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# Optimal Production Opportunities in a Remanufacturing System

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# Optimal Production Opportunities in a Remanufacturing System

## ABSTRACT

In recent years, remanufacturing has emerged in various industrial sectors and it adds an additional production alternative apart from the conventional manufacturing process. With remanufacturing, a used return product is often disassembled, reprocessed and then reassembled into a new product. The remanufactured product is often of the same quality as those from regular manufacturing. Even though remanufacturing saves a substantial cost of materials, it often requires more labour and other inputs. Thus it becomes important to examine the expanded production opportunity of the system. In this paper we therefore make a first attempt to model such a system with respect to its optimal production. This model is based on the Cobb-Douglas (Wicksell) production function. More specifically, we focus on how labour costs, material costs and the budget influence optimal production decisions. Explicit properties of the optimal plan are derived and presented as theorems. Results of this study are intended to provide a guideline for managers to make sound decisions, when dealing with the increasingly important remanufacturing systems.

## 1 INTRODUCTION

In recent years, due to stricter environmental legislations, such as the EU directives on waste electrical and electronic equipment (WEEE) and end-of-life vehicles (The European Parliament and The Council of the European Union, 2000, 2003), an increasing number of companies have started to pay attention to remanufacturing processes and to investigating the associated activities of remanufacturing, such as production, logistics, purchasing and planning. The remanufacturing

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4 activity has been transmitted from the automotive industry to several other business sectors, for  
5 instance those involving various types of electrical apparatuses, toner cartridges, or household  
6 appliances. According to a survey study by Lund (1996), it was estimated that in 1998, there  
7 were around 73 000 companies in the remanufacturing industry in USA. Its turnover reached 53  
8 billion US\$, equivalent to that of the steel industry in USA. This industry, on the other hand, is  
9 more labour intensive, for instance its total number of employees is twice the number of those in  
10 the steel industry.  
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24 The literature on remanufacturing and its relevant areas, such as closed loop supply chains and  
25 reverse logistics, has been growing rapidly in recent years. Research includes many aspects, from  
26 product design, product recovery, to reuse. Gungor and Gupta (1999) provide a classification of  
27 research issues in this area from an environmentally conscious manufacturing and product  
28 recovery perspective. Recovery definitions and process descriptions can also be found in the  
29 early work by Thierry et al (1995), Fleishmann (1997), Guide (2000), and the recent book by  
30 Dekker et al (2004). The research has also been extended from operations management issues,  
31 such as inventory control policies and disassembly scheduling (van der Laan et al, 1999,  
32 Lambert, 2003), to microeconomic issues such as the market segmentation problem with  
33 manufactured and remanufactured products (Debo et al 2005). This line of research also provides  
34 us with a motive to study the production function from the microeconomic perspective of a  
35 remanufacturing system, an item which has yet not been dealt with in the literature according to  
36 our knowledge.  
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57 In various industrial sectors, remanufacturing today adds an additional production alternative,  
58 apart from the conventional manufacturing process. With remanufacturing, a used return product  
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4 is often disassembled, reprocessed and then reassembled into a new component/product,  
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6 essentially having the same quality as those coming directly from manufacturing. Even though  
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8 remanufacturing saves a substantial cost of materials, it often requires more labour and other  
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10 factor inputs. With this additional opportunity in mind, it becomes important to examine how an  
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12 optimal production plan can be achieved by taking into account the substitution effect of  
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14 materials and labour. In short, the system performance and production expansion depend on  
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16 properties of the production function and on factor prices, which need to be appropriately  
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18 described and presented.  
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26 Apart from this background, our study is also motivated by the authors' working experiences  
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28 with two remanufacturing companies in Sweden. Both companies remanufacture parts of  
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30 automobiles such as engines, water pumps, brake callipers and steering systems for the service  
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32 market. As much as possible, parts are recovered from returned cores, and they are then  
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34 reassembled with newly purchased components into finished products. Performance measures  
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36 such as productivity and profitability are used in the companies to evaluate, control and improve  
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38 the production process. Again, the companies have to make decisions such as if they should  
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40 spend more labour hours to recover returned cores and thereby use more returned materials, or  
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42 instead purchase new materials (components) so as to reduce labour input. In addition, due to the  
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44 End of Life Vehicles Directive, both of these two companies foresee a growing market and  
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46 production capability in remanufacturing. But still the remaining question is to identify the  
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48 growth path of production so that the right amount of materials and labour are used to improve  
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50 their performance. From this aspect, investigating the production function definitely helps to  
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52 make sound strategic decisions.  
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4 In this paper, we therefore make a first attempt to develop a model for investigating a production  
5 system with an existing remanufacturing alternative. Our model is based on the non-linear Cobb-  
6 Douglas production function, widely used in production-economic and econometric applications.  
7  
8 More specifically, we focus on how the costs of labour and materials influence the production  
9 decisions. The results of this study are hoped to provide insights for managers to make sound  
10 decisions when dealing with the increasingly important remanufacturing systems, as well as to  
11 provide grounds for creating further more detailed models adding realism to this theory.  
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24 Following this introduction we present a description of the system, including the flows of the  
25 production system and details of the production function. In Section 3 we then derive  
26 consequences from our model, and provide properties of the solved optimal values of the  
27 decision variables concerned. The model is then illustrated in Section 4 by numerical examples.  
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29 We conclude the paper by indicating some directions for future research.  
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## 38 2 SYSTEM DESCRIPTION

