New induction relations for complete functions in Jucys-Murphy elements

Valentin Féray

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ABSTRACT. The problem of computing the class expansion of some symmetric functions evaluated in Jucys-Murphy elements appears in different contexts, for instance in the computation of matrix integrals. Recently, M. Lassalle gave a unified algebraic method to obtain some induction relations on the coefficients in this kind of expansion. In this paper, we give a simple purely combinatorial proof of his result. Besides, using the same type of argument, we obtain new simpler formulas. We also prove an analogous formula in the Hecke algebra of (S_{2n}, H_n) and use it to solve a conjecture of S. Matsumoto on the subleading term of orthogonal Weingarten function. Finally, we propose a conjecture for a continuous interpolation between both problems.

1. INTRODUCTION

1.1. **Background.** The Jucys-Murphy elements J_i lie in the symmetric group algebra $\mathbb{Z}[S_n]$. Despite their beautiful properties, their definition is very elementary:

$$J_i = \sum_{j<i} (j\ i)$$

where $(j\ i)$ is the transposition in S_n exchanging i and j. They have been introduced separately by A. Jucys [Juc66, Juc74] and G. Murphy [Mur81] and have played since a quite important role in representation theory. Indeed, they act diagonally on the Young basis of any irreducible representation V_λ: the eigenvalue of J_i on an element e_T of this basis (T is a standard tableau of shape λ) is simply given by the content (*i.e.* the difference between the column-index and the row-index) of the box of T containing i.

In fact, representation theory of symmetric groups S_n can be constructed entirely using this property (see [OV96]). We also refer to papers of Biane [Bia98] and Okounkov [Oko00] for nice applications of Jucys-Murphy elements to asymptotic representation theory.

A fundamental property, already observed by Jucys and Murphy, is that elementary symmetric functions evaluated in the J_i's have a very nice expression (this evaluation is well-defined because Jucys-Murphy elements commute with each

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other). More precisely, if $\kappa(\sigma)$ denotes the number of cycles of a permutation $\sigma \in S_n$, then

$$e_k(J_1, \dots, J_n) = \sum_{\substack{\sigma \in S_n \\ \kappa(\sigma) = n-k}} \sigma.$$

As this is a central element in the group algebra, all symmetric functions evaluated in Jucys-Murphy elements are also central. Therefore it is natural to wonder what their class expansion is. In other terms, given some symmetric function F, can we compute the coefficients a_λ^F defined by:

$$F(J_1, \dots, J_n) = \sum_{\lambda \vdash n} a_\lambda^F \mathcal{C}_\lambda,$$

where the sum runs over all partitions of n and $\mathcal{C}_\lambda$ denotes the sum of all permutations of cycle-type λ? This problem may seem anecdotal at first sight, but it in fact appears in different domains of mathematics:

- When F is a power sum p_k, it is linked with mathematical physics via vertex operators and Virasoro algebra (see [LT01]).
- When F is a complete symmetric function h_k, the coefficients appearing are exactly the coefficients in the asymptotic expansion of unitary Weingarten functions. The latter is the elementary brick to compute polynomial integrals over the unitary group (see [Nov10, ZJ10]).
- The inverse problem (how can we write a given conjugacy class $\mathcal{C}_\lambda$ as a symmetric function in Jucys-Murphy element) is equivalent to try to express character values as a symmetric function of the contents. This question has been studied in some papers [CGS04, Las08a] but never using the combinatorics of Jucys-Murphy elements.

1.2. **Previous and new results.** As mentioned in the paragraph above, the class expansion of elementary functions in Jucys-Murphy elements is very simple and was first established by A. Jucys. The next result of this kind was obtained by A. Lascoux and J.-Y. Thibon via an algebraic method: they gave the coefficients of the class expansion of power sums in Jucys-Murphy elements as some coefficients of an explicit series [LT01].

Then S. Matsumoto and J. Novak [MN09] computed the coefficients of the permutations of maximal absolute length in any monomial function in Jucys-Murphy elements. Their proof is purely combinatorial but does not seem to be extendable to all coefficients. As monomial functions form a linear basis of symmetric functions, one can deduce from their result a formula for the top coefficients for any symmetric function, in particular for complete functions (see also [Mur04, CM09]). To be comprehensive, let us add that the authors also obtained all coefficients of cycles in complete symmetric functions using character theory (their approach works for all cycles, not only the ones of maximal length).

Recently, M. Lassalle [Las10] gave a unified method to obtain some induction relations for the coefficients of the class expansion of several families of symmetric functions in Jucys-Murphy elements. These induction relations allow to compute any coefficient quite quickly. Besides, it is possible to use them to recover the

results of A. Jucys, A. Lascoux and J.-Y. Thibon and also the top component of complete symmetric functions. Therefore, the work of M. Lassalle unifies most of the results obtained until now on the subject.

He proves his result with a sophisticated algebraic machinery: he begins by translating the problem in terms of shifted symmetric functions and then introduces some relevant differential operators.

In this paper, we give a simple combinatorial proof of his induction formulas. Our method of proof can also be adapted to find another formula (Theorem 2.6). The latter is new and quite simple. An example of application is the following: using Matsumoto's and Novak's result on cycles, we are able to compute more generating series of coefficients.

1.3. **Generalizations.** An analogous problem can be considered in the Hecke algebra of the Gelfand pair (S_{2n}, H_n) (here, H_n is the hyperoctahedral group seen as a subgroup of S_{2n}). Definitions are given in section 3. In this algebra, it is relevant to consider symmetric functions in odd-indexed Jucys-Murphy elements.

It is a remarkable fact that complete symmetric functions evaluated in these elements are also linked with integrals over groups of matrices, but the complex unitary group should be replaced by the real orthogonal group (see [ZJ10, Mat10]).

In paper [Mat10], S. Matsumoto computed the coefficients of permutations of maximal length in the case of monomial symmetric functions (hence obtaining an analog of his result with J. Novak).

Our new induction formula extends quite easily to this framework. A consequence is a proof of a conjecture of S. Matsumoto (see paragraph 3.4.2).

In fact, one can even define a generalization of the problem with a parameter α which interpolates between both frameworks:

- the class expansion of symmetric functions in Jucys-Murphy elements corresponds to the case $\alpha = 1$;
- the analogue in the Hecke algebra of (S_{2n}, H_n) corresponds to the case $\alpha = 2$.

We recall this construction in section 4.

A very interesting point in Lassalle's method to obtain induction formulas is that it works almost without changing anything with a general parameter α [Las10, section 11]. Unfortunately, we are not (yet) able to extend our work to this general setting. However, computer exploration suggests that some of the results still hold in the general case and we present a conjecture in this direction in section 4.

1.4. **Organization of the paper.** In section 2, we present our results in the symmetric group algebra.

Then, in section 3, we look at the analogous problem in the Hecke algebra of (S_{2n}, H_n).

Finally, in section 4, we present a conjecture for the continuous deformation between these two models.

2. INDUCTION RELATIONS

2.1. **Definitions and notations.** The combinatorics of integer partitions is important in this work as they index the conjugacy classes in symmetric groups. A partition λ of $n \geq 0$ (we note $\lambda \vdash n$) is a non-increasing finite sequence of positive integers (called parts) of sum n. Its number of elements is denoted $\ell(\lambda)$. We use the notation $\lambda \backslash i$ for the partition obtained from λ by erasing one part equal to i (we only use this notation when λ has at least one part equal to i). In a similar fashion, $\lambda \cup i$ is the partition obtained by adding a part equal to i (in an appropriate place such that the sequence remains non-increasing).

Let us denote by S_n the symmetric group of size n and by $\mathbb{Z}[S_n]$ its group algebra over the integer ring. Throughout the paper, the coefficient of a permutation $\sigma \in S_n$ in an element $x \in \mathbb{Z}[S_n]$ will be denoted $[\sigma]x$. If this coefficient is non-zero, we say that σ is in x (this is a small abuse of language, where we consider x as its support).

Definition 2.1. The Jucys-Murphy elements J_i (for $1 \leq i \leq n$) are defined by:

$$J_i = (1\ i) + (2\ i) + \cdots + (i-1\ i) \quad \in \mathbb{Z}[S_n].$$

Note that $J_1 = 0$ but we include it in our formulas for aesthetic reasons.

Proposition 2.2.
- *Jucys-Murphy elements commute with each other.*
- *If F is a symmetric function, $F(J_1, J_2, \ldots, J_n)$ lie in the center of the symmetric group algebra $Z(\mathbb{Z}[S_n])$.*

We recall that the cycle-type of a permutation σ in S_n, which we will denote $\mathrm{type}(\sigma)$, is by definition the non-increasing sequence of the lengths of its cycle. This is an integer partition of n, which determines the conjugacy class of the permutation in the group S_n.

A basis of the center of the group algebra $Z(\mathbb{Z}[S_n])$ is given by the sums of the conjugacy classes, that is the family of elements

$$\mathcal{C}_\lambda = \sum_{\substack{\sigma \in S_n \\ \sigma \text{ has cycle-type } \lambda}} \sigma,$$

where λ runs over all partitions of n. Therefore, for any symmetric function F, there exists some integer numbers a_λ^F such that:

$$F(J_1, \ldots, J_n) = \sum_{\lambda \vdash n} a_\lambda^F \mathcal{C}_\lambda.$$

In other terms, a_λ^F is the coefficient of any permutation σ of type λ in $F(J_1, \ldots, J_n)$.

We will here focus on the case where F is a complete symmetric function (so $a_\lambda^{h_k}$ will be denoted a_λ^k) because of the link with some integrals over unitary groups mentioned in introduction. All the results of this section could be easily adapted to elementary and power-sum symmetric functions. Unfortunately, we are not able to deal with a linear basis of symmetric functions (as the coefficients a_λ^F depend linearly on F, this would solve the problem for all symmetric functions).

Example 2.3. As an illustration, let us look at the case $k = 2$ and $n = 3$:

$$\begin{aligned} h_2(J_1, J_2, J_3) &= (1\ 2)^2 + \big((1\ 3) + (2\ 3)\big)^2 + (1\ 2) \cdot \big((1\ 3) + (2\ 3)\big); \\ &= \mathrm{Id} + 2\,\mathrm{Id} + (1\ 2\ 3) + (1\ 3\ 2) + (1\ 2\ 3) + (1\ 3\ 2); \\ &= 3\mathcal{C}_{1^3} + 2\mathcal{C}_3. \end{aligned}$$

Note that the coefficient of a permutation at the end of the computation does depend only on its cycle-type, although 1, 2 and 3 play different roles in the computation.

In other terms, we have computed the following coefficients:

$$a^2_{(1^3)} = 3, \quad a^2_{(2\ 1)} = 0, \quad a^2_{(3)} = 2.$$

2.2. **A combinatorial proof of Lassalle's formula.** In this paragraph, we give an elementary proof of the following theorem, which has been proved by M. Lassalle [Las10] using sophisticated algebraic tools.

