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On the cube of the equivariant linking pairing for knots and 3–manifolds of rank one

Christine Lescop *

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Abstract

Let M be a closed oriented 3-manifold with first Betti number one. Its equivariant linking pairing may be seen as a two-dimensional cohomology class in an appropriate infinite cyclic covering of the space of ordered pairs of distinct points of M. We show how to define the equivariant cube $\mathcal{Q}(\mathbb{K})$ of this Blanchfield pairing with respect to a framed knot $\mathbb{K}$ that generates $H_1(M;\mathbb{Z})/\text{Torsion}$.

This article is devoted to the study of the invariant $\mathcal{Q}$. We prove many properties for this invariant including two surgery formulae.

Via surgery, the invariant $\mathcal{Q}$ is equivalent to an invariant $\hat{\mathcal{Q}}$ of null-homologous knots in rational homology spheres, that coincides with the two-loop part of the Kricker rational lift of the Kontsevich integral, at least for knots with trivial Alexander polynomial in integral homology spheres.

The invariant $\mathcal{Q}$ takes its values in a polynomial ring R over $\mathbb{Q}$. We determine the rational sub-vector space of R generated by the variations $(\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}))$ for two framed knots that generate $H_1(M;\mathbb{Z})/\text{Torsion}$, and we show that the invariant $\overline{\mathcal{Q}}(M)$ of M, that is the class of $\mathcal{Q}(\mathbb{K})$ in the quotient of R by this subspace, detects the connected sums with rational homology spheres with non trivial Casson-Walker invariant, when the Alexander polynomial of M has no multiple roots. Conjecturally, the invariant $\overline{\mathcal{Q}}(M)$ refines the two-loop part of an invariant recently defined by Ohtsuki.

Keywords: configuration space integrals, finite type invariants of knots and 3-manifolds, homology spheres, two-loop polynomial, rational lift of Kontsevich integral, equivariant Blanchfield linking pairing, Casson-Walker invariant, LMO invariant, clasper calculus, Jacobi diagrams, perturbative expansion of Chern-Simons theory, surgery formula.
MSC: 57M27 57N10 57M25 55R80 57R20 57R56 57R91

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1 Introduction

1.1 Short introduction

The study of 3–manifold invariants built from integrals over configuration spaces started after the work of Witten on Chern-Simons theory in 1989 [Wi], with work of Axelrod, Singer [AS1, AS2], Kontsevich [Ko], Bott, Cattaneo [BC1, BC2], Taubes [T].

The simplest non-trivial of these invariants is an invariant of homology 3-spheres N (equipped with an appropriate trivialisation) that may be written as the integral of the cube of a closed 2-form ω over a configuration space $C_2(N)$, where the cohomology class of ω represents the linking form of N. This invariant is associated to the θ–graph, G. Kuperberg and D. Thurston proved that it is the Casson invariant [KT].

We shall present a similar invariant for closed oriented 3-manifolds with first Betti number one, in an equivariant setting, already investigated by Julien Marché in [Ma] in the knot case.

Let M be such a closed oriented 3-manifold with first Betti number one. Its equivariant linking pairing may be seen as a two-dimensional cohomology class in an appropriate infinite cyclic covering of the space of ordered pairs of distinct points of M. We shall show how to define the equivariant cube $\mathcal{Q}(\mathbb{K})$ of this Blanchfield pairing with respect to a framed knot $\mathbb{K}$ that generates $H_1(M;\mathbb{Z})/\text{Torsion}$. We show many properties for the invariant $\mathcal{Q}$ including two surgery formulae.

Via surgery, the invariant $\mathcal{Q}$ is equivalent to an invariant $\hat{\mathcal{Q}}$ of null-homologous knots in rational homology spheres. For a knot $\hat{K}$ with trivial Alexander polynomial in an integral homology sphere, one of our surgery formulae combined with results of Garoufalidis and Rozansky [GR] shows that $\hat{\mathcal{Q}}(\hat{K})$ coincides with the two–loop part of the Kricker rational lift of the Kontsevich integral of $\hat{K}$. This lift was defined in [Kr, GK], from the LMO invariant of Le, Murakami and Ohtsuki [LMO], following conjectures of Rozansky [Ro], who may have had in mind the construction that is presented in this article. Its two-loop part is also called the two-loop polynomial (up to normalization). It was introduced by Rozansky in [Ro, Section 6] and has been extensively studied by Ohtsuki in [O1].

I think that $\hat{\mathcal{Q}}$ is always equivalent to the two–loop polynomial in the sense that if two knots with equivalent equivariant linking pairings are distinguished by one of these two invariants, then they are distinguished by the other one.

The invariant $\mathcal{Q}$ takes its values in a polynomial ring R over $\mathbb{Q}$. We determine the rational subvector space of R generated by the variations $(\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}))$ for pairs of framed knots that generate $H_1(M;\mathbb{Z})/\text{Torsion}$, and we define an invariant $\overline{\mathcal{Q}}(M)$ of M that is the class of $\mathcal{Q}(\mathbb{K})$ in the quotient of R by this subspace.

Conjecturally, the invariant $\overline{\mathcal{Q}}(M)$ refines the two-loop part of the invariant recently defined by Ohtsuki using the LMO invariant in [O3]. It detects connected sums with rational homology spheres with non trivial Casson-Walker invariant, when the Alexander polynomial of M has no multiple roots.

The constructions contained in this article apply to the higher loop degree case, and allow

us to define an invariant conjecturally equivalent to the whole rational lift of the Kontsevich integral for null-homologous knots in rational homology spheres. This is discussed in [L4].

I started to work on this project after a talk of Tomotada Ohtsuki at a workshop at the CTQM in Århus in Spring 2008. I wish to thank Joergen Andersen and Bob Penner for organizing this very stimulating meeting. Theorem 13.1 answers a question that George Thompson asked me at the conference *Chern-Simons Gauge theory : 20 years after, Hausdorff center for Mathematics* in Bonn in August 2009. I thank him for asking and I thank the organizers Joergen Andersen, Hans Boden, Atle Hahn, Benjamin Himpel of this great conference.

1.2 On the equivariant configuration space $\tilde{C}_2(M)$

Consider a closed oriented 3-manifold M with first Betti number one, and its standard infinite cyclic covering $\tilde{M}$. Let θ_M denote one of the two homeomorphisms that generate the covering group of $\tilde{M}$, and let $\widetilde{M^2}$ be the quotient of $\tilde{M}^2$ by the equivalence relation that identifies (p,q) to $(\theta_M(p),\theta_M(q))$. It is an infinite cyclic covering of M^2 whose covering group is generated by θ that acts as follows on the equivalence class $\overline{(p,q)}$ of (p,q)

$$\theta\overline{(p,q)} = \overline{(\theta_M(p),q)} = \overline{(p,\theta_M^{-1}(q))}.$$

The diagonal of $\tilde{M}^2$ projects to a preferred lift of the diagonal of M^2 in $\widetilde{M^2}$.

The configuration space $C_2(M)$ is the compactification of $(M^2 \setminus \text{diagonal})$ that is obtained from M^2 by *blowing-up* the diagonal of M^2 in the following sense: The diagonal of M^2 is replaced by the total space of its unit normal bundle. Thus, $C_2(M)$ is diffeomorphic to the complement of an open tubular neighborhood of the diagonal of M^2.

The normal bundle (TM^2/diag) of the diagonal of M^2 in M^2 is identified to the tangent bundle TM of M via $((x,y)\mapsto y-x)$. Therefore, $\partial C_2(M)$ is canonically identified to the unit tangent bundle $ST(M)$ of M.

The *equivariant configuration space* $\tilde{C}_2(M)$ is the $\mathbb{Z}$-covering of $C_2(M)$ that is obtained from $\widetilde{M^2}$ by blowing-up all the lifts of the diagonal of M^2 as above.

$$\partial\tilde{C}_2(M) = \mathbb{Z}\times ST(M)$$

where $n\times ST(M)$ stands for $\theta^n(ST(M))$ and $ST(M)$ is the preimage of the preferred lift of the diagonal under the blow-up map.

The configuration space $\tilde{C}_2(M)$ is a smooth 6–dimensional manifold in which we can define the equivariant algebraic intersections of two chains C and A, with complementary dimensions (i.e. the sum of dimensions is 6), whose projections in $C_2(M)$ are transverse, as follows:

$$\langle C,A\rangle_e = \sum_{i\in\mathbb{Z}} \langle \theta^{-i}(C),A\rangle_{\tilde{C}_2(M)} t^i \in \mathbb{Q}[t^{\pm 1}]$$

where $\langle\cdot,\cdot\rangle$ stands for the algebraic intersection and a c-dimensional *chain* is a linear combination of oriented compact smooth c-submanifolds with boundaries and corners. Definitions and conventions about algebraic intersections are recalled in Subsection 3.1.

We can similarly define the equivariant algebraic triple intersection of 3 codimension 2 rational chains C_X, C_Y, C_Z of $\tilde{C}_2(M)$, whose projections in $C_2(M)$ are transverse, as the following polynomial in $\mathbb{Q}[y^{\pm 1}, z^{\pm 1}]$:

$$\langle C_X, C_Y, C_Z \rangle_e = \sum_{(i,j)\in\mathbb{Z}^2} \langle C_X, \theta^{-i}(C_Y), \theta^{-j}(C_Z) \rangle_{\tilde{C}_2(M)} y^i z^j \in \mathbb{Q}[y^{\pm 1}, z^{\pm 1}].$$

1.3 Some other notation

In this article, all manifolds are oriented.

- Boundaries are oriented by the *outward normal first* convention.
- K is a knot in M whose homology class generates $H_1(M;\mathbb{Z})/\text{Torsion}$ and acts on $\tilde{M}$ by θ_M.
- S is a closed surface of M that intersects K transversally at one point with a positive sign.
- Unless otherwise mentioned, homology coefficients are in $\mathbb{Q}$.
- $\Delta = \Delta(M)$ is the Alexander polynomial of M and $\delta = \delta(M)$ is the annihilator of $H_1(\tilde{M})$. (The polynomial $\delta(M)$ divides $\Delta(M)$ and it has the same roots.) These polynomials are normalized so that $\Delta(1) = \delta(1) = 1$, $\Delta(t_M) = \Delta(t_M^{-1})$, and $\delta(t_M) = \delta(t_M^{-1})$. The definitions of Δ and δ are recalled in Lemma 2.2.
- $$I_\Delta = I_\Delta(t) = \frac{1+t}{1-t} + \frac{t\Delta'(M)(t)}{\Delta(M)(t)}.$$
- $$R_\delta = \frac{\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}, \frac{1}{\delta(x)}, \frac{1}{\delta(y)}, \frac{1}{\delta(z)}]}{(xyz = 1)}.$$
- A *rational homology sphere* or $\mathbb{Q}$*–homology sphere* is a closed 3-manifold with the same rational homology as the standard sphere S^3.
- The Casson-Walker invariant will be denoted by λ and normalized like the Casson invariant in [AM, GM, M]. If λ_W denotes the Walker invariant normalized as in [W], then $\lambda = \frac{\lambda_W}{2}$.

1.4 Definition of the invariant $\mathcal{Q}(\mathbb{K})$

The invariant that we shall study in this article is defined by the following theorem that is a direct consequence of Propositions 5.2, 5.8, 12.1 and 12.7.

Theorem 1.1 *Let $\tau : TM \to M \times \mathbb{R}^3$ be a trivialisation of TM and let $p_1(\tau)$ be its first Pontrjagin class ($p_1(\tau)$ is an integer whose definition is recalled in Subsection 5.5). Assume that τ maps oriented unit tangent vectors of K to some fixed $W \in S^2$. Then τ induces a parallelisation of K. Let $K_\parallel$ be a parallel of K with respect to this parallelisation. Let K_X, K_Y, K_Z be three disjoint parallels of K, on the boundary $\partial N(K)$ of a tubular neighborhood of K, that induce the same parallelisation of K as $K_\parallel$. Consider the continuous map*

$$\begin{array}{rll} \check{A}(K)\colon & (S^1 = [0,1]/(0 \sim 1)) \times [0,1] & \to \quad C_2(M) \\ & (t, u \in]0,1[) & \mapsto \quad (K(t), K(t+u)), \end{array}$$

and its lift $A(K)\colon S^1 \times [0,1] \to \tilde{C}_2(M)$ such that the lift of $(K(t), K(t+\varepsilon))$ is in a small neighborhood of the blown-up canonical lift of the diagonal, for a small positive ε. Let $A(K)$ also denote the 2–chain $A(K)(S^1 \times [0,1])$.

For $V \in S^2$, let $s_\tau(M;V) = \tau^{-1}(M \times V) \subset ST(M) \subset \partial\tilde{C}_2(M)$.

Let X, Y, Z be three distinct points in $S^2 \setminus \{W, -W\}$.

There exist three transverse rational 4–dimensional chains G_X, G_Y and G_Z of $\tilde{C}_2(M)$ whose boundaries are

$$\partial G_X = (\theta - 1)\delta(\theta)\left(s_\tau(M;X) - I_\Delta(\theta)ST(M)_{|K_X}\right),$$
$$\partial G_Y = (\theta - 1)\delta(\theta)\left(s_\tau(M;Y) - I_\Delta(\theta)ST(M)_{|K_Y}\right) \text{ and}$$
$$\partial G_Z = (\theta - 1)\delta(\theta)\left(s_\tau(M;Z) - I_\Delta(\theta)ST(M)_{|K_Z}\right)$$

and such that the following equivariant algebraic intersections vanish

$$\langle G_X, A(K)\rangle_e = \langle G_Y, A(K)\rangle_e = \langle G_Z, A(K)\rangle_e = 0.$$

Then

$$\mathcal{Q}(K, K_\parallel) = \frac{\langle G_X, G_Y, G_Z\rangle_e}{(x-1)(y-1)(z-1)\delta(x)\delta(y)\delta(z)} - \frac{p_1(\tau)}{4} \in R_\delta$$

only depends on the isotopy class of the knot K and on its parallelisation.

Fix

$$\mathbb{K} = (K, K_\parallel).$$

We shall furthermore show that

$$\delta(M)(x)\delta(M)(y)\delta(M)(z)\mathcal{Q}(\mathbb{K}) \in \frac{\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]}{(xyz = 1)}$$

in Proposition 12.1, and that

$$\mathcal{Q}(\mathbb{K})(x,y,z) = \mathcal{Q}(\mathbb{K})(y,x,z) = \mathcal{Q}(\mathbb{K})(z,y,x) = \mathcal{Q}(\mathbb{K})(x^{-1}, y^{-1}, z^{-1})$$

in Subsection 5.4. We shall also see that $\mathcal{Q}((S^1 \times W, S^1 \times W') \subset S^1 \times S^2) = 0$ in Remark 5.9.

Remark 1.2 In the definition of $\mathcal{Q}$ by the statement of Theorem 1.1, replace the assumption that the knots K_X, K_Y, K_Z are parallel to K by the assumption that these three knots are rationally homologous to K and that K_X, K_Y, K_Z and K are disjoint. This defines an invariant of the 4-uples $(K_X, K_Y, K_Z, \mathbb{K})$ that could be interesting to study. See Subsection 5.2.

1.5 On the equivariant linking number and its cube

By definition, the *equivariant linking number* $lk_e(\alpha, \beta)$ of two cycles α and β in $\tilde{M}$ of respective dimensions $\dim(\alpha)$ and $\dim(\beta)$, such that

- $\dim(\alpha) + \dim(\beta) = 2$ and
- the projections of α and β in M do not intersect

is equal to the equivariant intersection

$$lk_e(\alpha, \beta) = \langle \alpha, B \rangle_{e,\tilde{M}} = \sum_{i \in \mathbb{Z}} \langle \theta_M^{-i}(\alpha), B \rangle_{\tilde{C}_2(M)} t_M^i \in \mathbb{Q}[t_M^{\pm 1}]$$

if $\beta = \partial B$. In general, $(\theta_M - 1)\delta(\theta_M)(\beta)$ bounds a chain $(t_M - 1)\delta(t_M)B$ and

$$lk_e(\alpha, \beta) = \frac{\langle \alpha, (t_M - 1)\delta(t_M)B \rangle_{e,\tilde{M}}}{(t_M^{-1} - 1)\delta(t_M^{-1})}$$

and, for a polynomial $P \in \mathbb{Q}[t_M, t_M^{-1}]$,

$$lk_e(P(\theta_M)(\alpha), \beta) = P(t_M) lk_e(\alpha, \beta) = lk_e(\alpha, P(\theta_M^{-1})(\beta)).$$

See Section 3 for more definitions and properties of lk_e.

Let the multiplication by t stand for morphisms induced by θ and allow division by polynomials in t. We shall see (Proposition 3.3 and Theorem 4.8) that the assumptions on

$$F_X = \frac{G_X}{(t-1)\delta(t)},$$

$F_Y = \frac{G_Y}{(t-1)\delta(t)}$ and $F_Z = \frac{G_Z}{(t-1)\delta(t)}$ in Theorem 1.1 imply that

$$lk_e(\alpha, \beta) = \langle \alpha \times \beta, F_X \rangle_e = \langle \alpha \times \beta, F_Y \rangle_e = \langle \alpha \times \beta, F_Z \rangle_e$$

so that the 4-dimensional chains F_X, F_Y, F_Z may be seen as three parallel representatives of the equivariant linking number, and $\mathcal{Q}(\mathbb{K})$ may be thought of as the cube of the equivariant linking number with respect to $\mathbb{K}$.

1.6 Introduction to the properties of $\mathcal{Q}$

According to Theorem 1.1, $\mathcal{Q}(\mathbb{K} \subset M)$ is an invariant of framed knots $\mathbb{K}$ generating $\frac{H_1(M;\mathbb{Z})}{\text{Torsion}}$.

Let $M_{\mathbb{K}}$ denote the rational homology sphere obtained from M by surgery along $\mathbb{K}$. This manifold is obtained from M by replacing a tubular neighborhood $N(K)$ of K by another solid torus $N(\hat{K})$ whose meridian is the given parallel of K. The core $\hat{K}$ of the new torus $N(\hat{K})$ is a null-homologous knot in $M_{\mathbb{K}}$. Since M is obtained from $M_{\mathbb{K}}$ by 0-surgery on $\hat{K}$, the data $(M, \mathbb{K})$ are equivalent to $(M_{\mathbb{K}}, \hat{K})$. In particular, the formula

$$\hat{\mathcal{Q}}(\hat{K} \subset M_{\mathbb{K}}) = \mathcal{Q}(\mathbb{K} \subset M)$$

defines an invariant of null-homologous knots $\hat{K}$ in rational homology spheres that will be shown to share a lot of properties with the two–loop polynomial. For example, it will be shown to have the same variation under a surgery on the Garoufalidis and Rozansky degree 2 null-claspers of [GR] in Lemma 9.3.

We shall see in Propositions 5.6 and 5.7 that $\mathcal{Q}(\mathbb{K})$ is independent of the orientation of K, and that

$$\mathcal{Q}(\mathbb{K} \subset -M) = -\mathcal{Q}(\mathbb{K} \subset M).$$

The following behaviour of $\mathcal{Q}$ under connected sum with a rational homology sphere will be established in Subsection 6.2.

Proposition 1.3 *Let N be a rational homology sphere, then*

$$\mathcal{Q}(\mathbb{K} \subset M \sharp N) = \mathcal{Q}(\mathbb{K} \subset M) + 6\lambda(N)$$

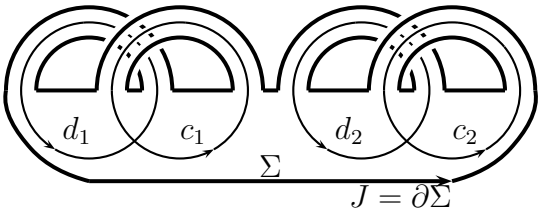
where $\sharp$ stands for the connected sum and λ is the Casson-Walker invariant.

The following property will be proved in Section 13.

$$\mathcal{Q}(\mathbb{K} \subset M)(1,1,1) = 6\lambda(M_{\mathbb{K}}).$$

The invariant $\mathcal{Q}$ satisfies the following surgery formula for surgeries on knots. The Dehn surgery formula of Theorem 1.4 will be proved in Subsection 9.3 from an LP surgery formula (Theorem 9.1) for Lagrangian-Preserving replacements of rational homology handlebodies, that generalize the null borromean surgeries. The LP surgery formula is precisely stated in Subsection 9.1. According to results of Garoufalidis and Rozansky [GR], it implies that $\hat{Q}$ lifts the (primitive) two-loop part of the Kontsevich integral for knots with trivial Alexander polynomials in integral homology spheres. See Theorem 9.2.

Theorem 1.4 *Let J be a knot of M that bounds a Seifert surface Σ disjoint from K such that $H_1(\Sigma)$ is mapped to 0 in $H_1(M)$.*
Let p/q be a nonzero rational number. Let $(c_i, d_i)_{i=1,\dots,g}$ be a symplectic basis of $H_1(\Sigma)$.



Let

$$\lambda'_e(J) = \frac{1}{12} \sum_{(i,j)\in\{1,\dots,g\}^2} \sum_{\mathfrak{S}_3(x,y,z)} \big(\alpha_{ij}(x,y) + \alpha_{ij}(x^{-1},y^{-1}) + \beta_{ij}(x,y)\big)$$

where

$$\alpha_{ij}(x,y) = lk_e(c_i,c_j^+)(x)lk_e(d_i,d_j^+)(y) - lk_e(c_i,d_j^+)(x)lk_e(d_i,c_j^+)(y),$$

$$\beta_{ij}(x,y) = \big(lk_e(c_i,d_i^+)(x) - lk_e(d_i^+,c_i)(x)\big)\big(lk_e(c_j,d_j^+)(y) - lk_e(d_j^+,c_j)(y)\big)$$

and $\sum_{\mathfrak{S}_3(x,y,z)}$ *stands for the sum over the* 6 *terms obtained by replacing* x *and* y *by their images under the* 6 *permutations of* $\{x,y,z\}$*, then*

$$\mathcal{Q}(\mathbb{K}\subset M(J;p/q)) - \mathcal{Q}(\mathbb{K}\subset M) = 6\frac{q}{p}\lambda'_e(J) + 6\lambda(S^3(U;p/q))$$

where $S^3(U;p/q)$ *is the lens space* $L(p,-q)$ *obtained from* S^3 *by* p/q*–surgery on the unknot* U*.*

Since $H_1(\Sigma)$ goes to 0 in $H_1(M)$ in the above statement, Σ lifts as homeomorphic copies of Σ in $\tilde{M}$ and $lk_e(c_i,c_j^+)$ denotes the equivariant linking number of a lift of c_i in $\tilde{M}$ in some lift of Σ and a lift of c_j^+ near the same lift of Σ. The superscript $+$ means that c_j is pushed in the direction of the positive normal to Σ.

When the above knot J is inside a rational homology ball, $\lambda'_e(J)$ coincides with $\frac{1}{2}\Delta''(J)$, where $\Delta(J)$ is the Alexander polynomial of J, and the right-hand side is nothing but 6 times the variation of the Casson-Walker invariant under a p/q–surgery on J. This is consistent with Proposition 1.3.

Recall $I_\Delta(t) = \frac{1+t}{1-t} + \frac{t\Delta'(t)}{\Delta(t)}$. The following theorem will be proved in Section 8.

Theorem 1.5 *Let* $\mathbb{K}'$ *be another framed knot of* M *such that* $H_1(M)/Torsion = \mathbb{Z}[K']$*. Then there exists an antisymmetric polynomial* $\mathcal{V}(\mathbb{K},\mathbb{K}')$ *in* $\mathbb{Q}[t,t^{-1}]$ *such that*

$$\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}) = \sum_{\mathfrak{S}_3(x,y,z)} \frac{\mathcal{V}(\mathbb{K},\mathbb{K}')(x)}{\delta(x)} I_\Delta(y).$$

Furthermore, for any $k\in\mathbb{Z}$*, there exists a pair of framed knots* $(\mathbb{K},\mathbb{K}')$ *such that* $\mathcal{V}(\mathbb{K},\mathbb{K}') = q(t^k - t^{-k})$ *for some nonzero rational number* q*.*

The following two propositions use the notation of Theorem 1.5. The first one is a direct consequence of Proposition 7.4, and the second one is proved right after Lemma 8.12.

Proposition 1.6 *If $\mathbb{K} = (K, K_\parallel)$ and if $\mathbb{K}' = (K, K'_\parallel)$, where $K'_\parallel$ is another parallel of K such that the difference $(K'_\parallel - K_\parallel)$ is homologous to a positive meridian of K in $\partial N(K)$, then*

$$\mathcal{V}(\mathbb{K}, \mathbb{K}')(t) = -\frac{\delta(t)}{2}\frac{t\Delta'(t)}{\Delta(t)}.$$

Proposition 1.7 *If K and K' coincide along an interval, if $(K' - K)$ bounds a surface B that lifts in $\tilde{M}$ such that $(K'_\parallel - K_\parallel)$ is homologous to a curve of B in the complement of $(\partial B \cup K)$ in a regular neighborhood of B, and if $(a_i, b_i)_{i\in\{1,\dots,g\}}$ is a symplectic basis of $H_1(B;\mathbb{Z})$, then*

$$\frac{\mathcal{V}(\mathbb{K}, \mathbb{K}')(t)}{\delta(t)} = \sum_{i=1}^{g}\left(lk_e(a_i, b_i^+) - \overline{lk_e(a_i, b_i^+)}\right).$$

1.7 The derived 3-manifold invariant

Definition 1.8 *Definition of an invariant for 3-manifolds of rank one:*
Let $Q_k(\delta, \Delta) = \sum_{\mathfrak{S}_3(x,y,z)} \frac{x^k - x^{-k}}{\delta(x)} I_\Delta(y)$ for $k \in (\mathbb{N} \setminus \{0\})$. For a fixed (δ, Δ), define $\overline{\mathcal{Q}}(M)$ in the quotient of R_δ by the vector space generated by the $Q_k(\delta, \Delta)$ for $k \in (\mathbb{N} \setminus \{0\})$ as the class of $\mathcal{Q}(\mathbb{K} \subset M)$.

For 3-manifolds M such that $H_1(M;\mathbb{Z}) = \mathbb{Z}$ and $\Delta(M) = 1$, the invariant $\overline{\mathcal{Q}}$ coincides with the two-loop part of the invariant defined by Ohtsuki in 2008, combinatorially in [O3] for 3-manifolds of rank one, up to normalization. I think that the two invariants are equivalent, in the sense that they distinguish the same pairs of manifolds with isomorphic equivariant pairings, when $\Delta = \delta$, and that $\overline{\mathcal{Q}}$ refines the two-loop part of the Ohtsuki invariant when $\Delta \neq \delta$.

The following proposition is a direct consequence of Proposition 14.2 that is proved in Section 14.

Proposition 1.9 *If Δ has only simple roots and if N is a rational homology sphere such that $\lambda(N) \neq 0$, then $\overline{\mathcal{Q}}(M) \neq \overline{\mathcal{Q}}(M \sharp N)$.*

1.8 On the derived knot invariant $\hat{Q}$

Here, we list a few properties of the invariant $\hat{Q}$ of null-homologous knots in rational homology spheres that was defined in the beginning of Subsection 1.6. These properties obviously follow from the corresponding properties of $\mathcal{Q}$. Let us first recall the definition of $\hat{Q}$. Let $\hat{K}$ be a null-homologous knot in a rational homology sphere N. Let $N(\hat{K};0)$ be the 3-manifold obtained by 0-surgery on $\hat{K}$. Let K be the core of the solid torus in $N(\hat{K};0)$ that is glued during the surgery, and let $K_\parallel$ be the meridian of $\hat{K}$ on the boundary of this torus. Then

$$\hat{\mathcal{Q}}(\hat{K} \subset N) = \mathcal{Q}((K, K_\parallel) \subset N(\hat{K};0)) \in R_{\delta(N(\hat{K};0))}.$$

The *Alexander polynomial* of $\hat{K}$ is defined like the Alexander polynomial of M in Lemma 2.2, it is normalized so that $\Delta(\hat{K}) = \Delta(N(\hat{K};0))$.

Proposition 1.10 *The invariant $\hat{\mathcal{Q}}$ satisfies the following properties for any null-homologous knot $\hat{K}$ in a rational homology sphere N.*

1. $\delta(M)(x)\delta(M)(y)\delta(M)(z)\hat{\mathcal{Q}}(\hat{K}) \in \frac{\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]}{(xyz=1)}$.
2. $\hat{\mathcal{Q}}(\hat{K})(x,y,z) = \hat{\mathcal{Q}}(\hat{K})(y,x,z) = \hat{\mathcal{Q}}(\hat{K})(z,y,x) = \hat{\mathcal{Q}}(\hat{K})(x^{-1},y^{-1},z^{-1})$.
3. $\hat{\mathcal{Q}}(\hat{K})(1,1,1) = 6\lambda(N)$.
4. $\hat{\mathcal{Q}}(-\hat{K}) = \hat{\mathcal{Q}}(\hat{K})$, $\hat{\mathcal{Q}}(\hat{K} \subset (-N)) = -\hat{\mathcal{Q}}(\hat{K} \subset N)$.
5. *For any other rational homology sphere N_2, $\hat{\mathcal{Q}}(\hat{K} \subset N \sharp N_2) = \hat{\mathcal{Q}}(\hat{K} \subset N) + 6\lambda(N_2)$.*
6. *For any knot J of N that bounds a Seifert surface Σ disjoint from $\hat{K}$ such that $H_1(\Sigma)$ maps to 0 in $H_1(N \setminus \hat{K})$, for any r in $\mathbb{Q} \setminus \{0\}$,*
$$\hat{\mathcal{Q}}(\hat{K} \subset N(J;r)) - \hat{\mathcal{Q}}(\hat{K} \subset N) = \frac{6}{r}\lambda'_e(J) + 6\lambda(S^3(U;r))$$
where $\lambda'_e(J)$ is defined like in Theorem 1.4, and the other used notation can be found there, too.
7. *If J is the parallel of $\hat{K}$ that bounds in $N \setminus \hat{K}$, then*
$$\hat{\mathcal{Q}}(\hat{K} \subset N(J;1)) - \hat{\mathcal{Q}}(\hat{K} \subset N) = \sum_{\mathfrak{S}_3(x,y,z)} \frac{x\Delta'(x)}{2\Delta(x)} I_\Delta(y).$$
8. *The invariant $\hat{\mathcal{Q}}$ satisfies the LP-surgery formula of Theorem 9.1 when A and B are disjoint rational homology handlebodies of $N \setminus \hat{K}$ whose H_1 map to 0 in $H_1(N \setminus \hat{K})$.*
9. *If N is an integral homology sphere, and if $\Delta(\hat{K}) = 1$, then $\hat{\mathcal{Q}}(\hat{K})$ lifts the (primitive) two-loop part of the Kontsevich integral. See Theorem 9.2.*

Note that the fourth property implies that $\hat{\mathcal{Q}}(U) = 0$ for the trivial knot U of S^3.

1.9 Organization of the article

The claims contained in the statement of Theorem 1.1 are mostly homological:

The existence of the chain G_X will be proved as soon as we have proved that the 3–cycle
$$(\theta - 1)\delta(\theta)\left(s_\tau(M;X) - I_\Delta(\theta)ST(M)_{|K_X}\right)$$
vanishes in $H_3(\tilde{C}_2(M))$.

The fact that the equivariant triple algebraic intersection $\langle G_X, G_Y, G_Z\rangle_e$ is well-defined by the statement comes from the fact that G_X is well-defined up to the boundary of a 5-chain in the interior of $\tilde{C}_2(M)$, and from the similar facts for G_Y and G_Z.

Therefore, we shall begin with a study of the homology of $\tilde{C}_2(M)$ in Section 2. We shall recall known facts about equivariant intersections and equivariant linking numbers in Section 3. Then we shall compare various homology classes of $\tilde{C}_2(M)$ and we shall prove the existence of G_X, G_Y and G_Z (but not yet their rationality) in Section 4. In Section 5, we shall define an invariant $\mathcal{Q}(K_X, K_Y, K_Z, \mathbb{K}, \tau)$, determine the dependence on τ to complete the definition of $\mathcal{Q}(\mathbb{K})$, and study an example with $M = S^1 \times S^2$.

In Section 6, we give a configuration space definition for the Casson-Walker invariant λ, and we prove that $\mathcal{Q}(\mathbb{K} \subset M\sharp N) = \mathcal{Q}(\mathbb{K} \subset M) + 6\lambda(N)$, for a rational homology sphere N. The dependence of $\mathcal{Q}$ on the parallelisation of K is discussed in Section 7.

In Section 8, we compute $(\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}))$ when $\mathbb{K}'$ is another framed knot whose homology class generates $H_1(M)/\text{Torsion}$. From Section 9 to 11, we prove our two surgery formulae that generalize properties of the two-loop polynomial described in [O1]. In order to prove the LP surgery formula in full generality in Section 11, we give a more flexible definition of $\mathcal{Q}$ in Section 10 by introducing *pseudo-trivialisations* that generalize the trivialisations used in Theorem 1.1.

In Section 12, we specialize our chains G_X, G_Y and G_Z in the preimage of $K \times M \cup M \times K$. We say that we show how K *interacts* via the chains G_X, G_Y and G_Z. This allows us to prove that these chains are rational, in Proposition 12.7, and that $\delta(x)\delta(y)\delta(z)\mathcal{Q}(\mathbb{K})$ is in $\mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. To prove this result, we give an alternative definition of $\mathcal{Q}(\mathbb{K})$ as an equivariant triple algebraic intersection in an equivariant configuration space associated with the complement of K in Subsection 12.2. This alternative definition is very close to work of Julien Marché [Ma]. This definition also allows us to compute the evaluation at $(1, 1, 1)$ of $\mathcal{Q}(\mathbb{K})$ and to find that it is $6\lambda(M_{\mathbb{K}})$, in Section 13. Finally, in Section 14, we prove that the induced invariant of 3-manifolds detects the connected sums with rational homology spheres when Δ has no multiple roots. The various sections are as independent as possible.

More precisely, we could partition this big article into 5 separate blocks. The first block is the *definition block* from Section 1 to Section 5, together with Section 6 that provides useful links with the construction of the Casson invariant. The other blocks rely on this first block but they do not rely on other blocks except for a few specific statements, so that they can be read independently of each other. The *dependence on the framed knot K in M block* goes from Section 7 to Section 8, the *surgery formula block* goes from Section 9 to Section 11, then Section 12 and Section 13 also form a block even if Section 13 also depends on the properties of pseudo-trivialisations studied in Section 10, and Section 14 is the final small block.

2 On the homology of $\tilde{C}_2(M)$

2.1 On the homology of $M \setminus S$

Recall that S is a closed surface of M that intersects K transversally at one point with a positive sign. Identify $H_*(S^- = S \times \{-1\})$, $H_*(S^+ = S \times \{+1\})$ and $H_*(S \times [-1,1])$, naturally, and, write the element of $H_*(S^+)$ (resp. $H_*(S^-)$) corresponding to an element c of $H_*(S \times [-1,1])$ as c^+ (resp. c^-). Let $*$ be a basepoint of S.

Lemma 2.1 *$H_i(M \setminus S; \mathbb{Z}) = 0$ for any $i \geq 3$, $H_2(M \setminus S; \mathbb{Z}) = \mathbb{Z}[S^+]$ and $H_0(M \setminus S; \mathbb{Z}) = \mathbb{Z}[*^+]$. Let $(z_i)_{i=1,\dots 2g}$ and $(z_i^*)_{i=1,\dots,2g}$ be two dual bases of $H_1(S;\mathbb{Z})$ such that*

$$\langle z_i, z_j^* \rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

Then

$$H_1(M \setminus S; \mathbb{Q}) = \bigoplus_{i=1}^{2g} \mathbb{Q}[z_i^+ - z_i^-]$$

and for any $v \in H_1(M \setminus S; \mathbb{Q})$,

$$v = \sum_{i=1}^{2g} lk(v, z_i^*)(z_i^+ - z_i^-).$$

Proof: Let $\mathcal{B}$ be a basis of $H_{i-1}(S)$ such that any $[b] \in \mathcal{B}$ is represented by an $(i-1)$-submanifold b of S. Then

$$H_i(M, M \setminus S; \mathbb{Z}) \cong H_i(S \times [-1,1], S^+ \sqcup S^-; \mathbb{Z})$$

is freely generated by the classes $[b \times [-1,1]]$, for $[b] \in \mathcal{B}$. The natural maps

$$H_i(M;\mathbb{Z}) \to H_i(M, M \setminus S; \mathbb{Z})$$

are therefore surjective for $i = 1$ and $i = 3$, and the kernel is the torsion of $H_1(M;\mathbb{Z})$ when $i = 1$. In particular, the natural map $H_2(M \setminus S; \mathbb{Z}) \to (H_2(M;\mathbb{Z}) = \mathbb{Z}[S])$ is injective, and since it is obviously surjective, it is an isomorphism. Then the boundary map

$$\partial_2 \colon H_2(M, M \setminus S; \mathbb{Z}) \to H_1(M \setminus S; \mathbb{Z})$$

is injective, and it is an isomorphism if we take coefficients in $\mathbb{Q}$. This gives the announced expression for $H_1(M \setminus S; \mathbb{Q})$.

Recall that the linking number of two disjoint null-homologous links makes sense in M, and if u is a curve of S, $lk(z_i^+ - z_i^-, u) = \langle z_i, u \rangle$. This allows us to express the coordinates of an element $v \in H_1(M \setminus S; \mathbb{Q})$ as in the statement. $\diamond$

2.2 On the homology of $\tilde{M}$, the Alexander polynomial and the annihilator

Let $p_M\colon \tilde{M} \to M$ be the infinite cyclic covering of M.

$$p_M^{-1}(S) = \sqcup_{n\in\mathbb{Z}} \theta_M^n(\hat{S})$$

where $\hat{S}$ is a homeomorphic lift of S in $\tilde{M}$. Let $* \in S$ and let $\hat{*}$ be its lift in $\hat{S}$. Here, we compute the homology of $\tilde{M}$ with coefficients in $\mathbb{Q}$ and we see it as endowed with a structure of $\mathbb{Q}[t_M, t_M^{-1}]$-module where the multiplication by t_M is induced by the action of θ_M on $\tilde{M}$. Set

$$\Lambda_M = \mathbb{Q}[t_M, t_M^{-1}].$$

For a polynomial P of $t_M^{1/2}\Lambda_M$, $\frac{\Lambda_M}{(P)}$ will denote the Λ_M-module $\frac{\Lambda_M}{(t_M^{1/2}P)}$ quotient of Λ_M by the Λ_M-ideal generated by $t_M^{1/2}P$. (We may need polynomials of $t_M^{1/2}\Lambda_M$ when $H_1(M;\mathbb{Z})$ has 2–torsion. See Remark 2.3.) The homology of $\tilde{M}$ is given by the following lemma.

Lemma 2.2 *Let $(z_i)_{i=1,\dots 2g}$ be a basis of $H_1(S;\mathbb{Z})$. Set*

$$\Delta(M) = det\left([t_M^{1/2} lk(z_j^+, z_i) - t_M^{-1/2} lk(z_j^-, z_i)]_{(i,j)\in\{1,2,\dots,2g\}^2}\right).$$

$$H_0(\tilde{M}) = \frac{\Lambda_M}{(t_M - 1)}[\hat{*}], \quad H_2(\tilde{M}) = \frac{\Lambda_M}{(t_M - 1)}[\hat{S}], \quad H_3(\tilde{M}) = 0$$

and

$$H_1(\tilde{M}) \cong \oplus_{i=1}^k \frac{\Lambda_M}{(\delta_i(M))}$$

for an integer $k \geq 0$ and polynomials $\delta_i(M)$ of $\Lambda_M \cup t_M^{1/2}\Lambda_M$ such that

- $\delta_i(M)(1) = 1$,
- $\delta_i(M)(t_M) = \delta_i(M)(t_M^{-1})$,
- $\delta_i(M)$ *divides* $\delta_{i+1}(M)$,
- $\delta_1(M)$ *is not a unit,*
- $\Delta(M) = \prod_{i=1}^k \delta_i(M)$.

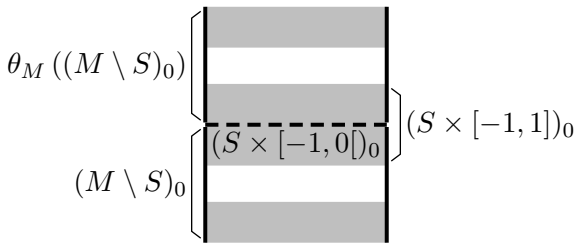
Then the Alexander polynomial *of M is $\Delta(M)$, it satisfies*

$$\Delta(M)(t_M) = \Delta(M)(t_M^{-1})$$

and the annihilator *of $H_1(\tilde{M})$ is $\delta(M) = \delta_k(M)$ (or 1 if $\Delta(M) = 1$, when $k = 0$).*

Like the previous one, the following proof is classical but it is useful to recall it to introduce notation.

Proof of Lemma 2.2: In $\tilde{M}$, both S and $M \setminus S$ lift homeomorphically. Consider a lift $(M \setminus S)_0$ of $M \setminus S$ and a lift $(S \times [-1,1])_0$ of $S \times [-1,1]$ that intersect along a lift of $S \times [-1,0[$ so that $\theta_M((M \setminus S)_0)$ intersects $(S \times [-1,1])_0$ along a lift of $S \times]0,1]$.