### 39 2.1 The Production System

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44 We study a production system with a remanufacturing opportunity, as illustrated in Figure 1. The  
45 material flow is based on the case of the two companies mentioned above. Remanufacturing  
46 starts with a disassembling and reconditioning process (hereafter referred to as the reconditioning  
47 process). In the two case study companies, engine cores will pass through disassembly, cleaning,  
48 reprocessing and testing processes in order for for reconditioned components to be obtained, such  
49 as cylinder heads, crankshafts, cylinder liner kits. These processes involve a high intensity of  
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4 labour input. The yield rate of the processes thus depend on the labour input (below written as  
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6  $L_r$ ). Hence, the output of reconditioned components will depend on  $L_r$  as well. After this  
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8 reconditioning process, reconditioned materials and components, such as cylinder heads,  
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10 crankshafts or cylinder liner kits, are then fed into the reassembly process in parallel with similar  
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12 components acquired from a regular component production. The result is a remanufactured  
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14 engine being obtained. We refer to this process as *manufacturing*, since it has the same  
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16 production property as ~~in~~ a normal manufacturing process.  
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24 A portion of the output from the manufacturing process returns to the system after it has been  
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26 used and consumed by customers, for instance engine cores that are returned by dealers. This  
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28 return will then be used as part of the input in the reconditioning process. The cost of returned  
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30 cores is usually minor (the logistics cost is often the major part), but the disassembly and  
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32 reconditioning process is often labour intensive. Therefore, in this study, besides materials,  
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34 labour is considered to be the second major input into the two production processes considered.  
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42 Nowadays, this type of **production** structure widely exists in **the** manufacturing industry. In the  
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44 automotive industry, typically returned cars are disassembled in some detail, and then the engine,  
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46 steering system, gear box, etc, will be reprocessed to become a competitive alternative to a  
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48 direct-manufactured subsystem.  
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## 53 2.2 Notation

### 54 Decision variables

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58  $L_m$ : Labour input into manufacturing process

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60  $L_r$ : Labour input into **reconditioning** process



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5  $M_m$ : (Fresh) material input into manufacturing process  
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7  $R$ : Return, input into **reconditioning**  
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10  $\lambda$ : Lagrange multiplier for budget constraint (Eq. 4 below)  
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12  $\mu$ : Lagrange multiplier for return constraint (Eq. 5 below)  
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18 Outputs of the decision variables  
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20  $M_r$ : **Reconditioned** material input into manufacturing process  
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22  $M$ : Total material input into manufacturing process,  $M = M_m + M_r$   
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25  $Q$ : Output from manufacturing  
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28 Parameters  
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30  $\gamma$ : Return ratio,  $\gamma < 1$ .  $\gamma Q$  is the maximum input to **the recovery of components**, when returns  
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32 are limited  
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35  $P_l$ : Price of input labour  
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38  $P_m$ : Price of input material  
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41  $P_r$ : Price of return product  
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44  $C$ : Budget limit  
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46  $\alpha, a, b$  and  $\beta, c, d$ : **Positive parameters of the two Cobb-Douglas functions introduced**  
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51 \*\*\* Insert Figure 1 \*\*\*  
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56 **2.3 Production Functions**  
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4 The production opportunities in the two processes are represented by the Cobb-Douglas function,  
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6 cf Eqs. 1 and 2. This nonlinear production function was first introduced by the Swedish  
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8 economist Knut Wicksell in 1900 and later became popular due to publications by Charles W.  
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10 Cobb and **Senator** Paul H. Douglas in the *American Economic Review* in 1927 and 1948  
11  
12 (Sandelin 1976 and Grubbström 1995).  
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18 In the Cobb-Douglas function, all parameters are positive.  $\alpha$  and  $\beta$  are *technical efficiency*  
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20 *parameters* indicating the state of technology in manufacturing and reconditioning, respectively.  
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22 The exponents determine the marginal rate of technical substitution of inputs. Usually a  
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24 production process has a diminishing return to scale property. Thus we let the sum of the  
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26 exponents be less than unity, i. e.  $a+b < 1$  and  $c+d < 1$ .  
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$$Q = \alpha L_m^a M^b \quad (1)$$