Theorem 2.4 (Lassalle, 2010)**.** *For any partition ρ and integer k, one has:*

$$a^{(k)}_{\rho\cup 1} = a^{(k)}_{\rho} + \sum_{i=1}^{\ell(\rho)} \rho_i a^{(k-1)}_{\rho\setminus(\rho_i)\cup(\rho_i+1)}; \tag{1}$$

$$\begin{aligned} \sum_{i=1}^{\ell(\rho)} \rho_i a^{(k)}_{\rho\setminus(\rho_i)\cup(\rho_i+1)} = &\sum_{\substack{1\le i,j\le \ell(\rho)\\ i\neq j}} \rho_i\rho_j a^{(k-1)}_{\rho\setminus(\rho_i,\rho_j)\cup(\rho_i+\rho_j+1)} \\ &+ \sum_{i=1}^{\ell(\rho)} \rho_i \sum_{\substack{r+s=\rho_i+1\\ r,s\ge 1}} a^{(k-1)}_{\rho\setminus(\rho_i)\cup(r,s)}. \end{aligned} \tag{2}$$

Proof. We start from the obvious induction relation

$$h_k(J_1, \ldots, J_{n+1}) = h_k(J_1, \ldots, J_n) + J_{n+1} h_{k-1}(J_1, \ldots, J_{n+1}) \tag{3}$$

and we apply to it the following operator:

$$\begin{array}{rccl} & \mathbb{Z}[S_{n+1}] & \to & \mathbb{Z}[S_n] \\ \mathbb{E}: & \sigma & \mapsto & \begin{cases} \sigma/\{1,\ldots,n\} \text{ if } \sigma(n+1) = n+1; \\ 0 \text{ else.} \end{cases} \end{array}$$

Then we look at the coefficient of a permutation σ of type $\rho \vdash n$ (in the following, σ' is the image of σ by the canonical embedding of S_n into S_{n+1}, which means

that we add $n+1$ as fixed point).

$$[\sigma]\mathbb{E}\big(h_k(J_1,\dots,J_{n+1})\big) = [\sigma']h_k(J_1,\dots,J_{n+1}) = a^{(k)}_{\rho\cup 1}, \tag{4}$$

$$[\sigma]\mathbb{E}\big(h_k(J_1,\dots,J_n)\big) = [\sigma]h_k(J_1,\dots,J_n) = a^{(k)}_{\rho}, \tag{5}$$

$$\begin{aligned}[\sigma]\mathbb{E}\big(J_{n+1}h_{k-1}(J_1,\dots,J_{n+1})\big) &= [\sigma']\sum_{j\leq n}(j\ n+1)h_{k-1}(J_1,\dots,J_{n+1})\\ &= \sum_{j\leq n}[(j\ n+1)\sigma']h_{k-1}(J_1,\dots,J_{n+1})\\ &= \sum_{j\leq n} a^{(k-1)}_{\mathrm{type}((j\ n+1)\sigma')},\end{aligned}$$

Let us label the cycles of σ with the numbers $1,2,\dots,\ell(\rho)$ such that the i-th cycle of σ has length ρ_i. It is easy to see that $(j\ n+1)\sigma'$ has exactly the same cycle decomposition as σ except that $n+1$ has been added right before j. Therefore, if j is in the i-th cycle of σ, then $(j\ n+1)\sigma'$ has cycles of length $\rho_1,\rho_2,\dots,\rho_i+1,\dots,\rho_{\ell(\rho)}$. In other terms, its type is $\rho\setminus(\rho_i)\cup(\rho_i+1)$. As there are ρ_i elements in the i-th cycle of σ, one obtains:

$$[\sigma]\mathbb{E}\big(J_{n+1}h_{k-1}(J_1,\dots,J_{n+1})\big) = \sum_{1\leq i\leq \ell(\rho)} \rho_i a^{(k-1)}_{\rho\setminus(\rho_i)\cup(\rho_i+1)}. \tag{6}$$

Putting together equations (3), (4), (5) and (6), we obtain the first part of the theorem.

The second equality is obtained the same way except that we multiply equation (3) by J_{n+1} before applying the operator $\mathbb{E}$. One obtains:

$$\begin{aligned}\mathbb{E}\big(J_{n+1}h_k(J_1,\dots,J_{n+1})\big) = \mathbb{E}\big(J_{n+1}h_k(J_1,\dots,J_n)\big)\\ + \mathbb{E}\big(J^2_{n+1}h_{k-1}(J_1,\dots,J_{n+1})\big).\end{aligned} \tag{7}$$

The coefficient of σ in the left-hand side has been computed: see equation (6). Let τ be a permutation in $h_k(J_1,\dots,J_n)$. It fixes $n+1$ and, hence, $(j\ n+1)\tau$ can not fix $n+1$ for $j=1,\dots,n$. Therefore,

$$\mathbb{E}\big(J_{n+1}h_k(J_1,\dots,J_n)\big) = 0. \tag{8}$$

For the last term, we write:

$$\begin{aligned}[\sigma]\mathbb{E}\big(J^2_{n+1}h_{k-1}(J_1,\dots,J_{n+1})\big)&\\ = [\sigma']\sum_{j_1,j_2\leq n}(j_1\ n+1)\cdot(j_2\ n+1)\cdot h_{k-1}(J_1,\dots,J_{n+1})&\\ = \sum_{j_1,j_2\leq n}[(j_2\ n+1)\cdot(j_1\ n+1)\cdot\sigma']h_{k-1}(J_1,\dots,J_{n+1})&\\ = \sum_{j_1,j_2\leq n} a^{(k-1)}_{\mathrm{type}((j_2\ n+1)\cdot(j_1\ n+1)\cdot\sigma')}.&\end{aligned}$$

As before, we label the cycles of σ. We split the sum in two parts, depending on whether j_1 and j_2 are in the same cycle of σ or not:

- Suppose that both j_1 and j_2 are in the i-th cycle of σ. That implies that $j_2 = \sigma^m(j_1)$ for some integer m between 1 and ρ_i (eventually $j_1 = j_2$, which corresponds to $m = \rho_i$). Then $(j_1\ n+1)\cdot(j_2\ n+1)\cdot\sigma'$ has the same cycles as σ except for its i-th cycle, as well as two other cycles:

  $$\big(j_1, \sigma(j_1), \dots \sigma^{m-1}(j_1)\big) \text{ and } \big(j_2, \sigma(j_2), \dots \sigma^{\rho_i - m - 1}(j_2), n+1\big).$$

  Thus it has cycle-type $\rho \setminus (\rho_i) \cup (m, \rho_i - m + 1)$. There are ρ_i elements in the i-th cycle of σ and, hence, ρ_i possible values for j_1. For each value of j_1, there is exactly one value of j_2 corresponding to each value of m between 1 and ρ_i. Therefore, one has:

  $$\begin{aligned}\sum_{\substack{j_1, j_2 \le n \\ j_1 \sim_\sigma j_2}} a^{(k-1)}_{\mathrm{type}((j_2\ n+1)\cdot(j_1\ n+1)\cdot\sigma')} &= \sum_{i \le \ell(\rho)} \rho_i \sum_{m=1}^{\rho_i} a^{(k-1)}_{\rho\setminus(\rho_i)\cup(m,\rho_i - m+1)} \\ &= \sum_{i \le \ell(\rho)} \rho_i \sum_{\substack{r+s=\rho_i+1 \\ r,s\ge 1}} a^{(k-1)}_{\rho\setminus(\rho_i)\cup(r,s)},\end{aligned}$$

  where $j_1 \sim_\sigma j_2$ means that j_1 and j_2 are in the same cycle of σ.
- Let us suppose now that j_1 and j_2 are respectively in the i_1-th and i_2-th cycles of σ with $i_1 \neq i_2$. In this case $(j_1\ n+1)\cdot(j_2\ n+1)\cdot\sigma'$ has the same cycles as σ except for its i_1-th and i_2-th cycles, as well as one new cycle:

  $$\big(j_1, \sigma(j_1), \dots \sigma^{\rho_{i_1}-1}(j_1), n+1, j_2, \sigma(j_2), \dots \sigma^{\rho_{i_2}-1}(j_2)\big).$$

  Thus $(j_1\ n+1)\cdot(j_2\ n+1)\cdot\sigma'$ has cycle-type $\rho\setminus(\rho_{i_1},\rho_{i_2})\cup(\rho_{i_1}+\rho_{i_2}+1)$. As there are ρ_{i_1} (resp. ρ_{i_2}) elements in the i_1-th (resp. i_2-th) cycle of σ, one obtains:

  $$\sum_{\substack{j_1, j_2 \le n \\ j_1 \nsim_\sigma j_2}} a^{(k-1)}_{\mathrm{type}((j_2\ n+1)\cdot(j_1\ n+1)\cdot\sigma')} = \sum_{\substack{i_1, i_2 \le \ell(\rho) \\ i_1 \neq i_2}} \rho_{i_1}\rho_{i_2} a^{(k-1)}_{\rho\setminus(\rho_{i_1},\rho_{i_2})\cup(\rho_{i_1}+\rho_{i_2}+1)},$$

  where $j_1 \nsim_\sigma j_2$ means that j_1 and j_2 are not in the same cycle of σ.

Finally,

$$\begin{aligned}(9)\quad [\sigma]\mathbb{E}\big(J_{n+1}^2 h_{k-1}(J_1, \dots, J_{n+1})\big) = \\ \sum_{i \le \ell(\rho)} \rho_i \sum_{\substack{r+s=\rho_i+1 \\ r,s\ge 1}} a^{(k-1)}_{\rho\setminus(\rho_i)\cup(r,s)} + \sum_{\substack{i,j \le \ell(\rho) \\ i \neq j}} \rho_i\rho_j a^{(k-1)}_{\rho\setminus(\rho_{i_1},\rho_{i_2})\cup(\rho_{i_1}+\rho_{i_2}+1)}.\end{aligned}$$

Putting together equations (7), (6), (8) and (9), we obtain the second part of the theorem. □

Remark 2.5. This theorem allows to compute inductively the coefficients a^k_ρ, see [Las10, end of page 13].

2.3. **New relations.** In this paragraph, we prove new induction relations on the coefficients a^k_ρ, using the same kind of method as above.

Theorem 2.6. *For any partition ρ and positive integers k, m one has:*

$$a^k_{\rho\cup(m)} = \delta_{m,1} a^k_\rho + \sum_{1\leq i\leq \ell(\rho)} \rho_i a^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{\substack{r+s=m\\ r,s\geq 1}} a^{k-1}_{\rho\cup(r,s)}. \tag{10}$$

Proof. The case $m = 1$ corresponds to equation (1) and has already been proved.

Suppose $m > 1$. Once again, we begin with equation (3) and we will look at the coefficient of some permutation σ on both sides.