$\tilde{M} = p_M^{-1}(M \setminus S) \cup p_M^{-1}(S \times [-1,1])$ where

$$p_M^{-1}(M \setminus S) = \sqcup_{i \in \mathbb{Z}} \theta_M^i((M \setminus S)_0) \quad \text{and} \quad p_M^{-1}(S \times [-1,1]) = \sqcup_{i \in \mathbb{Z}} \theta_M^i((S \times [-1,1])_0).$$

$$H_*(p_M^{-1}(M \setminus S)) = H_*(M \setminus S) \otimes_{\mathbb{Q}} \Lambda_M, \quad H_*(p_M^{-1}(S \times [-1,1])) = H_*(S \times [-1,1]) \otimes_{\mathbb{Q}} \Lambda_M,$$

$$H_*(\tilde{M}, p_M^{-1}(M \setminus S)) = H_*(p_M^{-1}(S \times [-1,1]), p_M^{-1}(S^+ \sqcup S^-)) = H_*((S \times [-1,1])_0, S^+ \sqcup S^-) \otimes_{\mathbb{Q}} \Lambda_M.$$

Consider the boundary maps

$$\partial_i \colon H_i(\tilde{M}, p_M^{-1}(M \setminus S)) \to H_{i-1}(M \setminus S) \otimes_{\mathbb{Q}} \Lambda_M.$$

Recall that $*$ is a basepoint of S, let $*^+$ and $*^-$ denote the corresponding basepoints of S^+ and S^-. Let $\hat{*}^+$ and $\hat{*}^-$ be their respective lifts in $(M \setminus S)_0$. The map ∂_1 maps the preferred generator of $H_1(\tilde{M}, p_M^{-1}(M \setminus S))$ to $t_M[\hat{*}^+] - [\hat{*}^-] = (t_M - 1)[\hat{*}^+]$. It is therefore injective and

$$H_0(\tilde{M}) = \frac{\Lambda_M}{(t_M - 1)}[\hat{*}^+](\cong \mathbb{Q}).$$

Similarly, ∂_3 maps the preferred generator of $H_3(\tilde{M}, p_M^{-1}(M \setminus S))$ to $t_M[\hat{S}^+] - [\hat{S}^-] = (t_M - 1)[\hat{S}^+]$ and is injective. The map ∂_2 maps a basis of $H_*(\tilde{M}, p_M^{-1}(M \setminus S))$ to the system

$$(t_M \hat{z}_i^+ - \hat{z}_i^-)_{i \in \{1,2,\ldots,2g\}}$$

of $H_{i-1}(M \setminus S) \otimes_{\mathbb{Q}} \Lambda_M$. Its determinant reads

$$\det(\partial_2) = \frac{\bigwedge_{j=1}^{2g}(t_M \hat{z}_j^+ - \hat{z}_j^-)}{\bigwedge_{j=1}^{2g}(\hat{z}_j^+ - \hat{z}_j^-)}.$$

In particular, this determinant maps $t_M = 1$ to 1 and it does not vanish. Therefore, ∂_2 is injective, too.

Hence, for any i, $H_i(\tilde{M})$ is the cokernel of ∂_{i+1}. This gives the result for $i = 2$, and we are left with the computation of $H_1(\tilde{M})$.

According to the theory of modules over principal domains (see [Bou, Chapter 7]), $H_1(\tilde{M})$ reads $\oplus_{i=1}^{k} \frac{\Lambda_M}{\delta_i}$ for polynomials δ_i of Λ_M such that δ_i divides δ_{i+1} and δ_1 is not a unit, and these polynomials are well-defined up to multiplications by units ($qt_M^{\pm k}$) of Λ_M. Furthermore, $\prod_{i=1}^{k} \delta_i = \det(\partial_2)$ up to units of Λ_M. Since $\det(\partial_2)$ maps $t_M = 1$ to 1, we can assume that $\delta_i(1) = 1$ for any i. The δ_i are now well-defined up to multiplications by units ($t^{\pm k}$) of $\mathbb{Q}[t^{\pm 1}]$.

The matrix of ∂_2 is a presentation matrix for $H_1(\tilde{M})$, it is equivalent to the matrix

$$[t_M lk(z_j^+, z_i) - lk(z_j^-, z_i)]_{(i,j)\in\{1,2,\ldots,2g\}^2},$$

it is also equivalent to its transposed matrix $[t_M lk(z_i^+, z_j) - lk(z_i^-, z_j)]$ and by multiplication by the unit $(-t_M^{-1})$ to $[t_M^{-1} lk(z_j, z_i^-) - lk(z_j, z_i^+)]$ which is the initial matrix where t_M has been changed into t_M^{-1}. Therefore, $\delta_i(t_M) = \delta_i(t_M^{-1})$ up to units and it suffices to multiply δ_i by some $t_M^{r/2}$ to normalize it as wanted. $\diamond$

Remark 2.3 We could be more precise and notice that

$$\delta_i(M) \in \Lambda_M \cup (t_M^{1/2} + t_M^{-1/2})\Lambda_M$$

and that if $H_1(M;\mathbb{Z})$ has no 2–torsion, then $\delta_i(M) \in \Lambda_M$, for any i.

Indeed, if the degree (difference between highest degree and lowest degree) of $\delta_i(M)$ is even, then $\delta_i(M) \in \Lambda_M$. Since $\delta_i(M)(t_M) = \delta_i(M)(t_M^{-1})$, if x is a root of $\delta_i(M)$, then x^{-1} is a root with the same order. In particular, if $x \neq x^{-1}$ for all the roots x of $\delta_i(M)$, then the degree of $\delta_i(M)$ is even. Since 1 cannot be a root of $\delta_i(M)$, the only annoying root could be (-1) and this leads to the first part of the statement. To prove the second part, note that (-1) cannot be a root of Δ if there is no two-torsion in $H_1(M)$. Indeed, in this case, there exists an odd integer r such that $r\Delta(M)$ has integral coefficients. Then $r\Delta(M)(1) - r\Delta(M)(-1)$ is even and since $r\Delta(M)(1) = r$, $\Delta(M)(-1) \neq 0$. See [Mo] for more precise results about the occuring polynomials.

2.3 First remarks on the homology of $\widetilde{M^2}$

The homology of $\widetilde{M^2}$ with coefficients in $\mathbb{Q}$ is endowed with a structure of $\mathbb{Q}[t, t^{-1}]$-module where the multiplication by t is induced by the action of θ on $\widetilde{M^2}$. Set

$$\Lambda = \mathbb{Q}[t, t^{-1}].$$

Recall that $\delta = \delta(M)$ is the annihilator of $H_1(\tilde{M})$. Since δ has the same roots as Δ, $\delta\frac{\Delta'}{\Delta} \in \Lambda \cup t^{1/2}\Lambda$.

Proposition 2.4 *For any element $x \in H_*(\widetilde{M^2};\mathbb{Q})$, $(t-1)^2\delta(M)(t)^2 x = 0$.*

To prove this proposition -which would be sufficient to define the invariant $\mathcal{Q}$ but not sufficient to get the correct denominators-, we shall use the following lemma.

Lemma 2.5 *Let $\tilde{M}_1$ and $\tilde{M}_2$ be two identified copies of $\tilde{M}$. Let $S^{I-} = (S \times [-1,0[)$ and $S^{I+} = (S \times]0,1])$. Then*

$$\widetilde{M^2} = \Big((S \times [-1,1]) \times \tilde{M}_1\Big) \sqcup \Big((M \setminus S) \times \tilde{M}_2\Big) / \sim$$

where $\sim$ identifies

$$(\sigma^- \in (S^{I-} \subset S \times [-1,1]), \mu \in \tilde{M}_1) \sim (\sigma^- \in M \setminus S, \mu \in \tilde{M}_2)$$

and

$$(\sigma^+ \in (S^{I+} \subset S \times [-1,1]), \mu \in \tilde{M}_1) \sim (\sigma^+ \in M \setminus S, \theta_M^{-1}(\mu) \in \tilde{M}_2) = \theta(\sigma^+, \mu) = (\theta_M(\sigma^+), \mu).$$

Proof: $\widetilde{M^2}$ can be constructed as the union of

$$\widetilde{M^2} = \Big((S \times [-1,1]) \times \tilde{M}_1\Big) \cup \Big((M \setminus S) \times \tilde{M}_2\Big)$$

where $\tilde{M}_1$ and $\tilde{M}_2$ are two copies of $\tilde{M}$ that contain a preferred copy $(S \times [-1,1])_1$ of $S \times [-1,1]$, and, a preferred copy $(M \setminus S)_2$ of $M \setminus S$, respectively. These preferred copies are such that the canonical lift of the diagonal of $(S \times [-1,1])^2$ is inside $(S \times [-1,1]) \times (S \times [-1,1])_1$ and the canonical lift of the diagonal of $(M \setminus S)^2$ is inside $(M \setminus S) \times (M \setminus S)_2$.

In $\tilde{M}_2$, $(M \setminus S)_2$ intersects a lift $(S \times [-1,1])_2$ of $S \times [-1,1]$ along a lift of $S \times [-1,0[$ and the lift $\theta_M^{-1}((S \times [-1,1])_2)$ along a lift of $S \times]0,1]$. Identify $(\tilde{M}_2, (S \times [-1,1])_2)$ with $(\tilde{M}_1, (S \times [-1,1])_1)$.

Then $(S \times [-1,1]) \times \tilde{M}_1$ and $(M \setminus S) \times \tilde{M}_2$ intersect along $(S^{I-} \times \tilde{M}_1) \sqcup (S^{I+} \times \tilde{M}_1)$ that are seen as natural parts of $(S \times [-1,1]) \times \tilde{M}_1$ and that map to $(M \setminus S) \times \tilde{M}_2$ by sending the diagonals of S^{I+} and S^{I-} to $(M \setminus S) \times (M \setminus S)_2$ and therefore to $(M \setminus S) \times \theta_M^{-1}((S \times]0,1])_2)$, and to $(M \setminus S) \times ((S \times [-1,0[)_2)$, respectively. This gives the result. $\diamond$

Proof of Proposition 2.4 : Let $p\colon \widetilde{M^2} \to M^2$ be the covering map.

$$\widetilde{M^2} = p^{-1}((M \setminus S) \times M) \cup p^{-1}((S \times [-1,1]) \times M)$$

where

$$p^{-1}((M \setminus S) \times M) = (M \setminus S) \times \tilde{M} \quad \text{and} \quad p^{-1}((S \times [-1,1]) \times M) = (S \times [-1,1]) \times \tilde{M}.$$

Thanks to the Künneth formula,

$$\begin{aligned} H_r((M \setminus S) \times \tilde{M}) &= \Big(H_*(M \setminus S) \otimes_{\mathbb{Q}} H_*(\tilde{M})\Big)_r \\ &= \bigoplus_{p,q;p+q=r} \Big(H_p(M \setminus S) \otimes_{\mathbb{Q}} H_q(\tilde{M})\Big) \end{aligned}$$

is a torsion module over $\mathbb{Q}[t,t^{-1}]$. More precisely, all its homogeneous elements are either $(t-1)$-torsion elements or $\delta(t)$-torsion elements.

$$\begin{aligned} H_i(\widetilde{M^2},(M\setminus S)\times\tilde{M};\mathbb{Q}) &= H_i((S\times[-1,1])\times\tilde{M},(S^+\sqcup S^-)\times\tilde{M};\mathbb{Q}) \\ &\cong \Big(H_*(S\times[-1,1],S^+\sqcup S^-)\otimes_{\mathbb{Q}} H_*(\tilde{M})\Big)_i \end{aligned}$$

is also a torsion module over $\mathbb{Q}[t,t^{-1}]$ whose homogeneous elements are annihilated by $(t-1)$ or $\delta(t)$. Then the long exact sequence associated to the pair $(\widetilde{M^2},(M\setminus S)\times\tilde{M})$ allows us to conclude that $(t-1)^2\delta(M)^2x=0$ for any $x\in H_*(\widetilde{M^2};\mathbb{Q})$. $\diamond$

2.4 More on the homology of $\widetilde{M^2}$

The involution that exchanges the two factors in $\tilde{M}^2$ induces an involution ι of $\widetilde{M^2}$. See S^1 as the unit circle of $\mathbb{C}$. Define

$$\begin{array}{llll} f\colon & M & \to & S^1 \\ & (M\setminus(S\times]-1,1[)) & \mapsto & -1 \\ & (\sigma,u)\in S\times[-1,1] & \mapsto & \exp(i\pi u) \end{array}$$

Assume that K meets $S\times[-1,1]$ as $*\times[-1,1]$, and define a homeomorphism $f_K\colon K\to S^1$ that coincides with f on $*\times[-1/2,1/2]$.

Define $\mathrm{diag}_u(K\times M)$ as the lift in $\widetilde{M^2}$ of $(f_K\times f)^{-1}(\mathrm{diag}((S^1)^2)$ that meets the preferred lift of the diagonal of M^2. Define $\mathrm{diag}_u(M\times K)$, similarly.

$$\mathrm{diag}_u(M\times K)=\pm\iota(\mathrm{diag}_u(K\times M)).$$

Note that if γ is a curve homotopic to the diagonal of $(S^1)^2$, then the preimage of the homotopy yields a cobordism between $\mathrm{diag}_u(K\times M)$ and a lift of the preimage of γ under $(f_K\times f)$. In particular the homotopy

$$(u\in[0,1],z\in S^1)\mapsto(z\exp(2i\pi u),z)$$

shows that $\theta\,(\mathrm{diag}_u(K\times M))$ is homologous to $\mathrm{diag}_u(K\times M)$, that is homologous to all the lifts of $(f_K\times f)^{-1}(\gamma)$.

Similarly, all the lifts in $\widetilde{M^2}$ of the preimages $(f\times f)^{-1}(\gamma)$ for the curves γ of $(S^1)^2$ homotopic to the diagonal of $(S^1)^2$, such that $(f\times f)^{-1}(\gamma)$ is a 5–chain, are homologous to each other. Denote their homology class by $[\mathrm{diag}_u(M\times M)]$ and note that

$$t[\mathrm{diag}_u(M\times M)]=[\mathrm{diag}_u(M\times M)].$$

Recall that boundaries are oriented by the *outward normal first* convention.

Proposition 2.6 *Recall that*

$$H_1(\tilde{M}) = \oplus_{i=1}^{k} \frac{\Lambda_M}{(\delta_i)}[c_i]$$

where δ_1 is not a unit, $\delta_i(t_M) = \delta_i(t_M^{-1})$, δ_i divides δ_{i+1} for any i such that $i < k$ and $\delta_k = \delta$. Then the rational homology of $\widetilde{M^2}$ reads as follows

$$\begin{array}{lcl}
H_0(\widetilde{M^2}) & = & \frac{\Lambda}{(t-1)}[* \times *] \\
H_1(\widetilde{M^2}) & = & \frac{\Lambda}{(t-1)}[\mathrm{diag}(K^2)] \\
H_2(\widetilde{M^2}) & = & \frac{\Lambda}{(t-1)}[S \times *] \oplus \frac{\Lambda}{(t-1)}[* \times S] \oplus \oplus_{(i,j)\in\{1,2,\ldots,k\}^2} \frac{\Lambda}{\left(\delta_{\min(i,j)}(t)\right)}[c_i \times c_j] \\
H_3(\widetilde{M^2}) & = & \frac{\Lambda}{(t-1)}[\mathrm{diag}_u(K \times M)] \oplus \frac{\Lambda}{(t-1)}[\mathrm{diag}_u(M \times K)] \oplus \oplus_{(i,j)\in\{1,2,\ldots,k\}^2} \frac{\Lambda}{\left(\delta_{\min(i,j)}\right)}[C(\Sigma_i \times \Sigma_j)] \\
H_4(\widetilde{M^2}) & = & \frac{\Lambda}{(t-1)}[S \times S] \\
H_5(\widetilde{M^2}) & = & \frac{\Lambda}{(t-1)}[\mathrm{diag}_u(M \times M)]
\end{array}$$

where $* \times *$, $S \times *$, $* \times S$, $c_i \times c_j$ and $S \times S$ abusively denote the projections in $\widetilde{M^2}$ of $\hat{*} \times \hat{*}$, $\hat{S} \times \hat{*}$, $\hat{*} \times \hat{S}$, $c_i \times c_j$ and $\hat{S} \times \hat{S}$, respectively, Σ_i is a rational 2–chain of $\tilde{M}$ whose boundary is $\delta_i c_i$ and $C(\Sigma_i \times \Sigma_j)$ denotes

$$C(\Sigma_i \times \Sigma_j) = \frac{\delta_i(t)}{\delta_{\min(i,j)}(t)} c_i \times \Sigma_j \cup \frac{\delta_j(t)}{\delta_{\min(i,j)}(t)} \Sigma_i \times c_j$$

or its projection in $\widetilde{M^2}$.

Proof: First notice that all the mentioned chains are cycles that are annihilated by the corresponding polynomials. For example if $i \leq j$, $\partial(\Sigma_i \times \Sigma_j) = \delta_i C(\Sigma_i \times \Sigma_j)$. Now, continue the proof of Proposition 2.4 to determine $H_i(\widetilde{M^2})$ from the short exact sequences associated with the long exact sequence of the pair $(\widetilde{M^2}, (M \setminus S) \times \tilde{M})$

$$0 \to \mathrm{Coker}(\partial_{i+1}) \to H_i(\widetilde{M^2}) \to \mathrm{Ker}(\partial_i) \to 0$$

where

$$\partial_{i+1} \colon H_{i+1}(\widetilde{M^2}, (M \setminus S) \times \tilde{M}) \to H_i((M \setminus S) \times \tilde{M})$$

will be rewritten as

$$F_i = \oplus_{(r,s);r+s=i} F_{r,s} \colon H_{i+1}((S \times [-1,1]) \times \tilde{M}, (S^+ \sqcup S^-) \times \tilde{M}) \to H_i((M \setminus S) \times \tilde{M})$$

thanks to the excision isomorphism with

$$\begin{array}{rlcl}
F_{r,s}\colon & H_r(S) \otimes_{\mathbb{Q}} H_s(\tilde{M}) & \to & H_r(M \setminus S) \otimes_{\mathbb{Q}} H_s(\tilde{M}) \\
 & \sigma \otimes x & \mapsto & \sigma^+ \otimes t_M^{-1} x - \sigma^- \otimes x
\end{array}$$

thanks to Lemma 2.5. Thus, the above short exact sequence may be rewritten as

$$0 \to \oplus_{(r,s);r+s=i} \mathrm{Coker}(F_{r,s}) \to H_i(\widetilde{M^2}) \to \oplus_{(r,s);r+s=i-1} \mathrm{Ker}(F_{r,s}) \to 0$$

and we are left with the computation of the $F_{r,s}$.

Lemma 2.7 *Let $\sigma_0 = [*]$ and $\sigma_2 = [S]$.*
When r and s belong to $\{0,2\}$, $F_{r,s}\colon \frac{\Lambda}{(t-1)}[\sigma_r]\otimes[\sigma_s] \to \frac{\Lambda}{(t-1)}[\sigma_r]\otimes[\sigma_s]$ is the multiplication by $t-1$ that is zero. Therefore, $Coker(F_{r,s}) = \frac{\Lambda}{(t-1)}[\sigma_r]\otimes[\sigma_s]$. Furthermore, there is a section

$$s_{r,s}\colon Ker(F_{r,s}) = \frac{\Lambda}{(t-1)}[\sigma_r]\otimes[\sigma_s] \to H_{r+s+1}(\widetilde{M^2})$$

and

$$\begin{array}{rcl} s_{0,0}(Ker(F_{0,0})) & = & \frac{\Lambda}{(t-1)}[\mathrm{diag}(K^2)] \\ s_{0,2}(Ker(F_{0,2})) & = & \frac{\Lambda}{(t-1)}[\mathrm{diag}_u(K\times M)] \\ s_{2,0}(Ker(F_{2,0})) & = & \frac{\Lambda}{(t-1)}[\mathrm{diag}_u(M\times K)] \\ s_{2,2}(Ker(F_{2,2})) & = & \frac{\Lambda}{(t-1)}[\mathrm{diag}_u(M^2)]. \end{array}$$

Proof: The first assertions are obvious. Let us prove the existence of a section $s_{0,2}$ such that

$$s_{0,2}(\mathrm{Ker}(F_{0,2})) = \frac{\Lambda}{(t-1)}[\mathrm{diag}_u(K\times M)].$$

The class $[\mathrm{diag}_u(K\times M)]$ is the class of a lift of the preimage under $f_K\times f$ of any curve γ of $(S^1)^2$ homologous to the diagonal. There exists such a curve γ that intersects $\exp(i\pi[-1/2,1/2])\times S^1$ as $\exp(i\pi[-1/2,1/2])\times 1$. This makes clear that the image of $[\mathrm{diag}_u(K\times M)]$ in $H_3(\widetilde{M^2},(M\setminus S)\times\tilde{M})$ comes from $\pm[(\sigma_0\times[-1/2,1/2])\times\sigma_2]\in H_3((S\times[-1/2,1/2])\times\tilde{M},(S^+\sqcup S^-)\times\tilde{M})$, and this shows how to define the wanted section. The other cases can be treated similarly. $\diamond$

Lemma 2.8 *If $r\in\{0,2\}$, then $F_{r,1}$ is an isomorphism so that $Ker(F_{r,1})=0$ and $Coker(F_{r,1})=0$.*

Proof:

$$F_{r,1}\colon \mathbb{Q}\otimes H_1(\tilde{M}) \to H_1(\tilde{M})$$

is the multiplication by $(t-1)$ that is an isomorphism on each $\frac{\Lambda}{(\delta_i)}$ since $(t-1)$ is coprime with δ_i. $\diamond$

To treat the symmetric case $r=1$, we shall use the following standard diagonalisation lemma over principal domains, see [Bou, Chapter 7].

Lemma 2.9 *Recall*

$$H_1(\tilde{M}) = \oplus_{i=1}^k \frac{\Lambda_M}{(\delta_i)}[c_i].$$

There exist a basis $(d_i)_{i\in\{1,2,\dots,2g\}}$ of $H_1(S)\otimes_{\mathbb{Q}}\Lambda_M$ and a basis $(c_i)_{i\in\{1,2,\dots,2g\}}$ of $H_1(M\setminus S)\otimes_{\mathbb{Q}}\Lambda_M$ such that

$$\begin{array}{rll} t_M d_i^+ - d_i^- & = \delta_i c_i & \text{if } i\le k \\ & = c_i & \text{if } i>k \end{array}$$

in $H_1(M\setminus S)\otimes_{\mathbb{Q}}\Lambda_M$.

$\diamond$

Using these bases, if $s \in \{0,2\}$, $F_{1,s}$ reads

$$\begin{array}{rlcl} F_{1,s}\colon & \bigoplus_{i=1}^{2g} \frac{\Lambda}{(t-1)}[d_i] \otimes [\sigma_s] & \to & \bigoplus_{i=1}^{2g} \frac{\Lambda}{(t-1)}[c_i] \otimes [\sigma_s] \\ & [d_i] \otimes [\sigma_s] & \mapsto & \delta_i [c_i] \otimes [\sigma_s] \end{array}$$

where $\delta_i = 1$ if $i > k$. Therefore we have the following lemma.

Lemma 2.10 *If $s \in \{0,2\}$, then $F_{1,s}$ is an isomorphism, $Ker(F_{1,s}) = 0$ and $Coker(F_{1,s}) = 0$.*

$\diamond$

Lemma 2.11 *$F_{1,1}$ reads $\bigoplus_{i=1}^{2g} \bigoplus_{j=1}^{k} f_{i,j}$ where*

$$\begin{array}{rlcl} f_{i,j}\colon & \frac{\Lambda}{(\delta_j)}[d_i \times c_j] & \to & \frac{\Lambda}{(\delta_j)}[c_i \times c_j] \\ & [d_i \times c_j] & \mapsto & \delta_i [c_i \times c_j] \end{array}$$

so that $Ker(F_{1,1}) = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{k} Ker(f_{i,j})$, $Coker(F_{1,1}) = \bigoplus_{i=1}^{k} \bigoplus_{j=1}^{k} Coker(f_{i,j})$, $Coker(f_{i,j}) = \frac{\Lambda}{(\delta_{\min(i,j)})}[c_i \times c_j]$ and there is a section

$$\begin{array}{rlcl} u_{i,j}\colon & Ker(f_{i,j}) = \frac{\Lambda}{(\delta_{\min(i,j)})} \frac{\delta_j}{\delta_{\min(i,j)}}[d_i \times c_j] & \to & H_3(\widetilde{M^2}) \\ & \frac{\delta_j}{\delta_{\min(i,j)}}[d_i \times c_j] & \mapsto & C(\Sigma_i \times \Sigma_j). \end{array}$$

Proof: Assume that the c_i are outside $p_M^{-1}(S \times [-1,1])$. Since $S \times c_i$ is rationally null-homologous (because $(t-1)S \times c_i$ and $\delta_i S \times c_i$ bound), the homology class of $C(\Sigma_i \times \Sigma_j)$ is independent of the chains Σ_i and Σ_j that have the given boundaries. Since $\delta_i c_i$ cobounds with $(\theta_M(d_i^+) - d_i^-)$ in $p_M^{-1}(M \setminus S)$, there is a 2-chain $\pm\Sigma_i$ in $\tilde{M}$ with boundary $\pm\delta_i c_i$ that intersects $p_M^{-1}(S \times [-1,1])$ as $(d_i \times [-1,1])$.

Then $C(\Sigma_i \times \Sigma_j)$ intersects $(S \times [-1,1]) \times \tilde{M}$ as $\pm\frac{\delta_j}{\delta_{\min(i,j)}}[(d_i \times [-1,1]) \times c_j]$ and its class in $H_3(\widetilde{M^2}, (M \setminus S) \times \tilde{M})$ is therefore the image of this element under the excision isomorphism. Therefore the class $[C(\Sigma_i \times \Sigma_j)]$ in $H_3(\widetilde{M^2})$ is mapped to the given generator of $\mathrm{Ker}(f_{i,j})$. Since $\delta_{\min(i,j)} C(\Sigma_i \times \Sigma_j) = \partial(\Sigma_i \times \Sigma_j)$, the wanted section exists. $\diamond$

Now, Proposition 2.6 is proved. $\diamond$

2.5 On the homology of $\tilde{C}_2(M)$

Recall from Subsection 1.2 that $\tilde{C}_2(M)$ is obtained from $\widetilde{M^2}$ by blowing-up the lifts of the diagonal, and that $\tilde{C}_2(M)$ is diffeomorphic to the complement of an open tubular neighborhood of the lifts of the diagonal. Again we consider the homology of $\tilde{C}_2(M)$ with coefficients in $\mathbb{Q}$ endowed with the structure of $\mathbb{Q}[t,t^{-1}]$-module where the multiplication by t is induced by the action of θ on $\widetilde{M^2}$ or $\tilde{C}_2(M)$. Recall $\Lambda = \mathbb{Q}[t,t^{-1}]$ and let $\mathbb{Q}(t)$ be the field of fractions of Λ.

Then $H_*(\tilde{C}_2(M))$ is a graded Λ-module, set

$$H_*(C_2(M);\mathbb{Q}(t)) = H_*(\tilde{C}_2(M);\mathbb{Q}) \otimes_\Lambda \mathbb{Q}(t).$$

Also recall from Subsection 1.2 that $\partial\tilde{C}_2(M) = \mathbb{Z} \times ST(M)$ where $ST(M)$ is the unit tangent bundle of M that is diffeomorphic to $M \times S^2$. When N is a submanifold of M, $ST(N)$ will denote the restriction of $ST(M)$ to N viewed either as a submanifold of $\partial C_2(M)$ or as a submanifold of $\partial\tilde{C}_2(M)$ that sits inside the preimage of the preferred lift of the diagonal under the blow-up map.

Proposition 2.12

$$H_i(C_2(M);\mathbb{Q}(t)) = H_{i-2}(M;\mathbb{Q}) \otimes_\mathbb{Q} \mathbb{Q}(t)$$

for any $i \in \mathbb{Z}$.

$$H_2(C_2(M);\mathbb{Q}(t)) = \mathbb{Q}(t)[ST(*)(\cong * \times S^2)]$$
$$H_3(C_2(M);\mathbb{Q}(t)) = \mathbb{Q}(t)[ST(K)(\cong K \times S^2)]$$
$$H_4(C_2(M);\mathbb{Q}(t)) = \mathbb{Q}(t)[ST(S)(\cong S \times S^2)]$$
$$H_5(C_2(M);\mathbb{Q}(t)) = \mathbb{Q}(t)[ST(M)(\cong M \times S^2)].$$

Proof: We know that

$$H_*(M^2;\mathbb{Q}(t)) = H_*(\widetilde{M^2}) \otimes_\Lambda \mathbb{Q}(t) = 0,$$

according to Proposition 2.4, and we compute $H_*(C_2(M);\mathbb{Q}(t)) = H_*(\tilde{C}_2(M)) \otimes_\Lambda \mathbb{Q}(t)$. In this proof, think of $C_2(M)$ as $(M^2 \setminus \text{diag})$ that has the same homotopy type.

$$H_i(\widetilde{M^2}, \tilde{C}_2(M);\mathbb{Q}) = H_i(M \times B^3, M \times S^2;\mathbb{Q}) \otimes_\mathbb{Q} \Lambda = H_{i-3}(M;\mathbb{Q}) \otimes_\mathbb{Q} \Lambda$$

This module is always free and therefore the natural maps from $H_i(\widetilde{M^2};\mathbb{Q})$ to $H_i(\widetilde{M^2}, \tilde{C}_2(M);\mathbb{Q})$ vanish. Thus, we have the exact sequence of Λ-modules

$$0 \to H_{i+1}(\widetilde{M^2}, \tilde{C}_2(M);\mathbb{Q}) \to H_i(\tilde{C}_2(M);\mathbb{Q}) \to H_i(\widetilde{M^2};\mathbb{Q}) \to 0.$$

that gives rise to an isomorphism

$$H_{i+1}(M^2, C_2(M);\mathbb{Q}(t)) \cong H_i(C_2(M);\mathbb{Q}(t)).$$

that proves the result. $\diamond$

3 On the equivariant linking number

3.1 On the equivariant intersection

In this text, unless otherwise mentioned, all the diffeomorphisms preserve the orientation, and the order of appearance of coordinates induces the orientation. Recall that boundaries are oriented with the *outward normal first* convention.

The fiber $N_u(A)$ of the normal bundle $N(A)$ of an oriented submanifold A in an oriented manifold C at $u \in A$ is oriented so that $T_uC = N_u(A) \oplus T_uA$ as oriented vector spaces. For two oriented transverse submanifolds A and B of C, $A \cap B$ is oriented so that $N_u(A \cap B) = N_u(A) \oplus N_u(B)$. In particular, when the sum of the dimensions of A and B is the dimension of C, the sign of an intersection point u of A and B is positive if and only if $T_uC = N_u(A) \oplus N_u(B)$, that is if and only if $T_uC = T_uA \oplus T_uB$ as oriented vector spaces.

When C is equipped with a free action of an abelian group G, the equivariant intersection has its coefficients in the group ring

$$\mathbb{Z}[G] = \oplus_{g \in G} \mathbb{Z} \exp(g)$$

and reads

$$A \cap_e B = \cup_{g \in G} \exp(g) A \cap g.B,$$

when A is transverse to $g.B$ for any g, so that

$$g_1.A \cap_e g_2.B = \exp(g_1 - g_2) g_1.(A \cap_e B).$$

When the sum of the dimensions of A and B is the dimension of C, the *equivariant intersection number* $\langle A, B \rangle_{e,C}$ is the sum of the coefficients of the points. Using the linear involution

$$\exp(g) \mapsto \overline{\exp(g)} = \exp(-g),$$

$$\langle A, B \rangle_{e,C} = (-1)^{\dim(A)\dim(B)} \overline{\langle B, A \rangle_{e,C}}.$$

3.2 On the linking number

Proposition 3.1 *Let C be a compact connected manifold of dimension c. Let A and B be two transverse submanifolds of C with boundaries, and with respective dimensions $dim(A) = a$ and $dim(B) = b$, such that $a + b = c + 1$, and $\partial A \cap \partial B = \emptyset$. Then the* linking number *of ∂A and ∂B can be defined in the following equivalent ways:*

$$lk(\partial A, \partial B) = \langle \partial A, B \rangle_C = (-1)^a \langle A, \partial B \rangle_C = (-1)^{ab} lk(\partial B, \partial A).$$

Let $* \in C$, and let $S(*)$ denote the oriented boundary of a ball that contains $*$. The homology class of $* \times S(*)$ in $H_{c-1}(C^2 \setminus diag(C^2))$ is denoted by $[S^{c-1}(C)]$. Then $\partial A \times \partial B$ is homologous to $lk(\partial A, \partial B)[S^{c-1}(C)]$ in $H_{c-1}(C^2 \setminus diag(C^2))$.

Note that $lk(\partial A, \partial B) = (-1)^a lk(\partial B, \partial A)$ when c is odd. Also note that the above formulae show that $\langle \partial A, B\rangle_C$ is independent of B and that $\langle A, \partial B\rangle_C$ is independent of A.
Proof of Proposition 3.1: Since $N(\partial A)$ is oriented as $N(A) \oplus N_{ext,A}(\partial A)$,

$$\partial(A \cap B) = (-1)^{c-b}\partial A \cap B + A \cap \partial B.$$

Therefore,

$$\langle \partial A, B\rangle_C = (-1)^{c-b+1}\langle A, \partial B\rangle_C = (-1)^a \langle A, \partial B\rangle_C.$$

Let us now prove that with this definition, $\partial A \times \partial B$ is homologous to $lk(\partial A, \partial B)[S^{c-1}(C)]$ in $H_{c-1}(C^2 \setminus \mathrm{diag}(C^2))$. The manifold A induces a cobordism between ∂A and boundaries of neighborhoods in A of points of $A \cap \partial B$ in $C \setminus \partial B$. This cobordism allows us to reduce the proof to the case where A is a disk of dimension a that intersects ∂B once. Using a similar cobordism induced by B allows us to reduce the proof to the case when B is also a small disk of dimension b, and $A \cap B$ is an interval. We compute the homology class of $\partial A \times \partial B$ in this case of generalized Hopf links where $C = \mathbb{R}^c$, $A = [-1,1]^{a-1} \times [-2,0] \times (0)^{c-a}$, $B = (0)^{a-1} \times [-1,1]^b$.

$$\partial A = \left(\partial[-1,1]^{a-1} \times [-2,0] \times (0)^{c-a}\right) \bigcup \left((-1)^{a-1}[-1,1]^{a-1} \times (\partial[-2,0] = 0 - (-2)) \times (0)^{c-a}\right)$$

$$\langle \partial A, B\rangle_C = (-1)^{a-1}.$$

Split ∂A into two topological disks

$$D_1^{a-1} = (-1)^{a-1}[-1,1]^{a-1} \times (0)^b \text{ and } D_2^{a-1} = \partial A \setminus D_1^{a-1},$$

so that D_2^{a-1} does not intersect B and $\partial(D_2^{a-1} \times B) = \partial(D_2^{a-1}) \times B + (-1)^{a-1}(D_2^{a-1} \times \partial B)$.

Then $\partial A \times \partial B$ is homologous to

$$D_1^{a-1} \times \partial B + (-1)^a \partial(D_2^{a-1}) \times B = D_1^{a-1} \times \partial B + (-1)^{a-1}\partial(D_1^{a-1}) \times B.$$

Now, change $\partial(D_1^{a-1}) \times B = \{((x,0),(0,\beta \in B)) \in C^2; x \in \partial(D_1^{a-1}), \beta \in B\}$ to

$$E = \{((x,0),(-xd(\beta,\partial B),\beta) \in C^2 \setminus \mathrm{diag}(C^2); x \in \partial(D_1^{a-1}), \beta \in B\}$$

by the obvious homotopy where $d(\beta, \partial B)$ denotes the distance between β and ∂B (for an arbitrary continuous distance of B). Now, the first factor in

$$D_1^{a-1} \times \partial B + (-1)^{a-1}E$$

can be contracted to 0 in D_1^{a-1} without meeting the diagonal. Therefore $\partial A \times \partial B$ is homologous to

$$(-1)^{a-1}\{0\} \times \{(-xd(\beta,\partial B),\beta); (x,\beta) \in \partial(D_1^{a-1} \times B)\}$$

that is homologous to $\{0\} \times \partial(D_1^{a-1} \times B)$, that is in turn homologous to $(-1)^{a-1}[S^{c-1}(C)]$. $\diamond$

Example 3.2 *In $\mathbb{R}^3$, let B be the unit ball, let $x \in Int(B)$ and let $y \notin B$, then*

$$lk(x-y, \partial B) = 1 = -lk(\partial B, x-y).$$

3.3 On the considered equivariant linking number

Proposition 3.3 *Let α and β be two submanifolds of $\tilde{M}$ of respective dimensions $dim(\alpha)$ and $dim(\beta)$, such that $dim(\alpha) + dim(\beta) = 2$, whose projections in M do not intersect. There exist rational chains A and B such that $\Delta(\theta_M)(\theta_M - 1)\alpha = \partial A$ and $\Delta(\theta_M)(\theta_M - 1)\beta = \partial B$.*

The two following definitions for the equivariant linking number $lk_e(\alpha, \beta)$ of α and β are equivalent.

1. $$lk_e(\alpha, \beta)(t_M) = \frac{\langle \alpha, B\rangle_{e,\tilde{M}}}{\Delta(t_M^{-1})(t_M^{-1} - 1)} = (-1)^{dim(\alpha)+1} \frac{\langle A, \beta\rangle_{e,\tilde{M}}}{\Delta(t_M)(t_M - 1)}$$
   where $\exp(\theta_M^n)$ is denoted by t_M^n so that $\langle \alpha, B\rangle_{e,\tilde{M}} = \sum_{n\in\mathbb{Z}} \langle \alpha, \theta_M^n(B)\rangle_{\tilde{M}} t_M^n$.

2. *The class of $\alpha \times \beta$ in $H_2(\tilde{C}_2(M); \mathbb{Q}) \otimes_\Lambda \mathbb{Q}(t)$ is equal to $lk_e(\alpha, \beta)(t)[ST(*)]$.*

The equivariant linking number has the following properties

$$lk_e(\alpha, \beta) = (-1)^{dim(\alpha)+1} \overline{lk_e(\beta, \alpha)}$$

and, when $\lambda \in \mathbb{Q}(t_M)$,

$$lk_e(\lambda\alpha, \beta) = \lambda lk_e(\alpha, \beta) = lk_e(\alpha, \overline{\lambda}\beta).$$

Proof: Applying Proposition 3.1 to A and $\theta_M^n(B)$, for any n, shows that

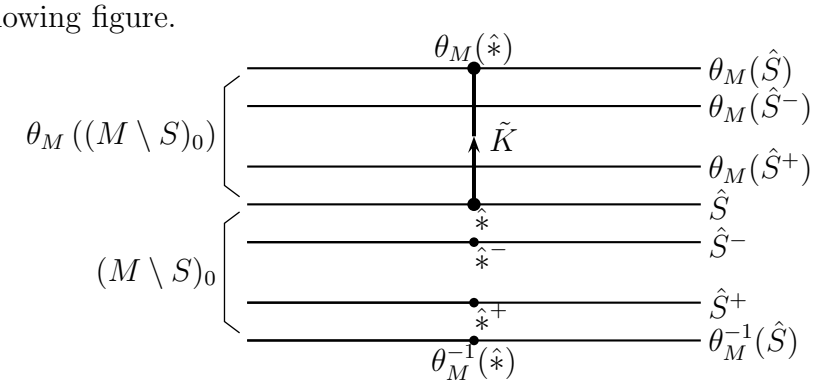
$$\langle \partial A, B\rangle_{e,\tilde{M}} = (-1)^{\dim(\alpha)+1} \langle A, \partial B\rangle_{e,\tilde{M}}$$

and allows us to define $lk_e(\partial A, \partial B)$ consistently by this expression so that for any $\lambda \in \Lambda_M$

$$lk_e(\lambda\partial A, \partial B) = \lambda lk_e(\partial A, \partial B) = lk_e(\partial A, \overline{\lambda}\partial B).$$

Extend the definition of lk_e so that this proposition holds for $\lambda \in \mathbb{Q}(t_M)$. The proof of Proposition 3.1 easily adapts to show that the class of $\alpha \times \beta$ in $H_2(C_2(M); \mathbb{Q}(t))$ is equal to $lk_e(\alpha, \beta)(t)[ST(*)]$. According to Proposition 2.12, this yields a definition of $lk_e(\alpha, \beta)$. $\diamond$

Example 3.4 The lifts $\hat{S}^+$, $\hat{S}^-$, $\hat{*}^+$ and $\hat{*}^+$ of submanifolds of S^+, S^-, $*^+$ and $*^+$ are in $(M \setminus S)_0$. The surface $\hat{S}$ is between $\hat{S}^-$ and $\theta_M(\hat{S}^+)$. Following the preimage of K under the covering map according to the orientation of K, we successively meet $\theta_M^{-1}(\hat{*})$, $\hat{*}^+$, $\hat{*}^-$, $\hat{*}$, $\theta_M(\hat{*}^+)$ as in the following figure.



In $\tilde{M}$, there is a lift $\tilde{K}$ of K whose boundary is $\partial\tilde{K} = (\theta_M - 1)\hat{*}$. The boundary of the closure of some lift of $(M \setminus S)$ is $(\theta_M - 1)\hat{S}$. Then

$$lk_e(\hat{*}, t_M\hat{S}^+) = \frac{1}{1 - t_M}$$

and

$$lk_e(\hat{S}, t_M\hat{*}^+) = \frac{1}{1 - t_M}.$$

The following lemma follows from Blanchfield duality [Bl].

Lemma 3.5 *The equivariant linking number is non-degenerate in the following sense. For any class β of $H_1(\tilde{M})$ of order $\delta(\beta) \in \Lambda_M$, there exists a two-component link (J, J^*) of $\tilde{M}$ such that J represents β, $p_M(J) \cap p_M(J^*) = \emptyset$ and*

$$lk_e(J, J^*) = \frac{q}{\delta(\beta)}$$

for some nonzero rational number q.

Proof: The polynomial $\delta(\beta)$ is defined up to units $qt_M^{\pm n}$ of Λ_M. Realize β by a link L of $p_M^{-1}(M \setminus (S\times] - 1, 1[))$. By assumption, $\delta(\beta)L$ is rationally homologous to

$$\sum_{i=1}^{2g} \alpha_i \left(z_i \times \partial[-1, 1] \subset p_M^{-1}(S \times [-1, 1])\right)$$

in $p_M^{-1}(M \setminus (S\times] - 1, 1[))$, for some polynomials α_i of Λ_M, where $\delta(\beta)$ is coprime with the greatest common divisor d of the α_i. Then the Bézout Identity allows us to find a link L^* of $p_M^{-1}(S\times] - 1, 1[)$ such that

$$lk_e(L, L^*) = \frac{q_1 d}{\delta(\beta)}$$

for some $q_1 \in \mathbb{Q} \setminus \{0\}$. The Bézout Identity also implies the existences of u and v in $\mathbb{Z}[t_M, t_M^{-1}]$ such that

$$u\delta(\beta) + vd = q_2$$

for some $q_2 \in \mathbb{Q} \setminus \{0\}$. Then there exists a link L'^* homologous to $\overline{v}L^*$ such that

$$lk_e(L, L'^*) = \frac{q_1 q_2}{\delta(\beta)} - uq_1.$$

Adding to L'^* meridians of $\theta_M^n(L)$ transforms L'^* into L''^* such that $lk_e(L, L''^*) = \frac{q_1 q_2}{\delta(\beta)}$. Finally make L''^* connected by connecting components by bands that don't intersect $\cup_{n\in\mathbb{Z}}\theta_M^n(L)$ and make L connected in a similar way. $\diamond$

4 Comparing homology classes in $\tilde{C}_2(M)$

4.1 On the logarithmic derivative of the Alexander polynomial

Recall that $(z_i)_{i=1,\dots 2g}$ and $(z_i^*)_{i=1,\dots,2g}$ are two dual bases of $H_1(S;\mathbb{Z})$ such that $\langle z_i, z_j^*\rangle = \delta_{ij}$. We shall need the following proposition.