$$M_r = \beta L_r^c R^d \quad (2)$$

### 3 MODEL FORMULATION

#### 3.1 General

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55 The optimisation model is presented in this section. The objective is to maximise the total output  
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57  $Q$  (Eq. 3), subject to a budget constraint (Eq. 4). In addition, the input of returned items  $R$  is  
58  
59 limited by a maximum input flow  $\gamma R$  constraining the **flow of reconditioned components** (Eq. 5).  
60

$$Q = \alpha L_m^a (M_m + \beta L_r^c R^d)^b \quad (3)$$

$$P_l(L_r + L_m) + P_m M_m + P_r R \leq C \quad (4)$$

$$R \leq \gamma Q \quad (5)$$

Furthermore, we require that there is no separate second-hand market for **reconditioned** material, preventing  $M_m$  from becoming negative, i.e.  $M_m \geq 0$ .

The Lagrangian function of our basic maximisation problem may be written as

$$\begin{aligned} \mathcal{L} &= \alpha L_m^a (M_m + \beta L_r^c R^d)^b + \lambda (C - P_l(L_r + L_m) - P_m M_m - P_r R) + \mu (\gamma Q - R) \\ &= \alpha L_m^a (M_m + \beta L_r^c R^d)^b (1 + \mu \gamma) + \lambda (C - P_l(L_r + L_m) - P_m M_m - P_r R) - \mu R, \end{aligned} \quad (6)$$

where  $\lambda$  and  $\mu$  are non-negative multipliers. Before we develop our main results, we state the following lemma:

*Lemma 1*

*For any positive budget  $C$ , the budget constraint in the optimum is binding.*

To prove this, assume that this were not so. Then there would be an opportunity to increase **the**

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4 manufacturing labour input  $L_m$ , which would increase output  $Q$  according to Eq. 1, contradicting  
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6 **the system** being in an optimal state. As a consequence  $\lambda > 0$ , and we must always have  $L_m > 0$   
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8  
9 for a positive budget.

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12 In the continuing, we always assume a positive  $C$  and that there is some return opportunity  $\gamma > 0$ .  
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### 15 16 17 18 19 **3.2 The reconditioning process**

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21 We first study the **reconditioning** sub-system alone. For any given budget for this sub-system  $C_R$ ,  
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23 the returned material output  $M_r = \beta L_r^c R^d$  must be maximised, subject to the return input  
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25 limitation. For a given  $Q$ , we state the Lagrangian  
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$$35 \mathcal{L}_R = \beta L_r^c R^d + \lambda_R (C_R - P_l L_r - P_r R) + \mu_R (Q - \gamma R) \quad (7)$$

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40 Obviously, in the optimum, the budget constraint will be binding so  $\lambda_R > 0$ , and also  $L_r > 0$  and  
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44  $M_r > 0$ . The first-order Kuhn-Tucker optimisation conditions are then  
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$$49 \frac{\partial \mathcal{L}_R}{\partial L_r} = \beta c L_r^{c-1} R^d - \lambda_R P_l = 0 \quad (8)$$

$$50 \frac{\partial \mathcal{L}_R}{\partial R} = \beta d L_r^c R^{d-1} - \lambda_R P_r - \mu_R = 0 \quad (9)$$

$$\frac{\partial \mathcal{L}_R}{\partial \lambda_R} = C_R - P_l L_r - P_r R = 0 \quad (10)$$

$$\frac{\partial \mathcal{L}_R}{\partial \mu_R} = \gamma Q - R \geq 0 \quad (11)$$

$$\frac{\partial \mathcal{L}_R}{\partial \mu_R} \mu_R = \mu_R (\gamma Q - R) = 0 \quad (12)$$

Using the abbreviation  $y_R = P_r (P_r + \mu_R / \lambda_R)^{-1}$ , the solution is found to be

$$L_r^* = C_R c P_l^{-1} (c + d y_R)^{-1} \quad (13)$$

$$R^* = C_R d P_r^{-1} y_R (c + d y_R)^{-1} \quad (14)$$

and the maximal output

$$M_r^* = C_R^{c+d} \beta c^c d^d P_l^{-c} P_r^{-d} y_R^d (c + d y_R)^{-(c+d)} \quad (15)$$

We distinguish between the two sub-cases that the return constraint is not binding with  $\mu_R = 0$

and  $y_R = 1$  (the free case), and when it is binding with  $\mu_R > 0$  and  $y_R < 1$ . The latter case

requires  $R = \gamma Q$ , which gives  $y_R = \frac{\gamma Q P_r c / d}{C_R - \gamma Q P_r} < 1$  and thus holds for  $Q < \frac{d C_R}{\gamma P_r (c + d)}$ .