Let $n = |\rho| + m - 1$ and σ be a permutation in S_{n+1} of type $\rho \cup (m)$ such that $n + 1$ is in a cycle of σ of length m (in particular, as $m > 1$, $n + 1$ is not a fixed point of σ). By definition,

$$[\sigma] h_k(J_1, \dots, J_{n+1}) = a^k_{\rho\cup(m)}. \tag{11}$$

Besides, as all permutations in $h_k(J_1, \dots, J_n)$ fix $n + 1$, but not σ, one has:

$$[\sigma] h_k(J_1, \dots, J_n) = 0. \tag{12}$$

We shall now compute

$$\begin{aligned}
[\sigma] J_{n+1} h_{k-1}(J_1, \dots, J_{n+1}) &= [\sigma] \sum_{j\leq n} (j\ n+1) h_{k-1}(J_1, \dots, J_{n+1}) \\
&= \sum_{j\leq n} [(j\ n+1)\sigma] h_{k-1}(J_1, \dots, J_{n+1}) \\
&= \sum_{j\leq n} a^{k-1}_{\mathrm{type}((j\ n+1)\sigma)}.
\end{aligned}$$

As before, we label the cycles of σ: the cycle containing $n+1$ gets the label $\ell(\rho)+1$, the others are labelled such that the i-th cycle has length ρ_i (for $1 \leq i \leq \ell(\rho)$). We distinguish two cases:

- Suppose that j is in the $\ell(\rho) + 1$-th cycle of σ (as $n + 1$). This implies that $j = \sigma^h(n + 1)$ for some h between 1 and $m - 1$ (as $j \neq n + 1$, h can not be equal to m). Then $(j\ n + 1)\sigma$ has the same cycles than σ except for its $\ell(\rho) + 1$-th cycle, as well as two new cycles:

  $$\big(n + 1, \sigma(n + 1), \dots, \sigma^{h-1}(n + 1)\big) \text{ and } \big(j, \sigma(j), \dots, \sigma^{m-h-1}(j)\big).$$

  Thus its cycle-type is $\rho \cup (h, m - h)$. Exactly one value of j corresponds to each integer h between 1 and $m - 1$. One has:

  $$\sum_{\substack{j\leq n\\ j\sim_\sigma n+1}} a^{k-1}_{\mathrm{type}((j\ n+1)\sigma)} = \sum_{h=1}^{m-1} a^{k-1}_{\rho\cup(h,m-h)} = \sum_{\substack{r+s=m\\ r,s\geq 1}} a^{k-1}_{\rho\cup(r,s)}.$$

- Otherwise, j is in the i-th cycle of σ for some $i \leq \ell(\rho)$ (in particular, it is not in the same cycle as $n+1$). In this case, $(j\ n+1)\sigma$ has the same cycles than σ except for its i-th and $\ell(\rho)+1$-th cycles, as well as one new cycle:

  $$\big(j, \sigma(j), \dots, \sigma^{\rho_i - 1}(j), n+1, \sigma(n+1), \dots, \sigma^{m-1}(n+1)\big).$$

  Thus its cycle-type is $\rho \setminus (\rho_i) \cup (\rho_i + m)$. As there are ρ_i elements in the i-th cycle of σ for each i, one obtains:

  $$\sum_{\substack{j \leq n \\ j \nsim_\sigma n+1}} a^{k-1}_{\mathrm{type}((j\ n+1)\sigma)} = \sum_{1 \leq i \leq \ell(\rho)} \rho_i a^{k-1}_{\rho \setminus (\rho_i) \cup (\rho_i + m)}.$$

Finally,

$$[\sigma] J_{n+1} h_{k-1}(J_1, \dots, J_{n+1}) = \sum_{\substack{r+s=m \\ r,s \geq 1}} a^{k-1}_{\rho \cup (r,s)} + \sum_{1 \leq i \leq \ell(\rho)} \rho_i a^{k-1}_{\rho \setminus (\rho_i) \cup (\rho_i + m)}. \tag{13}$$

The theorem follows from equations (3), (11), (12) and (13). □

Remark 2.7. This type of case distinctions, depending on whether some elements are in the same cycle or not, is quite classical and leads often to the same kind of induction relations, called *cut-and-join* equations: see for instance [GJ97].

Remark 2.8. This theorem implies Theorem 2.4. Indeed, equation (2) can be written as a linear combination of specializations of equation (10), but the converse is not true.

Remark 2.9. Our new induction relation allows to compute a^k_ρ by induction over $|\rho|$ and k in several different ways. Indeed, a given partition λ can be written as $\rho \cup (m)$ in several different ways. It is not evident *a priori* that the final result does not depend on this choice. This relies on the initial conditions:

$$a^1_\rho = \begin{cases} 1 \text{ if } \rho = 21^i \text{ for some } i; \\ 0 \text{ else.} \end{cases}$$

2.4. **Taking care of the dependance in n.** As mentioned by Lassalle [Las10, paragraph 2.7], the coefficients $a^k_{\rho \cup 1^{n-|\rho|}}$, seen as functions of n, have a very nice structure. More precisely, let us define c^k_λ, where λ is a partition, by induction on $|\lambda|$ by the formula:

$$a^k_\rho = \sum_{i=0}^{m_1(\rho)} c^k_{\bar{\rho} \cup 1^i} \binom{m_1(\rho)}{i}, \tag{14}$$

where $m_1(\rho)$ is the number of parts equal to 1 in ρ and $\bar{\rho}$ is obtained from ρ by erasing its parts equal to 1.

The interesting fact now is that c^k_ρ is equal to 0 as soon as $|\rho| - \ell(\rho) + m_1(\rho)$ is bigger than k, while, for a given k, one has infinitely many non-zero a^k_ρ (this fact is explained in paragraph 2.4.2). As a consequence, coefficients c are convenient to compute simultaneously the class expansion of $h_k(J_1, \dots, J_n)$ for all positive integers n (the integer k being fixed): see Example 2.15 at the end of this paragraph.

Using equation (14), one can translate Theorems 2.4 and 2.6 into relations over the c's, but it is rather technical (see [Las10, section 12]). We prefer here to explain the combinatorial meaning of the c's and derive directly relations over the c's using this interpretation.

2.4.1. *Algebra of partial permutations.* A good tool for that are the partial permutations introduced by Ivanov and Kerov in [IK99]. Let $\mathcal{B}_\infty$ be the following $\mathbb{Z}$-algebra:

- A partial permutations *i.e.* is a couple (d, σ) where d is a finite set of positive integers and σ a permutation of d. As a $\mathbb{Z}$-module, $\mathcal{B}_\infty$ is the set of infinite linear combinations of partial permutations.
- the product on partial permutations is given by:

$$(d, \sigma) \cdot (d', \sigma') = (d \cup d', \tilde{\sigma} \cdot \tilde{\sigma'}), \tag{15}$$

  where $\tilde{\sigma}$ (resp. $\tilde{\sigma'}$) is the canonical continuation of σ (resp. σ') to $d \cup d'$ (*i.e.* we add fixed points, we will use this notation throughout the paper). It extends to $\mathcal{B}_\infty$ by biliearity:

$$\left(\sum_{(d,\sigma)} c_{d,\sigma}(d, \sigma)\right) \cdot \left(\sum_{(d',\sigma')} c_{d',\sigma'}(d', \sigma')\right) = \sum_{(d,\sigma),(d',\sigma')} c_{d,\sigma} c_{d',\sigma'} (d, \sigma) \cdot (d', \sigma').$$

  It is easy to see that in the formula above, only a finite number of term can contribute to the coefficient of a given partial permutation (d'', σ'') (indeed, the indices of such terms must fulfill $d, d' \subset d''$). Therefore the right-hand side is a well-defined element of $\mathcal{B}_\infty$.

The infinite symmetric group S_∞ acts naturally on $\mathcal{B}_\infty$: if τ belong to S_∞, that is τ is a permutation of $\mathbb{N}^\star$ with finite support, we define

$$\tau \bullet (d, \sigma) = (\tau(d), \tau\sigma\tau^{-1}).$$

The invariants by the action of S_∞ form a subalgebra $\mathcal{A}_\infty$ of $\mathcal{B}_\infty$. As explained in [IK99, § 6], a basis of this subalgebra is

$$\big(\mathcal{PC}_\lambda\big)_{\lambda \text{ partition}} \text{ where } \mathcal{PC}_\lambda = \sum_{\substack{d \subset \mathbb{N}^\star,\ |d| = |\lambda| \\ \sigma \in S_d,\ \text{cycle-type}(\sigma) = \lambda}} (d, \sigma).$$

The nice property of this construction is that, for each n, there exists a morphism φ_n from $\mathcal{B}_\infty$ to the symmetric group algebra $\mathbb{Z}[S_n]$ defined by:

$$\varphi_n(d, \sigma) = \begin{cases} \tilde{\sigma} \text{ if } d \subset \{1, \dots, n\}; \\ 0 \text{ else.} \end{cases}$$

These morphisms restrict to morphisms $\mathcal{A}_\infty \to Z(\mathbb{Z}[S_n])$. The image of vectors of the basis is given by [IK99, equation (4.3)]:

$$\varphi_n(\mathcal{PC}_\lambda) = \binom{n - |\lambda| + m_1(\lambda)}{m_1(\lambda)} \mathcal{C}_{\lambda \cup 1^{n-|\lambda|}}.$$

We shall need a last property of the algebra $\mathcal{B}_\infty$. Let us define, for a partial permutation (d,σ) its degree to be

$$\deg(d,\sigma) = |d| - \#\text{ cycles of }\sigma + \#\text{ fixed points of }\sigma.$$

We consider the subspace $(\mathcal{B}_\infty)_{\leq\delta}$ to be the set of infinite linear combinations of partial permutations of degree smaller or equal to δ.

Lemma 2.10. *The decomposition* $\mathcal{B}_\infty = \bigcup_{\delta\geq 1}(\mathcal{B}_\infty)_{\leq\delta}$ *defines an algebra filtration.*

Remark 2.11. Consider $\deg'$ defined by

$$\deg'(d,\sigma) = |d| - \#\text{ cycles of }\sigma,$$

$\deg'$ is the minimal number of factors needed to write σ (or $\tilde{\sigma}$) as a product of transpositions. It is known to define a filtration of $\mathbb{Z}[S_n]$ and hence of $\mathcal{B}_\infty$ (see [IK99, equation (10.3)]).

Proof. We have to prove that if $(\pi,f) = (\sigma,d)\cdot(\tau,e)$, then

$$\deg(\pi,f) \leq \deg(\sigma,d) + \deg(\tau,e).$$

We make an induction on the number m_1 of fixed points of π.

If $m_1 = 0$, then

$$\deg(\pi,f) = \deg'(\pi,f) \leq \deg'(\sigma,d) + \deg'(\tau,e) \leq \deg(\sigma,d) + \deg(\tau,e).$$

Otherwise, let $i \in f$ be a fixed point of π. We consider the linear operator F_i

$$F_i : \begin{array}{ccc} \mathcal{B}_\infty & \longrightarrow & \mathcal{B}_\infty \\ (\sigma,d) & \longmapsto & \begin{cases} (\sigma_{\setminus i}, d\setminus\{i\}) \text{ if } i\in d \\ (\sigma,d) \text{ else,} \end{cases} \end{array}$$

where $\sigma_{\setminus i}$ is the permutation obtained by erasing i in the expression of σ as a product of cycles of disjoint supports. Equivalenty, by definition, $\sigma_{\setminus i}(j) = \sigma(j)$ if $j \neq \sigma^{-1}(i)$ and $\sigma_{\setminus i}(\sigma^{-1}(i)) = \sigma(i)$. It is immediate to check that $\deg(F_i(\sigma,d)) = \deg(\sigma,d) - 1$ unless i is in a cycle of length 2 in σ, in which case $\deg(F_i(\sigma,d)) = \deg(\sigma,d)$.

