with respect to the average WIP. The asymptotic function value for very high average WIP is the assumed constant FGI.

H. Jodlbauer

21

$$\begin{aligned}
 \lim_{Y_{(B_1, M_1)} \rightarrow \infty} L_2 &= \lim_{Y_{(B_1, M_1)} \rightarrow 0} \frac{Y_2}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{Y_{(B_1, M_1)}^2})}(y) dy} = \\
 &= \frac{Y_2}{1 - \frac{0}{P}} = Y_2 \\
 \lim_{Y_{(B_1, M_1)} \rightarrow 0} L_2 &= \lim_{Y_{(B_1, M_1)} \rightarrow 0} \frac{Y_2}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{Y_{(B_1, M_1)}^2})}(y) dy} = \\
 &= \lim_{Y_{(B_1, M_1)} \rightarrow 0} \frac{Y_2}{1 - \frac{P - Y_{(B_1, M_1)}}{P}} = \infty
 \end{aligned}
 \tag{41}$$

The average queuing time of end item goes to infinity for very small average WIP.

In addition formula (39) says that a higher WIP causes a shorter queuing time of end items and that the end item queuing time does not depend on the deviation of the finished goods inventory.

5. Discussion of the whole system S

Combining theorem 3 and theorem 4 the average lead time of the system S can be expressed by the average WIP, WIP deviation and average finished goods inventory.

Theorem 5

For a production system with inventory equation (7) or (8), lead time definition (9) and (10) and bounded lead time according to (23) the following identities hold approximately true:

Average Lead time with respect to average WIP, WIP deviation and finished goods inventory

22 *H. Jodlbauer*

$$L = \frac{Y}{1 - \frac{1}{P} \int_0^P F_{(y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} \tag{42}$$

Proof

$$L = \frac{Y}{X_1} = \frac{Y}{1 - \frac{1}{P} \int_0^P F_{(y_{M_i}, \sigma_{M_i}^2)}(y) dy} \tag{43}$$

Formula (42) says that a smaller WIP deviation, shorter average processing time or lower FGI, cause a shorter lead time. The influence of the average WIP on the lead time is more complicated, because there are two divergent influences: A higher WIP increases the queuing time before the machine but decreases the queuing time of end items.

The next three figures illustrate the result of theorem 4.

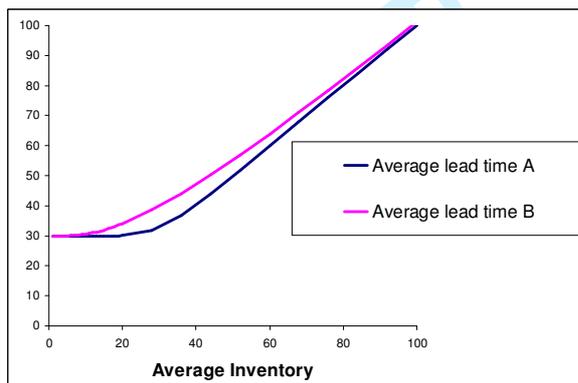


Fig. 7. Illustration of the relationship average lead time and average inventory (=WIP+FGI) for a constant ratio average WIP over average FGI

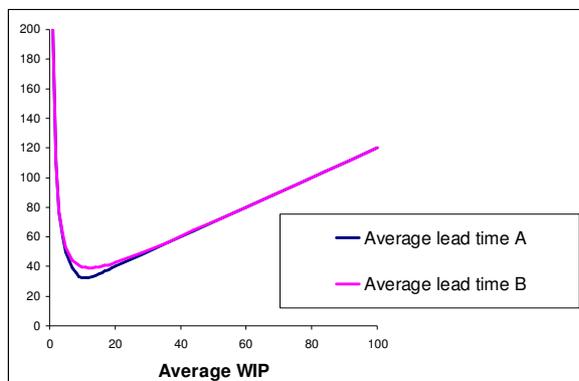


Fig. 8. Illustration of the relationship average lead time and average WIP for a constant average FGI

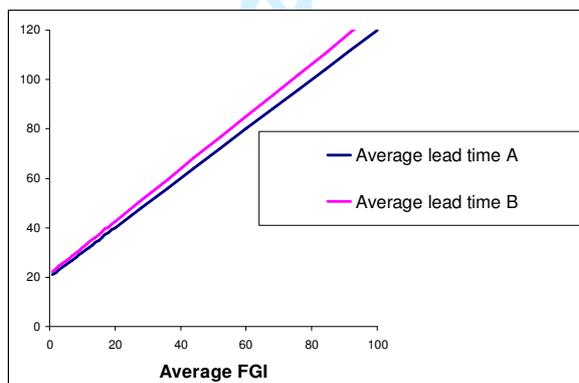


Fig. 9. Illustration of the relationship average lead time and average FGI for a constant average WIP