We therefore have the cost function for any given  $Q$

$$C_R = \begin{cases} (c + d) M_r^{\frac{1}{c+d}} \beta^{\frac{-1}{c+d}} (P_l / c)^{\frac{c}{c+d}} (P_r / d)^{\frac{d}{c+d}}, & \text{if } M_r \leq \beta (\gamma Q)^{c+d} (c / d)^c (P_r / P_l)^c, \\ \gamma Q P_r + (M_r / \beta)^{\frac{1}{c}} P_l (\gamma Q)^{\frac{-d}{c}}, & \text{if } M_r > \beta (\gamma Q)^{c+d} (c / d)^c (P_r / P_l)^c. \end{cases} \quad (16)$$

This function is easily shown to be continuous at the point  $M_r = \beta (\gamma Q)^{c+d} (c / d)^c (P_r / P_l)^c$ .

Furthermore, its derivative  $\frac{\partial C_R}{\partial M_r}$  is also continuous here. It is progressively increasing in  $M_r$ ,

since  $c + d < 1$ . We note that for a small output, the marginal cost is

$$\frac{\partial C_R}{\partial M_r} = M_r^{\frac{1-c-d}{c+d}} \beta^{\frac{-1}{c+d}} (P_l / c)^{\frac{c}{c+d}} (P_r / d)^{\frac{d}{c+d}}, \quad (17)$$

which is zero for a zero output  $M_r = 0$ . Therefore, the initial marginal production of **reconditioned components** is infinitely cheap and will always compete successfully with any other similar source of supply (such as  $M_m$ ). That  $M_r$  must be positive in the optimum, is thus stated in

*Theorem 1*

*For a positive budget  $C$ , we always have  $M_r > 0$ .* ■

A more formal proof is easily established.

That  $M_r$  always is positive in the optimum of the system, applies irrespective of the level of the return price  $P_r$ . This might seem counter-intuitive. A popular conclusion is therefore that we should always use some returns when they exist.

### 3.3 Detailed consequences in the manufacturing process

Knowing that  $M_r > 0$ ,  $L_r > 0$ ,  $R > 0$ , and returning to our Lagrangian in Eq 6, the first-order Kuhn-Tucker optimisation conditions are obtained as

$$\frac{\partial \mathcal{L}}{\partial L_r} = \alpha \beta b c L_m^a L_r^{c-1} R^d (M_m + \beta L_r^c R^d)^{b-1} (1 + \mu \gamma) - \lambda P_l = 0, \quad (18)$$

$$\frac{\partial \mathcal{L}}{\partial L_m} = \alpha a L_m^{a-1} (M_m + \beta L_r^c R^d)^b (1 + \mu \gamma) - \lambda P_l = 0, \quad (19)$$

$$\frac{\partial \mathcal{L}}{\partial M_m} = \alpha b L_m^a (M_m + \beta L_r^c R^d)^{b-1} (1 + \mu \gamma) - \lambda P_m \leq 0, \quad (20)$$

$$\frac{\partial \mathcal{L}}{\partial M_m} M_m = M_m \left( \alpha b L_m^a (M_m + \beta L_r^c R^d)^{b-1} (1 + \mu \gamma) - \lambda P_m \right) = 0 \quad (21)$$

$$\frac{\partial \mathcal{L}}{\partial R} = \alpha \beta b d L_m^a L_r^c R^{d-1} (M_m + \beta L_r^c R^d)^{b-1} (1 + \mu \gamma) - \lambda P_r - \mu = 0, \quad (22)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = (C - P_l(L_r + L_m) - P_m M_m - P_r R) = 0, \quad (23)$$

$$\frac{\partial \mathcal{L}}{\partial \mu} = \alpha L_m^a (M_m + \beta L_r^c R^d)^b \gamma - R \geq 0, \quad (24)$$

$$\frac{\partial \mathcal{L}}{\partial \mu} \mu = \mu (\alpha L_m^a (M_m + \beta L_r^c R^d)^b \gamma - R) = 0, \quad (25)$$

$$\lambda > 0, \quad (26)$$

$$\mu \geq 0. \quad (27)$$

The inequality remaining in the conditions, is Eq. 20 concerning whether or not  $M_m$  should be zero. We distinguish between two basic cases. In Case I, there is no input of direct materials, i.e.  $M_m = 0$ . Case II covers the situation that both  $M_m > 0$  and  $M_r > 0$ . The case that there is no input of return materials  $M_r = 0$  has been ruled out by Theorem 1. In each of the two cases, the return constraint may be loose or binding, and these sub-cases are denoted a and b, respectively.