Note that F_i is *not* an algebra morphism. However, as $i \in \mathrm{Fix}(\pi)$, one has:

$$F_i(\pi,f) = F_i(\sigma,d)\cdot F_i(\tau,e). \tag{16}$$

Let us explain why this is true. First, it is obvious that

$$(d\setminus\{i\}) \cup (e\setminus\{i\}) = (d\cup e)\setminus\{i\} = f\setminus\{i\}.$$

Then, if $j \neq i, \tau^{-1}(i)$, then $\tau_{\setminus i}(j) = j$ and thus

$$\sigma_{\setminus i}\big(\tau_{\setminus i}(j)\big) = \sigma_{\setminus i}\big(\tau(j)\big) = \sigma\big(\tau(j)\big) = \pi(j) = \pi_{\setminus i}(j)$$

because $\sigma(\tau(j)) = \pi(j)$ is different from i (indeed, $\pi(i) = i$ and $j \neq i$). Finally, one only has to check that:

$$\sigma_{\setminus i}\big(\tau_{\setminus i}(\tau^{-1}(i))\big) = \pi_{\setminus i}(\tau^{-1}(i)).$$

But $\tau_{\setminus i}(\tau^{-1}(i)) = \tau(i)$ and, as $\sigma(\tau(i)) = \pi(i) = i$, the left-hand side is equal to $\sigma_{\setminus i}(\tau(i)) = \sigma(i)$. But, as i is a fixed point of π, the permutation $\pi_{\setminus i}$ is simply $\pi/(f\setminus\{i\})$ and thus the right-hand side is equal to $\pi(\tau^{-1}(i)) = \sigma(i)$. This ends the proof of equation (16).

As $F_i(\pi, f)$ has one less fixed point than π, f, we can apply the induction hypothesis and one has:

$$\deg(F_i(\pi, f)) \leq \deg(F_i(\sigma, d)) + \deg(F_i(\tau, e)).$$

As mentioned above:

$$\begin{aligned} \deg(F_i(\pi, f)) &= \deg(\pi, f) + 1; \\ \deg(F_i(\sigma, d)) &= \deg(\sigma, d) + 1 - \delta_1; \\ \deg(F_i(\tau, e)) &= \deg(\tau, e) + 1 - \delta_2, \end{aligned}$$

where δ_1 (resp. δ_2) is equal to 1 if i is in a cycle of length 2 in σ (resp. τ) and 0 else.

If one of the δ's is equal to 0, one has

$$\begin{aligned} \deg((\pi, f)) = \deg(F_i(\pi, f)) + 1 &\leq \deg(F_i(\sigma, d)) + \deg(F_i(\tau, e)) + 1 \\ &\leq \deg(\sigma, d) + \deg(\tau, e) \end{aligned}$$

and the proof is over in this case. So the only case we have to study is when i is in cycles of length 2 in σ and τ. Of course, as $\sigma(\tau(i)) = i$, both $\sigma(i)$ and $\tau(i)$ are equal to the same number j. In this case, we have:

$$\begin{aligned} F_j(F_i(\pi, f)) &= F_j(F_i(\sigma, d)) \cdot F_j(F_i(\tau, e)); \\ \deg(\pi, f) &= \deg\big(F_j(F_i(\pi, f))\big) + 2; \\ \deg(\sigma, d) &= \deg\big(F_j(F_i(\sigma, d))\big) + 1; \\ \deg(\tau, e) &= \deg\big(F_j(F_i(\tau, e))\big) + 1 \end{aligned}$$

and we can conclude by induction. □

Remark 2.12. In their paper, V. Ivanov and S. Kerov considered a large family of filtrations on $\mathcal{B}_\infty$ [IK99, Proposition 10.3], but it does not contain this one.

2.4.2. *Complete functions in partial Jucys-Murphy elements.* It has been observed in [Fér09, Section 2] that, if we define natural analogs of Jucys-Murphy elements in $\mathcal{B}_\infty$ by:

$$X_i = \sum_{j<i} \big(\{j,\ i\},\ (j\ i)\big) \quad \text{for } i \geq 1,$$

- then they still commute with each other;
- besides, the evaluation $F(X_1, X_2, X_3, \dots)$ of any symmetric function F in the infinite sequence of partial Jucys-Murphy elements is well-defined and lies in $\mathcal{A}_\infty$.

Therefore there exist coefficients c_λ^k such that

$$h_k(X_1, X_2, X_3, \dots) = \sum_\lambda c_\lambda^k \mathcal{PC}_\lambda.$$

In other terms, c_λ^k is the coefficient of any partial permutation (d,σ) with $|d|=|\lambda|$ and σ of cycle-type λ in $h_k(X_1,X_2,X_3,\dots)$. Applying φ_n, one obtains:

$$\begin{aligned} h_k(J_1,\dots,J_n) &= \sum_\lambda c_\lambda^k \binom{n-|\lambda|+m_1(\lambda)}{m_1(\lambda)} \mathcal{C}_{\lambda\cup 1^{n-|\lambda|}} \\ &= \sum_{\rho\vdash n} \left(\sum_{\substack{\lambda \text{ such that} \\ \lambda\cup 1^{n-|\lambda|}=\rho}} c_\lambda^k \binom{n-|\lambda|+m_1(\lambda)}{m_1(\lambda)} \right) \mathcal{C}_\rho \\ &= \sum_{\rho\vdash n} \left(\sum_{i=1}^{m_1(\rho)} c^k_{\bar\rho\cup 1^i} \binom{m_1(\rho)}{i} \right) \mathcal{C}_\rho \end{aligned}$$

Therefore, the numbers c_λ^k fulfill equation (14) and this definition is equivalent to the one given at the beginning of the subsection. Note that with this construction, it is obvious that the c's are non-negative integers (fact which was observed numerically by Lassalle, private communication).

The fact that c_ρ^k is equal to 0 as soon as $|\rho|-\ell(\rho)+m_1(\rho)$ is bigger than k is also natural because each X_i is in $(\mathcal{B}_\infty)_{\leq 1}$ and hence $h_k(X_1,X_2,X_3,\dots)$ lies in $(\mathcal{B}_\infty)_{\leq k}$. This can of course be generalized to any symmetric function. In terms of a's, using equation (14), this implies the following property:

Proposition 2.13. *Let ρ be a partition and F a symmetric function of degree k. The function $t\mapsto a^F_{\rho\cup 1^t}$ is a polynomial in t of degree smaller or equal to*

$$k-(|\rho|-\ell(\rho)).$$

The fact that this function is a polynomial is already known [MN09, Theorem 4.4], but not the bound on the degree.

Besides, we can obtain induction relations on the c's with the same kind of argument we used for the a's:

Theorem 2.14. *For any partition ρ and positive integers m and k, one has*

$$\begin{aligned} c^k_{\rho\cup 1} &= \sum_i \rho_i c^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+1)}; \\ c^k_{\rho\cup 2} &= \sum_i \rho_i c^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+2)} + c^{k-1}_{\rho\cup(1,1)} + 2c^{k-1}_{\rho\cup(1)} + c^{k-1}_\rho; \\ c^k_{\rho\cup m} &= \sum_i \rho_i c^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{\substack{r+s=m \\ r,s\geq 1}} c_{\rho\cup(r,s)} + 2c^{k-1}_{\rho\cup(m-1)} \text{ if } m\geq 3. \end{aligned}$$

Proof. Let $n+1=|\rho|+m$ and fix a partial permutation (d,σ) with:

- $d=\{1,\dots,n+1\}$;
- σ has cycle-type $\rho\cup(m)$ and $n+1$ is in a cycle of length m.

Let us look at the coefficient $c^k_{\rho\cup m}$ of (d,σ) in $h_k(X_1,X_2,\dots)$. As $n+1$ is the biggest element in d, it implies that every monomials in the X_i's contributing to

the coefficient of (d, σ) contains no X_i with $i > n+1$ and contains at least one X_{n+1}. Thus:

$$\begin{aligned} c^k_{\rho \cup m} &= [(d,\sigma)] h_k(X_1, X_2, \dots) = [(d,\sigma)] X_{n+1} h_{k-1}(X_1, \dots, X_{n+1}); \\ &= [(d,\sigma)] \sum_{j<n+1} \sum_{\nu} \sum_{\substack{(d',\tau),\ |d'|=|\nu| \\ \text{cycle-type}(\tau)=\nu}} c^{k-1}_{\nu} \cdot (d' \cup \{j, n+1\}, (j\ n+1)\tilde{\tau}); \\ &= \sum_{j<n+1} \sum_{\substack{(d',\tau) \\ \text{cond. } (1)}} c^{k-1}_{\text{type}(\tau)}, \end{aligned}$$

where condition (1) is the equality $(d' \cup \{j, n+1\}, (j\ n+1)\tilde{\tau}) = (d, \sigma)$. For a given integer j between 1 and n, we have to determine which sets d' and permutations $\tau \in S_{d'}$ fulfill $d' \cup \{j, n+1\} = d$ and $(j\ n+1)\tilde{\tau} = \sigma$. Of course, one must have $\tilde{\tau} = (j\ n+1)\sigma$. As in the previous paragraphs, we make a case distinction:

- If j is not in the same cycle of σ as $n+1$, then they are in the same cycle of $\tilde{\tau}$. In particular, neither j nor $n+1$ are fixed points of $\tilde{\tau}$, so both belong to d'. Therefore, necessarily, $d' = d$. The discussion on the possible cycle-types of $\tau = \tilde{\tau}$ is exactly the same than in paragraph 2.3 and one has:

$$\sum_{\substack{j<n+1 \\ j \nsim n+1}} \sum_{\substack{(d',\tau) \\ \text{cond. } (1)}} c^{k-1}_{\text{type}(\tau)} = \sum_{1 \leq i \leq \ell(\rho)} \rho_i c^{k-1}_{\rho \setminus (\rho_i) \cup (\rho_i + m)}.$$

- If j is in the same cycle of σ as $n+1$ (this implies $m > 1$), we write $j = \sigma^h(n+1)$. If $d' = d$, then $\tau = \tilde{\tau} = (j\ n+1)\sigma$ and its possible cycle-types has been discussed in paragraph 2.3, so one has:

$$\sum_{\substack{j<n+1 \\ j \sim n+1}} \sum_{\substack{(d',\tau) \\ \text{cond. } (1) \text{ and } d'=d}} c^{k-1}_{\text{type}(\tau)} = \sum_{\substack{r+s=m \\ r,s \geq 1}} c^{k-1}_{\rho \cup (r,s)}.$$

  But, in this case, d' is not necessarily equal to d. Indeed, when $h = 1$, the permutation $\tilde{\tau}$ has $n+1$ as a fixed point. If $m > 2$, j can not be a fixed point in this case so $j \in d'$. Therefore $d' = d$ or $d' = d \setminus \{n+1\}$. In the last case, τ is a permutation of cycle-type $\rho \cup (m-1)$. A similar phenomenom happens when $h = m-1$: j is a fixed point of $\tilde{\tau}$, but not $n+1$, so d' can be equal to $d \setminus \{j\}$ and the corresponding permutation τ has cycle-type $\rho \cup (m-1)$. Therefore, if $m > 2$, one has:

$$\sum_{\substack{j<n+1 \\ j \sim n+1}} \sum_{\substack{(d',\tau) \\ \text{cond. } (1)}} c^{k-1}_{\text{type}(\tau)} = \sum_{\substack{r+s=m \\ r,s \geq 1}} c^{k-1}_{\rho \cup (r,s)} + 2 c^{k-1}_{\rho \cup (m-1)}.$$

  If $m = 2$, the only possible value of j is $\sigma(n+1)$ and, in this case, $\tilde{\tau}$ fixes both j and $n+1$. Therefore d' can be equal either to d, $d \setminus \{n+1\}$, $d \setminus \{j\}$ or $d \setminus \{j, n+1\}$. It is easy to see that the cycle-types of the corresponding permutations τ are respectively $\rho \cup (1,1)$, $\rho \cup (1)$, $\rho \cup (1)$ and ρ. Thus, for