Proposition 4.1 *Let $\tilde{z}_i^*$, $\tilde{z}_i^+$ and $\tilde{z}_i^-$ denote the lifts of z_i^*, z_i^+ and z_i^- in the lift $(S\times[-1,1])_0$ of $S\times[-1,1]$ in $\tilde{M}$, respectively. Then*

$$\frac{t\Delta'(t)}{\Delta(t)} = \sum_{i=1}^{2g} lk_e(\frac{\tilde{z}_i^+ + \tilde{z}_i^-}{2}, \tilde{z}_i^*) = \sum_{i=1}^{2g} lk_e(\tilde{z}_i^+, \tilde{z}_i^*) - g = \sum_{i=1}^{2g} lk_e(\tilde{z}_i^-, \tilde{z}_i^*) + g.$$

Remark 4.2 The above proposition implies that if $(a_i, b_i)_{i\in\{1,\dots,g\}}$ is a symplectic basis of $H_1(S;\mathbb{Z})$, ($\langle a_i, a_j\rangle = \langle b_i, b_j\rangle = 0$, $\langle a_i, b_j\rangle = \delta_{ij}$), then

$$\frac{t\Delta'(t)}{\Delta(t)} = \sum_{i=1}^{g} \left(lk_e(\tilde{a}_i, \tilde{b}_i^+) - \overline{lk_e(\tilde{a}_i, \tilde{b}_i^+)} \right).$$

Indeed, according to the proposition,

$$\begin{aligned}\tfrac{t\Delta'(t)}{\Delta(t)} &= \textstyle\sum_{i=1}^g lk_e(\tilde{a}_i, \tilde{b}_i^+) - \sum_{i=1}^g lk_e(\tilde{b}_i, \tilde{a}_i^+) + g \\ &= \textstyle\sum_{i=1}^g lk_e(\tilde{a}_i, \tilde{b}_i^+) - \sum_{i=1}^g lk_e(\tilde{b}_i, \tilde{a}_i^+) + \sum_{i=1}^g lk_e(\tilde{b}_i, \tilde{a}_i^+ - \tilde{a}_i^-) \\ &= \textstyle\sum_{i=1}^g lk_e(\tilde{a}_i, \tilde{b}_i^+) - \sum_{i=1}^g lk_e(\tilde{b}_i^+, \tilde{a}_i).\end{aligned}$$

To prove Proposition 4.1, we shall prove the following proposition that gives another topological expression of the logarithmic derivative of the Alexander polynomial and the following lemma that may be of independent interest.

Proposition 4.3 *Let $\hat{z}_i^+$ and $\hat{z}_i^-$ denote the lifts of z_i^+ and z_i^- in the lift $(M\setminus S)_0$ of $M\setminus S$ in $\tilde{M}$, so that $\hat{z}_i^- = \tilde{z}_i^-$ and $\tilde{z}_i^+ = \theta_M(\hat{z}_i^+)$ with the notation of the previous statement. Then $(\hat{z}_i^+ - \hat{z}_i^-)_{i=1,\dots 2g}$ and $t_M^{-1/2}(\tilde{z}_i^+ - \tilde{z}_i^-)_{i=1,\dots 2g}$ are two bases of $H_1(p_M^{-1}(M\setminus S);\mathbb{Q})\otimes_{\Lambda_M}\mathbb{Q}(t_M)[t_M^{1/2}]$ and*

$$t_M^{-1/2}(\tilde{z}_j^+ - \tilde{z}_j^-) = \sum_{i=1}^{2g} a_{ij}(\hat{z}_i^+ - \hat{z}_i^-)$$

where

$$a_{ij} = t_M^{1/2} lk(z_j^+, z_i^*) - t_M^{-1/2} lk(z_j^-, z_i^*).$$

Let A denote the matrix $A = [a_{ij}]_{(i,j)\in\{1,\dots,2g\}^2}$. Then $\det(A) = \Delta(M)$, and the trace $tr(A^{-1})$ of A^{-1} is related to $\frac{\Delta'}{\Delta}$ as follows:

$$\frac{t_M\Delta'(t_M)}{\Delta(t_M)} = g\frac{t_M+1}{t_M-1} - \frac{tr(A^{-1})}{t_M^{1/2} - t_M^{-1/2}}.$$

Lemma 4.4 *Set* $A^{-1} = [b_{ij}]_{(i,j)\in\{1,\dots,2g\}^2}$, *then*

$$lk_e(\tilde{z}_j^+, \tilde{z}_i^*) = \frac{\delta_{ij}}{(1-t_M^{-1})} - \frac{b_{ij}}{(t_M^{1/2} - t_M^{-1/2})}, \quad lk_e(\tilde{z}_j^-, \tilde{z}_i^*) = \frac{\delta_{ij}}{(t_M - 1)} - \frac{b_{ij}}{(t_M^{1/2} - t_M^{-1/2})} \quad \textit{and}$$

$$lk_e(\frac{\tilde{z}_j^+ + \tilde{z}_j^-}{2}, \tilde{z}_i^*) = \frac{\delta_{ij}(t_M+1)}{2(t_M-1)} - \frac{b_{ij}}{(t_M^{1/2} - t_M^{-1/2})}.$$

Proof of Proposition 4.3: In $H_1(M \setminus S)$, $z_j^\varepsilon = \sum_{i=1}^{2g} lk(z_j^\varepsilon, z_i^*)(z_i^+ - z_i^-)$. Thus,

$$(\theta_M(\hat{z}_j^+) - \hat{z}_j^-) = \sum_{i=1}^{2g} \left(t_M lk(z_j^+, z_i^*) - lk(z_j^-, z_i^*)\right)(\hat{z}_i^+ - \hat{z}_i^-)$$

and $t_M^{-1/2}(\tilde{z}_j^+ - \tilde{z}_j^-) = \sum_{i=1}^{2g} a_{ij}(\hat{z}_i^+ - \hat{z}_i^-)$. Therefore, A is a transition matrix from one basis of the statement to the other one. Since it coincides with the matrix of the statement of Lemma 2.2, up to a linear transformation with determinant 1 on its rows, $\det(A) = \Delta(M)$.

Set $z = t_M^{1/2} - t_M^{-1/2}$.

$$\begin{aligned} a_{ij}(t_M) &= t_M^{1/2} lk(z_j^+, z_i^*) - t_M^{-1/2} lk(z_j^-, z_i^*) \\ &= z lk(z_j^+, z_i^*) + t_M^{-1/2} lk(z_j^+ - z_j^-, z_i^*) \\ &= z lk(z_j^+, z_i^*) + t_M^{-1/2}\delta_{ij}. \end{aligned}$$

$$\begin{aligned} t_M a'_{ij}(t_M) &= \tfrac{1}{2}(t_M^{1/2} + t_M^{-1/2}) lk(z_j^+, z_i^*) - \tfrac{1}{2} t_M^{-1/2}\delta_{ij} \\ &= \tfrac{t_M^{1/2}+t_M^{-1/2}}{2z} a_{ij}(t_M) - \tfrac{t_M^{1/2}+t_M^{-1/2}}{2z} t_M^{-1/2}\delta_{ij} - \tfrac{1}{2} t_M^{-1/2}\delta_{ij} \\ &= \tfrac{t_M^{1/2}+t_M^{-1/2}}{2z} a_{ij}(t_M) - \tfrac{1}{z}\delta_{ij}. \end{aligned}$$

Let Δ_{ii} denote the cofactor of (i,i) in the matrix A.

$$t_M \Delta'(M) = 2g \tfrac{t_M^{1/2}+t_M^{-1/2}}{2z} \Delta(M) - \tfrac{1}{z}\sum_{i=1}^{2g} \Delta_{ii}.$$

Recall $A^{-1} = [b_{ij}]_{(i,j)\in\{1,\dots,2g\}^2}$, $b_{ii} = \frac{\Delta_{ii}}{\Delta(M)}$, then

$$t_M \frac{\Delta'(M)}{\Delta(M)} = g\frac{t_M^{1/2} + t_M^{-1/2}}{z} - \frac{1}{z}\mathrm{tr}(A^{-1}).$$

Proposition 4.3 is proved. ◇

Proof of Lemma 4.4: We compute $lk_e(\tilde{z}_j^+, \tilde{z}_i^*)$, using that $lk_e(\tilde{z}_j^+ - \tilde{z}_j^-, \tilde{z}_i^*) = \delta_{ij}$.

$$\tilde{z}_j^+ = t_M \hat{z}_j^+ = \frac{\tilde{z}_j^+ - \tilde{z}_j^- - (\hat{z}_j^+ - \hat{z}_j^-)}{1 - t_M^{-1}}.$$

According to Proposition 4.3,

$$\hat{z}_j^+ - \hat{z}_j^- = \sum_{i=1}^{2g} b_{ij} t_M^{-1/2} (\tilde{z}_i^+ - \tilde{z}_i^-).$$

Thus, $(1 - t_M^{-1}) lk_e(\tilde{z}_j^+, \tilde{z}_i^*) = \delta_{ij} - b_{ij} t_M^{-1/2}$. The other expressions follow easily. $\diamond$

Proof of Proposition 4.1: Since $lk_e(\tilde{z}_i^+ - \tilde{z}_i^-, \tilde{z}_i^*) = 1$,

$$\sum_{i=1}^{2g} lk_e(\tilde{z}_i^+, \tilde{z}_i^*) - g = \sum_{i=1}^{2g} lk_e(\tilde{z}_i^-, \tilde{z}_i^*) + g = \sum_{i=1}^{2g} lk_e(\frac{\tilde{z}_i^+ + \tilde{z}_i^-}{2}, \tilde{z}_i^*).$$

According to Lemma 4.4,

$$\sum_{i=1}^{2g} lk_e(\frac{\tilde{z}_i^+ + \tilde{z}_i^-}{2}, \tilde{z}_i^*) = \frac{g(t_M + 1)}{(t_M - 1)} - \frac{\mathrm{tr}(A^{-1})}{(t_M^{1/2} - t_M^{-1/2})}$$

and Proposition 4.3 allows us to conclude. $\diamond$

4.2 Homology class of $[s_\tau(S)]$

Proposition 4.5 *Let $\tau : TM \to M \times \mathbb{R}^3$ be a trivialisation of TM, let S be a closed (oriented) surface of M, let $V \in S^2$ and let $s_\tau(S; V) = \tau^{-1}(S \times V) \subset ST(M)$ be a section induced by τ. Let $s_+(S)$ and $s_-(S)$ be the sections of $ST(M)_{|S}$ induced by the positive normal of S and the negative normal of S, respectively.*

$$[s_+(S)] = [s_\tau(S; V)] + \frac{\chi(S)}{2}[ST(*)]$$

$$[s_-(S)] = [s_\tau(S; V)] - \frac{\chi(S)}{2}[ST(*)]$$

Proof: First notice that the homology class of $s_\tau(S; V)$ in $ST(M)_\tau$ does not depend on V. It will be denoted by $[s_\tau(S)]$. Next, since the generator of $\pi_1(SO(3))$ can be realized by rotations around the V-axis and since $\pi_2(SO(3)) = 0$, the homology class of $s_\tau(S)$ in $ST(M)_{|S}$ does not depend on τ either.

Embed $S \times [-1, 1]$ in $\mathbb{R}^3$, then the tangent bundles of $S \times [-1, 1]$ in $\mathbb{R}^3$ and in M are isomorphic (they are both isomorphic to the direct sum of the tangent bundle of S and the trivial normal bundle).

Using the trivialisation τ of $ST(M)_{|S}$ induced by the standard trivialisation of $\mathbb{R}^3$, the positive normal section of $ST(S)$ is a map from S to S^2 that can be homotoped to a constant outside an open disk. Then $[s_+(S)] - [s_\tau(S)] \in H_2(D^2 \times S^2)$ and $[s_+(S)] - [s_\tau(S)] = c[ST(*)]$, where c is the degree of the Gauss map from S to S^2 that maps a point to the direction of the positive normal of S. $\diamond$

Proposition 4.6 *Let S be a closed (oriented) surface. Let S and S^+ be two copies of S, let $(z_i)_{i=1,\dots 2g}$ and $(z_i^*)_{i=1,\dots,2g}$ be two dual bases of $H_1(S;\mathbb{Z})$ such that $\langle z_i, z_j^*\rangle = \delta_{ij}$. Let $* \in S$. Let $diag(S\times S^+) = \{(x,x^+); x\in S\}$. We have the following equality in $H_2(S\times S^+)$*

$$[diag(S\times S^+)] = [*\times S^+] + [S\times *^+] + \sum_{i=1}^{2g}[z_i\times z_i^{*+}].$$

PROOF:

$$H_2(S\times S^+) = \mathbb{Z}[*\times S^+]\oplus\mathbb{Z}[S\times *^+]\oplus \bigoplus_{(i,j)\in\{1,2,\dots,2g\}^2}\mathbb{Z}[z_i\times z_j^{*+}].$$

The dual basis of the above basis with respect to the intersection form is

$$\left([S\times *^+],[*\times S^+],([z_i^*\times z_j^+])_{(i,j)\in\{1,2,\dots,2g\}^2}\right).$$

To get the coordinates of $[\mathrm{diag}(S\times S^+)]$ in the first decomposition we compute the intersection numbers with the second one. $\langle[\mathrm{diag}(S\times S^+)],[z_i^*\times z_i^+]\rangle = \pm 1$ where the tangent space to $\mathrm{diag}(S\times S^+)$ is naturally parametrized by (u_i,v_i^*,u_i,v_i^*) and the tangent space to $[z_i^*\times z_i^+]$ is naturally parametrized by $(0,w_i^*,x_i,0)$, so the intersection sign is the sign of the permutation $(u,v,w,x)\mapsto(u,w,x,v)$ which is $+1$. ◇

Theorem 4.7 *Let*

$$I_\Delta = \frac{1+t}{1-t} + \frac{t\Delta'(M)}{\Delta(M)}.$$

Let S be a surface generating $H_2(M)$, then

$$[s_\tau(S)] = I_\Delta[ST(*)].$$

PROOF: According to Propositions 3.3, 4.5 and 4.6, and to Example 3.4,

$$[s_\tau(S)] = \left(\frac{2}{1-t} - \frac{\chi(S)}{2} + \sum_{i=1}^{2g} lk_e(\tilde{z}_i,\tilde{z}_i^{*+})\right)[ST(*)],$$

with the notation of Proposition 4.1, according to which,

$$\sum_{i=1}^{2g} lk_e(\tilde{z}_i^-,\tilde{z}_i^*) = \frac{t\Delta'(t)}{\Delta(t)} - g.$$

◇

4.3 Homology classes of $H_*(C_2(M), \partial C_2(M); \mathbb{Q}(t))$

We use equivariant intersections with homology classes of $H_{(6-*)}(C_2(M), \partial C_2(M); \mathbb{Q}(t))$ to evaluate homology classes of $H_*(C_2(M); \mathbb{Q}(t))$.

The configuration space $C_2(M)$ is oriented like M^2, $\partial C_2(M)$ is oriented like $ST(M)$. Note that when A is in $H_{6-i}(C_2(M), \partial C_2(M); \mathbb{Q}(t))$ and when B is in $H_i(\partial C_2(M); \mathbb{Q}(t))$,

$$\langle A, B\rangle_{e,\tilde{C}_2(M)} = \langle \partial A, B\rangle_{e,\partial\tilde{C}_2(M)}.$$

Theorem 4.8 *Let $\tau : TM \to M \times \mathbb{R}^3$ be a trivialisation of TM and let $s_\tau(M) = \tau^{-1}(M \times V) \subset ST(M)$ be a section induced by τ. Then*

$$[s_\tau(M)] = I_\Delta[ST(K)].$$

In particular, there exists a 4–dimensional chain F with coefficients in $\mathbb{Q}(t)$ whose boundary is

$$\partial F = s_\tau(M) - I_\Delta ST(K).$$

Such a 4–chain satisfies

$$\langle ST(*), F\rangle_e = 1.$$

Proof: As a consequence of Theorem 4.7, $([s_\tau(S)] - I_\Delta[ST(*)])$ bounds a 3-dimensional chain $A_\tau(S)$ and

$$\begin{aligned}\langle ST(K), A_\tau(S)\rangle_{e,\tilde{C}_2(M)} &= -\overline{\langle A_\tau(S), [ST(K)]\rangle_{e,\tilde{C}_2(M)}} \\ &= -\overline{\langle [s_\tau(S)] - I_\Delta[ST(*)], [ST(K)]\rangle_{e,\partial\tilde{C}_2(M)}} = -1.\end{aligned}$$

On the other hand, since

$$\langle A_\tau(S), [s_\tau(M)]\rangle_{e,\tilde{C}_2(M)} = \langle [s_\tau(S)] - I_\Delta[ST(*)], [s_\tau(M)]\rangle_{e,\partial\tilde{C}_2(M)} = -I_\Delta,$$

$$\langle [s_\tau(M)], A_\tau(S)\rangle_{e,\tilde{C}_2(M)} = \overline{I_\Delta} = -I_\Delta.$$

This shows that $[s_\tau(M)] = I_\Delta[ST(K)]$ thanks to Proposition 2.12. The other assertions follow. $\diamond$

Lemma 4.9 *See the knot K as a map from $\mathbb{R}/\mathbb{Z}$ to M. Consider the continuous map*

$$\begin{array}{rll}\check{A}(K): & (S^1 = [0,1]/(0 \sim 1)) \times]0,1[& \to \quad C_2(M) \\ & (t, u \in]0,1[) & \mapsto \quad (K(t), K(t+u)),\end{array}$$

and its lift $A(K)\colon S^1 \times]0,1[\to \tilde{C}_2(M)$ such that the lift of $(K(t), K(t+\varepsilon))$ is in a small neighborhood of the canonical lift of the diagonal, for a small positive ε. This map $A(K)$ extends to the closed annulus $S^1 \times [0,1]$. Its extension, still denoted by $A(K)$, maps $S^1 \times \{0\}$ to the section

$s_{TK}(K)$ of $ST(K)$ given by the direction of the tangent vector to K and $A(K)(S^1 \times \{1\}) = \theta^{-1}(s_{(-TK)}(K))$. The image $A(K)$ of $A(K)$ is supported in $p^{-1}(K^2)$. Then the homology class $[A(K)]$ of $A(K)$ in $H_2(C_2(M), \partial C_2(M); \mathbb{Q}(t))$ satisfies:

$$\partial[A(K)] = [s_{TK}(K)] - t^{-1}[(s_{(-TK)}(K))]$$

and

$$\langle ST(S), A(K)\rangle_{e,\tilde{C}_2(M)} = 1 - t.$$

Proof:

$$\langle A(K), ST(S)\rangle_{e,\tilde{C}_2(M)} = \langle \partial A(K), ST(S)\rangle_{e,\partial\tilde{C}_2(M)} = 1 - t^{-1}.$$

◇

Proposition 4.10 *Let $\tau : TM \to M \times \mathbb{R}^3$ be a trivialisation of TM that maps the vectors of T^+K (tangent to K and directed by K) to $K \times \mathbb{R}^+\{W\}$, for some $W \in S^2$. Let K_X be a knot disjoint from K that is rationally homologous to K. Let $X \in (S^2 \setminus \{W, -W\})$ and let $s_\tau(M; X) = \tau^{-1}(M \times X) \subset \partial ST(M)$. Then there exists a 4–dimensional chain F_X with coefficients in $\mathbb{Q}(t)$ whose boundary is*

$$\partial F_X = s_\tau(M; X) - I_\Delta ST(K_X)$$

such that

$$\langle F_X, A(K)\rangle_{e,\tilde{C}_2(M)} = 0.$$

Furthermore, if F'_X is another such, then the class of $(F_X - F'_X)$ vanishes in $H_4(C_2(M); \mathbb{Q}(t))$.

Proof: According to Theorem 4.8, there exists a chain F such that $\partial F = s_\tau(M; X) - I_\Delta ST(K_X)$. Now,

$$F_X = F - \frac{\langle F, A(K)\rangle_{e,\tilde{C}_2(M)}}{(1-t)} ST(S)$$

is a 4– chain with the announced properties. The fact that $\langle F_X - F'_X, A(K)\rangle = 0$ guarantees that the second assertion is true, thanks to Lemma 4.9 and to Proposition 2.12. ◇

It will be proved in Section 12 that $\delta(M)(t-1)F_X$ can be furthermore assumed to be rational. See Proposition 12.7.

5 Definition of the invariant $\mathcal{Q}$

5.1 On the equivariant triple intersection in $\tilde{C}_2(M)$

Let C_X, C_Y and C_Z be three 4-dimensional rational chains in $\tilde{C}_2(M)$ whose boundaries are in $\partial\tilde{C}_2(M)$, whose projections $p(C_X)$, $p(C_Y)$ and $p(C_Z)$ in $C_2(M)$ are transverse, and such that (therefore) $p(C_X) \cap p(C_Y) \cap p(C_Z) \cap \partial C_2(M) = \emptyset$, define

$$\langle C_X, C_Y, C_Z \rangle_e = \sum_{(i,j)\in\mathbb{Z}^2} \langle C_X, \theta^{-i}(C_Y), \theta^{-j}(C_Z)\rangle_{\tilde{C}_2(M)} y^i z^j \in \mathbb{Q}[y^{\pm 1}, z^{\pm 1}]$$

This number does not change if C_X (resp. C_Y or C_Z) is replaced by a chain C'_X (resp. C'_Y or C'_Z) with the same boundary such that $(C'_X - C_X)$ (resp. $(C'_Y - C_Y)$ or $(C'_Z - C_Z)$) is a rationally null-homologous cycle.

Indeed, in this case $(C'_X - C_X)$ is the boundary of a 5-chain A whose projection can be assumed to be transverse to $p(C_Y)$ and $p(C_Z)$ and disjoint from $\partial C_2(M)$. Then

$$\big(C'_X \cap \theta^{-i}(C_Y) \cap \theta^{-j}(C_Z)\big) - \big(C_X \cap \theta^{-i}(C_Y) \cap \theta^{-j}(C_Z)\big) = \pm\partial(A \cap \theta^{-i}(C_Y) \cap \theta^{-j}(C_Z)).$$

Set

$$x = y^{-1}z^{-1}.$$

For any three one-variable polynomials f, g and h with rational coefficients,

$$\langle f(t)C_X, g(t)C_Y, h(t)C_Z\rangle_e = f(x)g(y)h(z)\langle C_X, C_Y, C_Z\rangle_e.$$

Therefore, the definition of the above *equivariant triple intersection* extends to chains Q_X, Q_Y and Q_Z of $C_2(M)$ with coefficients in $\mathbb{Q}(t)$, (whose boundaries are in $\partial\tilde{C}_2(M)$, and whose projections $p(Q_X)$, $p(Q_Y)$ and $p(Q_Z)$ in $C_2(M)$ are transverse) as follows. There exist $f(t)$, $g(t)$ and $h(t)$ in $\Lambda = \mathbb{Q}[t, t^{-1}]$ such that $f(t)Q_X$, $g(t)Q_Y$ and $h(t)Q_Z$ are rational chains and

$$\langle Q_X, Q_Y, Q_Z\rangle_e = \frac{1}{f(x)g(y)h(z)}\langle f(t)Q_X, g(t)Q_Y, h(t)Q_Z\rangle_e.$$

This equivariant triple intersection takes values in the field of fractions

$$\mathbb{Q}(y,z) = K(\mathbb{Q}[y^{\pm 1}, z^{\pm 1}]) = K\left(\frac{\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]}{xyz = 1}\right)$$

of $\mathbb{Q}[y^{\pm 1}, z^{\pm 1}]$, and it is such that for any f, g and h in $\mathbb{Q}(t)$,

$$\langle f(t)Q_X, g(t)Q_Y, h(t)Q_Z\rangle_e = f(x)g(y)h(z)\langle Q_X, Q_Y, Q_Z\rangle_e.$$

Lemma 5.1 *Set $P(y,z) = \langle Q_1, Q_2, Q_3\rangle_e$. Let σ be a permutation of $\{1,2,3\}$ and let β be the natural bijection from $\{1,2,3\}$ to $\{x,y,z\}$, $x = \beta(1)$, $y = \beta(2)$, $z = \beta(3)$. Then*

$$\langle Q_{\sigma(1)}, Q_{\sigma(2)}, Q_{\sigma(3)}\rangle_e = P(\beta\circ\sigma^{-1}(2), \beta\circ\sigma^{-1}(3)).$$

Proof: View the ring $\mathbb{Q}[y^{\pm 1}, z^{\pm 1}]$ as the quotient of the ring $\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ by the relation $xyz = 1$. Then if an equivariant intersection point ξ that contributes with a sign ε to $\langle Q_1, Q_2, Q_3 \rangle_e$ lifts as $\xi_0 \in \tilde{C}_2(M)$, and if $\xi_1 = \theta^{\alpha_1}(\xi_0) \in Q_1$, $\xi_2 = \theta^{\alpha_2}(\xi_0) \in Q_2$ and $\xi_3 = \theta^{\alpha_3}(\xi_0) \in Q_3$, then this point contributes as $c(\xi) = \varepsilon x^{\alpha_1} y^{\alpha_2} z^{\alpha_3}$. Note that ε does not depend on the order of Q_1, Q_2 and Q_3 and that this point will contribute to $\langle Q_{\sigma(1)}, Q_{\sigma(2)}, Q_{\sigma(3)} \rangle_e$ as $c_\sigma(\xi) = \varepsilon x^{\alpha_{\sigma(1)}} y^{\alpha_{\sigma(2)}} z^{\alpha_{\sigma(3)}}$. Then

$$c_\sigma(\xi) = \varepsilon \prod_{i=1}^{3} \beta(i)^{\alpha_{\sigma(i)}} = \varepsilon \prod_{i=1}^{3} \beta(\sigma^{-1}(i))^{\alpha_i}.$$

Assume without loss that $\alpha_1 = 0$. Then

$$c_\sigma(\xi) = \varepsilon \beta(\sigma^{-1}(2))^{\alpha_2} \beta(\sigma^{-1}(3))^{\alpha_3}.$$

$\diamond$

5.2 Equivariant triple intersection of representatives of the linking number

Proposition 5.2 *Assume that τ maps the vectors of T^+K to $K \times \mathbb{R}^+\{W\}$ for some $W \in S^2$. Let X, Y, Z be three distinct points in $(S^2 \setminus \{W, -W\})$, and let K_X, K_Y, K_Z and K be four disjoint knots in M that are rationally homologous to K. For $V \in S^2$, $s_\tau(M; V) = \tau^{-1}(M \times V) \subset \partial ST(M)$.*
There exist three 4–dimensional chains F_X, F_Y and F_Z of $C_2(M)$ with coefficients in $\mathbb{Q}(t)$, whose boundaries are $\partial F_X = s_\tau(M; X) - I_\Delta ST(K_X)$, $\partial F_Y = s_\tau(M; Y) - I_\Delta K_Y \times S^2$ and $\partial F_Z = s_\tau(M; Z) - I_\Delta K_Z \times S^2$, such that

$$\langle F_X, A(K) \rangle_e = \langle F_Y, A(K) \rangle_e = \langle F_Z, A(K) \rangle_e = 0.$$

Then $\langle F_X, F_Y, F_Z \rangle_e$ only depends on the isotopy class of the link (K_X, K_Y, K_Z, K), on the homotopy class of τ and on the trivialisation of the tubular neighborhood of K induced by $\tau_{|K}$. It will be denoted by $\mathcal{Q}(K_X, K_Y, K_Z, K, K_\parallel, \tau)$ where $K_\parallel$ is the parallel of K induced by $\tau_{|K}$. It will be denoted by $\mathcal{Q}(K, K_\parallel, \tau)$ when K_X, K_Y and K_Z are parallels of K with respect to the trivialisation induced by $K_\parallel$, living on the boundary of a tubular neighborhood of K.

Proof: The existence of the chain F_X and the fact that if a chain F'_X satisfies the same conditions, then $(F'_X - F_X)$ is null-homologous come from Proposition 4.10. Then according to the previous subsection, $\langle F_X, F_Y, F_Z \rangle_e$ only depends on ∂F_X, ∂F_Y, and ∂F_Z, and therefore on $(K_X, K_Y, K_Z, K, K_\parallel, \tau)$, W, X, Y and Z.

Hence, it is enough to see that $\langle F_X, F_Y, F_Z \rangle_e$ does not vary

- under a homotopy τ_t of τ such that $\tau_0(T^+K) = \tau_1(T^+K) = \mathbb{R}^+W$, and τ_0 and τ_1 induce the same parallelisation of K (where there is no loss in assuming that $\tau_t(T^+K) = \mathbb{R}^+W$ for any t, as we show at the end of this proof),

- under an isotopy of the link (K_X, K_Y, K_Z, K), extended to an isotopy $H\colon [0,1]\times M \to M$,
- under a rotation of $SO(3)$ that acts on W, X, Y, Z and τ, simultaneously,
- when (X, Y, Z) varies among triples of distinct points of $S^2 \setminus \{W, -W\}$.

Without loss, assume that F_X, F_Y and F_Z are products of their boundaries by the interval in a neighborhood $\partial\tilde{C}_2(M) \times [0,1]$ of $\partial\tilde{C}_2(M)$. Then each of these moves can be realized by global homeomorphisms of the whole picture in this neighborhood that change none of the intersection numbers $\langle F_X, A(K)\rangle_e$, $\langle F_Y, A(K)\rangle_e$, $\langle F_Z, A(K)\rangle_e$ and $\langle F_X, F_Y, F_Z\rangle_e$.

To conclude, let us prove that there is no loss in assuming that $\tau_t(T^+K) = \mathbb{R}^+W$ for any t, for the first item. It suffices to prove that –after a possible composition by a homotopy τ'' such that $\tau''_t(T^+K) = \mathbb{R}^+W$ for any t– τ_1 is obtained from τ_0 by a homotopy τ' such that $\tau'_t(T^+K) = \mathbb{R}^+W$ for any t.

The indermediate trivialisation τ_t is the composition of τ_0 with a map

$$\begin{array}{lcl} M\times\mathbb{R}^3 & \to & M\times\mathbb{R}^3 \\ (m,v) & \mapsto & (m, \rho(t,m)(v)) \end{array}$$

for a map $\rho\colon [0,1]\times M \to SO(3)$ that maps $\{0\}\times M$ to 1, and that maps $\{1\}\times K(S^1)$ to a loop of the group $SO(2)(W)$ of rotations with axis $\mathbb{R}W$. This loop is null-homotopic since τ_0 and τ_1 induce the same trivialisation of K. Then there is a homotopy τ'', such that $\tau''_t(T^+K) = \mathbb{R}^+W$ between τ_1 and a trivialisation τ'_1 such that $\tau'_1 = \tau_0$ on $TM_{|K}$, and we assume from now on that $\tau_1 = \tau_0$ on $TM_{|K}$ without loss. Change τ_t into $\rho(t, K(1))^{-1} \circ \tau_t$ again without loss so that $\tau_t = \tau_0$ at $K(1)$, for any t. Now, the restriction of ρ to $[0,1]\times K(S^1)$ maps

$$\big(\{0,1\}\times K(S^1)\big) \cup ([0,1]\times K(1))$$

to $1 \in SO(3)$, so that it can be seen as a map from $(S^2, *)$ to $(SO(3), 1)$, that is homotopic to the constant map with value 1 since $\pi_2(SO(3))$ is trivial.

See this homotopy as associated with a map

$$\begin{array}{rlcl} r\colon & [0,1]\times[0,1]\times K(S^1) & \to & SO(3) \\ & (0,t,K(z)) & \mapsto & \rho(t,K(z)) \\ & (1,t,K(z)) & \mapsto & 1 \\ & (u, t\in\{0,1\}, K(z)) & \mapsto & 1 \end{array}$$

as above. Let $N(K) = K\times D^2$ be a tubular neighborhood of K and see $D^2 = \{dz_D; d \in [0,1], z_D \in S^1\}$.

Then replace our homotopy induced by ρ by the homotopy induced by r' with

$$\begin{array}{rlcl} r'\colon & [0,1]\times M & \to & SO(3) \\ & (t, m\notin N(K)) & \mapsto & \rho(t,m) \\ & (t,(K(z),dz_D)) & \mapsto & r(d,t,K(z))^{-1}\circ\rho(t,(K(z),dz_D)). \end{array}$$

$\diamond$

5.3 An example in $S^1 \times S^2$

In this subsection, we compute $\mathcal{Q}(K, K_\parallel, \tau)$ when $M = S^1 \times S^2$, $K = S^1 \times W$, $K_\parallel = S^1 \times W'$, and $T(S^1 \times S^2) = S^1 \times \mathbb{R} \times TS^2 = S^1 \times (TS^2 \oplus \mathbb{R})$ is seen as the product by S^1 of the stabilisation $(TS^2 \oplus \mathbb{R})$ of the tangent bundle of S^2. The bundle $(TS^2 \oplus \mathbb{R})$ is the tangent bundle of $\mathbb{R}^3$ restricted to its unit sphere that is trivialised as such, using the obvious trivialisation of $T\mathbb{R}^3$.

Proposition 5.3 *Under the above assumptions,* $\mathcal{Q}(K, K_\parallel, \tau) = 0$.

Proof: In this case,

$$\widetilde{M^2} = S^1 \times (S^2 \times \mathbb{R} \times S^2)$$

where the projection map sends $(z, v \in S^2, \alpha \in \mathbb{R}, w \in S^2)$ to $((z, v), (z \exp(2i\pi\alpha), w)) \in M^2$. The structure of a product by S^1 extends to the configuration space $\tilde{C}_2(M)$ that reads $S^1 \times \tilde{C}_2(M)_0$. Note that the chain $A(K)$ of Lemma 4.9 also reads $S^1 \times A(K)_0$, for a 1-chain $A(K)_0$ in $\tilde{C}_2(M)_0$ (that is the closure of $\{W\} \times]0, 1[\times \{W\}$ in $\tilde{C}_2(M)_0$). For a point $X \in S^2 \setminus \{W, -W\}$, we are going to contruct a rational 3–chain f_X in $\tilde{C}_2(M)_0$ with boundary

$$\partial f_X = (t^{-1} - 1) s_\tau(\{1\} \times S^2; X) - (1 + t^{-1}) ST(\{1\} \times X)$$

explicitly, so that the boundary of $F_X = \frac{1}{1-t^{-1}}(S^1 \times f_X)$ will be $s_\tau(S^1 \times S^2; X) - \frac{1+t}{1-t} ST(S^1 \times \{X\})$. Furthermore, since the dimension of f_X is 3 and the dimension of $A(K)_0$ is 1 in the 5 dimensional-manifold $\tilde{C}_2(M)_0$, and since ∂f_X does not meet $A(K)_0$, we can assume that f_X does not meet $A(K)_0$. Therefore, we can use chains F_X, F_Y and F_Z that factor through S^1 to compute $\mathcal{Q}(K, K_\parallel, \tau)$. Since the projections of these chains in $\tilde{C}_2(M)_0$ are of codimension 2, again, we can assume that they have no equivariant triple intersection. Then F_X, F_Y and F_Z have no equivariant triple intersection. Thus, we are left with the construction of f_X to finish proving the proposition.

Consider the closure G_0 in $\tilde{C}_2(M)_0$ of (the natural lift of)

$$\{((1, v), (\exp(2i\pi\alpha), v)); \alpha \in]0, 1[, v \in S^2\}$$

$$\partial(G_0) = t^{-1} s_-(S^2) - s_+(S^2)$$

where $s_+(S^2)$ is the section of $ST(S^2 = \{1\} \times S^2)$ given by the direction of the S^1 factor and $s_-(S^2)$ is the opposite section.

We wish to change $s^+(S^2)$ to $s_\tau(S^2; X)$.

Let $v \in S^2$, note that $\tau(s_+(v)) = (v, v)$. If $v \neq -X$, there is a unique shortest arc $[X, v]$ of great circle from X to v. Let $s_{[X,+]}(S^2)$ denote the closure in $ST(S^2)$ of

$$\mathrm{Int}(s_{[X,+]}(S^2)) = \cup_{v \in S^2 \setminus \{-X\}} \tau^{-1}(\{v\} \times [X, v]).$$

Then

$$\partial s_{[X,+]}(S^2) = s_+(S^2) - s_\tau(S^2; X) - ST(-X).$$

About $ST(-X)$, let $D(-X)$ be a disk centered at $(-X)$, $D(-X) \setminus \{-X\} =]0,1] \times \partial D(-X)$, $ST(-X)$ meets $s_{[X,+]}(S^2)$ on its boundary as

$$-s_\tau(-X; lim_{\varepsilon \to 0} \cup_{v \in \partial(\varepsilon D(-X))} [X, v])$$

that is as $-ST(-X)$. Note that the sign in front of $ST(-X)$ can also be deduced from the fact that $s_+(S^2) - s_\tau(S^2; X)$ is homologous to $ST(-X)$.

Similarly, let $s_{[X,-]}(S^2)$ denote the closure in $ST(S^2)$ of $\cup_{v \in S^2 \setminus \{X\}} s_\tau(v; [X, -v])$.

$$\partial s_{[X,-]}(S^2) = s_-(S^2) - s_\tau(S^2; X) + ST(X).$$

Let α an arc of S^2 such that $\partial\alpha = \{-X\} - \{X\}$, $\partial ST(\alpha) = ST(-X) - ST(X)$.

Let

$$G = G_0 + s_{[X,+]}(S^2) - t^{-1} s_{[X,-]}(S^2) + ST(\alpha).$$
$$\partial G = (t^{-1} - 1) s_\tau(S^2; X) - ST(X) - t^{-1} ST(X)$$

and G can be transformed into a 3–chain f_X transverse to the boundary. $\diamond$

5.4 Symmetries

Proposition 5.4 *Under the hypotheses of Proposition 5.2, let $\mathcal{Q} = \mathcal{Q}(K_X, K_Y, K_Z, K, K_\parallel, \tau)$, then*

$$\mathcal{Q}(x, y, z) = \mathcal{Q}(x^{-1}, y^{-1}, z^{-1}).$$

Proof: Consider the involution ι of $\tilde{C}_2(M)$ induced by the involution that exchanges the two factors of $\tilde{M}^2$. This involution reverses the orientation of $\tilde{C}_2(M)$ and is such that $\iota\theta = \theta^{-1}\iota$. Let $\iota(F_X)$ inherit its orientation from F_X. Then

$$\partial\iota(F_X) = s_\tau(M; -X) - I_\Delta(t^{-1})(-ST(K_X)) = s_\tau(M; -X) - I_\Delta(t) ST(K_X)$$

and

$$\iota(A(K)) = -tA(K).$$

Therefore,

$$\mathcal{Q} = \langle \iota(F_X), \iota(F_Y), \iota(F_Z) \rangle_{e, \tilde{C}_2(M)}.$$

If an intersection point u contributes to $\langle F_X, F_Y, F_Z \rangle_{e, \tilde{C}_2(M)}$ as $\varepsilon x^a y^b z^c$, the point $\iota(u)$ contributes to $\langle \iota(F_X), \iota(F_Y), \iota(F_Z) \rangle_{e, \tilde{C}_2(M)}$ as $\varepsilon x^{-a} y^{-b} z^{-c}$. Let us check that the signs are indeed the same. The oriented normal of $\iota(F_X)$ in $\tilde{C}_2(M)$ is $-\iota(N(F_X))$. Therefore the sign for $\iota(u)$ will be positive if and only if $\iota(N(F_X)) \oplus \iota(N(F_Y)) \oplus \iota(N(F_Z))$ induces the orientation of $(-\tilde{C}_2(M))$ that is the orientation of $\iota(\tilde{C}_2(M))$. Hence, the sign for $\iota(u)$ will be positive if and only if the sign for u is positive. $\diamond$

Proposition 5.5 *Under the hypotheses of Proposition 5.2, let $\mathcal{Q} = \mathcal{Q}(K, K_\parallel, \tau)$, then*

$$\mathcal{Q}(x, y, z) = \mathcal{Q}(y, x, z) = \mathcal{Q}(x, z, y).$$

PROOF: This is a direct consequence of Lemma 5.1 and Proposition 5.2. $\diamond$

Proposition 5.6 *Under the hypotheses of Proposition 5.2, $\mathcal{Q}(K, K_\parallel, \tau)$ does not depend on the orientation of K.*

PROOF: When the orientation of K is reversed, t is changed into t^{-1}, and $A(K)$ is changed by a multiplication by a unit. Therefore, we may use the same chains F to compute $\mathcal{Q}$ and Proposition 5.4 allows us to conclude. $\diamond$

Proposition 5.7 *Let $(-M)$ denote the manifold obtained from M by reversing the orientation of M. Let τ_{-M} be a trivialisation of $T(-M)$ obtained from τ by a composition by a fixed orientation-reversing isomorphism of $\mathbb{R}^3$. Under the hypotheses of Proposition 5.2,*

$$\mathcal{Q}(K \subset (-M), K_\parallel, \tau_{-M}) = -\mathcal{Q}(K, K_\parallel, \tau).$$

PROOF:

$$\mathcal{Q}(K \subset (-M), K_\parallel, \tau_{-M}) = \langle -F_X, -F_Y, -F_Z \rangle_e$$

in $\tilde{C}_2(-M) = \tilde{C}_2(M)$. $\diamond$

5.5 Changing the manifold parallelisation

Any closed oriented 3-manifold M bounds an oriented compact 4-dimensional manifold W with signature 0. Then $TW_{|M} = \mathbb{R} \oplus TM$. A trivialisation τ of TM induces a trivialisation of $TW \otimes \mathbb{C}$ on M. The *first Pontrjagin class* $p_1(\tau)$ of such a trivialisation τ of the tangent bundle of M is the obstruction $p_1(W; \tau)$ to extend this trivialisation to W. It belongs to $H^4(W, M; \pi_3(SU(4))) \cong \mathbb{Z}$. We use the notation and conventions of [MS], see also [L1, Section 1.5].