The unit of the average inventory, average WIP, average finished goods inventory as well as of the average lead-time is the time unit. Case A and B are defined as before. For figure Fig. 7 in addition a constant ratio WIP over finished good inventory of 0.5 is assumed. For figure Fig. 8 a constant WIP equal to 20 and for figure Fig. 9 a constant FGI equal to 20 is assumed.

The average lead time of the whole system S is a increasing function with respect to the average inventory which is equal to the sum of average WIP plus average FGI assuming a constant ration average WIP over average FGI. A higher WIP deviation causes a longer average lead time. For very low total inventory the total lead time is approximately equal to the average processing time multiplied by the term one minus the ratio FGI over WIP (see formula (44)).

$$\begin{aligned}
 \lim_{Y \rightarrow 0} L &= \lim_{Y \rightarrow 0} \frac{Y}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} = \lim_{Y \rightarrow 0} \frac{Y}{1 - \frac{P - Y_{(B_1, M_1)}}{P}} = \\
 &= P \left(1 + \frac{Y_2}{Y_{(B_1, M_1)}} \right) \\
 \lim_{Y \rightarrow \infty} L &= \lim_{Y \rightarrow \infty} \frac{Y}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} = \lim_{Y \rightarrow \infty} Y
 \end{aligned}
 \tag{44}$$

For high average inventory the average lead time is approximately the average inventory.

$$\begin{aligned}
 \lim_{Y_{(B_1, M_1)} \rightarrow 0} L &= \lim_{Y_{(B_1, M_1)} \rightarrow 0} \frac{Y}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} = \\
 &= \lim_{Y_{(B_1, M_1)} \rightarrow 0} \frac{Y_{(B_1, M_1)} + Y_2}{1 - \frac{P - Y_{(B_1, M_1)}}{P}} = \infty \\
 \lim_{Y_{(B_1, M_1)} \rightarrow \infty} L &= \lim_{Y_{(B_1, M_1)} \rightarrow \infty} \frac{Y}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} = \\
 &= \lim_{Y_{(B_1, M_1)} \rightarrow \infty} Y_{(B_1, M_1)} + Y_2
 \end{aligned}
 \tag{45}$$

For a constant average FGI the average lead time grows to infinity for very small average WIP and is approximately a line for very large average WIP.

H. Jodlbauer

25

$$\begin{aligned}
 \lim_{Y_2 \rightarrow 0} L &= \lim_{Y_2 \rightarrow 0} \frac{Y}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} = \\
 &= \frac{Y_{(B_1, M_1)}}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} = L_{(B_1, M_1)} \\
 \lim_{Y_2 \rightarrow \infty} L &= \lim_{Y_2 \rightarrow \infty} \frac{Y}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} = \\
 &= \lim_{Y_2 \rightarrow \infty} \frac{Y_2 + Y_{(B_1, M_1)}}{1 - \frac{1}{P} \int_0^P F_{(Y_{(B_1, M_1)}, \sigma_{(B_1, M_1)}^2)}(y) dy} = \infty
 \end{aligned}
 \tag{46}$$

For a constant average WIP the average lead time goes to the production lead time for vanishing average FGI and goes to infinity for very large average FGI.

Before discussing the service level an additional relationship between the lead time and inventory is addressed. The next theorem shows that the lead time of the whole system is the lead time of the sub system (B_1, M_1) multiplied by the term one plus the ratio FGI/WIP or that the lead time of the whole system is equal to the queuing time of end items multiplied by the term one plus the ratio WIP/FGI . Analogous equations hold true for the inventory.