$m = 2$,

$$\sum_{\substack{j<n+1\\ j\sim n+1}} \sum_{\substack{(d',\tau)\\ \text{cond. (1)}}} c^{k-1}_{\mathrm{type}(\tau)} = c^{k-1}_{\rho\cup(1,1)} + 2c^{k-1}_{\rho\cup(1)} + c^{k-1}_{\rho}.$$

Summing the different contributions in the different cases, we obtain our theorem. □

Example 2.15. Here are the non-zero values of c^k_ρ for small values of k ($k \leq 3$). It is immediate that $c^1_{(2)}$ is equal to 1, while all other c^1_μ are 0. Then Theorem 2.14 allows to compute:

$$\begin{aligned}
c^2_{(1,1)} &= 1 \cdot c^1_{(2)} = 1;\\
c^2_{(2,2)} &= 2c^1_{(4)} + c^1_{(2,1,1)} + 2c^1_{(2,1)} + c^1_{(2)} = 1;\\
c^2_{(3)} &= 2c^1_{(2,1)} + 2c^1_{(2)} = 2;\\
c^3_{(2)} &= c^2_{(1,1)} = 1;\\
c^3_{(2,1)} &= 2c^2_{(3)} = 4;\\
c^3_{(2,1,1)} &= 2c^2_{(3,1)} + c^2_{(2,2)} = 1;\\
c^3_{(2,2,2)} &= c^2_{(2,2)} = 1;\\
c^3_{(3,2)} &= c^2_{(3)} = 2;\\
c^3_{(4)} &= c^2_{(2,2)} + 2c^2_{(3)} = 5.
\end{aligned}$$

Using equation (14), we can compute all coefficients a^k_ρ for $k = 2, 3$ and we find the following class expansion (true for any $n \geq 1$):

$$\begin{aligned}
h_2(J_1,\dots,J_n) &= \delta_{n\geq 3}\, 2\mathcal{C}_{(3,1^{n-3})} + \delta_{n\geq 4}\, \mathcal{C}_{(2,2,1^{n-4})} + \binom{n}{2}\mathcal{C}_{1^n};\\
h_3(J_1,\dots,J_n) &= \delta_{n\geq 4}\, 5\mathcal{C}_{(4,1^{n-4})} + \delta_{n\geq 5}\, 2\mathcal{C}_{(3,2,1^{n-5})} + \delta_{n\geq 6}\, \mathcal{C}_{(2,2,2,1^{n-6})}\\
&\quad + \delta_{n\geq 2}\left(\binom{n-2}{2} + 4\binom{n-2}{1} + \binom{n-2}{0}\right)\mathcal{C}_{2,1^{n-2}}.
\end{aligned}$$

This kind of results could also have been obtained with Theorem 2.6 but the computation is a little harder (it involves discrete integrals of polynomials).

2.5. **Generating series for some coefficients.** S. Matsumoto and J. Novak have computed, using character theory, the following generating function [MN09, Theorem 6.7].

Theorem 2.16 (Matsumoto, Novak, 2009)**.** *For any integer $n \geq 2$, one has:*

$$\sum_k a^k_{(n)} z^k = \frac{\mathrm{Cat}_{n-1}\, z^{n-1}}{(1-1^2z^2)(1-2^2z^2)\dots(1-(n-1)^2z^2)}, \tag{17}$$

where $\mathrm{Cat}_{n-1} = \frac{1}{n}\binom{2(n-1)}{n-1}$ is the usual Catalan number.

As $a^k_{(n)} = c^k_{(n)}$, the same result holds on the c's. Unfortunately, we are not able to find a proof of their formula *via* Theorem 2.14, but the latter can be used to derive new results of the same kind.

For instance, with $\rho = (n-1)$ and $m = 1$, our induction relation writes as $c^k_{(n-1,1)} = (n-1) \cdot c^{k-1}_{(n)}$ and thus

$$\sum_k c^k_{(n-1,1)} z^k = z \sum_k (n-1) c^{k-1}_{(n)} z^{k-1} \\ = \frac{(n-1)\operatorname{Cat}_{n-1} z^n}{(1-1^2z^2)(1-2^2z^2)\dots(1-(n-1)^2z^2)}.$$

In terms of a's, this result implies:

$$\sum_k a^k_{(n-1,1)} z^k = \sum_k \left(c^k_{(n-1,1)} + c^k_{(n-1)} \right) z^k \\ = \frac{(n-1)\operatorname{Cat}_{n-1} z^n + (1-(n-1)^2z^2)\operatorname{Cat}_{n-2} z^{n-2}}{(1-1^2z^2)(1-2^2z^2)\dots(1-(n-1)^2z^2)}. \tag{18}$$

This expression is simpler than the one obtained by Matsumoto and Novak for the same quantity [MN09, Proposition 6.9] and their equivalence is not obvious at all.

If we want to go further and compute other generating series, one has to solve linear systems. For instance, denoting $F_\mu = \sum_k c^k_\mu z^k$, Theorem 2.14 gives:

$$F_{(n-2,1,1)} = z\left((n-2)F_{(n-1,1)} + F_{(n-2,2)}\right); \\ F_{(n-2,2)} = z\left((n-2)F_{(n)} + F_{(n-2,1,1)} + F_{(n-2,1)} + F_{(n-2)}\right).$$

After resolution, one has:

$$F_{(n-2,1,1)} = \frac{z^2\left(n(n-2)F_{(n)} + z(n-2)F_{(n-1)} + F_{(n-2)}\right)}{1-z^2}; \\ F_{(n-2,2)} = \frac{z\left((n-2)F_{(n)} + F_{(n-2,1)} + F_{(n-2)}\right) + z^2(n-2)F_{(n-1,1)}}{1-z^2}.$$

Using results above, one can deduce an explicit generating series for the c's which can be easily transformed into series for the a's.

3. Analogues in the Hecke algebra of (S_{2n}, H_n)

In this section, we consider a slightly different problem, which happens to be the analog of the one of the previous section. It was first considered recently by P. Zinn-Justin [ZJ10] and S. Matsumoto [Mat10] in connection with integrals over orthogonal groups.

3.1. **Hecke algebra of** (S_{2n}, H_n)**.** The results of this section are quite classical. A good survey, with a more representation-theoretical point of view, can be found in I.G. Macdonald's book[Mac95, Chapter 7].

Let us consider the symmetric group of even size S_{2n}, whose elements are seen as permutations of the set $\{1, \bar{1}, \dots, n, \bar{n}\}$. It contains the hyperoctahedral group which is the subgroup formed by permutations $\sigma \in S_{2n}$ such that $\overline{\sigma(i)} = \sigma(\bar{i})$ (by convention, $\bar{\bar{i}} = i$). We are interested in the double cosets $H_n \backslash S_{2n} / H_n$, *i.e.* the equivalence classes for the relation:

$$\sigma \equiv \tau \text{ if and only if } \exists\, h, h' \in H_n \text{ s.t. } \sigma = h\tau h'.$$

Conjugacy classes in the symmetric group algebra can be characterized easily using cycle-types. We recall a similar result for the double cosets: they are characterized *via* coset-types.

Definition 3.1. Let σ be a permutation of S_{2n}. Consider the following graph G_σ:

- its $2n$ vertices are labelled by $\{1, \bar{1}, \dots, n, \bar{n}\}$;
- we put a solid edge between i and $\bar{i}$ and a dashed one between $\sigma(i)$ and $\sigma(\bar{i})$ for each i.

Forgetting the types of the edges, we obtain a graph with only vertices of degree 2. Thus, it is a collection of cycles. Moreover, due to the bicoloration of edges, it is easy to see that all these cycles have an even length.

We call coset-type of σ the partition μ such that the lengths of the cycles of G_σ are equal to $2\mu_1, 2\mu_2, \dots$

Example 3.2. Let $n = 4$ and σ be the following permutation:

$$1 \mapsto 3,\ \bar{1} \mapsto 1,\ 2 \mapsto \bar{4},\ \bar{2} \mapsto \bar{3},\ 3 \mapsto \bar{2},\ \bar{3} \mapsto 2,\ 4 \mapsto 4,\ \bar{4} \mapsto \bar{1}.$$

The corresponding graph G_σ is drawn on figure 1.

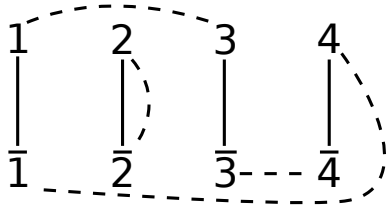

FIGURE 1. Example of graph G_σ

This graph is the disjoint union of one cycle of length 6 $(1, 3, \bar{3}, \bar{4}, 4, \bar{1})$ and one cycle of length 2 $(2, \bar{2})$. Thus the coset-type of σ is the integer partition $(3, 1)$.

Proposition 3.3. [Mac95, section 7.1] *Two permutations are in the same double coset if and only if their coset-types are the same.*

If μ is a partition of n, we denote

$$\mathcal{C}_\mu^{(2)} = \sum_{\substack{\sigma \in S_{2n} \\ \text{coset-type}(\sigma)=\mu}} \sigma \ \in \mathbb{Z}[S_{2n}].$$

It is immediate that the elements $\mathcal{C}_\mu^{(2)}$, when μ runs over partitions of n span linearly a subalgebra $Z_n^{(2)}$ of $\mathbb{Z}[S_{2n}]$. Equivalently, one can define $Z_n^{(2)}$ as the algebra of functions on S_{2n}, invariant by left and right multiplication by an element of H_n, endowed with the convolution product

$$f \star g(\sigma) = \sum_{\substack{\tau_1, \tau_2 \in S_{2n} \\ \tau_1 \tau_2 = \sigma}} f(\tau_1) g(\tau_2).$$

One can prove using representation theory [Mac95, section 7.2] that this algebra is commutative (in other terms, (S_{2n}, H_n) is a Gelfand pair).

3.2. **Odd Jucys-Murphy elements.** In this section we will look at symmetric functions in odd-indexed Jucys-Murphy elements in S_{2n}. Rewriting as permutations on the set $\{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}\}$ (ordered by $1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}$), these elements are:

$$J_i^{(2)} = \sum_{j=1,\bar{1},\dots,i-1,\overline{i-1}} (j\ i).$$

They were considered by P. Zinn-Justin [ZJ10] and then S. Matsumoto [Mat10].

Let us consider also the following element in $\mathbb{Q}[S_{2n}]$:

$$p_n = \frac{1}{|H_n|} \sum_{h \in H_n} h.$$

Then the following result holds, which may be seen as an analogue of the fact that symmetric functions in Jucys-Murphy elements are central in the symmetric group algebra.

Proposition 3.4. *If F is a symmetric function, then:*

$$x_{n,F} := F(J_1^{(2)}, \dots, J_n^{(2)}) p_n = p_n F(J_1^{(2)}, \dots, J_n^{(2)}).$$

Moreover $x_{n,F}$ belongs to the algebra $Z_n^{(2)}$.