Since $p_1(X) = 3\,\mathrm{signature}(X)$ for a closed (oriented) 4-manifold X, and since the signature of a closed 4-manifold obtained by gluing two 4-manifolds W_1 and W_2 along their common boundary is the sum of the signatures of W_1 and W_2 by Novikov additivity, $p_1(\tau)$ is well-defined.

Proposition 5.8 *Let τ and τ' be two trivialisations that induce the same parallelisation on K. Then*

$$\mathcal{Q}(K, K_\parallel, \tau') - \mathcal{Q}(K, K_\parallel, \tau) = \frac{p_1(\tau') - p_1(\tau)}{4}.$$

In particular

$$\mathcal{Q}(K, K_\parallel) = \mathcal{Q}(K, K_\parallel, \tau) - \frac{p_1(\tau)}{4}$$

does not depend on τ.

Remark 5.9 Since $\pi_2(SU(4)) = \{0\}$, the stabilised complexification of the trivialisation τ of $S^1 \times S^2$ involved in Proposition 5.3 extends to $S^1 \times B^3$ whose signature is zero. Therefore $p_1(\tau) = 0$, and

$$\mathcal{Q}((S^1 \times W, S^1 \times W') \subset S^1 \times S^2) = 0.$$

Proof of Proposition 5.8: Let $\tau: TM \to M \times \mathbb{R}^3$ be a parallelisation of M. Any other parallelisation τ' of M reads $\tau' = G \circ \tau$ for some

$$\begin{array}{rccc} G: & M \times \mathbb{R}^3 & \to & M \times \mathbb{R}^3 \\ & (m, v) & \mapsto & (m, (g(m))^{-1}(v)) \end{array}$$

associated with a map $g: M \to SO(3)$.

Since we work up to homotopy, we shall assume that G maps a tubular neighborhood $K \times D^2$ of K that contains K_X, K_Y and K_Z to 1.

Set $E = M \setminus \mathrm{Int}\,(K \times D^2)$. In $[0,3] \times ST(M)$, that is thought of as an external collar glued to $ST(M) = \{0\} \times ST(M)$, consider the 3-cycle

$$c_X = 1 \times G^{-1}(E \times X) - 0 \times (E \times X) - [0,1] \times (K \times S^1) \times X.$$

Lemma 5.10 *The homology class of c_X vanishes in $H_3([0,1] \times ST(E))$.*

Proof: The algebraic intersection with $[0,1] \times (S \setminus \mathrm{Int}(D^2)) \times_\tau Y$ determines the elements of $H_3\left([0,1] \times ST(E)\right) = \mathbb{Q}[\{1/2\} \times ST(K_\parallel)]$. In particular, to prove that the homology class of c_X is zero, it suffices to prove that the degree of $g(\cdot)(X): S \to S^2$ at Y is zero. Of course, this degree only depends on the homotopy class of the restriction of g to S. We may assume that Y is a regular point of $g(\cdot)(X)$. The changes of homotopy class of $g: S \to SO(3)$ may be realised by compositions of rotations with axis Y supported in neighborhoods of a geometric basis of $H_1(S)$ that avoid $g(\cdot)(X)^{-1}(Y)$ (since $\pi_2(SO(3)) = \{0\}$ and $\pi_1(SO(3))$ is generated by a path of rotations with axis Y). Therefore, the degree is independent of g and equal to the degree for a constant map to $1 \in SO(3)$ that is zero. This ends the proof of the lemma. $\diamond$

Back to the proof of Proposition 5.8, c_X is the boundary of a chain $C_{0,X}$ in $[0,1] \times ST\,(E)$, and

$$C_X = C_{0,X} + \big([0,1] \times (K \times D^2) \times X\big)$$

is a 4–chain in $[0,1] \times M \times_\tau S^2$ such that

$$\partial C_X = \{1\} \times s_{\tau'}(M; X) - \{0\} \times s_\tau(M; X)$$

and C_X meets $[0,1] \times (K \times D^2) \times_\tau S^2$ as $[0,1] \times (K \times D^2) \times_\tau X$.

Set

$$D_X = C_X + ([1,3] \times s_{\tau'}(M; X)) - I_\Delta(t)[0,3] \times ST(K_X),$$
$$D_Y = ([0,1] \times s_\tau(M; Y)) + C_Y + ([2,3] \times s_{\tau'}(M; Y)) - I_\Delta(t)[0,3] \times ST(K_Y),$$
$$D_Z = ([0,2] \times s_\tau(M; Z)) + C_Z - I_\Delta(t)[0,3] \times ST(K_Z),$$

where

- $C_Y \subset [1,2] \times ST(M)$, $C_Z \subset [2,3] \times ST(M)$
- $\partial C_Y = \{2\} \times s_{\tau'}(M;Y) - \{1\} \times s_\tau(M;Y)$, $\partial C_Z = \{3\} \times s_{\tau'}(M;Z) - \{2\} \times s_\tau(M;Z)$
- $C_Y \cap ([1,2] \times ST(K \times D^2)) = [1,2] \times (K \times D^2) \times_\tau \{Y\}$ and $C_Z \cap [2,3] \times ST(K \times D^2) = [2,3] \times (K \times D^2) \times_\tau \{Z\}$.

Now

$$\delta = \mathcal{Q}(K, K_\parallel, \tau') - \mathcal{Q}(K, K_\parallel, \tau) = \langle D_X, D_Y, D_Z \rangle_e.$$

First note that no triple intersection can occur in $[0,3] \times ST(K \times D^2)$. Then no triple intersection can occur in $([0,1] \cup [2,3]) \times ST(M)$ either and

$$\delta = \langle [1,2] \times s_{G \circ \tau}(M;X), C_Y, [1,2] \times s_\tau(M;Z) \rangle_e.$$

It does not depend on τ. Therefore δ will be denoted by $\delta(g)$. Then $\delta(g^2) = 2\delta(g)$. Since $\pi_1(SO(3)) = \mathbb{Z}/2\mathbb{Z}$ and $\pi_2(SO(3)) = 0$, g^2 is homotopic to a map that sends the complement of a 3-ball B^3 in M to the identity, that is homotopic to $[p^k_{SO(3)}]$ where $[p_{SO(3)}]$ is defined as follows. See

$$B^3 = [0, 2\pi] \times S^2/(0 \times S^2 \sim 0).$$

Then $p_{SO(3)}(\beta, V)$ is the rotation $R_{\beta,V}$ with angle β and whose axis is directed by V. (Indeed, S^3 is the quotient of B^3 by its boundary $\{2\pi\} \times S^2$, $p_{SO(3)}$ factors through S^3 to define the covering map $\overline{p_{SO(3)}}$ whose homotopy class generates $\pi_3(SO(3)) = \mathbb{Z}[\overline{p_{SO(3)}}]$.) Then $\delta(g) = \frac{k}{2}\delta(p_{SO(3)})$.

Lemma 5.11 $\delta(p_{SO(3)}) = 1$.

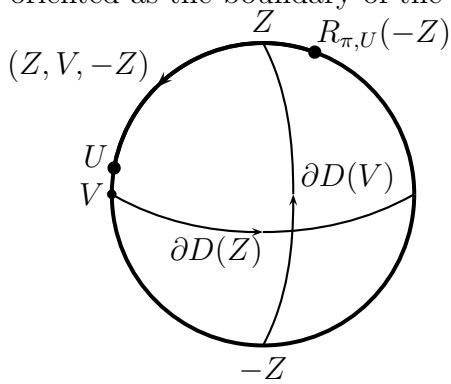
PROOF: When $g = p_{SO(3)}$, we can assume that

$$C_Y \cap \big([1,2] \times ST(M \setminus B^3)\big) = [1,2] \times (M \setminus B^3) \times_\tau \{Y\}.$$

Then

$$\delta(p_{SO(3)}) = \langle [1,2] \times B^3 \times_{\tau'} \{X\}, C_Y, [1,2] \times B^3 \times_\tau \{Z\} \rangle_{e, \mathbb{Z} \times [1,2] \times B^3 \times S^2}.$$

Assume that Z is the North Pole and that $X = -Z$ is the South Pole. Then $R_{\beta,V}(-Z) = Z$ if and only if $\beta = \pi$ and $V \in \partial D(Z)$, where $\partial D(Z)$ is the equator that is oriented as the boundary of the northern hemisphere $D(Z)$. Consider a point $V \in \partial D(Z)$, the great circle $(Z, V, -Z)$ containing V and the poles -oriented by $(Z, V, -Z)$- and the orthogonal great circle $\partial D(V)$ containing the poles oriented as the boundary of the hemisphere $D(V)$ containing V.



The normal bundle of $\pi \times \partial D(Z)$ in S^3 at V is $[0, 2\pi] \times (Z, V, -Z)$. Along $[0, 2\pi]$, near π, $R_{\beta,V}(-Z)$ moves along $\partial D(V)$. Along $(Z, V, -Z)$, near V, $R_{\pi,U}(-Z)$ moves along $(Z, V, -Z)$ near Z. Therefore the normal bundle of $\pi \times \partial D(Z)$ maps to S^2 in an orientation-preserving way and

$$\delta(p_{SO(3)}) = \langle [1, 2] \times \pi \times \partial D(Z) \times Z, C_Y \rangle_{[1,2] \times B^3 \times S^2}.$$

Set

$$C_{Y,\partial D(Z)} = C_Y \cap \big([1, 2] \times \pi \times \partial D(Z) \times S^2\big)$$

and $C_{Y,D(Z)} = C_Y \cap ([1, 2] \times \pi \times D(Z) \times S^2)$. Without loss, assume that the above intersections are transverse. $\delta(p_{SO(3)})$ is the degree of the projection of $C_{Y,\partial D(Z)}$ on S^2 at Z.

$$\partial C_{Y,\partial D(Z)} = (2 \times \pi \times \partial D(Z) \times_{\tau'} Y) - (1 \times \pi \times \partial D(Z) \times Y)\,.$$

Assume that $Y \in \partial D(Z)$. Then $\pi \times \partial D(Z) \times_{\tau'} Y$ projects to S^2 as the double cover of $\partial D(Z)$. In particular the degree of the projection of $C_{Y,\partial D(Z)}$ on S^2 is constant and equal to $\delta(p_{SO(3)})$ on the northern hemisphere, and, it is equal to $(\delta(p_{SO(3)}) - 2)$ on the southern hemisphere.

Since the degree of the projection on S^2 (via τ) of

$$\partial C_{Y,D(Z)} = \pm C_Y \cap \big((\partial[1, 2]) \times \pi \times D(Z) \times S^2\big) \pm C_{Y,\partial D(Z)}$$

vanishes, the degree of the projection of $C_{Y,\partial D(Z)}$ on S^2 coincides up to sign with the degree of the projection of $\pi \times D(Z) \times_{\tau'} Y$ that completely covers S^2 generically once. Therefore, $\delta(p_{SO(3)}) = 1$. $\diamond$

Similarly, define $p_1'(g) = p_1(G \circ \tau) - p_1(\tau)$. It is easy to see that this expression, that can be seen as an obstruction to extend a trivialisation on $[0, 3] \times M$, does not depend on τ and that p_1' is additive under the multiplication in $SO(3)$, so that $p_1'(g) = \frac{k}{2} p_1'(p_{SO(3)})$. According to Proposition 1.8 in [L1], $p_1'(p_{SO(3)}) = 4$. This ends the proof of Proposition 5.8. $\diamond$

6 Connected sum with a rational homology sphere

This section is devoted to prove Proposition 1.3 that asserts that, for a rational homology sphere N,

$$\mathcal{Q}(\mathbb{K} \subset M \sharp N) = \mathcal{Q}(\mathbb{K} \subset M) + 6\lambda(N).$$

The proof is based on the configuration space definition of the Casson-Walker invariant λ, that we summarize now.

6.1 A configuration space definition of the Casson-Walker invariant

For $r \in \mathbb{R}$, let $B(r)$ denote the ball of radius r in $\mathbb{R}^3$ that is equipped with its standard parallelisation τ_s. A rational homology sphere N may be written as $B_N \cup_{B(1)\setminus \mathrm{Int}(B(1/2))} B^3$ where B_N is a *rational homology ball*, that is a connected compact (oriented) smooth 3–manifold with boundary S^2 with the same rational homology as a point, B^3 is a 3-ball, B_N contains $(B(1) \setminus \mathrm{Int}(B(1/2)))$ as a neighborhood of its boundary $\partial B_N = \partial B(1)$, and B^3 contains $(B(1) \setminus \mathrm{Int}(B(1/2)))$ as a neighborhood of its boundary $\partial B^3 = -\partial B(1/2)$. Let $B(N) = B_N(3)$ be obtained from $B(3)$ by replacing the unit ball $B(1)$ of $\mathbb{R}^3$ by B_N. Equip $B(N)$ with a trivialisation τ_N that coincides with τ_s outside B_N.

Let W be a compact connected 4-manifold with signature 0 and with boundary

$$\partial W = B_N(3) \cup_{\{1\}\times\partial B(3)} (-[0,1] \times \partial B(3)) \cup_{\{0\}\times\partial B(3)} (-B(3)).$$

Define $p_1(\tau_N) \in (H^4(W, \partial W; \pi_3(SU(4))) = \mathbb{Z})$ as the obstruction to extend the trivialisation of $TW \otimes \mathbb{C}$ induced by τ_s and τ_N on ∂W to W. Again, we use the notation and conventions of [MS], see also [L1, Section 1.5].

Let $\mathbb{R}^3(N)$ be obtained from $\mathbb{R}^3$ by replacing its unit ball $B(1)$ by B_N. Let $C_2(\mathbb{R}^3(N))$ be obtained from $\mathbb{R}^3(N)^2$ by blowing-up the diagonal as in Subsection 1.2. Let $Q\colon C_2(\mathbb{R}^3(N)) \to \mathbb{R}^3(N)^2$ be the associated canonical projection and let $C_2(B(N)) = Q^{-1}(B(N)^2)$. Consider a smooth map $\chi\colon \mathbb{R} \to [0,1]$ that maps $]-\infty,-2]$ to 0 and $[-1,\infty[$ to 1. Define

$$\begin{array}{rlcl} p_{B(3)}\colon & B(3)^2 \setminus \text{diagonal} & \to & S^2 \\ & (U,V) & \mapsto & \frac{\chi(\|V\|-\|U\|)V - \chi(\|U\|-\|V\|)U}{\|\chi(\|V\|-\|U\|)V - \chi(\|U\|-\|V\|)U\|} \end{array}$$

This map extends to $C_2(B(3))$ to a map still denoted by $p_{B(3)}$, that reads as the projection to S^2 induced by τ_s (see Subsection 1.2) on the unit tangent bundle of $B(3)$. A similar map p_N can be defined on the boundary $\partial C_2(B(N))$: The map p_N is the projection to S^2 induced by τ_N on the unit tangent bundle of $B(N)$, and the map p_N is given by the above formula, where we set $\| U \| = 1$ when $U \in B_N$, for the other points of the boundary that are pairs (U,V) of $(B(N)^2 \setminus \text{diagonal})$ where U or V belongs to $\partial B(3)$ (therefore a possible point of B_N is replaced by $0 \in \mathbb{R}^3$ in the formula).

The following theorem, that gives a configuration space definition for the Casson-Walker invariant, is due to Kuperberg and Thurston [KT] for the case of integral homology spheres,

though it is stated in other words. It has been generalised to rational homology spheres in [L2, Section 6].

Theorem 6.1 *Let X, Y and Z be three distinct points of S^2. Under the above assumptions, for $V = X$, Y or Z, the 3–cycle $p_N^{-1}(V)$ of $\partial C_2(B(N))$ bounds a rational chain $F_{N,V}$ in $C_2(B(N))$, and*

$$\lambda(N) = \frac{\langle F_{N,X}, F_{N,Y}, F_{N,Z}\rangle_{C_2(B(N))}}{6} - \frac{p_1(\tau_N)}{24}.$$

Proof: This essentially comes from [L1, Sections 1.1 to 1.5] and [L2, Section 6.5] where is is shown that

$$\lambda(N) = \frac{\langle \Sigma_X, \Sigma_Y, \Sigma_Z\rangle_{C_2(N)}}{6} - \frac{p_1(\tau_N)}{24}$$

for chains Σ_V of a configuration space $C_2(N)$ with prescribed boundaries $p_N(\tau_N)^{-1}(V)$ associated to a given map $p_N(\tau_N)\colon \partial C_2(N) \to S^2$.

The configuration space $C_2(N)$ contains $C_2(B(N))$ and it has the same homotopy type; the map $p_N(\tau_N)$ plays the same role as the above map p_N. That is essentially why

$$\langle F_{N,X}, F_{N,Y}, F_{N,Z}\rangle_{C_2(B(N))} = \langle \Sigma_X, \Sigma_Y, \Sigma_Z\rangle_{C_2(N)}.$$

We nevertheless prove this with more details below.

The configuration space $C_2(N)$ is a compactification of

$$\mathrm{Int}(C_2(N)) = \left(\left(\mathbb{R}^3 \setminus \mathrm{Int}(B(1))\right) \cup_{\partial B(1)} B_N\right)^2 \setminus \text{diagonal}.$$

A point of $\partial C_2(N)$ may be viewed as a limit of a converging sequence $((U_n, V_n) \in \mathrm{Int}(C_2(N)))_{n\in\mathbb{N}}$. The map $p_N(\tau_N)$ maps such a limit to $\lim_{n\to\infty} \frac{V_n - U_n}{\|V_n - U_n\|}$. The compactification is defined so that this limit makes sense.

In order to finish our proof, it suffices to extend the map p_N to $C_2(N) \setminus \mathrm{Int}(C_2(B(N)))$ so that $p_N = p_N(\tau_N)$ on $\partial C_2(N)$. Indeed, this will allow us to set

$$\Sigma_V = F_{N,V} \cup p^{-1}_{N|C_2(N)\setminus \mathrm{Int}(C_2(B(N)))}(V)$$

and will make clear that the two algebraic intersections coincide.

Define a smooth map $\chi_2\colon \mathbb{R}^+ \to [0,1]$ such that $\chi_2(x) = 0$ if $x \le 3$, and $\chi_2(x) = 1$ if $x \ge 4$. and

$$\begin{array}{rccl} \chi_3\colon & (\mathbb{R}^+)^2 & \to & [0,1] \\ & (x,y) & \mapsto & (1-\chi_2(x))\chi(x-y) + \chi_2(x) \end{array}$$

so that $\chi_3(x,y) = 1$ if $x \ge 4$ or if $(x-y) \ge -1$. Then define

$$\begin{array}{rccl} p_N\colon & \mathrm{Int}(C_2(N)) \setminus \mathrm{Int}(C_2(B(N))) & \to & S^2 \\ & (U,V) & \mapsto & \frac{\chi_3(\|V\|,\|U\|)V - \chi_3(\|U\|,\|V\|)U}{\|\chi_3(\|V\|,\|U\|)V - \chi_3(\|U\|,\|V\|)U\|} \end{array}$$

where $\| U \| = 1$ when $U \in B_N$.

Then p_N coincides with the former p_N on $\partial C_2(B(N))$ because $\chi_2 = 0$ there. The pairs near the boundary of $C_2(N)$ are the pairs near the diagonal or the pairs of points where at least one point goes to ∞. For these pairs χ_3 is 1, unless one point is in $B_N(4)$ and the other one is near ∞, but in this case the limit does not see the point in $B_N(4)$, anyway. $\diamond$

6.2 Proof of Proposition 1.3

We keep all the notation from the previous subsection.

Consider the ball $B(3)$ of the previous subsection as a small ball embedded in M, outside a fixed neighborhood of K that contains all the needed parallels of K, so that the trivialisation τ of M coincides with the trivialisation τ_s on $B(3)$. We shall perform the connected sum by letting the rational homology ball B_N of the previous subsection replace $B(1)$. Therefore we normalize our chains F_V for M, over pairs of points that contain a point in $B(1)$.

Let $P \in \mathbb{Z}[t, t^{-1}]$ be such that $G_V = PF_V$ is an integral chain for $V = X$, Y and Z. Let m be the image of $0 \in B(3)$ under the implicit embedding of $B(3)$. Without loss, we successively assume that

- G_V coincides with $P.p_{B(3)}^{-1}(V)$ on $\coprod_{k\in\mathbb{Z}} \theta^k(C_2(B(3))) \subset \tilde{C}_2(M)$,
- G_V is transverse to the closures of $m \times (\tilde{M} \setminus p_M^{-1}(B(3)))$ and $(\tilde{M} \setminus p_M^{-1}(B(3))) \times m$,
- the intersection with these pieces read $\pm m \times \gamma_1(m; V)$ and $\pm\gamma_2(m; V) \times m$ for 1-manifolds $\gamma_1(m; V)$ and $\gamma_2(m; V)$ of $\tilde{M}$ whose boundaries are supported in $p_M^{-1}(3V)$ and $p_M^{-1}(-3V)$, respectively,
- G_V intersects the closures of $B(1) \times (\tilde{M} \setminus p_M^{-1}(B(3)))$ and $(\tilde{M} \setminus p_M^{-1}(B(3))) \times B(1)$ as $B(1) \times \gamma_1(m; V)$ and as $\gamma_2(m; V) \times B(1)$, respectively,
- the paths $\gamma_1(m; X)$, $\gamma_1(m; Y)$ and $\gamma_1(m; Z)$ are pairwise disjoint,
- the paths $\gamma_2(m; X)$, $\gamma_2(m; Y)$ and $\gamma_2(m; Z)$ are pairwise disjoint.

These conditions can be successively achieved by small perturbations without losing the former ones. Then let $F_{\sharp N,V}$ be the 4-chain of $\tilde{C}_2(M\sharp N)$ that coincides

- with $F_{N,V} \subset (C_2(B(N)) = \{0\} \times C_2(B(N))) \subset \Big(\mathbb{Z} \times C_2(B(N)) = \tilde{C}_2(B(N) = B_N(3))\Big)$ on $\tilde{C}_2(B(N))$,
- with $\frac{1}{P}(B_N \times \gamma_1(m; V))$ on the closure of $B_N \times (\tilde{M} \setminus p_M^{-1}(B(3)))$,
- with $\frac{1}{P}(\gamma_2(m; V) \times B_N)$ on the closure of $(\tilde{M} \setminus p_M^{-1}(B(3))) \times B_N$,
- with F_V, on the closure E of the complement of the above subsets.

Then $\mathcal{Q}(\mathbb{K} \subset M) = \langle F_X, F_Y, F_Z \rangle_e - \frac{p_1(\tau)}{4}$ where the F_V only intersect on E under the above assumptions, and

$$\mathcal{Q}(\mathbb{K} \subset M \sharp N) = \langle F_{\sharp N,X}, F_{\sharp N,Y}, F_{\sharp N,Z} \rangle_e - \frac{p_1(\tau_{M\sharp N})}{4}$$

where $\tau_{M\sharp N}$ is the trivialisation induced by τ and τ_N that satisfies $p_1(\tau_{M\sharp N}) = p_1(\tau) + p_1(\tau_N)$ and

$$\langle F_{\sharp N,X}, F_{\sharp N,Y}, F_{\sharp N,Z} \rangle_e = \langle F_X, F_Y, F_Z \rangle_{e,E} + \langle F_{N,X}, F_{N,Y}, F_{N,Z} \rangle.$$

This shows that $\mathcal{Q}(\mathbb{K} \subset M \sharp N) = \mathcal{Q}(\mathbb{K} \subset M) + 6\lambda(N)$. $\diamond$

7 First variations

7.1 Preliminary computations

To compute these triple intersections or their variations, we summarize some arguments that we shall frequently use. Recall that a neighborhood of $\partial\tilde{C}_2(M)$ in $\tilde{C}_2(M)$ is diffeomorphic to $\mathbb{Z} \times ST(M) \times [0,1]$ where $[0,1]$ is the inward normal, and $ST(M) =_\tau M \times S^2$. (All the diffeomorphisms preserve the orientation, and the order of appearance of coordinates induces the orientation.) Near its boundary, F_X can also be assumed to be diffeomorphic to $\partial F_X \times [0,1]$, that is $s_\tau(M;X)\times[0,1] - I_\Delta(t)ST(K_X)\times[0,1]$ where the oriented normal of $s_\tau(M;X)$ in $\partial\tilde{C}_2(M)$ coincides with the oriented normal of $s_\tau(M;X) \times [0,1]$ in $\tilde{C}_2(M)$ and is (the oriented tangent space of) $ST(*)$ and the oriented normal of $ST(K_X)$ is the surface S. In particular, the oriented intersection of $s_\tau(M;X) \times [0,1]$ and $ST(K_Y) \times [0,1]$ is $s_\tau(K_Y;X) \times [0,1]$, that is cooriented by $ST(S)$.

The following lemmas easily follow from these considerations.

Lemma 7.1

$$\langle ST(S), F_Y, F_Z\rangle_e = -I_\Delta(y) - I_\Delta(z).$$
$$\langle F_X, ST(S), F_Z\rangle_e = -I_\Delta(x) - I_\Delta(z)(= I_\Delta(yz) - I_\Delta(z)).$$
$$\langle F_X, F_Y, ST(S)\rangle_e = -I_\Delta(x) - I_\Delta(y)(= I_\Delta(yz) - I_\Delta(y)).$$

◇

Lemma 7.2 *Let C_X be a 4-dimensional chain in $\partial\tilde{C}_2(M)$ whose boundary does not meet*

$$\mathbb{Z} \times (s_\tau(K_Y;Z) \cup s_\tau(K_Z;Y) \cup s_{TK}(K) \cup s_{-TK}(K))$$

$$\langle C_X, F_Y, F_Z\rangle_e = -I_\Delta(y)\langle C_X, s_\tau(K_Y;Z)\rangle_{e,\mathbb{Z}\times ST(M)} - I_\Delta(z)\langle C_X, s_\tau(K_Z;Y)\rangle_{e,\mathbb{Z}\times ST(M)}$$

where the variable t in the equivariant intersection numbers of the right-hand side should be replaced by x.

$$\langle C_X, A(K)\rangle_e = \langle C_X, s_{TK}(K)\rangle_{e,\mathbb{Z}\times ST(M)} - t\langle C_X, s_{-TK}(K)\rangle_{e,\mathbb{Z}\times ST(M)}$$

Proof: Recall that $\partial A(K) = s_{TK}(K) - t^{-1}s_{-TK}(K)$ from Lemma 4.9. ◇

In the above statement, the roles of X, Y and Z can be permuted to give similar expressions for

$$\langle F_X, C_Y, F_Z\rangle_e = -I_\Delta(x)\langle C_Y, s_\tau(K_X;Z)\rangle_{e,\partial\tilde{C}_2(M)} - I_\Delta(z)\langle C_Y, s_\tau(K_Z;X)\rangle_{e,\partial\tilde{C}_2(M)}$$

and

$$\langle F_X, F_Y, C_Z\rangle_e = -I_\Delta(x)\langle C_Z, s_\tau(K_X;Y)\rangle_{e,\partial\tilde{C}_2(M)} - I_\Delta(y)\langle C_Z, s_\tau(K_Y;X)\rangle_{e,\partial\tilde{C}_2(M)}$$

since the triple intersection of codimension 2 chains does not depend on the order other than for assignments of the variables x, y and z.

Lemma 7.3 *Let Σ_X be a rational chain in M whose boundary $(K'_X - K_X)$ is disjoint from (K_Y, K_Z, K). Let*

$$F'_X = F_X - I_\Delta(t)ST(\Sigma_X) + I_\Delta(t)\langle \Sigma_X, K\rangle_M ST(S).$$

Then

$$\partial F'_X = s_\tau(M;X) - I_\Delta K'_X \times S^2 \ \text{ and } \ \langle F'_X, A(K)\rangle_e = 0.$$

$$\langle F'_X, F_Y, F_Z\rangle_e - \langle F_X, F_Y, F_Z\rangle_e =$$

$$I_\Delta(x)\langle \Sigma_X, K\rangle_M(-I_\Delta(y) - I_\Delta(z)) + I_\Delta(x)(I_\Delta(y)\langle \Sigma_X, K_Y\rangle_M + I_\Delta(z)\langle \Sigma_X, K_Z\rangle_M).$$

Proof: Recall that $\langle ST(S), A(K)\rangle_e = 1 - t$. $\diamond$

7.2 Changing the knot trivialisation

Consider the map

$$\begin{array}{llll} g\colon & S^1 & \to & SO(3) \\ & \exp(i\alpha) & \mapsto & \rho(W, \alpha + \pi) \end{array}$$

where $\rho(W, \alpha)$ is the rotation with oriented axis W and with angle α. The homotopy class of g generates $\pi_1(SO(3))$. Also consider the map f from M to S^1 whose restriction on K has degree one, such that $f(M \setminus (S\times]-1,1[)) = -1$ and $f(\sigma \in S, \beta \in]-1,1[) = \exp(i\pi\beta)$. This induces

$$\begin{array}{llll} h\colon & M \times \mathbb{R}^3 & \to & M \times \mathbb{R}^3 \\ & (m, v) & \mapsto & (m, (g \circ f(m))^{-1}(v)). \end{array}$$

Let $\tau' = h \circ \tau\colon TM \to M \times \mathbb{R}^3$. The parallel $K'_\parallel$ induced by τ' is obtained from the parallel induced by K by adding a positive meridian of K. Note that τ and τ' have the same Pontrjagin class since the change of parallelisations is associated with a map from S^1 to $SO(3)$ and $\pi_1(SU(4))$ is trivial.

In this subsection, we shall prove the following proposition.

Proposition 7.4 *Under the above hypotheses,*

$$\mathcal{Q}(K, K'_\parallel) - \mathcal{Q}(K, K_\parallel) = \mathcal{Q}(K, K'_\parallel, \tau') - \mathcal{Q}(K, K_\parallel, \tau) = \sum_{\mathfrak{S}_3(x,y,z)} \left(-\frac{x\Delta'(x)}{2\Delta(x)}\right) I_\Delta(y)$$

Remark 7.5 The triple $(S^1 \times S^2, S^1 \times \{W\}, S^1 \times \{W'\})$ is equivalent to the triple $(S^1 \times S^2, S^1 \times \{W\}, \{(\exp(i\beta) \in S^1, R_{\beta,W}(W'))\})$.

This is consistent with Proposition 7.4 that ensures that $\mathcal{Q}$ does not vary when $\Delta = 1$.

$$\mathcal{Q}(K, K_\parallel, \tau) = \mathcal{Q}(K_X, K_Y, K_Z, K, K_\parallel, \tau).$$

To prove the proposition, we compute

$$\delta_1 = \mathcal{Q}(K_X, K_Y, K_Z, K, K'_\parallel, \tau') - \mathcal{Q}(K, K_\parallel, \tau)$$

in Lemma 7.6 and

$$\delta_2 = \mathcal{Q}(K, K'_\parallel, \tau') - \mathcal{Q}(K_X, K_Y, K_Z, K, K'_\parallel, \tau')$$

in Lemma 7.7.

Lemma 7.6 *Let* $\delta_1 = \mathcal{Q}(K_X, K_Y, K_Z, K, K'_\parallel, \tau') - \mathcal{Q}(K, K_\parallel, \tau)$

$$\delta_1 = \sum_{\mathfrak{S}_3(x,y,z)} \frac{1+x}{2(1-x)} I_\Delta(y).$$

Proof: We first change F_X to $F_X + C_X$ where $\partial C_X = s_{\tau'}(M; X) - s_\tau(M; X)$. To construct such a C_X, define

$$\begin{array}{rlcl} g_X\colon & [0,1] \times S^1 & \to & S^2 \\ & (t, z) & \mapsto & g_X(t, z) \end{array}$$

where $g_X(0, z) = g_X(t, -1) = X$ and $g_X(1, z) = g(z)(X)$ and the image of g_X is the disk $D_X(W)$ of S^2 bounded by the circle with center W through X, g_X has degree one at W. Then the cobordism

$$\begin{array}{rlcl} C_X\colon & [0,1] \times M & \to & M \times S^2 \\ & (t, m) & \mapsto & (m, g_X(t, f(m))) \end{array}$$

in $ST(M) =_\tau M \times S^2$ satisfies

$$\partial C_X = s_{\tau'}(M; X) - s_\tau(M; X),$$

and, according to Lemma 4.9,

$$\langle C_X, A(K)\rangle_e = \langle C_X, s_\tau(K; W)\rangle_{e,\partial\tilde{C}_2(M)} - t\langle C_X, s_\tau(K; -W)\rangle_{e,\partial\tilde{C}_2(M)} = \langle C_X, s_\tau(K; W)\rangle_{e,\partial\tilde{C}_2(M)}$$

since the image of g_X does not meet $-W$. Assume without loss that K meets $S \times [-1, 1]$ as $\{\sigma\} \times [-1, 1]$. Then $\langle C_X, s_\tau(K; W)\rangle_{e,\partial\tilde{C}_2(M)}$ is the degree at W of the restriction of C_X to $[0, 1] \times \{\sigma\} \times [-1, 1]$ that is 1.

Let

$$F'_X = F_X + C_X - \frac{1}{1-t} ST(S).$$

Then $\langle F'_X, A(K)\rangle = 0$.

Assume that X, Y and Z lie on different circles around W so that $D_X(W) \subsetneq D_Y(W) \subsetneq D_Z(W)$ and define F'_Y and F'_Z, similarly. Then

$$\begin{aligned} \delta_1 &= \langle F'_X, F'_Y, F'_Z\rangle_e - \langle F_X, F_Y, F_Z\rangle_e \\ &= \langle F_X, F_Y, C_Z\rangle_e + \langle F_X, C_Y, F'_Z\rangle_e + \langle C_X, F'_Y, F'_Z\rangle_e + \delta'_1 \end{aligned}$$

where $\delta_1' = \sum_{\circlearrowleft} \frac{1}{1-x}(I_\Delta(y) + I_\Delta(z))$, according to Lemma 7.1. The symbol $\sum_{\circlearrowleft}$ stands for the sum of the three terms obtained from the written one by permuting X, Y and Z, cyclically.

Now, according to Lemma 7.2,

$$\langle C_X, F_Y', F_Z' \rangle_e = -I_\Delta(y)\langle C_X, s_{\tau'}(K_Y; Z)\rangle_{e,\mathbb{Z}\times ST(M)} - I_\Delta(z)\langle C_X, s_{\tau'}(K_Z; Y)\rangle_{e,\mathbb{Z}\times ST(M)} = 0$$

since the image of g_X does not meet the circles centered at W through Y and Z.

$$\begin{aligned}\langle F_X, F_Y, C_Z \rangle_e &= -I_\Delta(x)\langle C_Z, s_\tau(K_X; Y)\rangle_{e,\mathbb{Z}\times ST(M)} - I_\Delta(y)\langle C_Z, s_\tau(K_Y; X)\rangle_{e,\mathbb{Z}\times ST(M)} \\ &= -I_\Delta(x) - I_\Delta(y)\end{aligned}$$

since $\langle C_Z, s_\tau(K_X; Y)\rangle_{e,\mathbb{Z}\times ST(M)}$ and $\langle C_Z, s_\tau(K_Y; X)\rangle_{e,\mathbb{Z}\times ST(M)}$ are as before the degrees of g_Z, at Y and X, that are one. Similarly,

$$\langle F_X, C_Y, F_Z' \rangle_e = -I_\Delta(z)\langle C_Y, s_\tau(K_Z; X)\rangle_{e,\mathbb{Z}\times ST(M)} = -I_\Delta(z).$$

$$\delta_1 = \sum_{\circlearrowleft} \left(\frac{1}{1-x} - \frac{1}{2}\right)(I_\Delta(y) + I_\Delta(z)).$$

◇

Lemma 7.7 *Let* $\delta_2 = \mathcal{Q}(K, K_\parallel', \tau') - \mathcal{Q}(K_X, K_Y, K_Z, K, K_\parallel', \tau')$

$$\delta_2 = -\sum_{\mathfrak{S}_3(x,y,z)} \frac{1}{2} I_\Delta(x) I_\Delta(y).$$

Proof: Let $D^2 \times K$ be a tubular neighborhood of K trivialised by $\tau_{|K}$ where D^2 is the unit disk of $\mathbb{C}$. Assume $K_X = 1/4 \times K$, $K_Y = 1/2 \times K$ and $K_Z = 3/4 \times K$. Then let $K_X' \subset 1/4 S^1 \times K$ be a parallel of K such that $(K_X' - K_X)$ is homologous to a positive meridian of K in $D^2 \times K$, and let Σ_X be an annulus transverse to K in $1/4 D^2 \times K$ whose boundary is $(K_X' - K_X)$. Then $\langle \Sigma_X, K\rangle_M = 1$. Similarly define (Σ_Y, K_Y') and (Σ_Z, K_Z') by replacing $1/4$ by $1/2$ and $3/4$, respectively.

$$\mathcal{Q}(K, K_\parallel', \tau') = \mathcal{Q}(K_X', K_Y', K_Z', K, K_\parallel', \tau')$$

Use Lemma 7.3 and define

$$F_X'' = F_X' - I_\Delta(t) ST(\Sigma_X) + I_\Delta(t)\langle \Sigma_X, K\rangle_M ST(S).$$

and its twin brothers F_Y'', F_Z''. Then $\delta_2 = \sum_{\circlearrowleft} I_\Delta(x)(-I_\Delta(y) - I_\Delta(z)) + \delta_2'$ where

$$\begin{aligned}\delta_2' &= I_\Delta(x)(I_\Delta(y)\langle \Sigma_X, K_Y\rangle_M + I_\Delta(z)\langle \Sigma_X, K_Z\rangle_M) \\ &\quad + I_\Delta(y)(I_\Delta(x)\langle \Sigma_Y, K_X'\rangle_M + I_\Delta(z)\langle \Sigma_Y, K_Z\rangle_M) \\ &\quad + I_\Delta(z)(I_\Delta(x)\langle \Sigma_Z, K_X'\rangle_M + I_\Delta(y)\langle \Sigma_Z, K_Y'\rangle_M) \\ &= I_\Delta(y) I_\Delta(x) + I_\Delta(z)(I_\Delta(x) + I_\Delta(y)).\end{aligned}$$

◇

Remark 7.8 Proposition 7.4 could also be seen as a consequence of the Dehn surgery formula (Theorem 1.4 that will be proved independently later) by seeing the variation of $\mathcal{Q}$ under the addition of a meridian to the parallel of K as the result of a (-1)–surgery along the meridian of K. Then the Dehn surgery formula could be applied with the Seifert surface $S \setminus \mathrm{Int}(D^2)$. I checked that this gives the same result when the genus of S is one, but the above proof of Proposition 7.4 looks simpler even for the genus one case.

8 Variation of $\mathcal{Q}(\mathbb{K})$ under a general two–dimensional cobordism

This section is devoted to the proof of Theorem 1.5.

8.1 Beginning the proof of Theorem 1.5

In this subsection and the next one, we prove the first part of this theorem that is the following proposition, thanks to Proposition 5.6.

Proposition 8.1 *Let $\mathbb{K} = (K, K_\parallel)$ and $\mathbb{K}' = (K', K'_\parallel)$ be two framed knots that represent the preferred generator of $H_1(M)/Torsion$. Then there exists an antisymmetric polynomial $\mathcal{V}(\mathbb{K}, \mathbb{K}')$ in $\mathbb{Q}[t, t^{-1}]$ such that*

$$\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}) = \sum_{\mathfrak{S}_3(x,y,z)} \frac{\mathcal{V}(\mathbb{K}, \mathbb{K}')(x)}{\delta(M)(x)} I_\Delta(y).$$

Let K and K' be two knots that represent the preferred generator of $H_1(M)/\text{Torsion}$, and let τ be a trivialisation of TM that maps the unit tangent vectors of K and K' that induce the orientation of K and K' to $W \in S^2$. Let $K_\parallel$ and $K'_\parallel$ be the parallels of K and K' induced by τ. (The parallels can be chosen arbitrarily, thanks to Proposition 7.4.) Let $\mathbb{K} = (K, K_\parallel)$ and $\mathbb{K}' = (K', K'_\parallel)$.

This subsection is devoted to the computation of

$$\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}) = \mathcal{Q}(K', K'_\parallel, \tau) - \mathcal{Q}(K, K_\parallel, \tau).$$

There exists a rational 2–chain B such that

$$\partial B = K' - K.$$

Let $N(K)$ and $N(K')$ be tubular neighborhoods of K and K'. Then

$$[\partial(B \cap (M \setminus \text{Int}(N(K) \sqcup N(K')))] = [K'_\parallel + r'm(K')] - [K_\parallel + rm(K)]$$

where $m(K)$ and $m(K')$ denote meridians of K and K', respectively and the brackets stand for homology classes in $\partial(N(K) \sqcup N(K'))$.

Let $C(B; \tau)$ be the following 2–cycle of $\tilde{C}_2(M)$,

$$C(B; \tau) = A(K') - A(K) + t^{-1} s_\tau(B; -W) - s_\tau(B; W)$$

where $A(K)$ is defined in Lemma 4.9. Define $Q(B; \tau) \in \mathbb{Q}(t)$ so that

$$[C(B; \tau)] = (1 - t^{-1}) Q(B; \tau)(t^{-1})[ST(*)]$$

in $H_2(C_2(M); \mathbb{Q}(t))$.

Proposition 8.2 *Under the assumptions above,*

$$\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}) = \sum_{\mathfrak{S}_3(x,y,z)} \left(Q(B;\tau)(x) + \frac{r'-r}{2} I_\Delta(x) \right) I_\Delta(y).$$

Proof: Observe that $\langle B, K_X\rangle = \langle B, K_Y\rangle = \langle B, K_Z\rangle = r$ and that $\langle B, K'_X\rangle = \langle B, K'_Y\rangle = \langle B, K'_Z\rangle = -r'$. Recall $\partial A(K) = s_{TK}(K) - t^{-1}s_{-TK}(K)$ and $\langle ST(S), A(K)\rangle_e = 1-t$. Let Σ_X be a 2–chain obtained from B by a natural small isotopy such that $\partial\Sigma_X = K'_X - K_X$. Then $\langle \Sigma_X, K\rangle = r$ and $\langle \Sigma_X, K'\rangle = -r'$.