We introduce the abbreviations



$$y = \frac{P_r}{P_r + \mu / \lambda}, \quad (28)$$

$$G = \alpha \beta^b a^a b^{cb+bd-1} c^{cb} d^{bd-1} P_l^{-a-cb} P_r^{1-bd} C^{a+cb+bd-1} \quad (29)$$

$$H = \alpha \beta^{\frac{-1}{1-c-d}} a^a b^b (a+b)^{-(a+b)} c^{\frac{-c}{1-c-d}} d^{\frac{-(1-c)}{1-c-d}} \times P_l^{\frac{c-a(1-c-d)}{1-c-d}} P_m^{\frac{-1-b(1-c-d)}{1-c-d}} P_r^{\frac{1-c}{1-c-d}} C^{a+b} \quad (30)$$

$$h = \beta^{\frac{1}{1-c-d}} c^{\frac{c}{1-c-d}} d^{\frac{d}{1-c-d}} P_l^{\frac{-c}{1-c-d}} P_m^{\frac{1}{1-c-d}} P_r^{\frac{-d}{1-c-d}} C^{-1}, \quad (31)$$

all being positive and  $y \leq 1$  with equality when  $\mu = 0$ . We note that the aggregate parameters  $G$ ,  $H$  and  $h$  only depend on given parameters, and not on the multipliers. Applying the optimisation conditions Eqs. 18-20 and 22, the following solutions are derived:

Case I,  $M_m = 0$ ,  $M_r > 0$

$$Q^* = \frac{\alpha \beta^b a^a b^{bc+bd} c^{bc} d^{bd} P_l^{-a-bc} P_r^{-bd} C^{a+bc+bd} y^{bd}}{(a+bc+bdy)^{a+bc+bd}} \quad (32)$$

$$L_r^* = \frac{bc P_l^{-1} C}{a+bc+bdy} \quad (33)$$

$$L_m^* = \frac{aP_l^{-1}C}{a+bc+bdy} \quad (34)$$

$$R^* = \frac{bdP_r^{-1}Cy}{a+bc+bdy} \quad (35)$$

In Case I a, the return flow constraint is not binding, i.e.  $R^* < \gamma Q^*$  with  $\mu = 0$  and so  $y = 1$ , and in Case I b this constraint is binding, i.e.  $\mu > 0$  and so  $y < 1$ . In Case I a, the solution in  $\lambda$  is found explicitly

$$\lambda^* = G(a+bc+bd)^{1-a-cb-bd} P_r^{-1}, \quad (36)$$

whereas Case I b requires an implicit solution from

$$G = \frac{\gamma^{-1}y^{1-bd}}{(a+bc+bdy)^{1-a-bc-bd}}. \quad (37)$$

From this equation  $y = y_1$  is solved, and  $\lambda$  and  $\mu$  are then determined by

$$\lambda^* = \frac{\gamma^{-1}bdP_r^{-1}y_1}{1-bd(1-y_1)}, \quad (38)$$

$$\mu^* = \frac{\gamma^{-1}bd(1-y_1)}{1-bd(1-y_1)}. \quad (39)$$

The right-hand member of Eq. 37 is easily seen to be monotonically increasing in  $y$ , taking on its maximum at  $\gamma^{-1}(a+bc+bd)^{-(1-a-bc-bd)}$  for  $y = 1$ . When comparing which of the sub-cases that applies for given parameters, we thus find the result

### Lemma 2

In Case I, if  $G > \gamma^{-1}(a+bc+bd)^{-(1-a-bc-bd)}$ , then the free Case I a applies, and, otherwise the binding constraint Case I b.

This is equivalent to stating that in the latter sub-case, there is a solution  $y_1 < 1$ , but not in the former.

Case II,  $M_m > 0$ ,  $M_r > 0$

$$L_m^* = \frac{aP_l^{-1}C}{a+b} \left( 1 + hy^{\frac{d}{1-c-d}} (1-c-dy) \right) \quad (40)$$

$$M_m^* = \frac{bP_m^{-1}C}{a+b} \left( 1 - hy^{\frac{d}{1-c-d}} \left( \frac{a+bc}{b} + dy \right) \right) \quad (41)$$

$$L_r^* = cP_l^{-1}Chy^{\frac{d}{1-c-d}}, \quad (42)$$

$$R^* = dP_r^{-1}Chy^{\frac{1-c}{1-c-d}}, \quad (43)$$

$$Q^* = \frac{\alpha a^a b^b P_l^{-a} P_m^{-b} C^{a+b}}{(a+b)^{a+b}} \left( 1 + hy^{\frac{d}{1-c-d}} (1-c-dy) \right)^{a+b}. \quad (44)$$