Sketch of proof. The first step is to prove by induction that

$$e_k(J_1^{(2)}, \dots, J_n^{(2)}) p_n = p_n e_k(J_1^{(2)}, \dots, J_n^{(2)}) = \sum_{\substack{\mu \vdash n \\ |\mu| - \ell(\mu) = k}} \mathcal{C}_\mu^{(2)}.$$

The result follows for all F by multiplication and linear combination. See [ZJ10, Proposition 3] and [Mat10, Proposition 3.1] for details. □

Inspired by the results of section 2, we may look at the class expansion of $x_{n,F}$, *i.e.* the coefficients b_μ^F such that:

$$F(J_1^{(2)}, \dots, J_n^{(2)}) p_n = \sum_{\mu \vdash n} b_\mu^F \mathcal{C}_\mu^{(2)}.$$

As seen in the sketch of proof for the proposition above, the b's are easy to compute in the case of elementary functions.

In the following paragraph, we will establish some induction relations for the b's in the case of complete symmetric functions. We focus on this case (and thus use the short notation $b_\mu^k = b_\mu^{h_k}$) because these coefficients appear in the computation of the asymptotic expansion of some integrals over the orthogonal group [Mat10, Theorem 7.3].

3.3. **A simple induction relation.** In this paragraph, using the same method as in subsection 2.3, we prove the following induction formula for the b's.

Theorem 3.5. *For any partition ρ and positive integers k and m, one has:*

(19)
$$b^k_{\rho\cup(m)} = \delta_{m,1} b^k_\rho + 2\sum_{1\leq i\leq \ell(\rho)} \rho_i b^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{\substack{r+s=m\\ r,s\geq 1}} b^{k-1}_{\rho\cup(r,s)} + (m-1)b^{k-1}_{\rho\cup(m)}.$$

Proof. As before, the starting point of our proof is an induction relation on complete symmetric functions:

$$h_k(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)}) = h_k(J_1^{(2)},\dots,J_n^{(2)}) + J_{n+1}^{(2)} h_{k-1}(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)}).$$

Multiplying both sides by p_{n+1}, one has:

$$h_k(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})\cdot p_{n+1} = h_k(J_1^{(2)},\dots,J_n^{(2)})\cdot p_{n+1} + J_{n+1}^{(2)} h_{k-1}(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})\cdot p_{n+1}. \tag{20}$$

Let us begin with the case $m=1$. We choose a permutation $\sigma\in S_{2n}$ of coset-type ρ and we denote σ' its image by the canonical embedding $S_{2n}\hookrightarrow S_{2n+2}$. It has coset-type $\rho\cup(1)$. By definition,

$$[\sigma']h_k(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})p_{n+1} = b^k_{\rho\cup 1} \tag{21}$$

For the second term, we write:

$$h_k(J_1^{(2)},\dots,J_n^{(2)})\cdot p_{n+1} = h_k(J_1^{(2)},\dots,J_n^{(2)})\cdot p_n \cdot\left(1+(n+1\ \overline{n+1}) + \sum_{i=1,\bar 1,\dots,n,\bar n}(n+1\ i)(\overline{n+1}\ \bar i)\right).$$

Notice that $h_k(J_1^{(2)},\dots,J_n^{(2)})p_n$ lies in the algebra $\mathbb{Z}[S_{2n}]\subset\mathbb{Z}[S_{2n+2}]$ and hence is a linear combination of permutations fixing $n+1$ and $\overline{n+1}$. For such permutations τ, neither $\tau(n+1\ \overline{n+1})$ nor $\tau(n+1\ i)(\overline{n+1}\ \bar i)$ can be equal to σ' (these two permutations do not fix $n+1$ and $\overline{n+1}$). Therefore,

$$[\sigma']h_k(J_1^{(2)},\dots,J_n^{(2)})\cdot p_{n+1} = [\sigma]h_k(J_1^{(2)},\dots,J_n^{(2)})\cdot p_n = b^k_\rho. \tag{22}$$

We still have to compute:

$$[\sigma']J^{(2)}_{n+1}h_{k-1}(J^{(2)}_1,\dots,J^{(2)}_n,J^{(2)}_{n+1})\cdot p_{n+1} \tag{23}$$
$$= \sum_{j=1,\bar{1},\dots,n,\bar{n}} [(n+1\ j)\sigma']h_{k-1}(J^{(2)}_1,\dots,J^{(2)}_n,J^{(2)}_{n+1})\cdot p_{n+1}$$
$$= \sum_{j=1,\bar{1},\dots,n,\bar{n}} b^{k-1}_{\text{coset-type}((n+1\ j)\sigma')}.$$

Let us look at the coset-type of $(n+1\ j)\sigma'$. Denote by d_j (resp. d_{n+1}) the other extremity of the dashed edge of extremity j (resp. $n+1$) in $G_{\sigma'}$ (see definition 3.1). Then the graph $G_{(n+1\ j)\sigma'}$ has exactly the same edges as $G_{\sigma'}$, except for (j,d_j) and $(n+1,d_{n+1})$, which are replaced by (j,d_{n+1}) and $(n+1,d_j)$.

As $(n+1,\overline{n+1})$ is a loop of length 2 in $G_{\sigma'}$, if we assume that j was in a loop of size $2\rho_i$, then these two loops are replaced by a loop of size $2\rho_i+2$ in $G_{(n+1\ j)\sigma'}$ (it is a particular case of the phenomenon drawn on Figure 2).

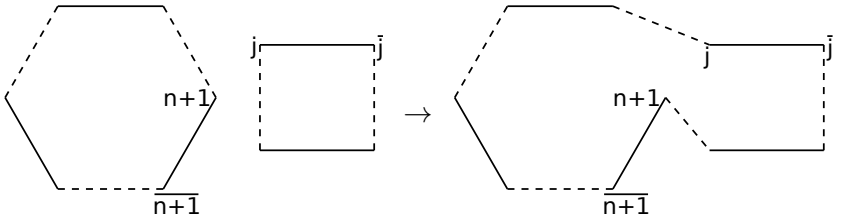

FIGURE 2. G_σ and $G_{(n+1\ j)\sigma}$ in the "join" case

Therefore $(n+1\ j)\sigma'$ has coset-type $\rho\backslash\rho_i\cup(\rho_i+1)$. As there are $2\rho_i$ elements in the i-th loop of G'_σ, one obtains:

$$[\sigma']J^{(2)}_{n+1}h_{k-1}(J^{(2)}_1,\dots,J^{(2)}_n,J^{(2)}_{n+1})\cdot p_{n+1} = 2\sum_{1\le i\le \ell(\rho)} \rho_i b^{k-1}_{\rho\backslash(\rho_i)\cup(\rho_i+1)} \tag{24}$$

Putting together equations (20), (21), (22) and (24), we obtain the case $m=1$ of the theorem.

Let us consider now the case $m>1$. We choose a permutation $\sigma\in S_{2n+2}$ of coset-type $\rho\cup(m)$ such that $n+1$ is in a loop of size $2m$ in G_σ. As $m>1$, this implies that $\overline{\sigma^{-1}(n+1)}\neq\sigma^{-1}(\overline{n+1})$. On the other hand, if τ lies in $\mathbb{Z}[S_{2n}]\subset\mathbb{Z}[S_{2n+2}]$ and $i=1,\bar{1},\dots,n,\bar{n}$, one has:

$$\big(\tau(i\ n+1)(\bar{i}\ \overline{n+1})\big)^{-1}(n+1)=i;$$
$$\big(\tau(i\ n+1)(\bar{i}\ \overline{n+1})\big)^{-1}(\overline{n+1})=\bar{i}.$$

Thus, σ can not be written as $\tau(i\ n+1)(\bar{i}\ \overline{n+1})$ with the conditions above. It can not be equal to τ or written as $\tau(n+1\ \overline{n+1})$ either. Therefore,

$$[\sigma]h_k(J^{(2)}_1,\dots,J^{(2)}_n)p_{n+1}=0$$

As a consequence, one has:

$$\begin{aligned} b^k_{\rho\cup(m)} &= [\sigma]h_k(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})\cdot p_{n+1}\\ &= [\sigma]J_{n+1}^{(2)}h_{k-1}(J_1^{(2)},\dots,J_n^{(2)},J_{n+1}^{(2)})\cdot p_{n+1}\\ &= \sum_{j=1,\bar 1,\dots,n,\bar n} b^{k-1}_{\text{coset-type}((n+1\ j)\sigma')}. \end{aligned}$$

and we have to look at the possible coset types of $(n+1\ i)\sigma$ (equation (23) is still true).

Let us number the loops of the graph G_σ with the integers $1,2,\dots,\ell(\rho)+1$ such that the i-th loop has length $2\rho_i$ for $i\le\ell(\rho)$ and the $\ell(\rho)+1$-th loop is the one containing $n+1$. As before, the graph $G_{(n+1\ j)\sigma}$ is obtained from G_σ by replacing edges (j,d_j) and $(n+1,d_{n+1})$ by (j,d_{n+1}) and $(n+1,d_j)$. We distinguish three cases:

"join": If j lies in the i-th loop of G_σ, then $G_{(n+1\ j)\sigma}$ is obtained from G_σ by erasing its i-th and $\ell(\rho)+1$-th loops and replacing them by a loop of size $2(\rho_i+m)$ (see figure 2). In this case, $(n+1\ i)\sigma$ has coset-type $\rho\backslash\rho_j\cup(\rho_j+m)$.

As there is $2\rho_i$ elements in the i-th loop of G_σ, one obtains:

$$\sum_{\substack{j=1,\bar 1,\dots,n,\bar n\\ j\nsim_{G_\sigma} n+1}} b^{k-1}_{\text{coset-type}((n+1\ i)\sigma')} = 2\sum_{1\le i\le\ell(\rho)}\rho_i b^{k-1}_{\rho\backslash(\rho_i)\cup(\rho_i+m)},$$

where $j\nsim_{G_\sigma} n+1$ means that j and $n+1$ lie in different loops of G_σ.

"twist": If j lies in the $\ell(\rho)+1$-th loop of G_σ and if the distance between j and $n+1$ is odd, then $G_{(n+1\ i)\sigma}$ is obtained from G_σ by the transformation drawn on figure 3. In particular, in this case, $(n+1\ j)\sigma$ has the same coset-type as σ, that is $\rho\cup(m)$.

As j can not be equal to $\overline{n+1}$, there is $m-1$ possible values for j in this case. Thus,

$$\sum_{\substack{j=1,\bar 1,\dots,n,\bar n\\ j\sim_{G_\sigma} n+1\\ d_G(j,n+1)\text{ odd}}} b^{k-1}_{\text{coset-type}((n+1\ j)\sigma')} = (m-1)b^{k-1}_{\rho\cup(m)}.$$

"cut": We consider now the case where j lies in the $\ell(\rho)+1$-th loop of G_σ and the distance between j and $n+1$ is even. We choose an arbitrary orientation of the $\ell(\rho)+1$-th loop of G_σ (we keep the same for all j in this situation) and we denote $2h$ ($1\le h\le m-1$) the distance between $n+1$ and j when following the loop along this direction. Then $G_{(n+1\ j)\sigma}$ is obtained from G_σ by erasing its $\ell(\rho)+1$-th loop and replacing it by two loops of length $2h$ and $2(m-h)$.(see figure 4). Thus, in this case, $(n+1\ j)\sigma$ has coset-type $\rho\cup(h,m-h)$.