Let $F^1_X = F_X - I_\Delta(t)ST(\Sigma_X)$ where $\langle A(K), F_X\rangle_e = 0$. Then

$$\begin{aligned}
\langle A(K'), F^1_X\rangle &= \langle A(K'), F_X\rangle_e + I_\Delta(t)\langle A(K'), ST(\Sigma_X)\rangle_e \\
&= \langle A(K'), F_X\rangle_e + I_\Delta(t)(1-t^{-1})\langle \Sigma_X, K'\rangle \\
&= \langle C(B;\tau) + A(K) - t^{-1}s_\tau(B;-W) + s_\tau(B;W), F_X\rangle_e - r'I_\Delta(t)(1-t^{-1}) \\
&= (1-t^{-1})\left(Q(B;\tau)(t^{-1}) - r'I_\Delta(t) + I_\Delta(t)\langle B, K_X\rangle\right) \\
&= (1-t^{-1})\left(Q(B;\tau)(t^{-1}) - (r-r')I_\Delta(t^{-1})\right).
\end{aligned}$$

Then

$$F'_X = F^1_X + ((r-r')I_\Delta(t) - Q(B;\tau)(t))\, ST(S)$$

satisfies $\langle A(K'), F'_X\rangle = 0$. Set $Q(t) = Q(B;\tau)(t) + (r'-r)I_\Delta(t)$. Define Σ_Y, Σ_Z, $F^1_Y = F_Y - I_\Delta(t)ST(\Sigma_Y)$ and $F^1_Z = F_Z - I_\Delta(t)ST(\Sigma_Z)$, similarly. Then

$$\begin{aligned}
\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}) &= \langle F'_X, F^1_Y - Q(t)ST(S), F^1_Z - Q(t)ST(S)\rangle_e - \langle F_X, F_Y, F_Z\rangle_e \\
&= \textstyle\sum_{\circlearrowleft} Q(x)(I_\Delta(y) + I_\Delta(z)) + \delta_3 = \sum_{\mathfrak{S}_3(x,y,z)} Q(x)I_\Delta(y) + \delta_3
\end{aligned}$$

according to Lemma 7.1, where

$$\begin{aligned}
\delta_3 &= -I_\Delta(x)\langle ST(\Sigma_X), F^1_Y, F^1_Z\rangle_e - I_\Delta(y)\langle F_X, ST(\Sigma_Y), F^1_Z\rangle_e - I_\Delta(z)\langle F_X, F_Y, ST(\Sigma_Z)\rangle_e \\
&= I_\Delta(x)\left(I_\Delta(y)\langle \Sigma_X, K'_Y\rangle + I_\Delta(z)\langle \Sigma_X, K'_Z\rangle\right) \\
&\quad + I_\Delta(y)\left(I_\Delta(x)\langle \Sigma_Y, K_X\rangle + I_\Delta(z)\langle \Sigma_Y, K'_Z\rangle\right) \\
&\quad + I_\Delta(z)\left(I_\Delta(x)\langle \Sigma_Z, K_X\rangle + I_\Delta(y)\langle \Sigma_Z, K_Y\rangle\right) \\
&= -r'I_\Delta(x)(I_\Delta(y) + I_\Delta(z)) + rI_\Delta(y)I_\Delta(x) - r'I_\Delta(y)I_\Delta(z) + rI_\Delta(z)(I_\Delta(x) + I_\Delta(y)) \\
&= \tfrac{r-r'}{2}\textstyle\sum_{\mathfrak{S}_3(x,y,z)} I_\Delta(x)I_\Delta(y).
\end{aligned}$$

◇

Remark 8.3 When B is changed to $B+\alpha S$, then $r' = r'(B;\tau)$ is changed to $(r'(B+\alpha S;\tau) = r'-\alpha)$, r is changed to $(r+\alpha)$ and $Q(B+\alpha S;\tau) = Q(B;\tau) + \alpha I_\Delta(t)$, according to Theorem 4.7, so that $Q(B;\tau)(t) + \frac{r'-r}{2}I_\Delta(t)$ is invariant as it must be.

Lemma 8.4 *Let $W \in S^2$. Let Φ be a map from the unit disk D^2 of $\mathbb{C}$ to $SO(3)$ such that $\Phi(\exp(i\beta))$ is the rotation $R_{2\beta,W}$ with axis directed by W and with angle 2β. Then the map*

$$\begin{array}{rccc}
\Phi_W = \Phi(\cdot)(W)\colon & D^2 & \to & S^2 \\
& z & \mapsto & \Phi(z)(W)
\end{array}$$

sends ∂D^2 to W and the degree of the induced map from $D^2/\partial D^2$ to S^2 is (-1).

Proof: First note that the above degree does not depend on Φ on the interior of D^2, since $\pi_2(SO(3)) = 0$. See the restriction of Φ_W to ∂D^2 as the path composition of the maps $(\beta \in [0, 2\pi] \mapsto R_{\beta,W})$ and the inverse of $(\beta \in [0, 2\pi] \mapsto R_{\beta,-W})$ (that is twice the first map). Consider an arc α of a great circle of S^2 from $-W$ to W, then Φ can be seen as the map from $\alpha \times [0, 2\pi]$ to $SO(3)$ that maps (V, β) to $R_{\beta,V}$ so that the only preimage of $-W$ under Φ_W reads (V_0, π) where $V_0 \perp W$ and the local degree is easily seen to be (-1). $\diamond$

Remark 8.5 Consider our framed knot $(K, K_\parallel)$ in M that generates $H_1(M; \mathbb{Z})/\text{Torsion}$. Let $K''_\parallel$ be the parallel of K obtained from $K_\parallel$ by adding two positive meridians. According to Proposition 7.4,

$$\mathcal{Q}(K, K''_\parallel) - \mathcal{Q}(K, K_\parallel) = \sum_{\mathfrak{S}_3(x,y,z)} \left(-\frac{x\Delta'(x)}{\Delta(x)} \right) I_\Delta(y)$$

We can see this fact as a consequence of Proposition 8.2 as follows. Assume that τ has the following natural properties on $K \times D^2$ (trivialised with respect to τ), it maps $TK \times \{z\}$ to $\mathbb{R}W$ for $z \in D^2$, and it does not depend on $k \in K$ on $\{k\} \times TD^2$. Perform the following radial homotopy of τ that changes τ into a trivialisation τ' that coincides with τ outside $K \times D^2$, and that induces the same parallelisation as $(K_\parallel + 2m(K))$.

$$\tau \circ \tau'^{-1}(k, z)(v) = \nu(k, |z|)(v)$$

where $\nu(k, |z|) \in SO(3)$, $\nu(k, 1) = \text{Id}$, $\nu(\exp(i\beta), 0) = R_{2\beta,W}$. Set $K = K \times \{0\}$, $K' = K \times \{1\}$ and $B = -K \times [0, 1]$. Then $r(\tau') = r - 2$ and $s_{\tau'}(B; W)$ and $s_\tau(B; W)$ coincide outside $K \times D^2$ and on K. The difference $[s_{\tau'}(B; W) - s_\tau(B; W)]$ reads $d[ST(*)]$ where d is the degree at $(-W)$ of $\tau \circ \tau'^{-1}(k, z)(W)$ restricted to B. Then d is the degree at $(-W)$ of the map

$$\begin{array}{lcl} [0, 1] \times K & \to & S^2 \\ (t, k) & \mapsto & \nu(k, t)(W) \end{array}$$

that is 1 according to Lemma 8.4. Then

$$[s_{\tau'}(B; W) - s_\tau(B; W)] = [ST(*)].$$

Since $s_\tau(B; -W)$ and $s_{\tau'}(B; -W)$ are obtained from $s_\tau(B; W)$ and $s_{\tau'}(B; W)$ by the involution ι, and since $[\iota(ST(*))] = -[ST(*)]$, we find that $[C(B; \tau')] - [C(B; \tau)] = -(1 + t^{-1})[ST(*)]$ so that $Q(B; \tau') - Q(B; \tau) = -\frac{1+t}{1-t}$. In particular,

$$\big(\mathcal{Q}(\mathbb{K}', \tau') - \mathcal{Q}(K, K''_\parallel, \tau')\big) - (\mathcal{Q}(\mathbb{K}', \tau) - \mathcal{Q}(\mathbb{K}, \tau)) = \sum_{\mathfrak{S}_3(x,y,z)} \left(-\frac{1 + x}{1 - x} + I_\Delta(x) \right) I_\Delta(y).$$

This is consistent with the expression of $\Big(\mathcal{Q}(\mathbb{K}, \tau) - \mathcal{Q}(K, K''_\parallel, \tau')\Big)$ coming from Proposition 7.4.

The computation of $(\mathcal{Q}(\mathbb{K}') - \mathcal{Q}(\mathbb{K}))$ is now reduced to the computation of $Q(B; \tau)$ that only depends on the parallelisations of K and K' and on the homology class of B.

8.2 Computation of $Q(B;\tau)$

Without loss, we now assume that B induces the framing of $\mathbb{K}$ i.e. that $r=0$. (It suffices to change B to $B-rS$.)

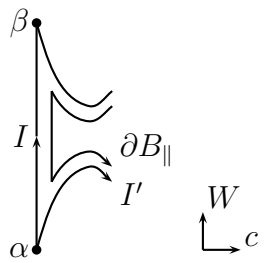
See K' as a band sum of K and of a rationally null-homologous knot. More precisely, write K as the union of two oriented intervals $I=[\alpha,\beta]$ and I_2 glued along their boundaries

$$K = I \cup_{\partial I_2} I_2 \quad \text{and} \quad K' = I' \cup_{\partial I_2} I_2$$

Then

$$\partial B = I' \cup_{\partial I} (-I)$$

where the gluing is not smooth at ∂I, it has two cusps there.



Near K, B reads $[0,1[\times I$ and its unit tangent space is mapped to a circle of S^2 that contains W and $(-W)$, and that is made of two half-great circles that we shall denote by $[-W,W]_c$ and $[-W,W]_{-c}$ when they are oriented from $-W$ to W. The half-circle $[-W,W]_c$ contains the direction c of the inward normal $[0,1[$ of B, while the half-circle $[-W,W]_{-c}$ contains the direction of the outward normal.

When a and b are real numbers such that $a<b$, $[a,b]\times_{\leq}[a,b]=\{(t,u)\in[a,b]^2; t\leq u\}$ and $[a,b]\times_{\geq}[a,b]=\{(t,u)\in[a,b]^2; t\geq u\}$. We shall use the notation $I\tilde{\times}_{\leq}I$ for the lift in $\tilde{C}_2(M)$ of the closure of $\{(t,u)\in[\alpha,\beta]^2; t<u\}$ in $C_2(M)$ that contains points of the blow-up of the preferred lift of the diagonal. The chains $(I'\cup_\beta(-I))\tilde{\times}_{\leq}(I'\cup_\beta(-I))$ and $(I'\cup_\beta(-I))\tilde{\times}_{\geq}(I'\cup_\beta(-I))$ should be understood similarly, they contain $\pm s_\tau(I';W)$ and $\pm s_\tau(I';-W)$, respectively. Furthermore, $(I'\cup_\beta(-I))\tilde{\times}_{\leq}(I'\cup_\beta(-I))$ contains the (lift of the) closure of the set of points (t,u) when t approaches α in I' and u approaches α in I. Hence, it contains $\pm s_\tau(\alpha;[-W,W]_{-c})$. It similarly contains $\pm s_\tau(\beta;[-W,W]_{-c})$.

We want to get rid of I_2 by letting ∂B bound. In general, B does not lift in $\tilde{M}$. However, $\delta(t_M)\partial B$ bounds a rational chain denoted by $\delta(t_M)B'$ in $\tilde{M}$ that reads $\delta(t)[0,1[\times I$ near the lifts of K. (Again, it suffices to add a combination of $\alpha_k t^k S$ to achieve this.) Let $\tilde{\alpha}$ be the preimage of α in $\tilde{M}$ that reads $\{0\}\times\alpha$ above. Then $\alpha\times B'$ denotes the closure in $\tilde{C}_2(M)$ of the projection in $\widetilde{M^2}$ of $\alpha\times(B'\setminus\tilde{\alpha})$. The chain $B'\times\alpha$ is also the most natural chain in $\tilde{C}_2(M)$ that fits with the notation, and that has the symmetric heavy definition.

Lemma 8.6

$$[C(B;\tau)] = t^{-1}[C_1(B';\tau)] - [C_0(B';\tau)]$$

in $H_2(C_2(M);\mathbb{Q}(t))$*, where*

$$C_0(B';\tau) = s_\tau(B;W) - s_\tau(I;[-W,W]_{-c}) - \alpha\times B' - B'\times\alpha - (I'\cup_\beta(-I))\tilde{\times}_{\leq}(I'\cup_\beta(-I)).$$

and

$$C_1(B';\tau) = s_\tau(B;-W) + s_\tau(I;[-W,W]_{c}) - \alpha\times B' - B'\times\alpha + (I'\cup_\beta(-I))\tilde{\times}_{\geq}(I'\cup_\beta(-I)).$$

Proof:

$$A(K) = [\alpha,\beta]\times I_2 + t^{-1}(I_2\times[\alpha,\beta]) + \left(I\tilde{\times}_{\leq}I\right) + t^{-1}\left(I\tilde{\times}_{\geq}I\right) + \left(I_2\tilde{\times}_{\leq}I_2\right) + t^{-1}\left(I_2\tilde{\times}_{\geq}I_2\right)$$

where $s_\tau(I;W)\subset\partial(I\tilde{\times}_{\leq}I)$, $s_\tau(I;-W)\subset-\partial(I\tilde{\times}_{\geq}I)$, and some attention should also be paid to points (α,α) and (β,β). For example, (α,α) goes to $t^{-1}s_\tau(\alpha;-W)$ in $[\alpha,\beta]\times I_2$ and to $t^{-1}s_\tau(\alpha;W)$ in $t^{-1}I_2\times[\alpha,\beta]$.

$$\begin{aligned} C(B;\tau) &= \partial B\times I_2 + t^{-1}I_2\times\partial B \\ &+ I'\tilde{\times}_{\leq}I' + t^{-1}I'\tilde{\times}_{\geq}I' - I\tilde{\times}_{\leq}I - t^{-1}I\tilde{\times}_{\geq}I \\ &+ t^{-1}s_\tau(B;-W) - s_\tau(B;W). \end{aligned}$$

Consider the closure $(B'\times(I\cup_\beta I_2))_{\tilde{C}_2(M)}$ in $\tilde{C}_2(M)$ of $(B'\times(I\cup_\beta I_2))\setminus\mathrm{diag}(I^2)$.

A point in $(B\times I)\setminus\mathrm{diag}(I^2)$ near the diagonal reads $(u+\varepsilon v,u)$, where $u\in I$, $\varepsilon\in[0,1[$, $v\in[-W,W]_c$, where the orientation is induced by (ε,v,u), or $(-(\varepsilon,(u,-v)))$ since $(v\mapsto(-v))$ preserves the orientation of S^1. Then $(B'\times(I\cup_\beta I_2))_{\tilde{C}_2(M)}$ is a 3–chain whose boundary is

$$\partial\left(B'\times(I\cup_\beta I_2)\right)_{\tilde{C}_2(M)} = \partial B\times(I\cup_\beta I_2) + (t^{-1}-1)(B'\times\alpha) - s_\tau(I;[-W,W]_{-c}).$$

(Again, $\partial B\times(I\cup_\beta I_2)$ abusively stands for the preferred lift in $\tilde{C}_2(M)$ of the closure in $C_2(M)$ of $(\partial B\times(I\cup_\beta I_2)\setminus\mathrm{diag}(I^2))$.) Similarly, the boundary of the 3–chain $t^{-1}\left((I\cup_\beta I_2)\times B'\right)_{\tilde{C}_2(M)}$ is

$$\partial\left(t^{-1}\left((I\cup_\beta I_2)\times B\right)_{\tilde{C}_2(M)}\right) = -t^{-1}(I\cup_\beta I_2)\times\partial B + (1-t^{-1})(\alpha\times B') + t^{-1}s_\tau(I;[-W,W]_c),$$

and $C(B;\tau)$ is homologous to

$$\begin{aligned} &-\partial B\times I + (1-t^{-1})(B'\times\alpha) + s_\tau(I;[-W,W]_{-c}) \\ &-t^{-1}I\times\partial B + (1-t^{-1})(\alpha\times B') + t^{-1}s_\tau(I;[-W,W]_c) \\ &+I'\tilde{\times}_{\leq}I' + t^{-1}I'\tilde{\times}_{\geq}I' - I\tilde{\times}_{\leq}I - t^{-1}I\tilde{\times}_{\geq}I + t^{-1}s_\tau(B;-W) - s_\tau(B;W) \end{aligned}$$

and to

$$\begin{aligned} &I'\times(-I) + (1-t^{-1})(B'\times\alpha) + s_\tau(I;[-W,W]_{-c}) \\ &+t^{-1}(-I)\times I' + (1-t^{-1})(\alpha\times B') + t^{-1}s_\tau(I;[-W,W]_c) \\ &+I'\tilde{\times}_{\leq}I' + t^{-1}I'\tilde{\times}_{\geq}I' + I\tilde{\times}_{\geq}I + t^{-1}I\tilde{\times}_{\leq}I + t^{-1}s_\tau(B;-W) - s_\tau(B;W). \end{aligned}$$

It is now easy to conclude. $\diamond$

We shall assume that $\delta(t)B'$ reads $\delta(t)I' \times [0,1[-R'M(I')$ near the lifts of K', where $M(I')$ is a small meridian disk of I', $R' \in \mathbb{Q}[t,t^{-1}]$ satisfies $R'(1) = r' \in \mathbb{Q}$ and $I' \times [0,1[$ induces the same parallelisation of K' as τ. Let $\partial B_{\|}$ be the parallel of ∂B induced by the parallels of K and K', obtained by pushing ∂B inside B near I and induced by the parallel of K' along I'.

$$lk_e(\partial B, \partial B_{\|}) = -\frac{R'}{\delta} = -\frac{\overline{R'}}{\delta}.$$

Lemma 8.7 *Set*

$$C_2(B';\tau) = C_0(B';\tau) - \frac{R'}{2\delta} ST(*).$$

Then $\delta C_2(B';\tau)$ is a rational cycle, and

$$[C_2(B';\tau)] = [C_1(B';\tau)] + \frac{R'}{2\delta}[ST(*)].$$

In particular,

$$[C(B;\tau)] = (t^{-1}-1)[C_2(B';\tau)] - (t^{-1}+1)\frac{R'}{2\delta}[ST(*)].$$

Proof: We only (need to) prove that the two expressions of $[C_2(B';\tau)]$ coincide.

$$\begin{aligned}\partial(\partial B \times (B' \setminus \mathrm{Int}(-\tfrac{R'}{\delta}M(I')))_{\tilde{C}_2(M)} = \; & -s_\tau(I';[-W,W]_{-c}) + s_\tau((-I);[-W,W]_c) \\ & -(I' \cup_\beta (-I))\tilde{\times}_{\leq}(I' \cup_\beta (-I)) \\ & -(I' \cup_\beta (-I))\tilde{\times}_{\geq}(I' \cup_\beta (-I)) - \partial B \times \tfrac{R'}{\delta} m(I')\end{aligned}$$

$$\begin{aligned}[C_0(B';\tau) - C_1(B';\tau) - \tfrac{R'}{\delta}ST(*)] \; & = [s_\tau(\partial B;[-W,W]_{-c}) + s_\tau(B;W) - s_\tau(B;-W)] \\ & = [\partial(s_\tau(B;[-W,W]_{-c}))] = 0.\end{aligned}$$

$\diamond$

Lemma 8.8 *Define $Q_2(B';\tau)(t)$ so that $[C_2(B';\tau)] = Q_2(B';\tau)(t)[ST(*)]$. Then*

$$Q(B;\tau)(t) = Q_2(B';\tau)(t) - \frac{1+t}{1-t}\frac{R'(t)}{2\delta}.$$

$$Q(B;\tau)(t) = -Q(B;\tau)(t^{-1}).$$

Proof: The involution that exchanges the two factors in $\tilde{M}^2$ induces an involution ι of $\tilde{C}_2(M)$ such that $\iota_*(P(t)[ST(*)]) = -P(t^{-1})[ST(*)]$ for any $P \in \Lambda$. Since $\iota(C_0(B';\tau)) = C_1(B';\tau)$,

$$\iota_*([C_2(B';\tau)]) = [C_2(B';\tau)]$$

and $Q_2(B';\tau)(t) = -Q_2(B';\tau)(t^{-1})$. Now, according to Lemma 8.7, and to the definition of $Q(B;\tau)$ before Proposition 8.2,

$$Q(B;\tau)(t^{-1}) = -Q_2(B';\tau)(t) - \frac{1+t^{-1}}{1-t^{-1}}\frac{R'(t)}{2\delta}.$$

This proves the formula that implies the antisymmetry of Q. $\diamond$

Lemma 8.9 *Any cycle Z of $H_2(\tilde{C}_2(M))$ reads $\frac{Q_Z}{\delta(t-1)}[ST(*)]$ in $H_2(C_2(M);\mathbb{Q}(t))$ for some Q_Z in $\mathbb{Q}[t,t^{-1}]$.*

Proof: Since $\delta(t-1)Z$ vanishes in $H_2(\widetilde{M^2})$ according to Proposition 2.6, it bounds a rational 3-chain, and $\pm Q_Z$ is the equivariant intersection of this 3-chain with the preferred lift of the diagonal of M. $\diamond$

Lemma 8.10 *Under the assumptions above, where $r = 0$, set*

$$[C_0(B';\tau)] = Q_0(B';\tau)[ST(*)].$$

Then Proposition 8.1 is satisfied with

$$\mathcal{V}(\mathbb{K},\mathbb{K}') = \delta Q(B;\tau) + \delta\frac{r'}{2}I_\Delta = \delta Q_0(B';\tau) - \frac{R'}{2} + \frac{1+t}{t-1}\frac{R'}{2} + \delta\frac{r'}{2}I_\Delta.$$

Proof: The first formula for $\mathcal{V}(\mathbb{K},\mathbb{K}')$ comes from Proposition 8.2, and the second one easily follows from Lemmas 8.7 and 8.8. It is enough to see that $\mathcal{V}(\mathbb{K},\mathbb{K}') \in \mathbb{Q}[t,t^{-1}] = \Lambda$. Recall that $\delta = \delta(M)$ is the annihilator of $H_1(\tilde{M})$. Since it has the same roots as $\Delta(M)$, $\delta\frac{\Delta'}{\Delta} \in \Lambda$. Since $(r'\delta - R')(1) = 0$, $(t-1)$ divides $(r'\delta - R')$ and $\frac{1+t}{2(1-t)}(r'\delta - R') \in \Lambda$. Therefore, it is enough to prove that $\delta Q_0(B';\tau) \in \Lambda$. Thanks to Lemma 8.9, $(1-t)^2\delta(t)Q(B;\tau)(t^{-1}) \in \Lambda$. Therefore, $(1-t)^2\delta Q_0(B';\tau) \in \Lambda$. According to Proposition 2.6, since $\delta C_0(B';\tau)$ is a rational cycle, it is rationally homologous in $\widetilde{M^2}$ to $u[S\times *] + v[*\times S]$ up to some elements of δ–torsion, where u and v are rational numbers that can be computed as the algebraic intersection with the preimage of $M \times K_\parallel$ (up to sign) or $K_\parallel \times M$, respectively. Since our assumptions imply that B' does not meet the preimage of $K_\parallel$, and since B does not meet $K_\parallel$ either, u and v are zero, and $\delta^2 C_0(B';\tau)$ is null-homologous in $H_2(\widetilde{M^2})$. Therefore $\delta^2 Q_0(B';\tau) \in \Lambda$, and since δ and $(1-t)^2$ are coprime, $\delta Q_0(B';\tau) \in \Lambda$. $\diamond$

8.3 An independent lemma

Lemma 8.11 *Let Σ be a compact oriented surface with one boundary component $J(S^1)$ equipped with a basepoint $* = J(1)$. Let Σ and Σ^+ be two copies of Σ, let $(z_i)_{i=1,\dots 2g}$ and $(z_i^*)_{i=1,\dots,2g}$ be two dual bases of $H_1(\Sigma;\mathbb{Z})$ such that*

$$\langle z_i, z_j^* \rangle = \delta_{ij}.$$

Set

$$J \times_{*,\leq} J^+ = \{(J(\exp(2i\pi t)), J(\exp(2i\pi u))); (t,u) \in [0,1]^2, t \leq u\}$$

and

$$J \times_{*,\geq} J^+ = \{(J(\exp(2i\pi t)), J(\exp(2i\pi u))); (t,u) \in [0,1]^2, t \geq u\}.$$

Let $diag(\Sigma \times \Sigma^+) = \{(x,x); x \in \Sigma\}$. Then the chains

$$C_{*,\leq}(\Sigma,\Sigma^+) = diag(\Sigma \times \Sigma^+) - * \times \Sigma^+ - \Sigma \times *^+ - J \times_{*,\leq} J^+$$

and

$$C_{*,\geq}(\Sigma,\Sigma^+) = diag(\Sigma \times \Sigma^+) - * \times \Sigma^+ - \Sigma \times *^+ + J \times_{*,\geq} J^+$$

are cycles and we have the following equality in $H_2(\Sigma \times \Sigma^+)$

$$[C_{*,\leq}(\Sigma,\Sigma^+)] = [C_{*,\geq}(\Sigma,\Sigma^+)] = \sum_{i=1}^{2g} [z_i \times z_i^{*+}].$$

Proof: Since

$$\partial(J \times_{*,\leq} J^+) = \mathrm{diag}(J \times J^+) - * \times J^+ - J \times *^+$$

$C_{*,\leq}(\Sigma,\Sigma^+)$ is a cycle. Consider the closed surface S obtained from Σ by gluing a disk D along J. According to Proposition 4.6, in $H_2(S \times S^+)$,

$$[\mathrm{diag}(S \times S^+)] = [* \times S^+] + [S \times *^+] + \sum_{i=1}^{2g} [z_i \times z_i^{*+}].$$

This implies that

$$[C_{*,\leq}(\Sigma,\Sigma^+) - C_{*,\leq}(-D,(-D)^+)] = \sum_{i=1}^{2g} [z_i \times z_i^{*+}]$$

in $H_2(S \times S^+)$. Since the cycle $C_{*,\leq}(-D,(-D)^+)$ lives in $D \times D^+$, it is null-homologous there, and since $H_2(\Sigma \times \Sigma^+)$ naturally injects into $H_2(S \times S^+)$, we can conclude that $[C_{*,\leq}(\Sigma,\Sigma^+)] = \sum_{i=1}^{2g} [z_i \times z_i^{*+}]$. The proof for $C_{*,\geq}(\Sigma,\Sigma^+)$ is the same. $\diamond$

8.4 Examples of variations

In this subsection, we shall keep the notation of the first two subsections of this section and assume that $r = r' = 0$. We shall compute $Q(B) = Q(B;\tau)$ in some examples where B lifts in $\tilde{M}$, so that $B = B'$ and $R' = 0$.

Lemma 8.12 *If B lifts in $\tilde{M}$, (if $R' = r' = r = 0$,) and if $(z_i)_{i=1,\dots 2g}$ and $(z_i^*)_{i=1,\dots,2g}$ are two dual bases of $H_1(B;\mathbb{Z})$ such that $\langle z_i, z_j^*\rangle = \delta_{ij}$, then*

$$[C_0(B)] = [C_0(B;\tau)] = g[ST(*)] + \sum_{i=1}^{2g}[z_i \times z_i^{*+}]$$

and

$$Q(B)(t) = \sum_{i=1}^{2g} lk_e(z_i, z_i^{*+}) + g.$$

Proof: The expression of $Q(B)$ follows from the expression of $[C_0(B)]$, thanks to Lemma 8.10. Thus, we are left with the computation of $[C_0(B)] = [C_0(B;\tau)]$ that was defined in Lemma 8.6. Our computation will rely on Lemma 8.11.

Set $d = W$. Along I, τ maps the inward normal of B to $c \in S^2$ and the positive normal of B to e so that (c, d, e) is an oriented orthonormal basis for $\mathbb{R}^3$. Let $\langle c, d, e\rangle$ denote the intersection of the sphere with the convex hull of 0, $3c$, $3d$ and $3e$ in $\mathbb{R}^3$ with the orientation of S^2 while $\langle d, c, e\rangle = -\langle c, d, e\rangle$. $[-W, W]_c = \langle -d, c\rangle + \langle c, d\rangle$. Consider B as $B \times 0$ in $B \times [0,1] \subset M$ and consider the closure $C_{*,\leq}(B, B \times [0,1])$ in $\tilde{C}_2(M)$ of $\cup_{t\in]0,1]} C_{*,\leq}(B, B \times t)$, where the basepoint $*$ is α, with the notation of Lemma 8.11. This is a 3–chain whose boundary contains $(-s_+(B))$ in the closure of $\cup_{t\in]0,1]}\mathrm{diag}(B \times B \times t)$. Let us consider the ways of approaching $ST(\alpha)$ and determine $ST(\alpha) \cap \partial C_{*,\leq}(B, B \times [0,1])$.

- In the closure of $\alpha \times (B\times]0,1])$, we only find $s_\tau(\alpha; \langle d, e\rangle)$.
- In the closure of $B \times (\alpha\times]0,1])$, we only have $s_\tau(\alpha; \langle -d, e\rangle)$.
- When a pair of points of I, (resp. of I') in $(I' \cup_\beta (-I))\tilde{\times}_{\leq}((I' \cup_\beta (-I))\times]0,1]$ approaches $ST(\alpha)$, it approaches $s_\tau(\alpha; \langle \pm d, e\rangle)$, again.
- When the first point is in I' and the second one is in $(-I)$, the vector between them is in $\langle d, -c, e\rangle \cup \langle -c, -d, e\rangle$ where I' gives the inward normal, $(-I)$ moves along $[d, -d]_{-c}$, and $[0,1]$ gives the e direction. In particular, the boundary of $-I' \times (-I) \times [0,1]$ is oriented like $(\langle d, -c, e\rangle \cup \langle -c, -d, e\rangle)$, that is like the sphere.

The only part that will matter to us is this last 2–dimensional part.

We similarly determine the 2–dimensional part of $ST(\beta) \cap \partial C_{*,\leq}(B, B \times [0,1])$ that comes from pairs of points where the first point is in I' and the second one is in $(-I)$, the vector

between them is again in $\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle$. Now, I' gives the outward normal, $(-I)$ moves along $[d,-d]_{-c}$, and $[0,1]$ gives the e direction. In particular the boundary of $-I'\times(-I)\times[0,1]$ is oriented like $(-(\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle))$ unlike the sphere.

We now compute

$$C_{0,+}=C_{*,\leq}(B,B\times 1)-\partial C_{*,\leq}(B,B\times[0,1])$$

forgetting the 1–dimensional parts,

$$\begin{aligned} C_{0,+} &= s_+(B)-\alpha\times(B\setminus\alpha)-(B\setminus\alpha)\times\alpha \\ &\quad -(I'\cup_\beta(-I))\tilde{\times}_{\leq}(I'\cup_\beta(-I))+s_\tau(I';\langle d,e\rangle)+s_\tau(-I;\langle -d,e\rangle)) \\ &\quad -s_\tau(\alpha;\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle)+s_\tau(\beta;\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle). \end{aligned}$$

where $d=W$ and e denotes the positive normal to B along I', and

$$[s_+(B)+s_\tau(\partial B;\langle d,e\rangle)-s_\tau(B;W)]=-g[ST(*)]$$

since this is an obstruction to homotope the positive normal to a point in S^2 via τ that would be zero if B were a disk, and that is similar to the obstruction computed in Proposition 4.5.

Then

$$\begin{aligned} [C_0(B)] &= [s_+(B)+s_\tau(\partial B;\langle d,e\rangle)+gST(*)-s_\tau(I;[-W,W]_{-c}) \\ &\quad -\alpha\times B-B\times\alpha-(I'\cup_\beta(-I))\tilde{\times}_{\leq}(I'\cup_\beta(-I))] \\ &= [C_{0,+}+gST(*)+s_\tau(\partial B;\langle d,e\rangle)-s_\tau(I;[-W,W]_{-c}) \\ &\quad -s_\tau(I';\langle d,e\rangle)-s_\tau(-I;\langle -d,e\rangle) \\ &\quad +s_\tau(\alpha;\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle)-s_\tau(\beta;\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle)]. \end{aligned}$$

where

$$\partial s_\tau((-I);\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle)$$

$$=s_\tau(\alpha;\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle)-s_\tau(\beta;\langle d,-c,e\rangle\cup\langle -c,-d,e\rangle)+s_\tau((-I);[-W,W]_{-c}-[-W,W]_e).$$

Since $[C_{0,+}]=[C_{*,\leq}(B,B\times 1)]=\sum_{i=1}^{2g}[z_i\times z_i^{*+}]$, according to Lemma 8.11, we find

$$[C_0(B)]=g[ST(*)]+\sum_{i=1}^{2g}[z_i\times z_i^{*+}].$$

◇

Proof of Proposition 1.7: It is a direct consequence of Proposition 8.2 and Lemma 8.12, using the same arguments as in Remark 4.2. ◇

Note that Proposition 1.7 implies that if the a_i and the b_i bound in the complement of S, then $Q(B)(t)=0$.

Example 8.13 According to Lemma 3.5, there exist a two-component link (a, b) in $\tilde{M} \backslash p_M^{-1}(K)$, and a rational number $q \neq 0$ such that $lk_e(a, b) = \frac{qt^k}{\delta(M)}$. Plumb two annuli around a and b so that $lk_e(a, b^+) = \frac{qt^k}{\delta(M)}$. Let K' be a band sum of K with the boundary of the genus one surface obtained by this plumbing. Then K' is homologous to K and $Q(t) = q\frac{t^k - t^{-k}}{\delta(M)}$.

This example concludes the proof of Theorem 1.5. $\diamond$

Remark 8.14 Consider a cobordism B between K and $1 \times K$ constructed from $[0, 1] \times K + S$ by surgery around the intersection segment.

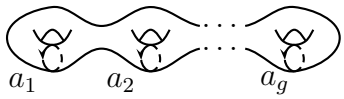
$$\partial B = 1 \times K - m(1 \times K) - K - m(K).$$

Then $\mathcal{Q}(K, K - m(1 \times K)) - \mathcal{Q}(K, K + m(1 \times K))$ is computed both in Proposition 7.4, and in Proposition 1.7. Thanks to Remark 4.2, these two computations lead to the same result and show that $Q(B)(t) = \frac{t\Delta'(t)}{\Delta(t)}$ in this case.

9 The Lagrangian-preserving surgery formula

9.1 The statement of the Lagrangian-preserving surgery formula

A *genus g* $\mathbb{Q}$*–handlebody* is an (oriented, compact) 3–manifold A with the same homology with rational coefficients as the standard (solid) handlebody H_g of the following figure.

$H_g =$ 

The boundary of such a $\mathbb{Q}$–handlebody A (or *rational homology handlebody*) is homeomorphic to the boundary ($\partial H_g = \Sigma_g$) of H_g.

For a $\mathbb{Q}$–handlebody A, the kernel $\mathcal{L}_A$ of the map induced by the inclusion

$$H_1(\partial A; \mathbb{Q}) \longrightarrow H_1(A; \mathbb{Q})$$

is a Lagrangian of $(H_1(\partial A; \mathbb{Q}), \langle , \rangle_{\partial A})$, that is called the *Lagrangian* of A. A *Lagrangian-preserving surgery* or *LP–surgery* (A, A') is the replacement of a $\mathbb{Q}$–handlebody A embedded in a 3–manifold by another such A' with identical (identified via a homeomorphism) boundary and Lagrangian. The manifold obtained by such an LP surgery (A, A') from a manifold M (with $A \subset M$) is denoted by $M(A'/A)$.

There is a canonical isomorphism

$$\partial_{MV} \colon H_2(A \cup_{\partial A} -A'; \mathbb{Q}) \to \mathcal{L}_A$$

that maps the class of a closed surface in the closed 3–manifold $(A \cup_{\partial A} -A')$ to the boundary of its intersection with A. This isomorphism carries the algebraic triple intersection of surfaces to a trilinear antisymmetric form $\mathcal{I}_{AA'}$ on $\mathcal{L}_A$.

$$\mathcal{I}_{AA'}(a_i, a_j, a_k) = \mathcal{I}_{AA'}(a_i \wedge a_j \wedge a_k) = \langle \partial_{MV}^{-1}(a_i), \partial_{MV}^{-1}(a_j), \partial_{MV}^{-1}(a_k) \rangle_{A \cup -A'}$$

Theorem 9.1 *Let A and B be two disjoint rational homology handlebodies of M disjoint from K. Assume that $H_1(A)$ and $H_1(B)$ are sent to 0 in $H_1(M; \mathbb{Q})$ by the maps induced by the inclusions. Let $(a_i, z_i)_{i=1,\dots,g_A}$ be a basis of $H_1(\partial A)$ such that $(a_i)_{i=1,\dots,g_A}$ is a basis of $\mathcal{L}_A$ and $\langle a_i, z_j \rangle_{\partial A} = \delta_{ij}$. Let $(b_i, y_i)_{i=1,\dots,g_B}$ be a basis of $H_1(\partial B)$ such that $(b_i)_{i=1,\dots,g_B}$ is a basis of $\mathcal{L}_B$ and $\langle b_i, y_j \rangle_{\partial B} = \delta_{ij}$. Let A' and B' be two rational homology handlebodies whose boundaries are identified to ∂A and to ∂B, respectively, so that $\mathcal{L}_{A'} = \mathcal{L}_A$ and $\mathcal{L}_{B'} = \mathcal{L}_B$. Set $M_A = M(A'/A)$, $M_B = M(B'/B)$, $M_{AB} = M(A'/A, B'/B)$ and let*

$$\mathcal{S} = \mathcal{Q}(\mathbb{K} \subset M_{AB}) + \mathcal{Q}(\mathbb{K} \subset M) - \mathcal{Q}(\mathbb{K} \subset M_A) - \mathcal{Q}(\mathbb{K} \subset M_B).$$

Then

$$\mathcal{S} = - \sum_{(i,j,k)\in\{1,\dots,g_A\}^3,(\ell,m,n)\in\{1,\dots,g_B\}^3} lk(i,j,k,\ell,m,n)\mathcal{I}(i,j,k,\ell,m,n)$$

where

$$\begin{aligned} lk(i,j,k,\ell,m,n) = \; & lk_e(z_i,y_\ell)(x)lk_e(z_j,y_m)(y)lk_e(z_k,y_n)(z) \\ & +lk_e(z_i,y_\ell)(x^{-1})lk_e(z_j,y_m)(y^{-1})lk_e(z_k,y_n)(z^{-1}) \\ & +\textstyle\sum_{\mathfrak{S}_3(x,y,z)} lk(z_i,y_\ell)lk_e(z_j,z_k)(x)lk_e(y_m,y_n)(y), \end{aligned}$$

and

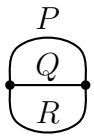
$$\mathcal{I}(i,j,k,\ell,m,n) = \mathcal{I}_{AA'}(a_i\wedge a_j\wedge a_k)\mathcal{I}_{BB'}(b_\ell\wedge b_m\wedge b_n).$$

Here, when the z_i (resp. the y_ℓ) are arguments of equivariant linking numbers, they are lifts of the z_i (resp. of the y_ℓ) sitting all in the same lift of A (resp. of B) in $\tilde{M}$.

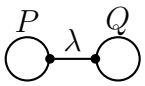
The proof of this theorem will be given in Section 11.

9.2 Relations with the clasper theory and with the Kontsevich integral

For P, Q and R in $\mathbb{Q}[t^{\pm 1}, \frac{1}{\delta(t)}]$, and for $\lambda \in \mathbb{Q}$, see the two *beaded graphs* below as the following elements of R_δ

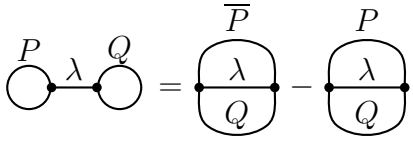


$$= \sum_{\mathfrak{S}_3(x,y,z)} \left(P(x)Q(y)R(z) + P(x^{-1})Q(y^{-1})R(z^{-1})\right)$$



$$\begin{aligned} &= \lambda \textstyle\sum_{\mathfrak{S}_3(x,y,z)} \left(P(x)Q(y^{-1}) + P(x^{-1})Q(y) - P(x)Q(y) - P(x^{-1})Q(y^{-1})\right) \\ &= \lambda \textstyle\sum_{\mathfrak{S}_3(x,y,z)} \left(P(x) - P(x^{-1})\right)\left(Q(y^{-1}) - Q(y)\right) \end{aligned}$$

so that the following IHX or Jacobi relation is satisfied.



Via the identification above, the invariants $\delta(M)(x)\delta(M)(y)\delta(M)(z)\mathcal{Q}(M)$ and

$$\delta(M)(x)\delta(M)(y)\delta(M)(z)\hat{\mathcal{Q}}(\hat{K})$$

take their values in the space $\mathcal{A}_2(\mathbb{Q}[t,t^{-1}])$ of 2-loop trivalent graphs with beads described in [GR, Definition 1.6]. (Here, the edges are oriented so that their polynomial labels are on their left-hand sides.) The lack of denominators follows from Proposition 12.1 that will be shown in an independent section. This space $\mathcal{A}_2(\mathbb{Q}[t,t^{-1}])$ maps to unitrivalent graphs via the map Hair described in [GR, Section 1.4]. The following result is a direct consequence of Theorem 9.1 and of the results in [GR].

Theorem 9.2 *For knots with trivial Alexander polynomials in integral homology spheres, the map Hair* $\circ\, \hat{\mathcal{Q}}$ *coincides with the* 2*-loop primitive part of the Kontsevich integral.*

Before saying more precisely how this theorem follows from Theorem 9.1 and [GR], let us relate Theorem 9.1 with clasper calculus and illustrate it by some examples.