In the sub-case of Case II a, the return flow constraint is not binding, i.e.  $R^* < \gamma Q^*$  with  $\mu = 0$  and so  $y = 1$ , and in Case II b, this constraint is binding, i.e.  $\mu > 0$  and so  $y < 1$ . In Case II a, the solution in  $\lambda$  is found explicitly

$$\lambda^* = \alpha a^a b^b P_l^{-a} P_m^{-b} \left( \frac{a+b}{C(1+h(1-c-d))} \right)^{1-a-b}, \quad (45)$$

and in Case II b, the binding return constraint requires

$$H = \gamma^{-1} y^{\frac{1-c}{1-c-d}} \left( 1 + hy^{\frac{d}{1-c-d}} (1-c-dy) \right)^{-(a+b)}. \quad (46)$$

The right-hand member is seen to be monotonically increasing in  $y$  and will take on the maximum

$$\gamma^{-1} (1 + h(1 - c - d))^{-(a+b)}. \quad (47)$$

If this value falls short of  $H$  in the left-hand member, Case II b is ruled out, and the constraint is loose with  $\mu = 0$ , whereas if it is larger, the constraint must be binding with a  $y$  smaller than unity. This solution we denote  $y_{II}$ . As in Case I b, values of  $\lambda$  and  $\mu$  are easily determined, once  $y$  is solved.

Studying the definitions of  $h$  and  $H$ , we therefore find that a small budget  $C$  or a low return price  $P_r$ , other parameters kept, creates a binding return flow constraint, quite in accordance with intuition.

We now turn our attention to what might create a solution with  $M_m = 0$ . The right-hand member of Eq. 41 is monotonically decreasing in  $y$

$$M_m^* = \frac{bP_m^{-1}C}{a+b} \left( 1 - hy^{\frac{d}{1-c-d}} \left( \frac{a+bc}{b} + dy \right) \right). \quad (48)$$

Sufficient for  $M_m$  to be positive at the optimum, is therefore that the right-hand member remains positive for the maximum  $y = 1$ .

Hence, we reach

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*Theorem 2a*

If  $h < b/(a + bc + bd)$ , then the Case II solution is optimal.

A small  $h$  obtains when  $P_m$  in combination with other parameters is small compared to the budget  $C$  and a sufficiently small  $h$  requires then that also materials be purchased directly, i.e.

$M_m > 0$ , quite as expected. Also, we have

*Lemma 2b*

If  $h \geq b/(a + bc + bd)$ , then either Case II is optimal with a binding return constraint  $y = y_{II} < 1$ , or Case I is optimal.

If the former applies, then  $M_m$  must be positive, which implies  $hy_{II}^{\frac{d}{1-c-d}}(a + bc + bdy_{II}) < b$ . But,

$hy_{II}^{\frac{d}{1-c-d}}(a + bc + bdy_{II}) \geq b$  implies  $h(a + bc + bd) \geq b$ . Hence,

*Theorem 2b*

If  $hy_{II}^{\frac{d}{1-c-d}}(a + bc + bdy_{II}) \geq b$ , then the Case I solution is optimal.

We now compare the Case I solution with the Case II solution assuming the optimal value of  $y$ .

Writing  $\Delta'$  for the difference in maximum output between the two cases (subtracting Eq. 44 from Eq. 32), we then obtain

$$\Delta'(h) =$$

$$= \frac{\alpha \beta^b a^a b^{b+cb+db-1} c^{cb} d^{db} P_l^{-a-cb} P_r^{-db} C^{cb+db+a} y^{db}}{(a+bc+bdy)^{a+bc+bd}} - \frac{\alpha a^a b^b P_l^{-a} P_m^{-b} C^{a+b}}{(a+b)^{a+b}} \left( 1 + hy^{\frac{d}{1-c-d}} (1-c-dy) \right)^{a+b}. \quad (49)$$

If common positive factors are cancelled, we have

$$\Delta''(h) = \frac{(a+b)^{a+b}}{b^{a+b}} \left( \frac{b}{a+bc+bdy} \right)^{a+bc+bd} y^{bd} - h^{-b(1-c-d)} \left( 1 + hy^{\frac{d}{1-c-d}} (1-c-dy) \right)^{a+b}, \quad (50)$$

where  $\Delta''(h)$  has the same sign as  $\Delta'(h)$ . Taking its derivative, it is easily shown that  $\Delta''(h)$  is unimodal with a maximum equal to

$$\Delta''(h^*) = \frac{(a+b)^{a+b} y^{bd}}{b^{b(1-c-d)}} (a+bc+bdy)^{-(a+bc+bd)} \times \left( 1 - \left( \frac{1-c-dy}{1-c-d} \right)^{b(1-c-d)} \right) \leq 0 \quad (51)$$

for  $h = h^* = \frac{y^{\frac{-d}{1-c-d}} b(1-c-d)}{(1-c-dy)(a+bc+bd)}$ . At most, this unique maximum of  $\Delta''(h)$  is zero, which

happens for  $y=1$ , in which case  $h^* = b/(a+bc+bd)$ . If  $y$  is smaller, then  $h^* > b/(a+bc+bd)$

and the maximum is less than zero as are all other values of  $\Delta''$ .