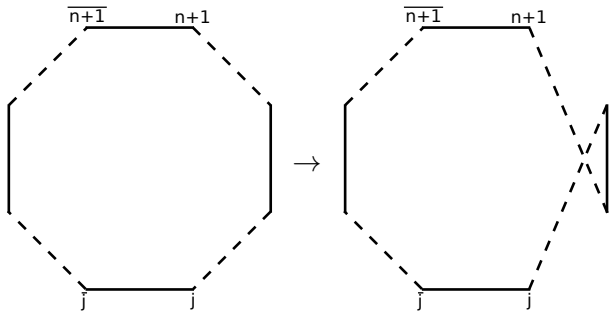

FIGURE 3. G_σ and $G_{(n+1\ j)\sigma}$ in the "twist" case

There is exactly one integer j for each integer h between 1 and $m-1$, so:

$$\sum_{\substack{j=1,\bar{1},\dots,n,\bar{n}\\ j\sim_{G_\sigma} n+1\\ d_G(j,n+1)\text{ even}}} b^{k-1}_{\text{coset-type}((n+1\ j)\sigma')} \quad = \quad \sum_{h=1}^{m-1} b^{k-1}_{\rho\cup(h,m-h)} \quad = \quad \sum_{\substack{r+s=m\\ r,s\geq 1}} b^{k-1}_{\rho\cup(r,s)}$$

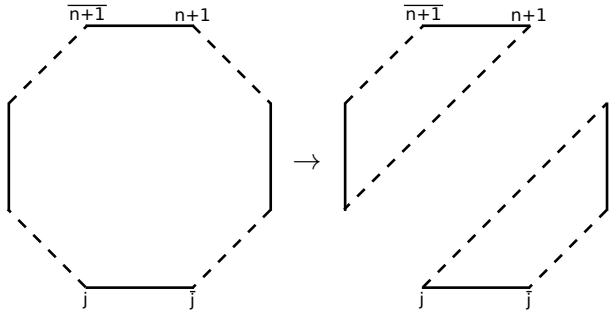

FIGURE 4. G_σ and $G_{(n+1\ j)\sigma}$ in the "cut" case

Putting the different cases together, one has

$$b^k_{\rho\cup(m)} \;=\; 2\sum_{1\leq i\leq \ell(\rho)} \rho_i b^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} \;+\; \sum_{\substack{r+s=m\\ r,s\geq 1}} b^{k-1}_{\rho\cup(r,s)} \;+\; (m\,-\,1) b^{k-1}_{\rho\cup(m)},$$

which is exactly what we wanted to prove. □

Remark 3.6. As in section 2, define coefficients d^k_ρ as solution of the sparse triangular system

$$b^k_\rho = \sum_{i=0}^{m_1(\rho)} d^k_{\bar{\rho}\cup 1^i} \binom{m_1(\rho)}{i}. \tag{25}$$

Then, for a given k, only finitely many d_ρ^k are non-zero. But, unfortunately, we have no combinatorial interpretation in this case to obtain directly induction relation on d. This raises the question of the existence of a partial Hecke algebra of (S_{2n}, H_n), out of the scope of this article.

A result similar to Theorem 3.5 could be obtained for power sum symmetric functions.

3.4. **Subleading term.** The induction relation proved in the previous paragraph is a good tool to study the leading and subleading terms of $h_k(J_1^{(2)}, \dots, J_n^{(2)})p_n$, that is the coefficients b_ρ^k with $|\rho| - \ell(\rho) = k$ or $k-1$. Indeed, an immediate induction shows that if the degree condition $|\rho| - \ell(\rho) \leq k$ is not satisfied, then $b_\rho^k = 0$. We can also recover the following result proved by S. Matsumoto [Mat10, Theorem 5.4].

Proposition 3.7. *If ρ is a partition and k an integer such that $|\rho| - \ell(\rho) = k$, then*

$$b_\rho^k = \prod \mathrm{Cat}_{\rho_i - 1}\,.$$

But our induction allows us to go further and to compute the subleading term (case $|\rho| - \ell(\rho) = k-1$), proving this way a conjecture of S. Matsumoto [Mat10, Conjecture 9.4] corresponding to the case where ρ is a hook.

Before stating and proving our result (in paragraph 3.4.2), we need a few definitions and basic lemmas on the total area of Dyck paths (paragraph 3.4.1).

3.4.1. *Area of Dyck paths.*

Definition 3.8. If $I = (i_1, \dots, i_r)$ is a weak composition (*i.e.* a sequence of non-negative integers), let us define $\mathcal{P}_I$ as the set of Dyck paths of length $k = i_1 + \cdots + i_r$ whose height after i_1, $i_1 + i_2$, ... steps is zero (such a path is the concatenation of Dyck paths of lengths i_1, i_2,...).

If C is a subset of Dyck paths of a given length, denote by $\mathfrak{A}_C$ the sum over the paths c in C of the area $\mathfrak{A}_c$ under c. In the case $C = \mathcal{P}_I$, we shorten the notation and denote $\mathfrak{A}_I = \mathfrak{A}_{\mathcal{P}_I}$.

For a weak composition $I = (k)$ of length 1, the set $\mathcal{P}_I$ is the set of all Dyck paths of lengths k. In this case, the area $\mathfrak{A}_k$ has a closed form, which has been computed by D. Merlini, R. Sprugnoli, and M. C. Verri in [MSV96]:

$$\mathfrak{A}_k = 4^k - \binom{2k+1}{k}.$$

The general case can be deduced easily, thanks to the following lemma:

Lemma 3.9. *Let C_1 and C_2 be respectively subsets of the set of Dyck paths of length $2m$ and $2n$. Define $C \simeq C_1 \times C_2$ the set of Dyck paths of length $2(m+n)$ which are the concatenation of a path in C_1 and a path in C_2. Then*

$$\mathfrak{A}_C = \mathfrak{A}_{C_1} \cdot |C_2| + |C_1| \cdot \mathfrak{A}_{C_2}.$$

Proof. The area between a concatenation $c_1 \cdot c_2$ of two Dick paths c_1 and c_2 is clearly equal to the sum of the ares under c_1 and c_2. Therefore:

$$\begin{aligned}\mathfrak{A}_C = \sum_{\substack{c_1 \in C_1 \\ c_2 \in C_2}} \mathfrak{A}_{c_1} + \mathfrak{A}_{c_2} &= \sum_{\substack{c_1 \in C_1 \\ c_2 \in C_2}} \mathfrak{A}_{c_1} + \sum_{\substack{c_1 \in C_1 \\ c_2 \in C_2}} \mathfrak{A}_{c_2} \\ &= |C_2| \sum_{c_1 \in C_1} \mathfrak{A}_{c_1} + |C_1| \sum_{c_2 \in C_2} \mathfrak{A}_{c_2} = \mathfrak{A}_{C_1} \cdot |C_2| + |C_1| \cdot \mathfrak{A}_{C_2}. \quad \square\end{aligned}$$

With an immediate induction, we obtain the following corollary.

Corollary 3.10. *For any weak composition I of length r, one has:*

$$\mathfrak{A}_I = \sum_{j=1}^{r} \mathfrak{A}_{i_j} \prod_{k \neq j} \mathrm{Cat}_{i_k}.$$

One will also need the following induction relation in the next paragraph.

Lemma 3.11 (Merlini, Sprugnoli and Verri [MSV96])**.** *If m is a positive integers, one has:*

$$\mathfrak{A}_{m-1} = (m-1)\,\mathrm{Cat}_{m-1} + \sum_{\substack{r+s=m \\ r,s \geq 1}} (\mathfrak{A}_{r-1}\,\mathrm{Cat}_{s-1} + \mathfrak{A}_{s-1}\,\mathrm{Cat}_{r-1}).$$

Proof. This is a consequence of the usual first return decomposition of Dyck paths. Indeed, let c be a Dyck path of length $2(m-1)$. We denote $2r$ the x-coordinate of the first point where the path touches the x-axis and $s = m - r$. Then c is the concatenation of one climbing step, a Dyck path c_1 of length $2(r-1)$, a down step and a Dyck path c_2 of length $2(s-1)$ and this decomposition is of course bijective.

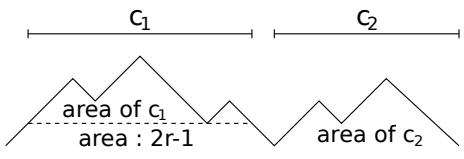

FIGURE 5. First-passage decomposition of a Dyck path

The area under c is the sum of the areas under c_1 and c_2, plus $2r-1$ (see figure 5). So we write:

$$\begin{aligned}\mathfrak{A}_{m-1} &= \sum_{\substack{r+s=m\\ r,s\geq 1}} \left[\sum_{\substack{c_1\in\mathcal{P}_{r-1}\\ c_2\in\mathcal{P}_{s-1}}} \mathfrak{A}_{c_1}+\mathfrak{A}_{c_2}+(2r-1)\right]\\ &= \sum_{\substack{r+s=m\\ r,s\geq 1}} \left[\sum_{\substack{c_1\in\mathcal{P}_{r-1}\\ c_2\in\mathcal{P}_{s-1}}} \mathfrak{A}_{c_1} + \sum_{\substack{c_1\in\mathcal{P}_{r-1}\\ c_2\in\mathcal{P}_{s-1}}} \mathfrak{A}_{c_2} + \sum_{\substack{c_1\in\mathcal{P}_{r-1}\\ c_2\in\mathcal{P}_{s-1}}} (2r-1)\right]\\ = \sum_{\substack{r+s=m\\ r,s\geq 1}} &\left[|\mathcal{P}_{s-1}| \sum_{c_1\in\mathcal{P}_{r-1}} \mathfrak{A}_{c_1} + |\mathcal{P}_{r-1}| \sum_{c_2\in\mathcal{P}_{s-1}} \mathfrak{A}_{c_2} + |\mathcal{P}_{s-1}|\cdot|\mathcal{P}_{r-1}|\cdot(2r-1)\right]\\ &= \sum_{\substack{r+s=m\\ r,s\geq 1}} \left[\mathfrak{A}_{r-1}\operatorname{Cat}_{s-1} + \mathfrak{A}_{s-1}\operatorname{Cat}_{r-1} + (2r-1)\operatorname{Cat}_{s-1}\operatorname{Cat}_{r-1}\right].\end{aligned}$$

The last part of the sum may be symmetrized in r and s:

$$\begin{aligned}\sum_{\substack{r+s=m\\ r,s\geq 1}} (2r-1)\operatorname{Cat}_{s-1}\operatorname{Cat}_{r-1} &= \sum_{\substack{r+s=m\\ r,s\geq 1}} \frac{1}{2}(2r-1+2s-1)\operatorname{Cat}_{s-1}\operatorname{Cat}_{r-1}\\ &= (m-1)\sum_{\substack{r+s=m\\ r,s\geq 1}} \operatorname{Cat}_{s-1}\operatorname{Cat}_{r-1} = (m-1)\operatorname{Cat}_{m-1},\end{aligned}$$

which ends the proof of the lemma. □

3.4.2. *Proof of a conjecture of Matsumoto.* Computing the subleading term of $h_k(J_1^{(2)},\dots,J_n^{(2)})\cdot p_n$ consists in computing the coefficient b_μ^k with $k=|\mu|-\ell(\mu)+1$. Therefore, for a partition μ, we denote:

$$\mathrm{SD}_\mu = b_\mu^{|\mu|-\ell(\mu)+1}.$$

Theorem 3.12. *Let μ be a partition. Then*

$$SD_\mu = \mathfrak{A}_{\mu-1}.$$

Proof. Let μ be a partition and $k=|\mu|-\ell(\mu)+1$.