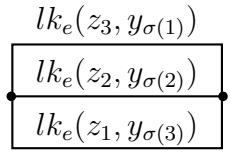
A special case of a Lagrangian-preserving surgery is the case where A is the regular neighborhood of a Y-graph, that is a genuine genus 3 handlebody and A' is obtained from A by a borromean surgery. See [AL, Section 2.2]. Then the three curves z_1, z_2 and z_3 can be chosen as longitudes of the leaves, the three curves a_1, a_2 and a_3 are the corresponding meridians and

$$\mathcal{I}_{AA'}(a_i \wedge a_j \wedge a_k) = \pm 1$$

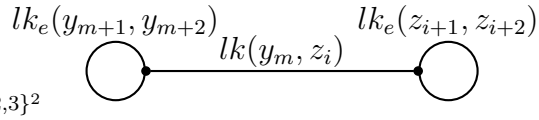
See [AL, Lemma 4.2]. The fact that $H_1(A)$ is sent to 0 in $H_1(M;\mathbb{Q})$ is equivalent to the fact that the leaves are null-homologous. Assume that (A, A') and (B, B') are both such borromean surgeries and that y_1, y_2 and y_3 are the longitudes of the leaves of the underlying Y–graph for B.

Lemma 9.3 *Under the hypotheses above, the expression* $\mathcal{S}$ *of Theorem 9.1 reads*

$$\mathcal{S} = \sum_{\sigma\in\mathfrak{S}_3} sign(\sigma)$$



$$+ \sum_{(i,m)\in\{1,2,3\}^2}$$

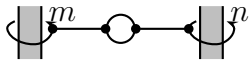


where the indices are integers mod 3 *in the latter expression.*

◇

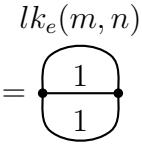
Proof of Theorem 9.2: It follows from Theorem 9.1 that $\mathcal{Q}$ and $\hat{\mathcal{Q}}$ are of null-type 2 in the sense of [GR, Definition 1.1] since the considered LP-surgeries do not change the equivariant linking numbers. According to [GR, Theorem 2], a null-type 2 invariant of knots with trivial Alexander polynomial in homology spheres, that is valued in a rational vector space, is determined by its value on the trivial knot of S^3 and by its values on the degree 2 null-claspers (that are special cases of the surgeries considered in Theorem 9.1). Then since for such a clasper G, $\mathcal{S}$ is the contraction of G in the sense of [GR, Theorem 4], the value of Hair $\circ\, \hat{\mathcal{Q}}(G)$ coincides with the primitive 2-loop part of the Kontsevich integral of G according to [GR, Theorem 4], and we are done since $\hat{\mathcal{Q}}$ vanishes for the trivial knot, and since the Kontsevich integral of the trivial knot is made of wheels [BLT]. ◇

Let us see some more examples of applications. Applying Theorem 9.1 to the case of a clasper in the complement of a (gray in the picture) Seifert surface of $\hat{K}$ like



like in Proposition 4.17 in [O1] yields

$$\mathcal{S} = \hat{\mathcal{Q}}(\hat{K} \subset (M_{\mathbb{K}})_{AB}) - \hat{\mathcal{Q}}(\hat{K} \subset M_{\mathbb{K}}) =$$



The following lemma gives an expression for $lk_e(m,n)$ that finishes to show that the two-loop polynomial described in [O1] behaves like $12\Delta(x)\Delta(y)\Delta(z)\hat{\mathcal{Q}}$ in this case.

Lemma 9.4 *Let m and n be two meridian curves of 1-handles of a Seifert surface Σ of $\hat{K}$ as above. Consider a basis $(z_i)_{i\in\{1,\dots,2g_\Sigma\}}$ of $H_1(\Sigma;\mathbb{Z})$, the matrix $C=[(c_{ij})_{(i,j)\in\{1,\dots,2g_\Sigma\}^2}]$ where*

$$c_{ij}=t_M^{1/2}lk(z_i^+,z_j)-t_M^{-1/2}lk(z_i^-,z_j)$$

and its inverse $D=C^{-1}=[(d_{ij})_{(i,j)\in\{1,\dots,2g_\Sigma\}^2}]$. Then

$$lk_e(m,n)=-(t_M^{1/2}-t_M^{-1/2})\sum_{(i,j)\in\{1,\dots,2g_\Sigma\}^2}lk(m,z_i)d_{ij}lk(n,z_j)$$

where m and n stand for lifts of m and n in the same lift of $M_{\mathbb{K}}\setminus\Sigma$, when the former ones are arguments of $lk_e(m,n)$.

Proof: There exist x and y in $H_1(\Sigma)$ such that m and n are respectively homologous to (x^+-x^-) and (y^+-y^-) in $M_{\mathbb{K}}\setminus\Sigma$. Then for any $z\in H_1(\Sigma)$, $lk(m,z)=\langle x,z\rangle$. Without loss, assume that (x^+-x^-) sits in the boundary of a regular neighborhood of Σ and that (y^+-y^-) sits in the boundary of a regular neighborhood of Σ inside the former one. Then with similar notation as in Section 4.1,

$$lk_e(m,n)=lk_e(\hat{x}^+-\hat{x}^-,\hat{y}^+-\hat{y}^-)=(t_M-1)lk_e(\hat{x}^+-\hat{x}^-,\tilde{y})$$

where

$$\hat{x}=\sum_{i=1}^{2g}\langle z_i,x\rangle\hat{z}_i^*$$

and like in Proposition 4.3

$$t_M^{-1/2}(\tilde{z}_j^+-\tilde{z}_j^-)=-\sum_{i=1}^{2g}c_{ji}(\hat{z}_i^{*+}-\hat{z}_i^{*-}).$$

Then

$$lk_e(m,n)=(t_M-1)\sum_{i=1}^{2g}\langle z_i,x\rangle\left(\sum_{j=1}^{2g}d_{ij}lk_e(-t_M^{-1/2}(\tilde{z}_j^+-\tilde{z}_j^-),\tilde{y})\right)$$

where

$$lk_e(\tilde{z}_j^+-\tilde{z}_j^-,\tilde{y})=\langle z_j,y\rangle.$$

$\diamond$

Similarly, the two-loop polynomial described in [O1] behaves like $12\Delta(x)\Delta(y)\Delta(z)\hat{\mathcal{Q}}$ in the case of [O1, Proposition 4.18].

9.3 Proving the Dehn surgery formula from the LP surgery formula

In this subsection, we prove Theorem 1.4 from Theorem 9.1. Since the proof is completely similar to the proof given [L3, Section 9], we shall be very sketchy and refer to [L3, Section 9] for details.

Let $F = (\Sigma \setminus \mathrm{Int}(D^2))$ be obtained from the Seifert surface Σ of J by removing an open disk with boundary c, let $F \times [-1,2]$ be embedded in a neighborhood of this surface and let U be a trivial knot in M located in the exterior of $F \times [-1,2]$, that is a meridian of c. Let $U_\parallel$ denote the parallel of U in the exterior of $F \times [-1,2]$ that does not link U, $J_\parallel$ denotes the parallel of J that does not link J. In [L3, Section 9], we define two LP-surgeries $(A, A') = (A_F = F \times [-1,0], A'_F)$ and $(B, B') = (B_F = F \times [1,2], B'_F)$ such that

- performing both surgeries does not change $(M, U, U_\parallel)$,
- performing one of the surgeries changes $(M, U, U_\parallel)$ to $(M, \pm J, \pm J_\parallel)$.

This implies that

$$M(J; p/q) = M(U; p/q)(A'/A) = M(U; p/q)(B'/B)$$

and that

$$M \sharp S^3(U; p/q) = M(U; p/q) = M(U; p/q)(A'/A, B'/B).$$

Then, according to Proposition 1.3,

$$\mathcal{Q}(\mathbb{K} \subset M) = \mathcal{Q}(\mathbb{K} \subset M \sharp S^3(U; p/q)) - 6\lambda(S^3(U; p/q))$$

and Theorem 9.1 allows us to write

$$2\mathcal{Q}(\mathbb{K} \subset M(J; p/q)) - 2\mathcal{Q}(\mathbb{K} \subset M) = -\mathcal{S} + 12\lambda(S^3(U; p/q))$$

where

$$(-\mathcal{S}) = \sum_{(i,j,k) \in \{1,\dots,g_A\}^3, (\ell,m,n) \in \{1,\dots,g_B\}^3} lk(i,j,k,\ell,m,n)\mathcal{I}(i,j,k,\ell,m,n)$$

with notation consistent with the statement of Theorem 9.1. Here, we use the basis

$$\big(c = c \times \{0\}, (c_j^- = c_j \times \{-1\}, d_j = d_j \times \{0\})_{j=1,\dots,g}\big)$$

that plays the role of the basis z_i for $H_1(A)$, and we use the basis

$$\big(c^+ = c \times \{1\}, (c_j^+ = c_j \times \{1\}, d_j^{++} = d_j \times \{2\})_{j=1,\dots,g}\big)$$

that plays the role of the basis y_i for $H_1(B_F)$. Then according to [L3, Lemma 9.3] and to the notation of [L3, Section 3], the only triples (z_i, z_j, z_k) for which $\mathcal{I}_{AA'}(a_i, a_j, a_k) \neq 0$ are the triples that are obtained from a triple of type (c, c_r^-, d_r) by permutation, and for $(z_i, z_j, z_k) =$

(c, c_r^-, d_r), $\mathcal{I}_{AA'}(a_i, a_j, a_k) = -1$, while the only triples (y_ℓ, y_m, y_n) for which $\mathcal{I}_{BB'}(b_\ell, b_m, b_n) \neq 0$ are the triples that are obtained from a triple of type (c^+, c_s^+, d_s^{++}) by permutation, and for $(y_\ell, y_m, y_n) = (c^+, c_s^+, d_s^{++})$, $\mathcal{I}_{BB'}(b_\ell, b_m, b_n) = 1$. All the lifts in $\tilde{M}$ of the mentioned curves except c and c^+ bound in the complement of the preimage of a ball that contains U, c and c^+. Therefore the equivariant linking number of a curve among the $c_j^{(+)}$, $d_j^{(+)}$ with another mentioned curve is not affected by the surgery on U, and these curves do not link c or c^+. Furthermore,

$$lk_e((c, c^+) \subset M(U; p/q)) = -q/p$$

Therefore, with the above bases, if (i, j, k, ℓ, m, n) contributes, then either $(z_i, y_\ell) = (c, c^+)$, or $(z_j, y_m) = (c, c^+)$, or $(z_k, y_n) = (c, c^+)$, and

$$(-\mathcal{S}) = \frac{q}{p} \sum_{\mathfrak{S}_3(x,y,z)} \sum_{r=1}^{g} \sum_{s=1}^{g} \left(\alpha_{rs}(x, y) + \alpha_{rs}(x^{-1}, y^{-1}) + \beta_{rs}(x, y)\right)$$

where $\alpha_{rs}(x, y) = lk_e(c_r, c_s^+)(x) lk_e(d_r, d_s^+)(y) - lk_e(c_r, d_s^+)(x) lk_e(d_r, c_s^+)(y)$
and $\beta_{rs}(x, y) = (lk_e(c_r, d_r^+)(x) - lk_e(d_r^+, c_r)(x))\,(lk_e(c_s, d_s^+)(y) - lk_e(d_s^+, c_s)(y))$. $\diamond$

An example of application of Theorem 1.4 is the following one.

Example 9.5 By Blanchfield duality (Lemma 3.5), for any polynomial P of $\mathbb{Q}[t_M^{\pm 1}]$, there exists a link (d_1, d_2) such that $lk_e(d_1, d_2) = q\frac{P}{\delta(M)}$ for some $q \in \mathbb{Q} \setminus \{0\}$. Let (c_1, c_2) be a Hopf link that does not link (d_1, d_2). Frame c_1 and c_2 trivially so that $lk_e(c_1, c_2) = 1$, $lk_e(c_1, c_1^+) = 0$, $lk_e(c_2, c_2^+) = 0$. Construct a surface Σ by plumbing bands around c_1 and d_1 and bands around c_2 and d_2 and by connecting them. This can be achieved so that

$$\begin{aligned} \lambda_e'(\partial\Sigma) = \ & \tfrac{1}{3}(lk_e(d_1, d_2)(x^{-1}) + lk_e(d_1, d_2)(y^{-1}) + lk_e(d_1, d_2)(z^{-1})) \\ & + \tfrac{1}{3}(lk_e(d_1, d_2)(x) + lk_e(d_1, d_2)(y) + lk_e(d_1, d_2)(z)). \end{aligned}$$

The next section contains independent preliminaries for the proof of Theorem 9.1 that will be given in Section 11.

10 Pseudo-trivialisations

10.1 Introduction to pseudo-trivialisations

Let τ be a trivialisation of M. Let A be a rational homology handlebody embedded in M, and let A' be another rational homology handlebody such that $\partial A = \partial A'$ and $\mathcal{L}_A = \mathcal{L}_{A'}$. The restriction of τ on $M \setminus \mathrm{Int}(A)$ does not necessarily extend as an actual trivialisation to A'. (See [L2, Section 4.2].) It does for the LP surgeries that are involved in the proof of the Dehn surgery formula, and more generally for the Torelli sugeries, or when A and A' are integral homology handlebodies.

To prove Theorem 9.1 in its full generality, we shall make the chains F_V coincide as much as possible. Therefore, like in [L2], we shall introduce *pseudo-trivialisations* $\tilde{\tau}$ that will

- generalize trivialisations,
- always extend to rational homology handlebodies,
- induce genuine trivialisations $\tilde{\tau}_{\mathbb{C}}$ of $TM \otimes_{\mathbb{R}} \mathbb{C}$ that have Pontrjagin classes $p_1(\tilde{\tau}_{\mathbb{C}})$,
- define 3-dimensional pseudo-sections $s_{\tilde{\tau}}(M;V)$ of $ST(M)$, for $V \in S^2$, so that

  $$\langle s_{\tilde{\tau}}(M;X) \cap ST(\Sigma), s_{\tilde{\tau}}(M;Y)\rangle_{ST(M)} = 0$$

  for a two cycle Σ when $X \neq Y$,
- thus provide 3-dimensional cycles

  $$\partial F_V(\tilde{\tau}) = s_{\tilde{\tau}}(M;V) - I_\Delta ST(M)_{|K_V}$$

  for any $V \in S^2$, and allow us to define $\mathcal{Q}(\mathbb{K};\tilde{\tau}) = \langle F_X(\tilde{\tau}), F_Y(\tilde{\tau}), F_Z(\tilde{\tau})\rangle_e$ for generic X, Y, Z of S^2, as in Proposition 5.2,

so that for such pseudo-trivialisations $\tilde{\tau}$

$$\mathcal{Q}(\mathbb{K}) = \mathcal{Q}(\mathbb{K},\tilde{\tau}) - \frac{p_1(\tilde{\tau}_{\mathbb{C}})}{4}.$$

In this section, we define and prove all we need about these pseudo-trivialisations. This section can be avoided by the reader who is only interested by the proof of Theorem 9.1 in the cases where the trivialisations extend to the replacing rational homology handlebodies, except for the independent lemma 10.6 that will be used in Section 11. This section will also be used in Section 13.

10.2 Definitions of pseudo-trivialisations and associated notions

Definition 10.1 A *pseudo-trivialisation* $\tilde{\tau} = (N(c); \tau_e, \tau_b)$ of $(M, \mathbb{K})$ consists of

- a link c of M equipped with a neighborhood $N(c) = [a, b] \times c \times [-1, 1]$ that avoids K,
- a trivialisation τ_e of M outside $N(c)$ that sends the oriented tangent vectors of K to $\mathbb{R}^+W$ for a fixed $W \in S^2$,
- a trivialisation τ_b of $T(N(c))$ such that

$$\tau_b = \begin{cases} \tau_e & \text{over } \partial([a, b] \times c \times [-1, 1]) \setminus (\{a\} \times c \times [-1, 1]) \\ \mathcal{T}_c^{-1} \circ \tau_e & \text{over } \{a\} \times c \times [-1, 1]. \end{cases}$$

where

$$\mathcal{T}_c^{-1} \circ \tau_e \left(\tau_e^{-1}(t, \gamma \in c, u \in [-1, 1]; V \in S^2)\right) = (t, \gamma, u, R_{\alpha(-u)}(V))$$

where $R_{\alpha(-u)} = R_{\alpha(-u),(1,0,0)}$ denotes the rotation of $\mathbb{R}^3$ with axis directed by $(1, 0, 0)$ and with angle $\alpha(-u)$, and α is a smooth map from $[-1, 1]$ to $[0, 2\pi]$ that maps $[-1, -1 + \varepsilon]$ to 0, that increases from 0 to 2π on $[-1 + \varepsilon, 1 - \varepsilon]$, and such that $\alpha(-u) + \alpha(u) = 2\pi$ for any $u \in [-1, 1]$.

Lemma 10.2 *Let A be a $\mathbb{Q}$-handlebody and let τ be a trivialisation of TA defined on a collar $[-4, 0] \times \partial A$ of ∂A. Then there is a pseudo-trivialisation of A that extends the restriction of τ to $[-1, 0] \times \partial A$.*

Proof: Indeed, there exists a trivialisation τ' of TA on A. There exists a curve c of $\{-2\} \times \partial A$ such that $\tau' = \mathcal{T}_c \circ \tau$ on $\{-2\} \times \partial A$ (after a homotopy of τ around $\{-2\} \times \partial A$). Then equip c with the neighborhood $[-2, -1] \times c \times [-1, 1]$ and define τ_b as τ on $N(c)$, define τ_e as τ on $([-2, 0] \times \partial A) \setminus \mathrm{Int}(N(c))$ and as τ' on $A \setminus (]-2, 0] \times \partial A)$. $\diamond$

Definition 10.3 [Trivialisation $\tilde{\tau}_{\mathbb{C}}$ of $TM \otimes_{\mathbb{R}} \mathbb{C}$] Let F_U be a smooth map such that

$$\begin{array}{rccl} F_U : & [a, b] \times [-1, 1] & \longrightarrow & SU(3) \\ & (t, u) & \mapsto & \begin{cases} \text{Identity} & \text{if } |u| > 1 - \varepsilon \\ R_{\alpha(u)} & \text{if } t < a + \varepsilon \\ \text{Identity} & \text{if } t > b - \varepsilon. \end{cases} \end{array}$$

F_U extends to $[a, b] \times [-1, 1]$ because $\pi_1(SU(3))$ is trivial. Define the trivialisation $\tilde{\tau}_{\mathbb{C}}$ of $TM \otimes_{\mathbb{R}} \mathbb{C}$ as follows.

- On $T(M \setminus N(c))$, $\tau_{\mathbb{C}} = \tau_e \otimes 1_{\mathbb{C}}$,
- Over $[a, b] \times c \times [-1, 1]$, $\tau_{\mathbb{C}} \tau_b^{-1}(t, \gamma, u; V) = (t, \gamma, u; F_U(t, u)(V))$.

Since $\pi_2(SU(3))$ is trivial, the homotopy class of $\tau_{\mathbb{C}}$ is well-defined. The definition of p_1 for genuine trivialisations in Section 5.5 naturally extends to define $p_1(\tilde{\tau}_{\mathbb{C}})$.

Definition 10.4 [Pseudo-sections $s_{\tilde{\tau}}(.;V)$] Let $\varepsilon > 0$ be a small positive number and let F be a smooth map such that

$$\begin{array}{rccl} F: & [a,b]\times[-1,1] & \longrightarrow & SO(3) \\ & (t,u) & \mapsto & \left\{\begin{array}{ll} \text{Identity} & \text{if } |u| > 1-\varepsilon \\ R_{\alpha(u)} & \text{if } t < a+\varepsilon \\ R_{-\alpha(u)} & \text{if } t > b-\varepsilon \end{array}\right. \end{array}$$

where α has been defined in Definition 10.1. The map F extends to $[a,b]\times[-1,1]$ because its restriction to the boundary is trivial in $\pi_1(SO(3))$.

Let $F(c,\tau_b)$ be defined on $ST(N(c)) \stackrel{\tau_b}{=} [a,b]\times c\times[-1,1]\times S^2$ as follows

$$\begin{array}{rccl} F(c,\tau_b): & [a,b]\times c\times[-1,1]\times S^2 & \longrightarrow & S^2 \\ & (t,\gamma,u;V) & \mapsto & F(t,u)(V). \end{array}$$

Let $V \in S^2$ and let $S^1(V)$ be the circle (or point) in S^2 that lies in a plane orthogonal to the axis generated by $(1,0,0)$ and that contains V. There is a 2-dimensional chain $C_2(V)$ in $[-1,1]\times S^1(V)$ whose boundary is $\{\left(u, R_{-\alpha(u)}(V)\right), u\in[-1,1]\}+\{\left(u, R_{\alpha(u)}(V)\right), u\in[-1,1]\}-2[-1,1]\times\{V\}$. Then $s_{\tilde{\tau}}(M;V)$ is the following 3–cycle

$$s_{\tilde{\tau}}(M;V) = s_{\tau_e}(M\setminus \mathrm{Int}(N(c));V) + \frac{s_{\mathcal{T}_c\circ\tau_b}(N(c);V) + F(c,\tau_b)^{-1}(V) + \{b\}\times c\times C_2(V)}{2}.$$

When Σ is a 2–chain that intersects $N(c)$ along sections $N_\gamma(c) = [a,b]\times\{\gamma\}\times[-1,1]$,

$$s_{\tilde{\tau}}(\Sigma;V) = s_{\tilde{\tau}}(M;V)\cap ST(M)_{|\Sigma}$$

so that

$$s_{\tilde{\tau}}(N_\gamma(c);V) = \frac{s_{\mathcal{T}_c\circ\tau_b}(N_\gamma(c);V) + F(c,\tau_b)^{-1}(V)\cap ST(M)_{|N_\gamma(c)} - \{b\}\times\{\gamma\}\times C_2(V)}{2}.$$

10.3 Two properties of pseudo-sections.

The obvious property that *genuine sections $s_\tau(\Sigma;X)$ and $s_\tau(\Sigma;Y)$ corresponding to distinct X and Y of S^2 are disjoint* generalizes as follows for pseudo-sections.

Lemma 10.5 *Let Σ be a surface embedded in a 3-manifold equipped with a pseudo-trivialisation $\tilde{\tau} = (N(c);\tau_e,\tau_b)$, such that Σ only intersects $N(c)$ along sections $N_{\gamma_i}(c) = [a,b]\times\{\gamma_i\}\times[-1,1]$ in the interior of Σ. Then if $Y\in S^2$ and if $Z\in S^2\setminus S^1(Y)$, $s_{\tilde{\tau}}(\Sigma;Y)$ and $s_{\tilde{\tau}}(\Sigma;Z)$ do not intersect algebraically in $ST(\Sigma)$.*

Proof: Recall that $S^1(V)$ denotes the circle in S^2 that lies in a plane orthogonal to the axis generated by $e_1 = (1,0,0)$ and that contains V. Let us consider the contribution to $\langle s_{\tilde{\tau}}(\Sigma;Y), s_{\tilde{\tau}}(\Sigma;Z)\rangle_{ST(\Sigma)}$ of an intersection point γ of c with Σ. Such a contribution will read

$$\pm\frac{1}{4}\left(\langle s_{\mathcal{T}_c\circ\tau_b}(N_\gamma(c);Y), F(c,\tau_b)^{-1}(Z)\rangle_{N_\gamma(c)\times S^2} + \langle F(c,\tau_b)^{-1}(Y), s_{\mathcal{T}_c\circ\tau_b}(N_\gamma(c);Z)\rangle_{N_\gamma(c)\times S^2}\right).$$

We show that both intersection numbers are ± 1 and that their signs are opposite. Consider an arc ξ of great circle from e_1 to $-e_1$. When ξ stands for $[a,b]$, the map F of Definition 10.4 above can be seen as the map that maps $(V \in \xi, u \in [-1,1])$ to the rotation $R_{\alpha(u),V}$ with axis V and angle $\alpha(u)$. Then $\langle F(c,\tau_b)^{-1}(Z), s_{\mathcal{T}_c \circ \tau_b}(N_\gamma(c);Y)\rangle$ is the degree of the map

$$\begin{array}{rccl} f_Y\colon & \xi \times [-1,1] & \to & S^2 \\ & (V,u) & \mapsto & R_{\alpha(u),V} \circ R_{\alpha(-u),e_1}(Y) \end{array}$$

at Z while the other intersection is the degree of the map f_Z at Y. Since the boundary of the image of f_Y is $\pm 2S^1(Y)$, the degree jumps by ± 2 from one component of $S^2 \setminus S^1(Y)$ to the other one. On the other hand, the degree of f_Y on the component of e_1 is independent of $Y \neq e_1$, and the degree on the component of $-e_1$ is independent of $Y \neq (-e_1)$. Therefore, using Lemma 8.4, we easily see that the degrees are ± 1 and that they are opposite. $\diamond$

Lemma 10.6 *Let $e_1 = (1,0,0) \in \mathbb{R}^3$. Let Σ be a surface immersed in a 3-manifold M equipped with a trivialisation $\tau : TM \to M \times \mathbb{R}^3$ such that $\tau^{-1}(.;e_1)$ is a positive normal to Σ along the boundary $\partial\Sigma$ of Σ. Let $a^{(1)}, \ldots, a^{(k)}$ denote the k connected components of the boundary $\partial\Sigma$ of Σ. For $i = 1,\ldots,k$, the unit bundle of $T\Sigma_{|a^{(i)}}$ is an S^1-bundle over $a^{(i)}$ with a canonical trivialisation induced by $Ta^{(i)}$. For a trivialisation τ as above, let $d(\tau, a^{(i)})$ be the degree of the projection on the fiber S^1 of this bundle of the section $\tau^{-1}(a^{(i)} \times e_2)$. Let $diag(n)(\Sigma) \subset ST(M)$ be the section of $ST(M)_{|\Sigma}$ in $ST(M)$ associated with the positive normal field n to Σ, that coincides with $\tau^{-1}(\partial\Sigma \times e_1)$ on $\partial\Sigma$. Then*

$$2(diag(n)(\Sigma) - s_\tau(\Sigma;e_1)) - \left(\sum_{i=1}^{k} d(\tau, a^{(i)}) + \chi(\Sigma)\right) ST(*)$$

is a cycle that is null-homologous in $ST(\Sigma)$.

Proof: This comes from the fact that the relative Euler class of $s_\tau(\partial\Sigma;e_2)$ in $T\Sigma$ is

$$\chi(\Sigma) + \sum_{i=1}^{k} d(\tau, a^{(i)}),$$

up to sign. Let us give some more details. Proposition 4.5 gives the result when $\partial\Sigma = \emptyset$. Since the generator of $\pi_1(SO(3))$ can be realized by rotations around the e_1-axis and since $\pi_2(SO(3)) = 0$, the homology class of $(\mathrm{diag}(n)(\Sigma) - \tau^{-1}(\Sigma \times e_1))$ does not depend on the trivialisation τ of $(T\Sigma \oplus \mathbb{R})$ when the $d(\tau, a^{(i)})$ are fixed. Then it is easy to find an embedding of Σ in $\mathbb{R}^3$ for which $d(\tau, a^{(i)}) = 1$ for all i with the standard trivialisation of $\mathbb{R}^3$ (a standard one obtained from a closed surface by drilling small holes) and another one for which $d(\tau, a^{(i)}) = -1$ for all i by gluing the following pieces to the hole boundaries.

Thus, the lemma is easy to prove when τ is an actual trivialisation and when $d(\tau, a^{(i)})$ does not depend on i and is ± 1, by computing degrees of Gauss maps.

Let A be an annulus that is a regular neighborhood of a boundary component of Σ. Let $h_A\colon A \to SO(3)$ be a homotopy between the trivial loop of $SO(3)$ and a loop of rotations of $SO(3)$ around e_1 that has degree 2. Since the composition of τ by h_A sends $\tau^{-1}(A \times e_1)$ to the sphere with degree ± 1 (see Lemma 8.4), and because the previous argument fixes the correct sign, we easily conclude that the cycle of the statement is null-homologous when the $d(\tau, a^{(i)})$ are odd.

Then twisting the trivialisation around e_1 across paths in Σ from one boundary component to another one allows us to prove that it is null-homologous for arbitrary $d(\tau, a^{(i)})$. $\diamond$

Lemma 10.7 *Lemma 10.6 is true if τ is a pseudo-trivialisation that is a genuine trivialisation satisfying the assumptions of Lemma 10.6 around $\partial\Sigma$, such that Σ meets $N(c)$ along sections $N_\gamma(c)$, where $s_\tau(\Sigma; e_1)$ is defined in Definition 10.4.*

Proof: Since the formula of Lemma 10.6 behaves well under gluing (or cutting) surfaces along curves that satisfy the assumptions of Lemma 10.6 and since these assumptions are easily satisfied by isotopy (when the cutting process is concerned), we are left with the proof of the lemma for a meridian disk Σ of c in Definition 10.1. For such a disk $s_\tau(\Sigma; e_1)$ is the average of two genuine sections corresponding to trivialisations τ_1 and τ_2 so that

$$2(\mathrm{diag}(n)(\Sigma) - s_\tau(\Sigma; e_1)) - \left(\frac{d(\tau_1, \partial\Sigma) + d(\tau_2, \partial\Sigma)}{2} + \chi(\Sigma)\right) ST(*)$$

is a null-homologous cycle. Since the exterior trivialisation τ_e is such that

$$d(\tau_e, -\partial\Sigma) = \frac{d(\tau_1, -\partial\Sigma) + d(\tau_2, -\partial\Sigma)}{2},$$

we are done. $\diamond$

10.4 Making the definition of $\mathcal{Q}$ more flexible with pseudo-sections

Lemma 10.8 *Let $\tilde{\tau}$ be a pseudo-trivialisation as in Definition 10.1, and let τ be a genuine trivialisation of M that coincides with τ_e on a tubular neighborhood of K. For any $V \in S^2$, there exists a 4-chain $C_4(V)$ of $[0,1] \times ST(E = M \setminus Int(N(K)))$ such that $\partial C_4(V) = c_3(V)$ where*

$$c_3(V) = \{1\} \times s_{\tilde{\tau}}(ST(E); V) - \{0\} \times s_\tau(ST(E); V) + [0,1] \times \partial N(K) \times_\tau \{V\}.$$

Proof: Like in the proof of Lemma 5.10, it suffices to prove that the algebraic intersection

$$\langle c_3(V), [0,1] \times (S \setminus \mathrm{Int}(D^2)) \times_\tau W\rangle_{[0,1]\times ST(E)}$$

is zero for some $W \in S^2 \setminus S^1(V)$, where S is assumed to meet $N(c)$ along sections $N_\gamma(c)$. Up to sign, this intersection is

$$\langle s_{\tilde{\tau}}(S \setminus \mathrm{Int}(D^2); V), (S \setminus \mathrm{Int}(D^2)) \times_\tau W \rangle_{ST(S \setminus \mathrm{Int}(D^2))}.$$

Since Lemma 10.7 implies that the cycle $(s_{\tilde{\tau}}(S \setminus \mathrm{Int}(D^2); V) - s_\tau(S \setminus \mathrm{Int}(D^2); V))$ bounds in $ST(S \setminus \mathrm{Int}(D^2))$, this algebraic intersection vanishes. $\diamond$

Lemma 10.8 allows us to construct a 4–chain $F_V(\tilde{\tau})$ with the wanted boundary

$$\partial F_V(\tilde{\tau}) = s_{\tilde{\tau}}(M; V) - I_\Delta ST(M)_{|K_V}$$

from our 4-chain $F_V(\tau)$ previously associated to τ, (using modifications near $ST(E) \subset \partial C_2(M)$) such that $\langle F_V(\tilde{\tau}), A(K) \rangle_e = 0$.

Note that as soon as the circles $S^1(X)$, $S^1(Y)$ and $S^1(Z)$ are disjoint, the chains $F_X(\tilde{\tau})$, $F_Y(\tilde{\tau})$ and $F_Z(\tilde{\tau})$ do not have triple intersection on the boundary.

As before, we define

$$\mathcal{Q}(\mathbb{K}, \tilde{\tau}) = \langle F_X(\tilde{\tau}), F_Y(\tilde{\tau}), F_Z(\tilde{\tau}) \rangle_e.$$

Because of the specific construction outlined above,

$$\mathcal{Q}(\mathbb{K}, \tilde{\tau}) - \mathcal{Q}(\mathbb{K}, \tau) = \langle C_4(X), C_4(Y), C_4(Z) \rangle_{[0,1] \times ST(E)}.$$

We shall prove that the following lemma follows from [L2].

Lemma 10.9

$$\langle C_4(X), C_4(Y), C_4(Z) \rangle_{[0,1] \times ST(E)} = \frac{p_1(\tilde{\tau}_{\mathbb{C}}) - p_1(\tau)}{4}.$$

Together with Proposition 5.8, this lemma obviously implies the following proposition.

Proposition 10.10 *Under the assumptions above,*

$$\mathcal{Q}(\mathbb{K}) = \mathcal{Q}(\mathbb{K}, \tilde{\tau}) - \frac{p_1(\tilde{\tau}_{\mathbb{C}})}{4}.$$

Definition 10.11 A *pseudo-trivialisation* $\tilde{\tau} = (N(c); \tau_e, \tau_b)$ *of a rational homology ball* $B(N)$ associated with a rational homology sphere N (as in Subsection 6.1) consists of a framed link c of $B(N)$ equipped with a neighborhood $N(c)$ in the interior of B_N, a trivialisation τ_e of $(B(N) \setminus \mathrm{Int}(N(c)))$ that is standard on $(B(N) \setminus \mathrm{Int}(B_N) = B(3) \setminus \mathrm{Int}(B(1)))$, and a trivialisation τ_b of $T(N(c))$ where τ_e and τ_b are related like in Definition 10.1.

Then the *trivialisation* $\tilde{\tau}_{\mathbb{C}}$ of $TB(N) \otimes_{\mathbb{R}} \mathbb{C}$ is defined like in Definition 10.3 and the definition of the *Pontrjagin class* for genuine trivialisations standard on $B(N) \setminus \mathrm{Int}(B_N)$, outlined in Subsection 6.1 and given in details in [L1, Section 1.5], naturally extends to pseudo-trivialisations of rational homology balls.

Finally, define a 3-*cycle* $\partial F_{N,V}(\tilde{\tau})$ as $p_N^{-1}(V)$ on $\partial C_2(B(N)) \setminus \mathrm{Int}(ST(N(c)))$, with respect to the map p_N of Subsection 6.1, and as $s_{\tilde{\tau}}(N(c); V)$ on $ST(N(c))$, like in Definition 10.4.

Recall that $S^1(V)$ denotes the circle through V in S^2 that lies in a plane orthogonal to the axis generated by $(1,0,0)$.

Proposition 10.12 *Let X, Y and Z be three distinct points of S^2 such that the circles $S^1(X)$, $S^1(Y)$ and $S^1(Z)$ are disjoint. With the above notation, for $V = X$, Y or Z, $\partial F_{N,V}(\tilde{\tau})$ bounds a rational chain $F_{N,V}(\tilde{\tau})$ in $C_2(B(N))$, and*

$$\lambda(N) = \frac{\langle F_{N,X}(\tilde{\tau}), F_{N,Y}(\tilde{\tau}), F_{N,Z}(\tilde{\tau})\rangle_{C_2(B(N))}}{6} - \frac{p_1(\tilde{\tau})}{24}.$$

Proof: It is essentially a consequence of [L2, Proposition 4.8]. We nevertheless give a few details to see how this proposition applies. According to [L2, Theorem 2.6], $Z_1(N) = \frac{\lambda(N)}{2}[\theta]$. According to [L2, Proposition 4.8],

$$Z_1(N) = Z_1(N; \omega(c, \tau_e, \tau_b)) + \frac{p_1(\tilde{\tau})}{4}\xi_1 = \frac{\int_{C_2(B(N))} \omega(c, \tau_e, \tau_b)^3}{12}[\theta] + \frac{p_1(\tilde{\tau})}{4}\xi_1$$

where $\xi_1 = -\frac{1}{12}[\theta]$ according to [L1, Proposition 2.45], and $\omega(c, \tau_e, \tau_b)$ is defined as $p_N^*(\omega_{S^2})$ on $\partial C_2(B(N)) \setminus \mathrm{Int}(ST(N(c)))$, and like in [L2, Notation 4.9] on $ST(N(c))$ (that is like in [L2, Notation 4.9], except that $C_2(N)$ is replaced by $C_2(B(N))$ like in the proof of Theorem 6.1). Therefore $\lambda(N) = \frac{\int_{C_2(B(N))} \omega(c,\tau_e,\tau_b)^3}{6} - \frac{p_1(\tilde{\tau})}{24}$. Now, we can change $\omega(c, \tau_e, \tau_b)^3$ in the above integral to $\omega_X \wedge \omega_Y \wedge \omega_Z$ where the closed form ω_V is Poincaré dual to $F_{N,V}(\tilde{\tau})$ and supported in a small neighborhood of $F_{N,V}(\tilde{\tau})$, like in [L2, Lemma 6.15] for instance. $\diamond$

Proof of Lemma 10.9: The trivialisation τ of $M \setminus \mathrm{Int}(N(K))$ extends to a rational homology sphere M' obtained by Dehn filling with respect to a parallel of K (that differs from the given parallel of K by adding a meridian). Call this trivialisation $\tau(M')$ and assume without loss that, after removing from M' an open ball inside the new solid torus T' of M', the boundary of the obtained rational homology ball $B(M')$ can be identified with the boundary of the ball of radius 3 of $\mathbb{R}^3$ so that $\tau(M')$ coincides with the standard trivialisation of $\mathbb{R}^3$ near the boundary. Then $(p_1(\tilde{\tau}_{\mathbb{C}}) - p_1(\tau))$ is the obstruction to extend the $SU(4)$-trivialisation induced by

- $\tilde{\tau}_{\mathbb{C}}$ (and $\tau(M')$ on T') on $\{1\} \times M$, $\{1\} \times M'$ or $\{1\} \times B(M')$,
- τ (and $\tau(M')$ on T') on $\{0\} \times M$, $\{0\} \times M'$ or $\{0\} \times B(M')$,

to $[0,1] \times M$, $[0,1] \times M'$ or $[0,1] \times B(M')$, respectively. Now, as before

$$\langle F_{M',X}(\tilde{\tau}), F_{M',Y}(\tilde{\tau}), F_{M',Z}(\tilde{\tau})\rangle_{C_2(B(M'))} - \langle F_{M',X}(\tau), F_{M',Y}(\tau), F_{M',Z}(\tau)\rangle_{C_2(B(M'))}$$
$$= \langle C_4(X), C_4(Y), C_4(Z)\rangle_{[0,1]\times ST(E)}.$$

Thus, Proposition 10.12 allows us to conclude. $\diamond$

11 Proof of the LP surgery formula

11.1 Sketch of proof of Theorem 9.1

The restriction of any trivialisation τ of M to $M \setminus \mathrm{Int}(A \cup B)$ extends as a pseudo-trivialisation to $A' \cup B'$. This induces a pseudo-trivialisation $\tau(M_{AB})$ on M_{AB}, and pseudo-trivialisations $\tau(M_A)$ and $\tau(M_B)$ on M_A and M_B, respectively, that coincide with each other and with τ as much as possible. The chains $F_V(M_S)$ associated with these manifolds M_S will be associated with these pseudo-trivialisations so that

- $\partial F_V(M_S) = \partial F_V(\tau(M_S))$,
- $\mathcal{Q}(\mathbb{K} \subset M_S) = \langle F_X(M_S), F_Y(M_S), F_Z(M_S)\rangle_e - \frac{p_1(\tau(M_S)_{\mathbb{C}})}{4}$
- $p_1(\tau = \tau(M)) - p_1(\tau(M_A)_{\mathbb{C}}) = p_1(\tau(M_B)_{\mathbb{C}}) - p_1(\tau(M_{AB})_{\mathbb{C}})$

The last equality is proved like [L2, Lemma 3.2] (where Y is a 3-manifold with fixed rational homology –instead of a $\mathbb{Q}$-handlebody– and fixed Lagrangian). It implies that the Pontrjagin classes can be forgotten in the expression of the alternate sum $\mathcal{S}$ that we are about to compute using the above expressions of the $\mathcal{Q}(\mathbb{K} \subset M_S)$.

For $i = 1, \dots, g_A$, let $\Sigma(a_i)$ and $\Sigma'(a_i)$ be rational chains of A and A', respectively, such that $\partial(\Sigma(a_i)) = \partial(\Sigma'(a_i)) = a_i$. Then $\mathcal{I}_{AA'}(a_i \wedge a_j \wedge a_k)$ is the algebraic intersection

$$\langle \Sigma(a_i) \cup (-\Sigma'(a_i)), \Sigma(a_j) \cup (-\Sigma'(a_j)), \Sigma(a_k) \cup (-\Sigma'(a_k))\rangle_{A \cup_{\partial A} - A'}.$$

Similarly, for $i = 1, \dots, g_B$, let $\Sigma(b_i)$ and $\Sigma'(b_i)$ be rational chains of B and B', respectively, such that $\partial(\Sigma(b_i)) = \partial(\Sigma'(b_i)) = b_i$. For a curve c among the a_i or the b_i, let $\Sigma_X(c)$, $\Sigma_Y(c)$ and $\Sigma_Z(c)$ denote three parallel copies of $\Sigma(c)$, and let $\Sigma'_X(c)$, $\Sigma'_Y(c)$ and $\Sigma'_Z(c)$ similarly denote three parallel copies of $\Sigma'(c)$ such that $\partial(\Sigma_V(c)) = \partial(\Sigma'_V(c))$ is parallel to c, for $V = X$, Y or Z. Fix preferred lifts for A and B in $\tilde{M}$, the preferred lifts of the curves c and the chains $\Sigma_V(c)$ in the preferred lifts of A and B are also denoted by c or $\Sigma_V(c)$.

For $i = 1, \dots, g_A$, $\delta(t_M)z_i$ bounds a rational chain in $\left(\tilde{M} \setminus p_M^{-1}(K)\right)$. (The possible intersections with $p_M^{-1}(K)$ can be removed with the help of S.) Therefore, it cobounds a rational cycle $\delta(t_M)\Sigma(\check{z}_i)$ in $\left(\tilde{M} \setminus p_M^{-1}(K \cup \mathring{A})\right)$ with a combination of a_j that has its coefficients in Λ_M.

$$\partial\Sigma(\check{z}_i) = z_i - \sum_{j=1}^{g_A} lk_e(z_i, z_j^-)a_j = \check{z}_i.$$

We can furthermore assume that

$$\Sigma(\check{z}_i) \cap p_M^{-1}(B) = \sum_{j=1}^{g_B} lk_e(z_i, y_j)\Sigma(b_j).$$

Define $\Sigma'(\check{z}_i) \subset \left(\widetilde{M_B} \setminus p_{M_B}^{-1}(K \cup \mathring{A})\right)$ from $\Sigma(\check{z}_i)$ by replacing the pieces $\Sigma(b_j)$ by pieces $\Sigma'(b_j)$.