We thus arrive at the conclusion stated in

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5 *Theorem 3*

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7 If  $h(a+bc+bd) \geq b$  and  $hy_{II}^{\frac{d}{1-c-d}}(a+bc+bdy_{II}) < b$ , then the Case II solution is optimal with a  
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10 binding return constraint.  
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16 \*\*\* Insert Figure 2 \*\*\*  
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21 **4 ILLUSTRATIVE EXAMPLES**  
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26 Our model from above is illustrated in this section by some numerical examples. Due to the  
27 availability of the full analytical results from Section 3, our purpose of this section is not  
28 conducting an extensive numerical experiment to reach a managerial conclusion, but rather to  
29 demonstrate how the model can be implemented. The following parameters are assigned  $a = 0.1$ ,  
30  $b = 0.7$ ,  $c = 0.2$ ,  $d = 0.4$ ,  $\alpha = 10$ ,  $\beta = 5$ ,  $P_l = 3$ ,  $P_m = 5$ ,  $P_r = 1$ ,  $\gamma = 0.3$ . Since production processes  
31 overwhelmingly have a diminishing returns to scale property, we set  $(a + b) < 1$ . Furthermore,  
32 this property should be even more pronounced in a reconditioning process, we have  $(c + d) < 1$ .  
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45 As earlier argued, a reconditioning process is often more labour intensive compared to a  
46 conventional manufacturing process. Thus, we should assign a relatively higher ratio for  $c/d$   
47 compared with that of  $a/b$ . Usually, the conventional manufacturing process is also more  
48 productive than the reconditioning one. This gives the relation  $\alpha > \beta$ . Finally, the returned  
49 product has a lower cost (the major costs are ~~is~~ collection and transportation) so that we should  
50 set  $P_m > P_r$ .  
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4 The results are presented in Table 1, where, with three different budget levels, we have solutions  
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6 belonging to different solution cases. When the budget  $C$  is smaller than 118, there is no input of  
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8 regular material,  $M_m = 0$  and all returns are fed into the reconditioning process. As long as  $C$  is  
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10 between 118 and 624, all returns are used as input into the reconditioning process. However, the  
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12 output from reconditioning is not enough to supply all materials needed for manufacturing, thus  
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14  $M_m > 0$ . Since further enlarging the scale of reconditioning will increase the marginal cost of this  
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16 process,  $R$  is less than the total return when  $C$  is larger than 624. Part of the return  $(\gamma Q - R)$  needs  
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18 then to be disposed of.  
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26 The output from the manufacturing process increases smoothly with the budget  $C$  (Figure 3). The  
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28 reconditioning process, as we have discussed before, has a breakeven point at  $C = 624$  (Figure 4),  
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30 after which its input and output remain constant. This makes the solution belong to the type Case  
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32 IIa. As easily seen, with a given technology, involving known substitution effects between  
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34 materials, return products and labour, and with known prices and return ratio, we  
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36 straightforwardly derive the expansion curve of the intricate remanufacturing system. The  
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38 optimal strategy for handling return products, as well as material and labour inputs may also be  
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40 developed accordingly.  
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\*\*\* Insert Table 1 \*\*\*

\*\*\* Insert Figure 3 \*\*\*

\*\*\* Insert Figure 4 \*\*\*

## 5 CONCLUSIONS

In this paper we have presented a production model to illustrate the optimal production conditions for a remanufacturing system. The optimal production decisions are developed as explicit conditions of price parameters, technical efficiency parameters, the marginal rate of substitution of inputs and the budget level. It is shown under Cobb-Douglas properties of production (and decreasing marginal returns to scale) that some reconditioning is always necessary if such an opportunity exists.

Intervals, given by relatively simple expressions of the parameters, provide necessary and sufficient conditions for the optimal choice of production, whether or not all returns must be used, and whether or not fresh material must be purchased. Given the prices and production function properties, the type of solution is explicitly found. However, for determining intervals explaining which sub-case that is optimal, requires in certain cases that non-linear equations are to be solved. But the uniqueness of solutions is guaranteed by the theorems derived.