Suppose that we write $\mu=\rho\cup(m)$, for some partition ρ and integer m. We will write Theorem 3.5 for ρ and m.

(26)
$$b^k_{\rho\cup(m)} = \delta_{m,1}b^k_\rho + 2\sum_{1\leq i\leq\ell(\rho)} \rho_i b^{k-1}_{\rho\setminus(\rho_i)\cup(\rho_i+m)} + \sum_{\substack{r+s=m\\ r,s\geq 1}} b^{k-1}_{\rho\cup(r,s)} + (m-1)b^{k-1}_{\rho\cup(m)}.$$

If $m=1$, the partition ρ fulfills

$$|\rho|-\ell(\rho) = (|\mu|-1)-(\ell(\mu)-1) = k-1$$

and thus $b^k_\rho = \mathrm{SD}_\rho$.

For any $i \leq \ell(\rho)$, the partition $\lambda = \rho \setminus (\rho_i) \cup (\rho_i + m)$ fulfills

$$|\lambda| - \ell(\lambda) = |\mu| - (\ell(\mu) - 1) = k$$

and therefore, thanks to the degree condition, $b_\lambda^{k-1} = 0$.

In a similar way, as $|\mu| - \ell(\mu) = k - 1$, the coefficient b_μ^{k-1} is simply given by Proposition 3.7:

$$b_{k-1}^{\mu} = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i - 1} = \mathrm{Cat}_{m-1} \prod_{i=1}^{\ell(\rho)} \mathrm{Cat}_{\rho_i - 1}\,.$$

For the last term, for any $r, s \geq 1$ with $r + s = m$, the partition $\lambda = \rho \cup (r, s)$ fulfills:

$$|\lambda| - \ell(\lambda) = |\mu| - (\ell(\mu) + 1) = k - 2.$$

Therefore, the coefficient b_λ^{k-1} corresponds to a subleading term and $b_\lambda^{k-1} = \mathrm{SD}_\lambda$.

Finally, Theorem 3.5 becomes in this case:

$$SD_{\rho \cup (m)} = \delta_{m,1} SD_\rho + (m-1)\,\mathrm{Cat}_{m-1} \prod_{i=1}^{\ell(\rho)} \mathrm{Cat}_{\rho_i - 1} + \sum_{\substack{r,s \geq 1 \\ r+s=m}} \mathrm{SD}_{\rho \cup (r,s)} \tag{27}$$

This equation gives an induction relation on the coefficient SD_ρ. We will prove that $\mathrm{SD}_\mu = \mathfrak{A}_{\mu - 1}$ by a double induction, first on the size n of the partition μ and then on the smallest part of μ

For $n = 1$, one has only partition $\mu = (1)$ and $\mathrm{SD}_{(1)} = b_{(1)}^1 = 0 = \mathfrak{A}_0$.

Fix now some $n > 1$ and suppose that the theorem is true for all partitions of size smaller than n.

If $\mu = \rho \cup (1)$ is a partition of n with smallest part equal to 1, then, by equation (27) and the induction hypothesis, one has:

$$\mathrm{SD}_\mu = \mathrm{SD}_\rho = \mathfrak{A}_{\rho - 1} = \mathfrak{A}_{\mu - 1}.$$

Let μ be a partition of n with smallest part $m > 1$ and suppose that $\mathrm{SD}_\mu = \mathfrak{A}_{\mu-1}$ for all partitions of n with smallest part $m' < m$. We write $\mu = \rho \cup (m)$ (*i.e.* $\rho = \mu \backslash m$). By equation (27),

$$\mathrm{SD}_\mu = (m-1) \prod \mathrm{Cat}_{\rho_i - 1} + \sum_{\substack{r+s=m \\ r,s \geq 1}} \mathrm{SD}_{\rho \cup (r,s)}.$$

By induction,

$$\begin{aligned} \mathrm{SD}_{\rho \cup (r,s)} = \mathfrak{A}_{(\rho \cup (r,s)) - 1} = \mathrm{Cat}_{r-1}\,\mathrm{Cat}_{s-1} \left(\sum_i \mathfrak{A}_{\rho_i - 1} \prod_{j \neq i} \mathrm{Cat}_{\rho_j - 1} \right) \\ + \mathfrak{A}_{s-1}\,\mathrm{Cat}_{r-1} \prod_i \mathrm{Cat}_{\rho_i - 1} + \mathfrak{A}_{r-1}\,\mathrm{Cat}_{s-1} \prod_i \mathrm{Cat}_{\rho_i - 1}\,. \end{aligned}$$

If we make the substitution in the previous equation, we obtain:

$$\mathrm{SD}_\mu = \left(\sum_{\substack{r,s\geq 1 \\ r+s=m}} \mathrm{Cat}_{r-1}\,\mathrm{Cat}_{s-1} \right) \left(\sum_i \mathfrak{A}_{\rho_i - 1} \prod_{j\neq i} \mathrm{Cat}_{\rho_j - 1} \right)$$
$$+ \left((m-1)\,\mathrm{Cat}_{m-1} + \sum_{\substack{r+s=m \\ r,s\geq 1}} \left[\mathfrak{A}_{s-1}\,\mathrm{Cat}_{r-1} + \mathfrak{A}_{r-1}\,\mathrm{Cat}_{s-1} \right] \right) \prod_i \mathrm{Cat}_{\rho_i - 1}\,.$$

Therefore, using both Lemma 3.11 and the classical induction on Catalan number $\sum_{r+s=m} \mathrm{Cat}_{r-1}\,\mathrm{Cat}_{s-1} = \mathrm{Cat}_{m-1}$, one has:

$$\mathrm{SD}_\mu = \sum_i \mathfrak{A}_{\mu_i - 1} \prod_{j\neq i} \mathrm{Cat}_{\mu_j - 1} = \mathfrak{A}_\mu.$$

Finally, for any partition μ, one has $\mathrm{SD}_\mu = \mathfrak{A}_\mu$, which is exactly what we wanted to prove. □

S. Matsumoto established a deep connection between the coefficients b_μ^k and the asymptotic expansion of orthogonal Weingarten functions [Mat10, Theorem 7.3]. In particular, Theorem 3.12 gives the subleading term of some matrix integrals over orthogonal group when the dimension of the group goes to infinity.

4. Towards a continuous deformation?

The questions studied in sections 2 and 3 may seem quite different at first sight but there exists a continuous deformation from one to the other.

We denote by $\mathcal{Y}_n$ the set of all Young diagrams (or partition) of size n. For any $\alpha > 0$, we consider two families of functions on $\mathcal{Y}_n$.

- First, we call α-content of a box of the Young diagram λ the quantity $\alpha(j-1) - (i-1)$, where i is its row index and j its column index. If $A_\lambda^{(\alpha)}$ stands for the multiset of the α-contents of boxes ofλ, one can look at the evaluation of complete symmetric functions $h_k(A_\lambda^{(\alpha)})$.
- Second, we consider Jack polynomials, which is the basis of symmetric function ring indexed by partitions and depending of a parameter α (they are deformations of Schur functions). The expansion of Jack polynomials on the power sum basis

  $$J_\lambda^{(\alpha)} = \sum_\mu \Theta_\mu^{(\alpha)}(\lambda) p_\mu$$

  defines a family $\Theta_\mu^{(\alpha)}$ of functions on $\mathcal{Y}_n$ (we use the same normalization and notation as in [Mac95, Chapter 6] for Jack polynomials).

Proposition 4.1. *The functions $\Theta_\mu^{(\alpha)}$, when μ runs over the partitions of n, form a basis of the algebra Z_n of functions over $\mathcal{Y}_n$.*

Proof. As the cardinal of this family corresponds to the dimension of the space, it is enough to prove that it spans Z_n. Let f be a function on $\mathcal{Y}_n$.

For a fixed α, Jack polynomials form a basis of symmetric functions, therefore there exist some coefficients $d^{(\alpha)}_{\mu,\lambda}$ such that:

$$p_\mu = \sum_\lambda d^{(\alpha)}_{\mu,\lambda} J^{(\alpha)}_\lambda.$$

Let us define the scalar:

$$c_\mu = \sum_\lambda d^{(\alpha)}_{\mu,\lambda} f(\lambda).$$

Then one has:

$$\sum_\mu c_\mu \Theta^{(\alpha)}_\mu(\lambda) = \sum_{\mu,\nu} \left(d^{(\alpha)}_{\mu,\nu} \Theta^{(\alpha)}_\mu(\lambda) \right) f(\nu) = f(\lambda),$$

where the last equality comes from the fact that the matrices $(\Theta^{(\alpha)}_\mu(\lambda))$ and $(d^{(\alpha)}_{\mu,\lambda})$ are by definition inverse of each other.

Finally, any function f on $\mathcal{Y}_n$ can be written as a linear combination of $\Theta^{(\alpha)}_\mu$. □

Remark 4.2. This proposition is also a consequence of the fact that suitably chosen normalizations of $\Theta^{(\alpha)}_\mu$, when μ runs over all partitions, form a linear basis of the algebra of α-shifted symmetric functions (see [Las08b, Section 3]). However, such a sophisticated tool is not needed when n is fixed.

The proposition implies the existence of some coefficients $a^{k,(\alpha)}_\mu$ such that:

$$h_k(A^{(\alpha)}_\lambda) = \sum_\mu a^{k,(\alpha)}_\mu \Theta_\mu(\lambda),$$

For $\alpha = 1$, using the action of Jucys-Murphy element on the Young basis [Juc66] and the discrete Fourier transform of S_n, one can see that $a^{k,(1)}_\mu = a^k_\mu$.

For $\alpha = 2$, using the identification between Jack polynomials for this special value of the parameter and zonal polynomials for the Gelfand pair (S_{2n}, H_n) [Mac95, Chapter 7], as well as the spherical expansion of $h_k(J^{(2)}_1, \dots, J^{(2)}_n)p_n$ established by S. Matsumoto [Mat10, Theorem 4.1], one has $a^{k,(2)}_\mu = b^k_\mu$.

It is natural to wonder if there are results similar to Theorems 2.6 and 3.5 in the general setting. Computer exploration using Sage [S$^+$10] leads to the following conjecture:

Conjecture 4.3. *The coefficients $a^{k,(\alpha)}_\rho$ fulfill the linear relation: for any $m \geq 2$,*

$$a^{k,(\alpha)}_{\rho\cup(m)} = \sum_{\substack{r+s=m \\ r,s\geq 1}} a^{k-1,(\alpha)}_{\rho\cup(r,s)} + \alpha \sum_{1\leq i\leq \ell(\rho)} r a^{k-1,(\alpha)}_{\rho\setminus\rho_i\cup(\rho_i+m)} + (\alpha-1)\cdot(m-1)\, a^{k-1,(\alpha)}_{\rho\cup(m)}.$$

Unfortunately, as we do not have a combinatorial description of the basis $\theta^{(\alpha)}_\mu$ in the algebra Z_n, we are not able to prove it. With Lassalle's algebraic approach, one can prove a generalization of Theorem 2.4 (see [Las10, Section 11]) which

is weaker than Conjecture 4.3. Nevertheless, his formula is sufficient to compute inductively the $a_\rho^{k,(\alpha)}$ and has been used in our numerical exploration.

In the author's opinion, this conjecture is a hint towards the existence of combinatorial constructions for other values of the parameter α (like the conjectures of papers [GJ96, Las08b, Las09]).

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