Similarly, construct $\Sigma(\check{y}_i)$ in $\left(\widetilde{M} \setminus p_M^{-1}(K \cup \mathring{B})\right)$, for $i = 1, \ldots, g_B$, such that

$$\partial\Sigma(\check{y}_i) = y_i - \sum_{j=1}^{g_B} lk_e(y_i, y_j^-)b_j = \check{y}_i$$

and

$$\Sigma(\check{y}_i) \cap p_M^{-1}(A) = \sum_{j=1}^{g_A} lk_e(y_i, z_j)\Sigma(a_j)$$

and define $\Sigma'(\check{y}_i) \subset \left(\widetilde{M_A} \setminus p_{M_A}^{-1}(K \cup \mathring{B})\right)$ from $\Sigma(\check{y}_i)$ by replacing the pieces $\Sigma(a_j)$ by pieces $\Sigma'(a_j)$.

When A is a submanifold of a closed 3-manifold M with first Betti number one, $C_2(A)$ will denote the preimage of A^2 under the natural projection $Q\colon C_2(M) \to M^2$, and $\tilde{C}_2(A)$ will be the preimage of A^2 under the natural projection $\tilde{Q}\colon \tilde{C}_2(M) \to M^2$.

Proposition 11.1 *We can assume that the fundamental chains $F_V(M_S)$ for the manifolds M, $M_A = M(A'/A)$, $M_B = M(B'/B)$, $M_{AB} = M(A'/A, B'/B)$, and for $V = X$, Y or Z, satisfy the following properties.*

1. *The chains $F_V(M)$ intersect*
   - *$A \times B$ as $\sum_{(i,j)\in\{1,\ldots,g_A\}\times\{1,\ldots,g_B\}} \overline{lk_e(z_i, y_j)}\Sigma_V(a_i) \times \Sigma_V(b_j)$ and*
   - *$B \times A$ as $\sum_{(i,j)\in\{1,\ldots,g_A\}\times\{1,\ldots,g_B\}} lk_e(z_i, y_j)\Sigma_V(b_j) \times \Sigma_V(a_i)$.*

   and the chains $F_V(M_A)$, $F_V(M_B)$ and $F_V(M_{AB})$ have similar intersections with $A^{(\prime)} \times B^{(\prime)}$ and $B^{(\prime)} \times A^{(\prime)}$ where appropriate primes are added.

2. *Set*
$$ST(\Sigma_{AA'}) = \sum_{(i,j,k)\in\{1,\ldots g_A\}^3} \mathcal{I}_{AA'}(a_i \wedge a_j \wedge a_k) lk_e(z_k, z_j) ST(M)_{|p_M(\Sigma(\check{z}_i))}$$
   and
$$ST(\Sigma_{BB'}) = \sum_{(\ell,m,n)\in\{1,\ldots g_B\}^3} \mathcal{I}_{BB'}(b_\ell \wedge b_m \wedge b_n) lk_e(y_n, y_m) ST(M)_{|p_M(\Sigma(\check{y}_\ell))}$$
   Similarly define
$$ST(\Sigma'_{AA'}) \subset \tilde{C}_2\left(M_B \setminus (\mathring{A} \cup K)\right)$$
   and
$$ST(\Sigma'_{BB'}) \subset \tilde{C}_2\left(M_A \setminus (\mathring{B} \cup K)\right)$$

by replacing Σ by Σ' in the right-hand sides. Let $\mathbb{Z} \times ST(M) \times [0,1] = \partial\tilde{C}_2(M) \times [0,1]$ denote an equivariant neighborhood of $\partial\tilde{C}_2(M)$ in $\tilde{C}_2(M)$ and, for $\eta \in]0,1]$, let $ST_\eta(\Sigma_{..})$ denote $ST(\Sigma_{..}) \times \{\eta\}$ in this neighborhood.
Let $\eta \in]0, 1/6[$, let $r(X) = 0$, $r(Y) = 1$ and $r(Z) = 2$.

- $F_V(M_A) = F_V(M) + ST_{(3-r(V))\eta}(\Sigma_{AA'})$ *on* $\tilde{C}_2(M \setminus \mathring{A})$,
- $F_V(M_{AB}) = F_V(M_B) + ST_{(3-r(V))\eta}(\Sigma'_{AA'})$ *on* $\tilde{C}_2(M_B \setminus \mathring{A})$,
- $F_V(M_B) = F_V(M) + ST_{(6-r(V))\eta}(\Sigma_{BB'})$ *on* $\tilde{C}_2(M \setminus \mathring{B})$,
- $F_V(M_{AB}) = F_V(M_A) + ST_{(6-r(V))\eta}(\Sigma'_{BB'})$ *on* $\tilde{C}_2(M_A \setminus \mathring{B})$.
- *For any permutation σ of $\{X, Y, Z\}$,*
  - $p(ST_{(3-r(\sigma(X)))\eta}(\Sigma_{AA'})) \cap p(F_{\sigma(Y)}(M)) \cap p(F_{\sigma(Z)}(M)) = \emptyset$
  - $p(ST_{(6-r(\sigma(X)))\eta}(\Sigma_{BB'})) \cap p(F_{\sigma(Y)}(M)) \cap p(F_{\sigma(Z)}(M)) = \emptyset$
  - *The intersection of $ST_{(3-r(\sigma(X)))\eta}(\Sigma'_{AA'})$, $F_{\sigma(Y)}(M_B)$ and $F_{\sigma(Z)}(M_B)$ is located in the interior of $p^{-1}(C_2(B'))$ and their algebraic equivariant intersection vanishes.*
  - *The intersection of $ST_{(6-r(\sigma(X)))\eta}(\Sigma'_{BB'})$, $F_{\sigma(Y)}(M_A)$ and $F_{\sigma(Z)}(M_A)$ is located in the interior of $p^{-1}(C_2(A'))$, their algebraic equivariant intersection is*

$$\sum_{(i,j,k)\in\{1,\dots g_A\}^3,(\ell,m,n)\in\{1,\dots g_B\}^3} \mathcal{I}(i,j,k,\ell,m,n) L_\sigma(i,j,k,\ell,m,n)$$

where $\mathcal{I}(i,j,k,\ell,m,n) = \mathcal{I}_{AA'}(a_i \wedge a_j \wedge a_k)\mathcal{I}_{BB'}(b_\ell \wedge b_m \wedge b_n)$, σ permutes x, y and z like X, Y and Z and

$$L_\sigma(i,j,k,\ell,m,n) =$$

$$-lk_e(y_n, y_m)(\sigma(x))\,(lk_e(z_k, z_j)(\sigma(y)) + lk_e(z_k, z_j)(\sigma(z)))\,lk(z_i, y_\ell).$$

The proof of this key proposition will start in the next subsection and end at the end of the section. Assuming it, let us conclude the proof of Theorem 9.1.

For a part C of $C_2(M_S)$, let

$$\mathcal{I}_S(C) = \langle F_X(M_S), F_Y(M_S), F_Z(M_S)\rangle_{e,p^{-1}(C)}$$

denote the restriction of the intersection to this part. Then

$$\mathcal{I}_B(C_2(M_B \setminus \mathring{B}')) = \mathcal{I}(C_2(M \setminus \mathring{B}))$$

and the equivariant intersection points are actually the same in both sides because the pieces $ST_{(6-r(\sigma(X)))\eta}(\Sigma_{BB'})$ are pairwise disjoint and they do not meet $\big(F_{\sigma(Y)}(M) \cap F_{\sigma(Z)}(M)\big)$. Similarly,

$$\mathcal{I}_{AB}(C_2(M_{AB} \setminus \mathring{B}')) - \mathcal{I}_A(C_2(M_A \setminus \mathring{B})) = (\mathcal{I}_{AB} - \mathcal{I}_A)(C_2(A'))$$

$$= - \sum_{(i,j,k)\in\{1,\dots,g_A\}^3,\,(\ell,m,n)\in\{1,\dots,g_B\}^3,\,\sigma\in\mathfrak{S}_3(x,y,z)} \mathcal{I}(i,j,k,\ell,m,n) lk(z_i,y_\ell) lk_e(z_j,z_k)(x) lk_e(y_m,y_n)(y)$$

and

$$(\mathcal{I}_{AB} - \mathcal{I}_A - \mathcal{I}_B + \mathcal{I})\left(C_2(M_S \setminus \mathring{B}^{(\prime)})\right) = (\mathcal{I}_{AB} - \mathcal{I}_A)(C_2(A')).$$

(The parentheses around the prime mean that there can be a prime.)

Consider the part

$$C(B^{(\prime)}, M_S \setminus \mathring{A}^{(\prime)}) = C_2(M_S \setminus \mathring{A}^{(\prime)}) \cap \overline{C_2(M_S) \setminus C_2(M_S \setminus B^{(\prime)})}$$

of $C_2(M_S)$ made of the configurations that project to M_S^2 as pairs of points such that at least one of the points is in $B^{(\prime)}$ and both of them are in $M_S \setminus \mathrm{Int}(A^{(\prime)})$.

Note that $\mathcal{I}_A$ and $\mathcal{I}$ coincide there with the same actual equivariant intersection like in the whole $C_2(M_S \setminus \mathrm{Int}(A^{(\prime)}))$, while the equivariant intersection points contributing to $(\mathcal{I}_{AB} - \mathcal{I}_B)(C(B', M_B \setminus \mathrm{Int}(A)))$ project to the interior of $C_2(B')$ and yield a trivial contribution. Thus,

$$(\mathcal{I}_{AB} - \mathcal{I}_A - \mathcal{I}_B + \mathcal{I})\left(C_2(M_S \setminus \mathring{A}^{(\prime)}) \cup C_2(M_S \setminus \mathring{B}^{(\prime)})\right) = (\mathcal{I}_{AB} - \mathcal{I}_A)(C_2(A')).$$

Since the closure of the complement of $C_2(M_S \setminus \mathrm{Int}(A^{(\prime)})) \cup C_2(M_S \setminus \mathrm{Int}(B^{(\prime)}))$ in $C_2(M_S)$ is $\left(A^{(\prime)} \times B^{(\prime)}\right) \cup \left(B^{(\prime)} \times A^{(\prime)}\right)$,

$$(\mathcal{S} - (\mathcal{I}_{AB} - \mathcal{I}_A)(C_2(A')))$$

is the contribution of the intersection points that project to $A^{(\prime)} \times B^{(\prime)}$ or $B^{(\prime)} \times A^{(\prime)}$. Then the remaining contributions can be seen as contributions of intersections in $p^{-1}\left((A \cup (-A')) \times (B \cup (-B'))\right)$ and intersections in $p^{-1}\left((B \cup (-B')) \times (A \cup (-A'))\right)$. The intersections in $p^{-1}\left((A \cup (-A')) \times (B \cup (-B'))\right)$ contribute as

$$\sum_{((i,\ell),(j,m),(k,n))\in(\{1,\dots,g_A\}\times\{1,\dots,g_B\})^3} lk_e(z_i,y_\ell)(x^{-1}) lk_e(z_j,y_m)(y^{-1}) lk_e(z_k,y_n)(z^{-1}) \mathcal{I}_{ijk\ell mn}$$

where $\mathcal{I}_{ijk\ell mn}$ is the algebraic triple intersection of $(\Sigma(a_i) \cup -\Sigma'(a_i)) \times (\Sigma(b_\ell) \cup -\Sigma'(b_\ell))$, $(\Sigma(a_j) \cup -\Sigma'(a_j)) \times (\Sigma(b_m) \cup -\Sigma'(b_m))$ and $(\Sigma(a_k) \cup -\Sigma'(a_k)) \times (\Sigma(b_n) \cup -\Sigma'(b_n))$ that is

$$\mathcal{I}_{ijk\ell mn} = -\mathcal{I}_{AA'}(a_i \wedge a_j \wedge a_k)\mathcal{I}_{BB'}(b_\ell \wedge b_m \wedge b_n).$$

The intersections in $p^{-1}\left((B \cup (-B')) \times (A \cup (-A'))\right)$ give the conjugate expression. This gives the wanted expression for $\mathcal{S}$.

The proof of Theorem 9.1 is now reduced to the proof of Proposition 11.1 that will occupy the rest of this section. $\diamond$

11.2 Normalizing F_X with respect to one handlebody A

Let A be a rational homology handlebody embedded in M (outside a tubular neighborhood of K) whose H_1 goes to torsion in $H_1(M)$ so that A lifts in $\tilde{M}$. Again, we shall consider a preferred lift of A in $\tilde{M}$ and often identify A with its preferred lift, and submanifolds of A with submanifolds of the preferred lift, without mentioning it.

Let $(a_i, z_i)_{i=1,\dots g}$ be a basis of $H_1(\partial A)$ such that $a_i = \partial(\Sigma(a_i) \subset A)_{i=1,\dots,g}$ where $\Sigma(a_i)$ is a rational chain of A and $\langle a_i, z_j\rangle = \delta_{ij}$.

Let $[-4,4] \times \partial A \subset M$ be a neighborhood of $\partial A = 0 \times \partial A$. ($[-4,0] \times \partial A \subset A$.)

For $s \in [-4,4]$, set

$$A_s = \begin{cases} A \cup (\partial A \times [0,s]) & \text{if } s \geq 0 \\ A \setminus (]s,0] \times \partial A) & \text{if } s \leq 0 \end{cases}$$

and $(M \setminus A)_s = M \setminus \mathring{A}_s$.

$$\partial A_s = \{s\} \times \partial A = -\partial(M \setminus A)_s.$$

Assume that $\Sigma(a_i)$ intersects $[-4,0] \times \partial A$ as $[-4,0] \times a_i$ and construct a rational chain $\Sigma_4(a_i) = \Sigma(a_i) \cup [0,4] \times a_i$. Set $\Sigma_s(a_i) = A_s \cap \Sigma_4(a_i)$.

Let p_A be a point of ∂A outside the a_i and the z_i. Let $(1-t_M)\gamma_A$ denote a path in $p_M^{-1}((M \setminus A)_0)$ that intersects $\left(p_M^{-1}(A_4) \subset \tilde{M}\right)$ as $(1-\theta_M)([0,4] \times p_A)$ so that its boundary is $(\theta_M - 1)p_A$. The path $(1-t_M)\gamma_A$ can be constructed as follows: First join ∂A to a parallel of K by a path, and let $(1-t_M)\gamma_A$ be the conjugate of this parallel of K by this path.

We shall assume that $\Sigma(\check{z}_i)$ (defined just before Proposition 11.1) avoids the preimage of γ_A and (by removing the possible intersections with the help of ∂A) that $\Sigma(\check{z}_i)$ intersects $p_M^{-1}([0,4] \times \partial A)$ as $-[0,4] \times \check{z}_i$ and set $\Sigma_s(\check{z}_i) = \Sigma(\check{z}_i) \cap p_M^{-1}((M \setminus A)_s)$.

Set $\gamma_{A,s} = \gamma_A \cap p_M^{-1}((M \setminus A)_s)$.

Set $E = A \times (M \setminus A)_3$ and

$$\tilde{E} = p^{-1}(E) = A \times_e (M \setminus A)_3 = A \times p_M^{-1}((M \setminus A)_3).$$

Proposition 11.2 *We can assume that F_X intersects $\tilde{E} = A \times_e (M \setminus A)_3$ like*

$$A \times \gamma_{A,3} + \sum_{(i,j)\in\{1,\dots,g\}^2} \overline{lk_e(z_i, a_j^+)}\Sigma(a_i) \times \Sigma_3(\check{z}_j)$$

where $\overline{lk_e(z_i, a_j^+)} = lk_e(z_i, a_j^+)(t^{-1})$.

Proof: We shall check that the class of F_X in

$$H_4(C_2(M), C_2(M) \setminus \mathrm{Int}(E); \mathbb{Q}(t)) = H_4(E, \partial E; \mathbb{Q}(t))$$

coincides with the class F_1 of the chain of the statement by proving first that $H_4(E, \partial E; \mathbb{Q}(t))$ is the dual of $H_2(E; \mathbb{Q}(t))$ -as Poincaré duality prescribes- and next that F_X and F_1 intersect a

basis of $H_2(E;\mathbb{Q}(t))$ in the same way. This is enough to produce a 5–chain W in $\tilde{E}$, or rather in the homotopically equivalent $A_1 \times_e (M \setminus A)_2$, such that $(\partial W + F_X - F_1) \subset \partial\,(A_1 \times_e (M \setminus A)_2)$ and to conclude by adding ∂W to F_X. Details follow. $\diamond$

Lemma 11.3

$$H_*(M \setminus A;\mathbb{Q}(t_M)) = H_{*+1}(A,\partial A;\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}(t_M)$$
$$H_2(M \setminus A;\mathbb{Q}(t_M)) = \mathbb{Q}(t_M)[\partial A].$$
$$H_1(M \setminus A;\mathbb{Q}(t_M)) = \oplus_{i=1}^g \mathbb{Q}(t_M)[a_i].$$
$$H_0(M \setminus A;\mathbb{Q}(t_M)) = 0.$$

Proof: Since $H_*(M;\mathbb{Q}(t_M)) = 0$ by Lemma 2.2,

$$H_*(M \setminus A;\mathbb{Q}(t_M)) = H_{*+1}(M, M \setminus A;\mathbb{Q}(t_M)).$$

$\diamond$

Lemma 11.4

$$H_q(E;\mathbb{Q}(t)) = \left(H_*(A;\mathbb{Q}) \otimes (\mathbb{Q}[\partial A_3] \oplus \oplus_{i=1}^g \mathbb{Q}[a_i \subset \partial A_3])\right)_q \otimes_{\mathbb{Q}} \mathbb{Q}(t)$$

$$H_q(\partial A \times (M \setminus A)_3;\mathbb{Q}(t)) = \left(H_*(\partial A;\mathbb{Q}) \otimes (\mathbb{Q}[\partial A_3] \oplus \oplus_{i=1}^g \mathbb{Q}[a_i \subset \partial A_3])\right)_q \otimes_{\mathbb{Q}} \mathbb{Q}(t)$$

Proof: $\tilde{E} = p^{-1}(E) = A \times p_M^{-1}(M \setminus \mathrm{Int}(A_3))$. $\diamond$

Lemma 11.5

$$H_4(E,\partial E;\mathbb{Q}(t)) = \oplus_{(i,j)\in\{1,2,\ldots,g\}^2}\mathbb{Q}(t)[\Sigma(a_i) \times \Sigma_3(\check{z}_j)] \oplus \mathbb{Q}(t)[A \times \gamma_{A,3}].$$

Proof: Since $H_4(E;\mathbb{Q}(t)) = 0$, $H_4(E,\partial E;\mathbb{Q}(t))$ is isomorphic to the kernel of the natural map

$$H_3(\partial E;\mathbb{Q}(t)) \to H_3(E;\mathbb{Q}(t)).$$

Let $A_{ij} = \partial\,(\Sigma(a_i) \times \Sigma_3(\check{z}_j))$ and $A_{00} = \partial(A \times \gamma_{A,3})$. It suffices to prove that the A_{ij} and A_{00} form a basis of this kernel.

Let us compute $H_3(\partial E = A \times \partial A_3 \cup_{\partial A \times \partial A_3} \partial A \times (M \setminus A)_3;\mathbb{Q}(t))$ with the help of the Mayer-Vietoris sequence. The cokernel of the Mayer-Vietoris map

$$H_3(\partial A \times \partial A_3;\mathbb{Q}(t)) \to H_3(A \times \partial A_3;\mathbb{Q}(t)) \oplus H_3(\partial A \times (M \setminus A)_3;\mathbb{Q}(t))$$

is

$$\oplus_{i=1}^g \mathbb{Q}(t)[z_i \times \partial A_3].$$

The kernel of the Mayer-Vietoris map

$$H_2(\partial A \times \partial A_3; \mathbb{Q}(t)) \to H_2(A \times \partial A_3; \mathbb{Q}(t)) \oplus H_2(\partial A \times (M \setminus A)_3; \mathbb{Q}(t))$$

is freely generated over $\mathbb{Q}(t)$ by the $[a_i \times \check{z}_j]$, for $(i,j) \in \{1,2,\dots,g\}^2$ and $[\partial A \times p_{A_3}]$ that are the images of the $\pm A_{ij}$ and $\pm A_{00}$ via the Mayer-Vietoris boundary map. Therefore

$$H_3(\partial E; \mathbb{Q}(t)) = \oplus_{i=1}^g \mathbb{Q}(t)[z_i \times \partial A_3] \oplus \mathbb{Q}(t) A_{00} \oplus \oplus_{(i,j)\in\{1,2,\dots,g\}^2} \mathbb{Q}(t) A_{ij}.$$

On the other hand, we know

$$H_3(E; \mathbb{Q}(t)) = \oplus_{i=1}^g \mathbb{Q}(t)[z_i \times \partial A_3].$$

This easily leads to the result. ◇

End of proof of Proposition 11.2: Enlarge $\tilde{E}$ to the homeomorphic

$$\tilde{E}_1 = A_1 \times_e (M \setminus A)_2 = p^{-1}(E_1)$$

where $E_1 = A_1 \times (M \setminus A)_2$. The basis of $H_4(E_1, \partial E_1; \mathbb{Q}(t))$ is dual to the basis of $H_2(E_1; \mathbb{Q}(t))$ made of the $z_i \times (\{3\} \times a_j)$ and $p_A \times \partial A_3$, with respect to the equivariant algebraic intersection. According to Proposition 3.3 and to Theorem 4.8, $\langle z_i \times (\{3\} \times a_j), F_X \rangle_e = lk_e(z_i, \{3\} \times a_j)$ and $\langle p_A \times \partial A_3, F_X \rangle_e = 1$. Therefore the initial chain F_X and the wanted one F_1 (naturally extended to $\tilde{E}_1$) intersect this basis in the same way algebraically, and their classes in $H_4(E_1, \partial E_1; \mathbb{Q}(t))$ coincide. Then there exists a 5–chain W of $\tilde{E}_1$ such that $(\partial W + F_X - F_1) \subset \partial \tilde{E}_1$. Replace F_X by $\partial W + F_X$. ◇

Set

$$(M \setminus A)_3 \times_e A = p^{-1}\left((M \setminus A)_3 \times A\right) = -\iota\left(A \times_e (M \setminus A)_3\right).$$

Proposition 11.6 *We can furthermore assume that*

$$\begin{aligned} F_X \cap ((M \setminus A)_3 \times_e A) &= \iota\left(F_X \cap (A \times_e (M \setminus A)_3)\right) \\ &= -\gamma_{A,3} \times A + \textstyle\sum_{(i,j)\in\{1,\dots,g\}^2} lk_e(z_i, a_j^+) \Sigma_3(\check{z}_j) \times \Sigma(a_i). \end{aligned}$$

Proof: Since $((M \setminus A)_3 \times_e A)$ and $(A \times_e (M \setminus A)_3)$ are disjoint, this statement does not interfer with the statement of Proposition 11.2. Its proof is obtained from the other one by symmetry. ◇

Lemma 11.7 *We can furthermore assume that* $\langle F_X, \Sigma_3(a_i) \times p_A \rangle_e = 0$ *and that* $\langle F_X, p_A \times \Sigma_3(a_i) \rangle_e = 0$ *for any* i.

Proof: Indeed, adding to F_X multiples of $\partial(\Sigma_2(\check{z}_i) \times A_1)$ allows us to fix the first intersections arbitrarily and independently. Similarly, adding to F_X multiples of $\partial(A_1 \times \Sigma_2(\check{z}_i))$ allows us to fix the second intersections without changing the first ones, and these additions do not change F_X on $((M \setminus A)_3 \times_e A) \cup (A \times_e (M \setminus A)_3)$. ◇

11.3 Normalization with respect to a finite set of handlebodies

Assume that a set of n disjoint rational homology handlebodies $A^{(i)}$ in M is given, outside a tubular neighborhood of K.

In this article, $n = 2$ is enough and $A^{(2)} = B$, but it is not harder to study the general case that will be useful in a later study of the higher loop degree case. For each i, let $p_{A^{(i)}}$ be a point of $\partial A^{(i)}$ outside the $(a_j^{(i)})$ and the $(z_j^{(i)})$ that play similar roles as before for each $A^{(i)}$. Construct disjoint paths $(1-t_M)\gamma_{A^{(i)}}$ in $p_M^{-1}(M \setminus (\coprod_{j=1}^n \mathrm{Int}(A^{(j)})))$ as before. The path $(1-t_M)\gamma_{A^{(i)}}$ will intersect $p_M^{-1}(A_4^{(i)})$ as $(1-\theta_M)[0,4] \times p_{A^{(i)}}$ so that its boundary is $(\theta_M - 1)p_{A^{(i)}}$. All the $\gamma_{A^{(i)}}$ are supposed to be disjoint from each other and from a tubular neighborhood of K.

Then construct the $\Sigma(\check{z}_j^{(i)})$ as before Proposition 11.1 so that they are transverse and disjoint from the $(1-t_M)\gamma_{A^{(k)}}$. (Again, their possible intersections with the paths and K can be removed with the help of the $\partial A^{(k)}$ and S, respectively.) Also assume that the $\Sigma(\check{z}_k^{(i)})$ for $A^{(i)}$ intersects $A_4^{(j)}$ as copies of $\Sigma_4(a_\ell^{(j)})$.

Proposition 11.8 *The normalizations of the previous subsection can be achieved simultaneously. In other words, we can assume that*

- *For any $j = 1, \dots, n$, F_X intersects $A^{(j)} \times_e (M \setminus A^{(j)})_3$ like*

$$A^{(j)} \times \gamma_{A^{(j)},3} + \sum_{(i,k)\in\{1,\dots,g(A^{(j)})\}^2} \overline{lk_e(z_i^{(j)}, a_k^{(j)+})}\Sigma(a_i^{(j)}) \times \Sigma_3(\check{z}_k^{(j)}).$$

- *For any $j = 1, \dots, n$, F_X intersects $(M \setminus A^{(j)})_3 \times_e A^{(j)}$ like*

$$-\gamma_{A^{(j)},3} \times A^{(j)} + \sum_{(i,k)\in\{1,\dots,g(A^{(j)})\}^2} lk_e(z_i^{(j)}, a_k^{(j)+})\Sigma_3(\check{z}_k^{(j)}) \times \Sigma(a_i^{(j)}).$$

- *For any $j = 1, \dots, n$, for any $i = 1, \dots, g(A^{(j)})$, $\langle F_X, \Sigma_3(a_i^{(j)}) \times p_{A^{(j)}}\rangle_e = 0$ and*

$$\langle F_X, p_{A^{(j)}} \times \Sigma_3(a_i^{(j)})\rangle_e = 0.$$

As usual, $A^{(j)} \times p_{A^{(j)},3}$, $\Sigma(a_i^{(j)}) \times \check{z}_k^{(j)}$, $p_{A^{(j)},3} \times A^{(j)}$, $\check{z}_k^{(j)} \times \Sigma(a_i^{(j)})$ are in the preferred lift of $C_2(A_4^{(j)})$, the one that contains $ST(A_4^{(j)})$.

Proof: Let F_{k-1} be a 4–chain that satisfies the normalization conditions -with respect to $(A_1^{(j)}, (M \setminus A^{(j)})_2)$ instead of $(A^{(j)}, (M \setminus A^{(j)})_3)$ for the first two ones without loss- for $j < k$ and let us try to modify it into a 4–chain F_k that also satisfies them with respect to $A^{(k)}$. To do that, we would like to add some ∂W to F_{k-1} where W is a 5–chain of $A_1^{(k)} \times_e (M \setminus A^{(k)})_2$. Unfortunately, we need to control ∂W on $\partial\left(A_1^{(k)} \times_e (M \setminus A^{(k)})_2\right)$ since this boundary contains $\partial A_1^{(k)} \times_e A^{(j)}$ for $j < k$ where we cannot afford to lose our former modifications.

We know that F_{k-1} is already as wanted on $A_1^{(k)} \times_e \coprod_{j=1}^{k-1} A_1^{(j)}$. Set

$$D = A_1^{(k)} \times \left((M \setminus A^{(k)})_2 \setminus (\coprod_{j=1}^{k-1} \mathrm{Int}(A_1^{(j)})) \right).$$

$$E_1 = A_1^{(k)} \times (M \setminus A^{(k)})_2.$$

We shall consider the difference of F_{k-1} with the wanted chain $F_{k,D}$ in $H_4(D, D \cap \partial E_1; \mathbb{Q}(t))$ where

$$D \cap \partial E_1 = \partial A_1^{(k)} \times \left((M \setminus A^{(k)})_2 \setminus (\coprod_{j=1}^{k-1} \mathrm{Int}(A_1^{(j)})) \right) \cup A_1^{(k)} \times \partial A_2^{(k)}$$

$$= \partial E_1 \setminus \left(\partial A_1^{(k)} \times (\coprod_{j=1}^{k-1} \mathrm{Int}(A_1^{(j)})) \right).$$

We shall prove the following lemma at the end of this subsection.

Lemma 11.9 *$H_4(D, D \cap \partial E_1; \mathbb{Q}(t))$ is generated by chains of the following forms for $j < k$, $\Sigma_1(a_i^{(k)}) \times \partial A_1^{(j)}$, $A_1^{(k)} \times \left(\{1\} \times (a_i^{(j)})\right)$, $A_1^{(k)} \times \gamma_{A^{(k)},2}$, and combinations of $\Sigma_1(a_i^{(k)}) \times \Sigma_2(\check{z}_\ell^{(k)})$ that are in D.*

Assume this lemma for a moment and let us conclude the proof.

The chains of the statement will be detected by the following dual chains $\check{z}_i^{(k)} \times \gamma_{A^{(j)}}$, $p_{A^{(k)}} \times \Sigma(\check{z}_i^{(j)})$, $p_{A^{(k)}} \times \partial A_3^{(k)}$ and $z_i^{(k)} \times a_{\ell,3}^{(k)}$. The chains $F_{k,D}$ and F_{k-1} intersect the cycles of the last two kinds as prescribed by the linking form condition that is consistent with the prescription of the statement. Let us show that their algebraic intersections with the chains of the first two kinds also coincide. Since we know $F_{k,D}$ on $A_1^{(k)} \times_e (M \setminus A^{(k)})_2$, we see that $\gamma_{A^{(j)}}$ is not in the interaction locus of $A^{(k)}$, and $\langle F_{k,D}, \check{z}_i^{(k)} \times \gamma_{A^{(j)}} \rangle_e = 0$. Similarly, $p_{A^{(k)}}$ interacts only with $\gamma_{A^{(k)}}$ that does not meet $\Sigma(\check{z}_i^{(j)})$, and $\langle F_{k,D}, p_{A^{(k)}} \times \Sigma(\check{z}_i^{(j)}) \rangle_e = 0$. On the other hand,

$$\langle F_{k-1}, \partial \left(\Sigma(\check{z}_i^{(k)}) \times \gamma_{A^{(j)}} \right) = \check{z}_i^{(k)} \times \gamma_{A^{(j)}} - \Sigma(\check{z}_i^{(k)}) \times p_{A^{(j)}} \rangle_e = 0.$$

Therefore, since

$$\langle F_{k-1}, \Sigma(\check{z}_i^{(k)}) \times p_{A^{(j)}} \rangle_e = 0$$

(because $\langle F_{k-1}, \Sigma_3(a_\ell^{(j)}) \times p_{A^{(j)}} \rangle_e = 0$ for all ℓ), $\langle F_{k-1}, \check{z}_i^{(k)} \times \gamma_{A^{(j)}} \rangle_e = 0$.

Similarly,

$$\langle F_{k-1}, \partial \left(\gamma_{A^{(k)}} \times \Sigma(\check{z}_i^{(j)}) \right) = -p_{A^{(k)}} \times \Sigma(\check{z}_i^{(j)}) - \gamma_{A^{(k)}} \times \check{z}_i^{(j)} \rangle_e = 0,$$

and the prescriptions for F_{k-1} impose that $\langle F_{k-1}, \gamma_{A^{(k)}} \times \check{z}_i^{(j)} \rangle_e = 0$. Therefore, $\langle F_{k-1}, p_{A^{(k)}} \times \Sigma(\check{z}_i^{(j)}) \rangle_e = 0$.

In particular the class of $(F_{k,D} - F_{k-1})$ vanishes in $H_4(D, D \cap \partial E_1; \mathbb{Q}(t))$. Therefore, there exists a 5–chain W in $p^{-1}(D)$ such that $\partial W + F_{k-1} - F_{k,D} \subset p^{-1}(D \cap \partial E_1)$, and $F_k = \partial W + F_{k-1}$ satisfies the normalizations conditions on $p^{-1}(E_1)$. It can similarly be assumed to satisfy them on the symmetric part $\iota(p^{-1}(E_1))$. Now, we independently deal with the additional conditions $\langle F_k, \Sigma_3(a_i^{(k)}) \times p_{A^{(k)}} \rangle_e = 0$ and $\langle F_k, p_{A^{(k)}} \times \Sigma_3(a_i^{(k)}) \rangle_e = 0$ for any i, as follows.

We shall add a combination C of $\partial(\Sigma_2(\check{z}_i^{(k)}) \times A_1^{(k)})$ to F_k in order to satisfy the first condition, for example. But this combination must not intersect some $A^{(j)} \times_e (M \setminus A^{(j)})_3$, for $j < k$. Under our assumptions, this is equivalent to say that the algebraic equivariant intersection $\langle \check{z}_r^{(j)} \times \gamma_{A^{(k)}}, C \rangle_e$ must vanish.

Therefore, in order to prove that we may furthermore assume that $\langle F_k, \Sigma_3(a_i^{(k)}) \times p_{A^{(k)}} \rangle_e = 0$, it suffices to prove the following lemma.

Lemma 11.10 *The combination C of $\partial(\Sigma_2(\check{z}_i^{(k)}) \times A_1^{(k)})$ such that $\langle F_k + C, \Sigma_3(a_i^{(k)}) \times p_{A^{(k)}} \rangle_e = 0$ satisfies*

$$\langle \check{z}_r^{(j)} \times \gamma_{A^{(k)}}, C \rangle_e = 0$$

for any $j < k$ and for any $r = 1, \ldots, g^{(j)}$.

Proof: Define a linear form f on $H_1(M \setminus \mathring{A}^{(k)}; \mathbb{Q}(t_M))$ by its values on the generators $a_i^{(k)}$

$$f(a_i^{(k)}) = \langle \Sigma_3(a_i^{(k)}) \times p_{A^{(k)}}, F_k \rangle_e.$$

Then

$$f(a_i^{(k)}) = \langle a_{i,3}^{(k)} \times \gamma_{A^{(k)}}, F_k \rangle_e.$$

Now,

$$\langle \check{z}_r^{(j)} \times \gamma_{A^{(k)}}, F_k \rangle_e = \langle \big(\Sigma(\check{z}_r^{(j)}) \cap (M \setminus A^{(k)})_3\big) \times p_{A^{(k)}}, F_k \rangle_e + f(z_r^{(j)})$$

where $\langle \check{z}_r^{(j)} \times \gamma_{A^{(k)}}, F_k \rangle_e = 0$ because of the form of F_k on $A^{(j)} \times_e (M \setminus A^{(j)})_3$ and

$$\langle \big(\Sigma(\check{z}_r^{(j)}) \cap (M \setminus A^{(k)})_3\big) \times p_{A^{(k)}}, F_k \rangle_e = 0$$

because of the form of F_k on $(M \setminus A^{(k)})_3 \times_e A^{(k)}$. Therefore, $f(z_r^{(j)}) = 0$.

Our combination C is a boundary that is defined so that

$$f(a_i^{(k)}) = -\langle \Sigma_3(a_i^{(k)}) \times p_{A^{(k)}}, C \rangle_e = -\langle a_{i,3}^{(k)} \times \gamma_{A^{(k)}}, C \rangle_e.$$

Thus,

$$\langle \check{z}_r^{(j)} \times \gamma_{A^{(k)}}, C \rangle_e = \langle \big(\Sigma(\check{z}_r^{(j)}) \cap (M \setminus A^{(k)})_3\big) \times p_{A^{(k)}}, C \rangle_e - f(z_r^{(j)}).$$

Since $\partial(\Sigma_2(\check{z}_i^{(k)}) \times A_1^{(k)})$ does not meet $\Big(\Sigma(\check{z}_r^{(j)}) \cap (M \setminus A^{(k)})_3\Big) \times p_{A^{(k)}}$, $\langle \check{z}_r^{(j)} \times \gamma_{A^{(k)}}, C \rangle_e = 0$, and we are done.

$\diamond$

Similarly, we may furthermore assume that $\langle F_k, p_{A^{(k)}} \times \Sigma_3(a_i^{(k)})\rangle_e = 0$ for any i. This concludes the proof of Proposition 11.8 up to the proof of Lemma 11.9 that we give now. $\diamond$

Proof of Lemma 11.9: $D = A_1^{(k)} \times \left((M \setminus A^{(k)})_2 \setminus (\coprod_{j=1}^{k-1} \mathrm{Int}(A_1^{(j)}))\right)$.

$$E_1 = A_1^{(k)} \times (M \setminus A^{(k)})_2.$$

Since $H_4(D;\mathbb{Q}(t)) = 0$, $H_4(D, D \cap \partial E_1;\mathbb{Q}(t))$ is the kernel of the natural map $H_3(D \cap \partial E_1;\mathbb{Q}(t)) \to H_3(D;\mathbb{Q}(t))$ where

$$H_3(D;\mathbb{Q}(t)) = H_1(A_1^{(k)};\mathbb{Q}) \otimes_{\mathbb{Q}} \left(\oplus_{j=1}^{k-1}\mathbb{Q}(t)[\partial(A_1^{(j)})] \oplus \mathbb{Q}(t)[\partial(A_2^{(k)})]\right).$$

We compute $H_3(D \cap \partial E_1;\mathbb{Q}(t))$ using the Mayer-Vietoris sequence associated with the decomposition

$$D \cap \partial E_1 = D_1 \cup_{\partial A_1^{(k)} \times \partial A_2^{(k)}} D_2$$

where

$$D_1 = \partial A_1^{(k)} \times \left((M \setminus A^{(k)})_2 \setminus (\coprod_{j=1}^{k-1} \mathrm{Int}(A_1^{(j)}))\right) \text{ and } D_2 = A_1^{(k)} \times \partial A_2^{(k)}.$$

The cokernel of the map from $H_3(D_1 \cap D_2 = \partial A_1^{(k)} \times \partial A_2^{(k)};\mathbb{Q}(t))$ to $H_3(D_1;\mathbb{Q}(t)) \oplus H_3(D_2;\mathbb{Q}(t))$ is

$$\left(H_1(A_1^{(k)};\mathbb{Q}) \otimes \mathbb{Q}(t)[\partial A_2^{(k)}]\right) \oplus \left(H_1(\partial A_1^{(k)};\mathbb{Q}) \otimes \oplus_{j=1}^{k-1}\mathbb{Q}(t)[\partial A_1^{(j)}]\right) \oplus \bigoplus_{j=1}^{k-1} \bigoplus_{i=1}^{g(A^{(j)})} \mathbb{Q}(t)[\partial A_1^{(k)} \times a_i^{(j)}].$$

The kernel of the map from $H_2(D_1 \cap D_2;\mathbb{Q}(t))$ to $H_2(D_1;\mathbb{Q}(t)) \oplus H_2(D_2;\mathbb{Q}(t))$ is

$$(\oplus_i \mathbb{Q} a_i^{(k)}) \otimes \mathrm{Ker}\left(H_1(\partial A_2^{(k)};\mathbb{Q}(t_M)) \to H_1\left((M \setminus A^{(k)})_2 \setminus (\coprod_{j=1}^{k-1} \mathrm{Int}(A_1^{(j)}));\mathbb{Q}(t_M)\right)\right)$$
$$\oplus \mathbb{Q}(t)[\partial A_1^{(k)} \times p_{A_2^{(k)}}].$$

Therefore $H_3(D \cap \partial E_1;\mathbb{Q}(t))$ is generated by classes of the following form that vanish in $H_3(D;\mathbb{Q}(t))$

- $\partial\left(A_1^{(k)} \times \gamma_{A^{(k)},2}\right)$,
- combinations $\sum_{(i,j)} \alpha_{ijk}\partial(\Sigma_1(a_i^{(k)}) \times \Sigma_2(\check{z}_j^{(k)}))$ such that $\sum_{(i,j)} \alpha_{ijk}\Sigma_1(a_i^{(k)}) \times \Sigma_2(\check{z}_j^{(k)}) \subset D$
- $(\{1\} \times a_i^{(k)}) \times \partial A_1^{(j)}$, and
- $\partial A_1^{(k)} \times (\{1\} \times a_i^{(j)})$

and classes of $H_1(A_1^{(k)};\mathbb{Q}) \otimes_{\mathbb{Q}} \left(\oplus_{j=1}^{k-1}\mathbb{Q}(t)[\partial A^{(j)}] \oplus \mathbb{Q}(t)[\partial A_2^{(k)}]\right)$ that survive in $H_3(D;\mathbb{Q}(t))$. Only the first four ones contribute to $H_4(D, D \cap \partial E_1;\mathbb{Q}(t))$ as in the statement. $\diamond$

11.4 Almost concluding the proof of Proposition 11.1

With the notation of the previous subsection, for any $i = 1, \dots, n$, let $A^{(i)\prime}$ be another rational homology handlebody with the same boundary and the same Lagrangian as $A^{(i)}$, let $\Sigma_4'(a_j^{(i)})$ be a rational 4-chain with the same boundary $4 \times a_j^{(i)}$ as $\Sigma_4(a_j^{(i)})$ and that intersects $[-4,4] \times \partial A^{(i)}$ like $\Sigma_4(a_j^{(i)})$.

Without loss, in addition to the requirements of Proposition 11.8, assume that

- For any $j = 1, \dots, n$, for any $t \in [-4,0]$, F_X intersects $A_t^{(j)} \times_e (M \setminus A^{(j)})_{t+3}$ like $A_t^{(j)} \times \gamma_{A^{(j)},t+3} + \sum_{(i,k)\in\{1,\dots,g(A^{(j)})\}^2} \overline{lk_e(z_i^{(j)}, a_k^{(j)+})}\Sigma_t(a_i^{(j)}) \times \Sigma_{t+3}(\check{z}_k^{(j)})$.
- For any $j = 1, \dots, n$, for any $t \in [-4,0]$, F_X intersects $(M \setminus A^{(j)})_{t+3} \times_e A_t^{(j)}$ like $-\gamma_{A^{(j)},t+3} \times A_t^{(j)} + \sum_{(i,k)\in\{1,\dots,g(A^{(j)})\}^2} lk_e(z_i^{(j)}, a_k^{(j)+})\Sigma_{t+3}(\check{z}_k^{(j)}) \times \Sigma_t(a_i^{(j)})$.
- $F_X = \partial F_X \times [0,1]$ in an equivariant neighborhood $\partial\tilde{C}_2(M) \times [0,1]$ of $\partial\tilde{C}_2(M)$ in $\tilde{C}_2(M)$.