Further studies should be conducted towards applying general transformation opportunities, in which more unspecified production functions or production sets are used. It would be expected that leaving the Cobb-Douglas structure, at least having a positive initial marginal cost, this would create opportunities when positive return flows would not necessarily belong to the optimal solution. A future line of development could be to replace the disassembly/reconditioning unit in Figure 1 by a linear production-economic system with an arborescent product structure, having used end products and labour as inputs (as above) but with

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4 two or more reconditioned components of varying quality as outputs. Certain outputs of  
5 sufficient quality would be entered as input components into the manufacturing process, whereas  
6 those of low condition would be scrapped and possibly sold at some low price. An approach of  
7 this kind would be expected to add realism and detail, but at the same time the model  
8 transparency would be lowered (cf Grubbström, 2001).  
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19 There is also ample space for determining comparative-static investigations as to the effects of  
20 parameter changes on solution properties.  
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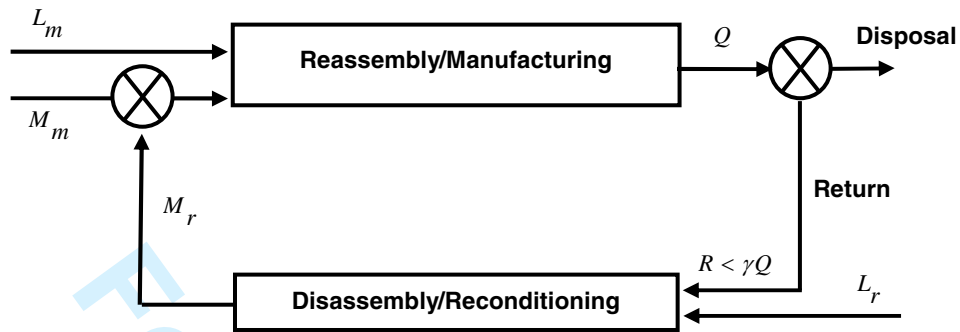


Figure 1. A remanufacturing system

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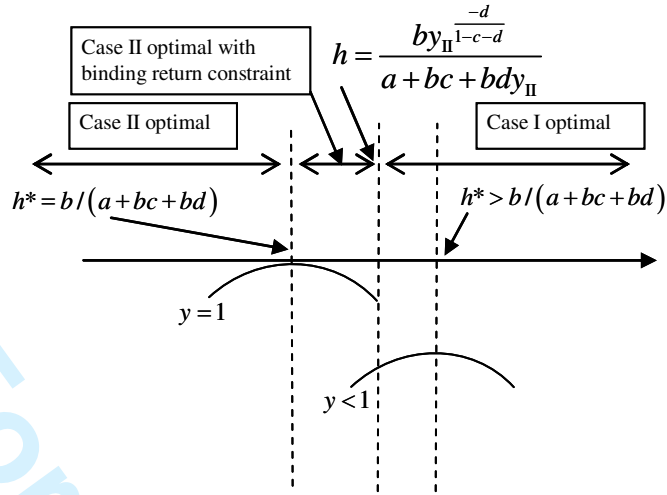


Figure 2. Influence of aggregate parameter  $h$  on solution.

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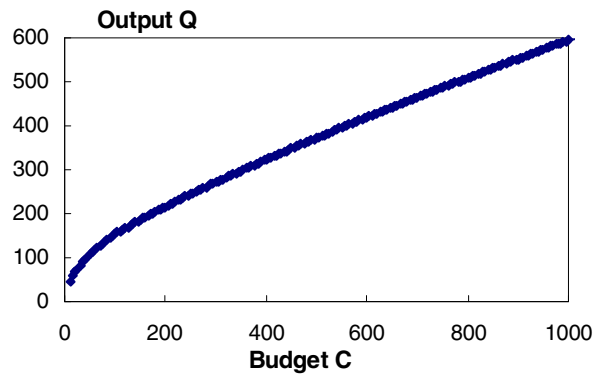


Figure 3. Output  $Q$  vs budget  $C$  in the **production** system.



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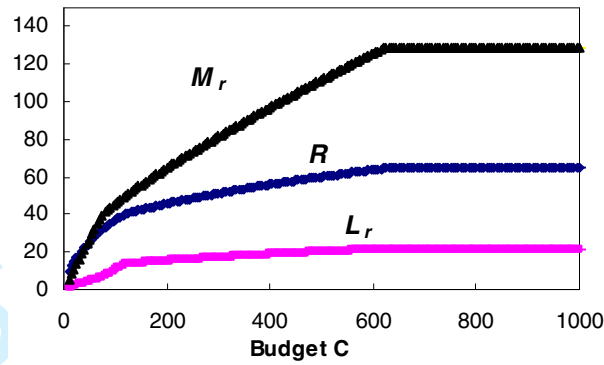


Figure 4. Optimal inputs and outputs in the **reconditioning** process with an increasing budget  $C$ .

Table 1: Optimal results

$C$	$L_r$	$R$	$L_m$	$M_m$	$M_r$	$Q$	Solution
100	10.51	45.96	7.51	0.00	37.00	153.21	Case I
500	20.01	111.65	26.19	49.95	60.03	372.16	Case IIb
1000	21.52	129.10	47.05	133.04	64.55	594.69	Case IIa