Theorem 6

$$L = L_{(B_1, M_1)} \left(1 + \frac{Y_2}{Y_{(B_1, M_1)}} \right) = L_2 \left(1 + \frac{Y_{(B_1, M_1)}}{Y_2} \right) \tag{47}$$

$$Y = Y_{(B_1, M_1)} \left(1 + \frac{L_2}{L_{(B_1, M_1)}} \right) = Y_2 \left(1 + \frac{L_{(B_1, M_1)}}{L_2} \right) \tag{48}$$

Proof

$$L - L_{(B_1, M_1)} \stackrel{\equiv}{=} L_2 \stackrel{\equiv}{=} \frac{Y_2 L_{(B_1, M_1)}}{Y_{(B_1, M_1)}} \quad (49)$$

$$\Rightarrow L = L_{(B_1, M_1)} \left(1 + \frac{Y_2}{Y_{(B_1, M_1)}} \right)$$

The other relationships are proven in the same way. \square

The next theorem discusses the service level. There are different approaches to defining service level. Spearman/Woodruff/Hopp (1990) define service level for instance as the fraction of jobs whose actual flow time is not greater than their planned lead time. This definition is preferably applied for push planning and control systems like MRP. A well-accepted general definition of the service level is the ratio number of fulfilled customer orders over all customer orders. Because for a general system we do not know the relationship between the statistical distribution of the inventory and the statistical distribution of the lead time, in this article the following definition for the service level is introduced.

Definition 3

The service level measured by inventory and backlog is defined by one minus the ratio average backlog over the sum average backlog plus average finished goods inventory.

$$s = 1 - \frac{B}{B + Y_2} \quad (50)$$

Definition 3 says that no backlog means service level one and no finished goods inventory cause a service level of zero. In general the service level measured by the inventory and backlog is not identical to the traditional definition ratio of fulfilled customer orders over all orders.

The next two figures illustrate the service level with respect to the backlog or finished goods inventory.

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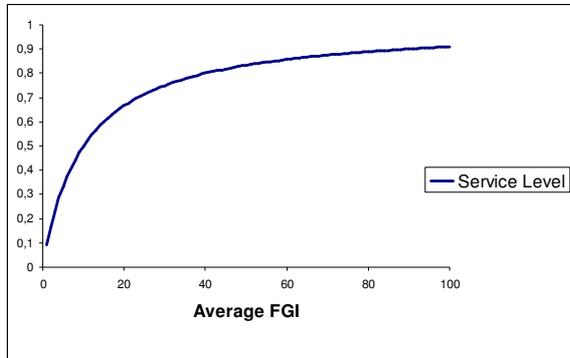


Fig. 10. Illustration of the relationship service level and average FGI for a constant average backlog

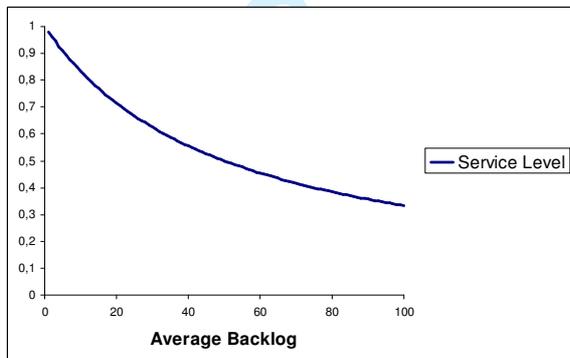


Fig. 11. Illustration of the relationship service level and average backlog for a constant average FGI

The unit of the average FGI and average backlog is the time unit. The service level is a dimensionless number between zero and one. The service level is an increasing function with respect to the FGI for a constant backlog. On the other hand the service level decreases if the backlog is reduced and the FGI is not changed.

8. Conclusion

In this article a time continuous model for a multi item production system describing relationships between lead time, WIP and WIP deviation, utilization, FGI, backlog and service level is developed and discussed. The fluctuations of a real system are modeled by the WIP deviation.

One interesting finding is that the structure of the inventory equation is the same as for the backlog equation. In addition it is shown that the waiting time of items in the final stock does not depend on the deviation of the WIP. Furthermore, an increase of the average WIP causes an increase of the average production lead time but a decrease of the average waiting time of end items. Based on the analytical formulas it is possible to calculate the logistical key figures for extreme cases. For instance it is shown that for very small total average inventory the total average lead time equals to average processing time multiplied by the term one plus fraction average FGI over average WIP.

The model does not consider scrap loss, rework, machine breakdowns, lot sizes, set ups nor multilevel systems. Further research should be carried out to extend the model in this direction.

One possible application of the findings of this article is the development of more efficient methods for estimating clearing functions, introduced by Graves (1986), see for more details Asmundsson/Rardin /Uzsoy (2006).

An additional interesting idea for further research is to combine this work with different planning and control strategies by applying the ideas of Takahashi/Hirotsani (2005). The goal of this is to get an analytical formula for both the lead time as well as for the service level with respect to the inventory for different strategies like MRP, KANBAN or CONWIP.

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