Let $M_i = M(A^{(i)\prime}/A^{(i)})$. Define $\tilde{F}(M_i)$ on $(\tilde{C}_2(M_i) \setminus \mathrm{Int}(\tilde{C}_2(A_{-1}^{(i)\prime})))$ so that

- $\tilde{F}(M_i) = F_X$ on $\tilde{C}_2((M \setminus A^{(i)\prime})_{-4})$
- For any $t \in [-4,0]$, $\tilde{F}(M_i)$ intersects $A_t^{(i)\prime} \times_e (M \setminus A^{(i)})_{t+3}$ like
$$\sum_{(j,k)\in\{1,\dots,g(A^{(i)})\}^2} \overline{lk_e(z_j^{(i)}, a_k^{(i)+})}\Sigma_t'(a_j^{(i)}) \times \Sigma_{t+3}(\check{z}_k^{(i)}) + A_t^{(i)\prime} \times \gamma_{A^{(i)},t+3}.$$
- For any $t \in [-4,0]$, $\tilde{F}(M_i)$ intersects $(M \setminus A^{(i)})_{t+3} \times_e A_t^{(i)\prime}$ like
$$\sum_{(j,k)\in\{1,\dots,g(A^{(i)})\}^2} lk_e(z_j^{(i)}, a_k^{(i)+})\Sigma_{t+3}(\check{z}_k^{(i)}) \times \Sigma_t'(a_j^{(i)}) - \gamma_{A^{(i)},t+3} \times A_t^{(i)\prime}.$$
- $\tilde{F}(M_i) = \partial F_X(M_i)$ on $ST(M_i)$.

In particular, $\tilde{F}(M_i)$ is well-defined outside the interior of $\tilde{C}_2(A_{-1}^{(i)\prime})$ and we wish to extend it there. Assume without loss that $\tilde{F}(M_i)$ is transverse to $\partial\tilde{C}_2(A^{(i)\prime})$, and consider the class of the 3–cycle
$$\tilde{F}_3 = \partial\tilde{C}_2(A^{(i)\prime}) \cap_e \tilde{F}(M_i).$$

The following lemma will be proved in Subsection 11.5.

Lemma 11.11 *Under the hypotheses above*
$$\tilde{F}_3 - \sum_{(j,k,\ell)\in\{1,\dots g(A^{(i)})\}^3} \mathcal{I}_{A^{(i)}A^{(i)\prime}}(a_j^{(i)} \wedge a_k^{(i)} \wedge a_\ell^{(i)}) lk_e(z_\ell^{(i)}, z_k^{(i)}) ST(p_M(\check{z}_j^{(i)}))$$
bounds a 4–chain $\tilde{F}_X(A^{(i)\prime})$ in $\tilde{C}_2(A^{(i)\prime})$.

Let $\eta < \frac{1}{3n}$. Define $ST([-4+3i\eta,0]\times\partial\Sigma_{A^{(i)}A^{(i)\prime}})$ in $\tilde{C}_2([-4,0]\times\partial A^{(i)\prime})$ as

$$\sum_{(j,k,\ell)\in\{1,\dots g(A^{(i)})\}^3} \mathcal{I}_{A^{(i)}A^{(i)\prime}}(a_j^{(i)}\wedge a_k^{(i)}\wedge a_\ell^{(i)})lk_e(z_\ell^{(i)},z_k^{(i)})ST(M)_{|[-4+3i\eta,0]\times p_M(\check{z}_j^{(i)})},$$

and set

$$ST(\Sigma_{A^{(i)}A^{(i)\prime},-4+3i\eta}) = ST(\Sigma_{A^{(i)}A^{(i)\prime}}) \cup -ST([-4+3i\eta,0]\times\partial\Sigma_{A^{(i)}A^{(i)\prime}}).$$

Assuming Lemma 11.11, we shall rather assume without loss that

$$\partial\tilde{F}_X(A^{(i)\prime}) = \tilde{F}_3 - \partial ST_1(\Sigma_{A^{(i)}A^{(i)\prime},-4+3i\eta}),$$

with notation consistent with the statement of Proposition 11.1, and that $\tilde{F}_X(A^{(i)\prime})$ coincides with $\tilde{F}(M_i)$ on $\tilde{C}_2(A^{(i)\prime})\setminus\tilde{C}_2(A_{-1}^{(i)\prime})$. Set

$$F_X(A^{(i)\prime}) = \tilde{F}_X(A^{(i)\prime}) + \Big(ST_{3i\eta}(\Sigma_{A^{(i)}A^{(i)\prime},-4+3i\eta})\cap\tilde{C}_2(A^{(i)\prime})\Big) - \big(ST(\partial\Sigma_{A^{(i)}A^{(i)\prime},-4+3i\eta})\times[3i\eta,1]\big)$$

(where $ST(\partial\Sigma_{A^{(i)}A^{(i)\prime},-4+3i\eta})\times[3i\eta,1]\subset\partial\tilde{C}_2(M)\times[0,1]$). Then

$$F_X(M_i) = \tilde{F}(M_i)\cap\overline{\tilde{C}_2(M)\setminus\tilde{C}_2(A^i)} + F_X(A^{(i)\prime}) + \Big(ST_{3i\eta}(\Sigma_{A^{(i)}A^{(i)\prime},-4+3i\eta})\cap\overline{\tilde{C}_2(M)\setminus\tilde{C}_2(A^{(i)\prime})}\Big)$$

is a 4-chain of $\tilde{C}_2(M_i)$ such that $\partial F_X(M_i)$ is the wanted fixed boundary. Without loss, we shall furthermore assume that

$$\begin{aligned}\partial F_X(M_i)\cap\Big(\partial\tilde{C}_2(M_i)\times[0,1]\Big) = \ & \partial F_X(M_i)\times[0,1] \\ & + ST_{3i\eta}(\Sigma_{A^{(i)}A^{(i)\prime},-4+3i\eta}) - ST(\partial\Sigma_{A^{(i)}A^{(i)\prime},-4+3i\eta})\times[3i\eta,1].\end{aligned}$$

Now, for any subset I of $\{1,2,\dots,n\}$, set $M_I = M((A^{(j)\prime}/A^{(j)})_{j\in I})$ and consider the chain $ST(\Sigma^I_{A^{(i)}A^{(i)\prime}})$ in $\tilde{C}_2(M_I)$ obtained from $ST(\Sigma_{A^{(i)}A^{(i)\prime}})$ by replacing the pieces $\Sigma(a_k^{(j)})$ by $\Sigma'(a_k^{(j)})$, for $j\in I$.

Define $\tilde{F}_X(M_I)$ on $\tilde{C}_2(M_I)\setminus\coprod_{i\in I}\mathring{\tilde{C}}_2(A^{(i)\prime})$ as

- F_X on $\tilde{C}_2\left(M\setminus\coprod_{i\in I}\mathring{A}_{-4}^{(i)}\right)$,
- $\sum_{(j,k)\in\{1,\dots,g(A^{(i)})\}^2}\overline{lk_e(z_j^{(i)},a_k^{(i)+})}\Sigma'_t(a_j^{(i)})\times\Sigma^I_{t+4}(\check{z}_k^{(i)}) + A_t^{(i)\prime}\times\gamma_{A^{(i)},t+4}$ on $A_t^{(i)\prime}\times_e(M_I\setminus A^{(i)\prime})_{t+4}$, for $i\in I$ and $t\in[-4,0]$,
- $\sum_{(j,k)\in\{1,\dots,g(A^{(i)})\}^2} lk_e(z_j^{(i)},a_k^{(i)+})\Sigma^I_{t+4}(\check{z}_k^{(i)})\times\Sigma'_t(a_j^{(i)}) - \gamma_{A^{(i)},t+4}\times A_t^{(i)\prime}$ on $(M_I\setminus A^{(i)\prime})_{t+4}\times_e A_t^{(i)\prime}$, for $i\in I$ and $t\in[-4,0]$.

$$\begin{aligned} F_X(M_I) = \ & \tilde{F}_X(M_I) \cap \left(\tilde{C}_2(M_I) \setminus \coprod_{i\in I} \mathring{\tilde{C}}_2(A^{(i)\prime})\right) \\ & + \sum_{i\in I} ST_{3i\eta}(\Sigma^I_{A^{(i)}A^{(i)\prime},-4+3i\eta}) \cap \left(\tilde{C}_2(M_I) \setminus \mathring{\tilde{C}}_2(A^{(i)\prime})\right) + \sum_{i\in I} F_X(A^{(i)\prime}). \end{aligned}$$

We construct F_Y and F_Z, similarly, taking parallel copies of the pieces Σ in the description of $\tilde{F}(M_I)$, and changing $3i\eta$ to $(3i-1)\eta$ and $(3i-2)\eta$, respectively. $F_V(M_I)$ reads $\partial F_V(M_I)\times[0,1]$ in $\partial\tilde{C}_2(M_I)\times[0,1]$, with the following two exceptions

- the pairwise disjoint $ST_{(3i-r(V))\eta}(\Sigma^I_{A^{(i)}A^{(i)\prime}})$ parts
- the additional parts $ST_{(3i-r(V))\eta}(\Sigma_{A^{(i)}A^{(i)\prime},-4+(3i-r(V))\eta}) \cap \tilde{C}_2(A^{(i)\prime})$ and
$$-ST(\partial\Sigma_{A^{(i)}A^{(i)\prime},-4+(3i-r(V))\eta}) \times [(3i-r(V))\eta, 1]$$
in the $p^{-1}(ST(A^{(i)\prime})\times[0,1])$.

Then since the $ST(\Sigma_{A^{(i)}A^{(i)\prime}})$ avoid $\tilde{C}_2(N(K))$, the $ST_{\cdot}(\Sigma_{A^{(i)}A^{(i)\prime}})$ avoid the loci of double intersections of two $\left(\tilde{F}_V(M_I) \cap \left(\tilde{C}_2(M_I) \setminus \coprod_{i\in I} \mathring{\tilde{C}}_2(A^{(i)\prime})\right)\right)$ parts, and they can only meet double intersections of $F_V(A^{(j)\prime})$, with $j \neq i$.

Now, forget about the superscript (i) and let us come back to our case, where $n=2$, and to the notation $A = A^{(1)}$, $B = A^{(2)}$. The $ST_{(3-r(V))\eta}(\Sigma_{AA'})$-pieces do not meet the loci of double intersections of two $F_V(B')$ because the $F_V(B')$ read as disjoint products on $ST(B')\times[0,3.5\eta]$, if we are dealing with actual trivialisations. For pseudo-trivialisations, they read as non disjoint products on $ST(B')\times[0,3.5\eta]$ whose double intersections will yield algebraically cancelling intersections with the $ST_{(3-r(V))\eta}(\Sigma_{AA'})$, thanks to Lemma 10.5.

Then we are left with the computation of the algebraic intersection of $ST_{(6-r(\sigma(X)))\eta}(\Sigma'_{BB'})$, $F_{\sigma(Y)}(M_A)$ and $F_{\sigma(Z)}(M_A)$, in $p^{-1}(ST(A')\times[0,1])$. We shall assume that σ is the trivial permutation and it will be clear that the other permutations yield a similar result. The possible intersections caused by pseudo-trivialisations algebraically cancel as above and we shall forget about them. Then the only remaining intersection will involve $ST_{(6-r(X))\eta}(\Sigma'_{BB'})$, $-ST(\partial\Sigma_{AA',-4+(3-r(Y))\eta})\times[(3-r(Y))\eta,1]$ and $s_{\tau_{A'}}(A';Z)\times[0,1]$, up to a permutation of Y and Z. Since $ST_{6\eta}(\Sigma'_{BB'})$ intersects $p^{-1}(ST(A')\times[0,1])$ as

$$\sum_{(\ell,m,n)\in\{1,\dots g_B\}^3, i\in\{1,\dots g_A\}} \mathcal{I}_{BB'}(b_\ell\wedge b_m\wedge b_n) lk_e(y_n,y_m) lk(y_\ell,z_i) ST_{6\eta}(\Sigma'(a_i))$$

and since

$$\langle ST_{6\eta}(\Sigma'(a_i)), -ST(z_{j,-4+2\eta})\times[2\eta,1], s_{\tau_{A'}}(A';Z)\times[0,1]\rangle_e = -\delta_{ij},$$

we are done.

This ends the proof of Proposition 11.1, where $ST_{(3i-r(V))\eta}(\Sigma_{A^{(i)}A^{(i)\prime}})$ should rather be defined as $ST_{(3i-r(V))\eta}(\Sigma_{A^{(i)}A^{(i)\prime},-4+(3i-r(V))\eta}) \cap \left(\tilde{C}_2(M_I) \setminus \mathring{\tilde{C}}_2(A^{(i)\prime})\right)$, up to the proof of Lemma 11.11. $\diamond$

11.5 Proof of Lemma 11.11

To prove this lemma, we shall first refine and restate in other words some results of [L2, Section 5.3].

Again, consider a rational homology handlebody A equipped with a collar $[-4,0]\times\partial A$. For $s\in[-4,0]$, $A_s = A\setminus(]s,0]\times\partial A)$, $\partial A_s = \{s\}\times\partial A$. Let $(a_i,z_i)_{i=1,\dots g_A}$ be a basis of $H_1(\partial A)$ such that $a_i = \partial(\Sigma(a_i)\subset A)_{i=1,\dots,g}$ where $\Sigma(a_i)$ is a rational chain of A and $\langle a_i,z_j\rangle = \delta_{ij}$.

Consider a curve a representing an element of $\mathcal{L}_A$ of order k in $H_1(A;\mathbb{Z})$, $k\in\mathbb{N}\setminus\{0\}$. Let $\Sigma = k\Sigma(a)$ be a surface of A immersed in A bounded by ka that intersects $[-1,0]\times\partial A$ as k copies of $[-1,0]\times a$, and that intersects $\mathrm{Int}(A_{-1})$ as an embedded surface.

Lemma 11.12 *Let $(c_i)_{i=1,\dots 2g}$ and $(c_i^*)_{i=1,\dots,2g}$ be two dual bases of $H_1(\Sigma;\mathbb{Z})/H_1(\partial\Sigma;\mathbb{Z})$, $\langle c_i,c_j^*\rangle = \delta_{ij}$.*

Then $\sum_{i=1}^{2g} c_i\times c_i^*$ is homologous to $\sum_{(j,\ell)\in\{1,\dots g_A\}^2}\langle\Sigma,\Sigma(a_j),\Sigma(a_\ell)\rangle z_j\times z_\ell$ in A^2.

Furthermore, $\sum_{i=1}^{2g} c_i\times c_i^{*+}$ is homologous to $\sum_{(j,\ell)\in\{1,\dots g_A\}^2}\langle\Sigma,\Sigma(a_j),\Sigma(a_\ell)\rangle z_j\times z_\ell - gST(*)$ in $C_2(A)$, where the z_j are pairwise disjoint representatives of the $[z_j]$ on ∂A.

Proof: For $(j,\ell)\in\{1,\dots g_A\}^2$, set $S_{j\ell} = \Sigma(a_j)\cap\Sigma(a_\ell)$, $S_{\Sigma j} = \Sigma\cap\Sigma(a_j)$ and $S_{\Sigma\ell} = \Sigma\cap\Sigma(a_\ell)$. Then in $H_1(A)$, $c_i = \sum_{j=1}^{g_A}\langle c_i,\Sigma(a_j)\rangle_A z_j = \sum_{j=1}^{g_A}\langle c_i,S_{\Sigma j}\rangle_\Sigma z_j$ and similarly,

$$c_i^* = \sum_{\ell=1}^{g_A}\langle c_i^*,S_{\Sigma\ell}\rangle_\Sigma z_\ell.$$

Thus, in $H_2(A^2)$,

$$c_i\times c_i^* = \sum_{(j,\ell)\in\{1,\dots g_A\}^2}\langle c_i,S_{\Sigma j}\rangle_\Sigma\langle c_i^*,S_{\Sigma\ell}\rangle_\Sigma z_j\times z_\ell$$

On the other hand in $H_1(\Sigma)/H_1(\partial\Sigma)$,

$$S_{\Sigma j} = \sum_{i=1}^{2g}\langle c_i,S_{\Sigma j}\rangle_\Sigma c_i^*$$

and,

$$S_{\Sigma\ell} = -\sum_{i=1}^{2g}\langle c_i^*,S_{\Sigma\ell}\rangle_\Sigma c_i.$$

Then

$$\langle\Sigma,\Sigma(a_j),\Sigma(a_\ell)\rangle = \langle S_{\Sigma j},S_{\Sigma\ell}\rangle_\Sigma = \sum_{i=1}^{2g}\langle c_i,S_{\Sigma j}\rangle_\Sigma\langle c_i^*,S_{\Sigma\ell}\rangle_\Sigma$$

and the first assertion is proved. Let us now prove that $\alpha = \sum_{i=1}^{2g} c_i\times c_i^{*+}$ is homologous to

$$\beta = \sum_{(j,\ell)\in\{1,\dots g_A\}^2}\langle\Sigma(a),\Sigma(a_j),\Sigma(a_\ell)\rangle z_j\times z_\ell - gST(*)$$

in $C_2(A)$. First note that the homology class of α in $C_2(A)$ is independent of the dual bases (c_i) and (c_i^*). Indeed, since both $a \times \sigma^+$ and $\sigma \times a^+$ are null-homologous in $C_2(A)$, the class of α in $C_2(A)$ only depends on the class of $\sum_{i=1}^{2g} c_i \times c_i^{*+}$ in $H_1(\Sigma)/H_1(\partial\Sigma) \otimes H_1(\Sigma^+)/H_1(\partial\Sigma^+)$ that is determined by the following property: For any two closed curves e and f of Σ, $\langle e \times f^+, \sum_{i=1}^{2g} c_i \times c_i^{*+}\rangle_{\Sigma\times\Sigma^+} = -\langle e, f\rangle_\Sigma$.

In particular, $[\alpha] = [\sum_{i=1}^{2g} c_i^* \times (-c_i^+)]$. The previous computation tells us that the difference $[\beta - \alpha]$ of the two classes is a rational multiple of $[ST(*)]$.

Then

$$\begin{aligned}
[\beta - \alpha] &= +\tfrac{1}{2}\textstyle\sum_{(j,\ell)\in\{1,\dots g_A\}^2} \langle \Sigma(a), \Sigma(a_j), \Sigma(a_\ell)\rangle (lk_e(z_j, z_\ell) + lk_e(z_\ell, z_j))[ST(*)] - g[ST(*)] \\
&\quad -\tfrac{1}{4}\textstyle\sum_{i=1}^{2g} \Big(lk_e(c_i, c_i^{*+}) - lk_e(c_i^{*-} \times c_i)) + \overline{lk_e(c_i, c_i^{*+}) - lk_e(c_i^{*-} \times c_i)}\Big)\,[ST(*)] \\
&= -g[ST(*)] - \tfrac{1}{4}(-2g - 2g)[ST(*)] = 0.
\end{aligned}$$

$\diamond$

We now define a cycle $F^2(a)$ of $\partial C_2(A)$ that is associated to Σ. Let $(a \times [-1, 1])$ be a tubular neighborhood of a in ∂A. Let $p(a) \in a$ and see a as the image of a map $a\colon [0, 1] \to a$ such that $a(0) = a(1) = p(a)$. For $s \in [-2, 0]$, $\Sigma_s = \Sigma \cap A_s$ Let $\Sigma^+ = \Sigma_{-1} \cup k\{(t-1, a(\alpha), t); (t, \alpha) \in [0, 1]^2\}$ so that $\partial\Sigma^+ = k(a \times \{1\})$.

Let $p(a)^+ = (p(a), 1) = (0, p(a), 1) \in a \times [-1, 1] \subset (\partial A = \{0\} \times \partial A)$.

Let $T(a) = \{((a(v), 0), (a(w), +1)); (v, w) \in [0, 1]^2, v \geq w\}$.

Let $A(a)$ be the closure of $\{((a(v), 0), (a(v), t)); (t, v) \in]0, 1] \times [0, 1]\}$.

Let $\mathrm{diag}(n)\Sigma$ be the positive normal section of $ST(A)_{|\Sigma}$, and let $e(\Sigma(a) = \frac{\Sigma}{k}) = \frac{g+k-1}{k}$ where g is the genus of Σ.

$$e(\Sigma(a)) = \frac{-\chi(\Sigma)}{2k} + \frac{1}{2}.$$

Lemma 11.13 *With the notation above*

$$\begin{aligned}
F^2(a) = \;& A(a) + T(a) - p(a) \times \tfrac{1}{k}\Sigma^+ - \tfrac{1}{k}\Sigma \times p(a)^+ \\
& + \tfrac{1}{k} diag(n)\Sigma + e(\Sigma(a))[ST(*)] \\
& - \textstyle\sum_{(j,\ell)\in\{1,\dots g_A\}^2} \langle \Sigma(a), \Sigma(a_j), \Sigma(a_\ell)\rangle z_j \times z_\ell
\end{aligned}$$

is null-homologous in $C_2(A)$.

Proof: For $k = 1$, (when we are dealing with integral homology handlebodies, for example) it is a direct consequence of Lemma 8.11 and Lemma 11.12 above. Note that $T(a) = a \times_{p(a),\geq} (a \times \{1\})$ with the notation of Lemma 8.11.

Let us now focus on the case $k > 1$. (This is similar to [L2, Lemma 5.3]. However, we present a simpler independent proof below.) Without loss, assume that

$$\Sigma \cap ([-2, -1] \times \partial A) = \{(t - 2, a(\alpha), \frac{(j-1)(1-t)}{k}); (t, \alpha) \in [0, 1]^2; j \in 1, 2, \dots, k\}$$

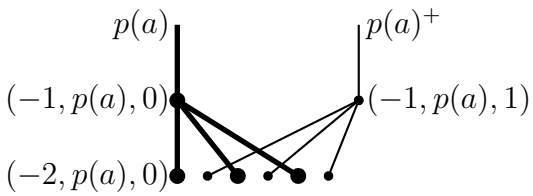
and change the definition of Σ^+ for the proof so that

$$\Sigma^+ \cap ([-1,0] \times \partial A) = k([-1,0] \times a \times \{1\})$$

and

$$\Sigma^+ \cap ([-2,-1] \times \partial A) = \{(t-2, a(\alpha), \frac{(j-\frac{1}{2})(1-t)}{k} + t); (t,\alpha) \in [0,1]^2; j \in 1,2,\ldots,k\}$$

as in the picture below that presents $\Sigma \cap ([-2,-1] \times p(a) \times [-1,1])$ as the thick lines and $\Sigma^+ \cap ([-2,-1] \times p(a) \times [-1,1])$ as the thin lines when $k = 3$.



Let $\Sigma_{-2} = \Sigma \cap A_{-2}$, $\partial\Sigma_{-2} = \cup_{j=1}^k \left(\{-2\} \times a \times \{\frac{j-1}{k}\}\right)$, and let Σ_{-2}^+ be a parallel copy of Σ on its positive side with boundary $\partial(-\Sigma^+ \cap ([-2,-1] \times \partial A))$.

Glue abstract disks D_j with respective boundaries $\{-2\} \times (-a) \times \{\frac{j-1}{k}\}$ on $\partial\Sigma_{-2}$ (resp. D_j^+ with boundaries $\{-2\} \times (-a) \times \{\frac{j-\frac{1}{2}}{k}\}$ on $\partial\Sigma_{-2}^+$), and let S (resp. S^+) be the obtained closed surface. For $j = 1,\ldots,k$, set $p_j = (-2, p(a), \frac{j-1}{k}) \in \partial A_{-2}$ and $p_j^+ = (-2, p(a), \frac{j-\frac{1}{2}}{k})$. Then it follows from Proposition 4.6 that

$$C(S) = \mathrm{diag}(S \times S^+) - p_1 \times S^+ - S \times p_k^+ - \sum_{i=1}^{2g} c_i \times c_i^{*+}$$

is null-homologous in $H_2(S \times S^+)$. Let $[p_1, p_j]$ (resp. $[p_j^+, p_k^+]$) denote a path in $\left(\Sigma_{-2} \setminus \cup_{i=1}^{2g} c_i\right)$ from p_1 to p_j (resp. in $\left(\Sigma_{-2}^+ \setminus \cup_{i=1}^{2g} c_i^+\right)$ from p_j^+ to p_k^+). Adding the null-homologous cycles

$$\partial(-[p_1,p_j] \times D_j^+) = p_1 \times D_j^+ - p_j \times D_j^+ + [p_1,p_j] \times \partial D_j^+,$$

$$\partial(D_j \times [p_j^+, p_k^+]) = D_j \times p_k^+ - D_j \times p_j^+ + \partial D_j \times [p_j^+, p_k^+],$$

for $j = 1,\ldots,k$, and the null-homologous cycles of Lemma 8.11

$$(-C_{*,\leq}(D_j, D_j^+))$$

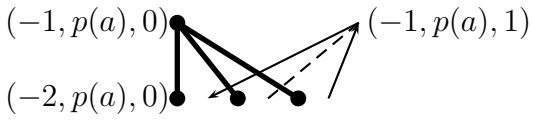
to $C(S)$ transforms it to the still null-homologous cycle

$$\begin{aligned} C(\Sigma_{-2}) = \ & \mathrm{diag}(\Sigma_{-2} \times \Sigma_{-2}^+) - p_1 \times \Sigma_{-2}^+ - \Sigma_{-2} \times p_k^+ - \textstyle\sum_{i=1}^{2g} c_i \times c_i^{*+} \\ & + \textstyle\sum_{j=1}^k (\partial D_j \times [p_j^+, p_k^+] + [p_1, p_j] \times \partial D_j^+ + \partial D_j \times_{p(a),\leq} \partial D_j^+). \end{aligned}$$

This cycle can be naturally continuously extended from the level $\{-2\}\times\Sigma$ to the level $\{0\}\times\Sigma$ to become naturally homologous to $kF^2(a)$ that will be therefore null-homologous provided that

$$\sum_{j=1}^{k}\left(lk(-a,\lim_{s\to -1}[\{s\}\times p_j^+,\{s\}\times p_k^+])+lk(\lim_{s\to -1}[\{s\}\times p_1,\{s\}\times p_j],-a^+)\right)=k-1.$$

To conclude, it suffices to prove this equality. When s approaches (-1), $[\{s\}\times p_j^+,\{s\}\times p_k^+]$ becomes a loop on $\Sigma_{-1}^+=\Sigma^+\cap A_{-1}$, its linking number with $(-a)$ is its intersection with $\frac{-1}{k}\Sigma$ that only occurs where Σ and Σ^+ intersect, in $([-2,-1]\times\partial A)$. The intersection can be seen in the picture below that $\lim_{s\to -1}[\{s\}\times p_j^+,\{s\}\times p_k^+]$ intersects as the oriented paths (for $j=1$ and $k=3$).



In particular, since the positive normal to Σ goes from left to right, we see that

$$lk(-a,\lim_{s\to -1}[\{s\}\times p_j^+,\{s\}\times p_k^+])=\frac{k-j}{k}.$$

Similarly, $lk(\lim_{s\to -1}[\{s\}\times p_1,\{s\}\times p_j],-a^+)=\frac{j-1}{k}$ and we are done. $\diamond$

Lemma 11.14 *If A is a rational homology handlebody such that $H_1(A)=\oplus_{j=1}^{g(A)}[z_j]$, then*

$$H_3(C_2(A);\mathbb{Q}(t))=\oplus_{j=1}^{g(A)}\mathbb{Q}(t)[ST(z_j)].$$

Proof: $C_2(A)$ and $C_2(\mathring{A})$ have the same homotopy type, that is the homotopy type of $\mathring{A}^2\setminus$diag. $H_3(\mathring{A}^2)=0$, $H_4(\mathring{A}^2,\mathring{A}^2\setminus\text{diag})=\oplus_{j=1}^{g(A)}\mathbb{Q}(t)[z_j\times B^3]$. $\diamond$

Proof of Lemma 11.11: We drop the useless superscripts (i) in this proof. Since

$$\langle ST(p_M(\check{z}_k)),F^2(a_j)\rangle_{e,\partial\tilde{C}_2(A)}=\delta_{jk},$$

according to the above two lemmas, it suffices to prove that

$$\langle\tilde{F}_3-\sum_{(i,k,\ell)\in\{1,\dots g(A)\}^3}\mathcal{I}_{AA'}(a_i\wedge a_k\wedge a_\ell)lk_e(z_\ell,z_k)ST(p_M(\check{z}_i)),F^2_{A'}(a_j)\rangle_{e,\partial\tilde{C}_2(A')}=0$$

for all j. Fix j and set $a=a_j$. Since $F^2(a)$ vanishes according to Lemma 11.13,

$$\langle F_X,F^2(a)\rangle_{e,\tilde{C}_2(M)}=0.$$

$$\langle F_X, \mathrm{diag}(n)\Sigma(a)+e(\Sigma(a))[ST(*)]\rangle_{e,\tilde{C}_2(M)} = \langle F_X, \frac{2k\mathrm{diag}(n)\Sigma(a)}{2k} - \frac{\chi(k\Sigma(a))}{2k}[ST(*)]\rangle_{e,\tilde{C}_2(M)} + \frac{1}{2}$$

Assume without loss that $\tau^{-1}(., e_1)$ is a positive normal to $k\Sigma(a)$ along a. Then, according to Lemma 10.6,

$$\begin{aligned}\langle F_X, \mathrm{diag}(n)\Sigma(a) + e(\Sigma(a))[ST(*)]\rangle_{e,\tilde{C}_2(M)} &= \langle F_X, \tfrac{2s_\tau(k\Sigma(a);e_1)}{2k}\rangle_{e,\tilde{C}_2(M)} + \tfrac{d(\tau,a)+1}{2} \\ &= \tfrac{d(\tau,a)+1}{2}.\end{aligned}$$

Similarly, according to Lemma 10.7,

$$\langle \tilde{F}_3, \mathrm{diag}(n)\Sigma'(a) + e(\Sigma'(a))[ST(*)]\rangle_{e,\partial\tilde{C}_2(A')} = \langle \tilde{F}_3, \frac{2s_{\tau(A')}(k'\Sigma'(a); e_1)}{2k'}\rangle_{e,\partial\tilde{C}_2(A')} + \frac{d(\tau, a) + 1}{2}.$$

Since this is equal to $\frac{d(\tau,a)+1}{2}$ according to Lemma 10.5,

$$\langle F_X, \mathrm{diag}(n)\Sigma(a) + e(\Sigma(a))[ST(*)]\rangle_{e,\tilde{C}_2(M)} = \langle \tilde{F}_3, \mathrm{diag}(n)\Sigma'(a) + e(\Sigma'(a))[ST(*)]\rangle_{e,\partial\tilde{C}_2(A')}.$$

Since $A(a)+T(a) \subset C_2(\partial A)$, $\langle \tilde{F}_3, A(a)+T(a)\rangle_{e,\partial\tilde{C}_2(A')} = \langle F_X, A(a)+T(a)\rangle_{e,\tilde{C}_2(M)}$. Similarly, setting $\Sigma_{[-4,0]} = \Sigma_0 \setminus \mathrm{Int}(\Sigma_{-4})$, and assuming here $\Sigma_{[-4,0]} = \Sigma'_{[-4,0]} = [-4, 0] \times a$ without loss,

$$\langle \tilde{F}_3, -p(a)\times\Sigma'^{+}_{[-4,0]}(a)-\Sigma'_{[-4,0]}(a)\times p(a)^{+}\rangle_{e,\partial\tilde{C}_2(A')} = \langle F_X, -p(a)\times\Sigma^{+}_{[-4,0]}(a)-\Sigma_{[-4,0]}(a)\times p(a)^{+}\rangle_{e,\tilde{C}_2(M)}$$

while the normalization conditions imply that

$$\langle \tilde{F}_3, -p(a) \times \Sigma'^{+}_{-4}(a) - \Sigma'_{-4}(a) \times p(a)^{+}\rangle_{e,\partial\tilde{C}_2(A')} = \langle F_X, -p(a) \times \Sigma^{+}_{-4}(a) - \Sigma_{-4}(a) \times p(a)^{+}\rangle_{e,\tilde{C}_2(M)}.$$

Therefore,

$$\langle \tilde{F}_3, F^2_{A'}(a_j)\rangle_{e,\partial\tilde{C}_2(A')} - \langle F_X, F^2(a_j)\rangle_{e,\tilde{C}_2(M)} = \sum_{(k,\ell)\in\{1,\dots g_A\}^2} \mathcal{I}_{AA'}(a_j \wedge a_k \wedge a_\ell) lk_e(z_\ell, z_k),$$

and we are done. $\diamond$

12 Interactions with K

12.1 Introduction

In this section, we shall prove

Proposition 12.1

$$\delta(M)(x)\delta(M)(y)\delta(M)(z)\mathcal{Q}(\mathbb{K}) \in \frac{\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]}{(xyz = 1)}.$$

In order to do this, we shall fix the chains F_X, F_Y and F_Z outside $\tilde{C}_2(M \setminus T(K))$, or equivalently we shall fix the interactions of $T(K)$ via F. More precisely, (in Proposition 12.5) we shall define chains Φ_X, Φ_Y and Φ_Z that are similar to F_X, F_Y and F_Z and that can replace them without loss (by Proposition 12.6) such that

$$\mathcal{Q}(\mathbb{K}, \tau) = \langle F_X, F_Y, F_Z \rangle_{e, \tilde{C}_2(M)} = \langle \Phi_X, \Phi_Y, \Phi_Z \rangle_{e, \tilde{C}_2(M)}.$$

We shall fix the chains Φ_V for $V = X$, Y and Z, so that Φ_X, Φ_Y and Φ_Z intersect only on $\tilde{C}_2(M \setminus T(K))$, and

$$C_V = \delta(M)(t)(\Phi_V \cap \tilde{C}_2(M \setminus T(K))$$

is rational for all V, so that

$$\delta(M)(x)\delta(M)(y)\delta(M)(z)\mathcal{Q}(\mathbb{K}, \tau) = \langle C_X, C_Y, C_Z \rangle_{e, \tilde{C}_2(M \setminus T(K))} \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$$

where C_V will be defined by boundary conditions (see Proposition 12.2) (like in the work [Ma] of Julien Marché, so that we get a definition of $\mathcal{Q}(\mathbb{K})$ in the spirit of this work). The precise definition is stated in Subsection 12.2.

12.2 An alternative definition of $\mathcal{Q}(\mathbb{K})$ with boundary conditions

Let us first introduce some notation. Write the sphere S^2 as the quotient of $[0, 8] \times S^1$ where $\{0\} \times S^1$ is identified to a single point (the North Pole of S^2) and $\{8\} \times S^1$ is identified to another single point (the South Pole of S^2). When $\alpha \subset [0, 8]$, D^2_α denotes the image of $\alpha \times S^1$ via the quotient map q. For example, $D^2_{[1,8]}$ is a disk. Embed $D^2_{[1,8]} \times S^1$ as a tubular neighborhood of K, so that $K = \{*_W\} \times S^1$ for some $*_W \in \partial D^2_{[0,6]}$, and $K_\parallel = \{q_X\} \times S^1$ for some $q_X \in \partial D^2_{[0,2]}$, and let $M_{[0,1]} = M \setminus (D^2_{]1,8]} \times S^1)$. More generally, let

$$\begin{array}{rlcl} r\colon & M & \to & [1, 8] \\ & x \in M_{[0,1]} & \mapsto & 1 \\ & (q(t, z_W), z) \in D^2_{[1,8]} \times S^1 & \mapsto & t. \end{array}$$

When $\alpha \subset [0, 8]$, $M_\alpha = r^{-1}(\alpha)$ and $\tilde{M}_\alpha = p_M^{-1}(M_\alpha)$.

Consider a map $f\colon M \to S^1$ that coincides with the projection onto S^1 on $D^2_{[1,8]} \times S^1$, and a lift of this map $\tilde{f}\colon \tilde{M} \to \mathbb{R}$. Embed $\tilde{M}_{]1,6]} = p_M^{-1}(D^2_{]1,6]} \times S^1)$ in $\mathbb{R}^3$, seen as $\mathbb{C} \times \mathbb{R}$, as $\{z \in \mathbb{C}; 1 < |z| \leq 6\} \times \mathbb{R}$ naturally so that the projection on $\mathbb{R}$ is $\tilde{f}$. Here $\mathbb{C}$ is thought of as horizontal and $\mathbb{R}$ is vertical. This embedding induces a trivialisation τ on $TM_{]1,6]}$ that we extend on $TM_{[0,6]}$. (The only obstruction to do so would be the obstruction to extend it to $(S \setminus D^2_{]1,8]})$ that vanishes.) This trivialisation respects the product structure with $\mathbb{R}$ on $\tilde{M}_{]1,6]}$, we also extend it on $\tilde{M}_{]1,8]}$ so that it still respects the product structure with $\mathbb{R}$, there.

Construction of a map $\pi\colon p^{-1}\left((M^2_{[0,5]} \setminus M^2_{[0,3[}) \setminus \mathrm{diag}(M^2_{[3,5]})\right) \to S^2$

Let

$$\begin{array}{rlcl} \chi\colon & [-4,4] & \to & [0,1] \\ & t \in [-0.5,4] & \mapsto & 1 \\ & t \in [-4,-1] & \mapsto & 0 \end{array}$$

be a smooth map. Recall that $\tilde{M}_{]1,5]}$ is embedded in $\mathbb{R}^3$ that is seen as $\mathbb{C} \times \mathbb{R}$. When $(u,v) \in \left(\tilde{M}^2_{[0,5]} \setminus \tilde{M}^2_{[0,3[}\right) \setminus \mathrm{diag}(\tilde{M}^2_{[0,5]})$, set

$$U(u,v) = (1 - \chi(r(u) - r(v)))(0, \tilde{f}(u)) + \chi(r(u) - r(v))u$$

$$V(u,v) = (1 - \chi(r(v) - r(u)))(0, \tilde{f}(v)) + \chi(r(v) - r(u))v$$

so that $(U(u,v), V(u,v)) \in (\mathbb{R}^3)^2 \setminus \mathrm{diag}$.

Define

$$\begin{array}{rlcl} \pi\colon & p^{-1}\left((M^2_{[0,5]} \setminus M^2_{[0,3[}) \setminus \mathrm{diag}(M^2_{[3,5]})\right) & \to & S^2 \\ & (u,v) & \mapsto & \frac{V(u,v)-U(u,v)}{\|V(u,v)-U(u,v)\|}. \end{array}$$

The map π extends naturally to $\tilde{C}_2(M_{[0,5]}) \setminus \tilde{C}_2(M_{[0,3[})$.

Proposition 12.2 *Let q_X, q_Y and q_Z be three distinct points on $\partial D^2_{[0,2]}$, and let X, Y and Z be three distinct vectors of S^2 whose vertical coordinate is in $]0, \frac{1}{50}[$. For $V = X, Y$ or Z, let $K_V = q_V \times S^1$, then there exist an element J_Δ of $\frac{1}{\delta(M)}\mathbb{Q}[t,t^{-1}]$, and a 4-dimensional rational chain C_V of $\tilde{C}_2(M_{[0,3]})$ whose boundary is*

$$\delta(M)\left(\pi^{-1}_{|\partial\tilde{C}_2(M_{[0,3]})\setminus\partial\tilde{C}_2(M_{[0,3[})}(V) \cup s_\tau(M_{[0,3]}; V) \cup (-J_\Delta)K_V \times_\tau S^2\right),$$

and that is transverse to $\partial\tilde{C}_2(M_{[0,3]})$.

Note that this proposition is true when $M = S^1 \times S^2$, with $J_\Delta = 0$, because in this case, π extends to $\tilde{C}_2(M_{[0,3]})$ and $C_V = \pi^{-1}_{|\tilde{C}_2(M_{[0,3]})}(V)$ fulfills the conditions. It will be proved in general in Subsection 12.6.

Proposition 12.3 *With the notation of Proposition 12.2,*

$$\delta(M)(x)\delta(M)(y)\delta(M)(z)\mathcal{Q}(\mathbb{K},\tau) = \langle C_X, C_Y, C_Z\rangle_{e,\tilde{C}_2(M_{[0,3]})}$$

Proposition 12.4 *With the notation of Proposition 12.2,*

$$J_\Delta = \frac{t\Delta'(t)}{\Delta(t)}.$$

These propositions, that will be proved later in this section, imply Proposition 12.1. They give an alternative definition of $\mathcal{Q}(\mathbb{K})$ together with Proposition 5.8.

12.3 Fixing the interactions with K

Let $*_X$, $*_Y$, $*_Z$ and $*_W$ be 4 distinct points in $D^2_{[6,8]}$, $*_W \in \partial D^2_{[6,8]}$, let p_X, p_Y and p_Z be three distinct points on $\partial D^2_{[0,5]}$ close to each other and such that the three distances between two of them are pairwise distinct, and let h be a positive number smaller than $1/4$. Consider the three points $X = \frac{(p_X,h)}{\|(p_X,h)\|}$, $Y = \frac{(p_Y,h)}{\|(p_Y,h)\|}$, $Z = \frac{(p_Z,h)}{\|(p_Z,h)\|}$ of S^2.

Let $P\colon \tilde{M}^2 \to \widetilde{M^2}$ be the canonical quotient map. Define $C^5_{[4,8]}$ as the closure in $\tilde{C}_2(M_{[4,8]})$ of $P\left(\{((z \in D^2_{[4,8]}, 0); (z_2 \in D^2_{[4,8]}, t \in \mathbb{R}\setminus\mathbb{Z}))\}\right)$ where $M_{[4,8]} = D^2_{[4,8]} \times S^1$. Then

$$\tilde{C}_2(M_{[4,8]}) = S^1 \times C^5_{[4,8]}$$

where

$$(\exp(2i\pi u), P\left((z,0);(z_2, t \notin \mathbb{Z})\right)) = P\left((z,u);(z_2, t+u)\right)$$

for $u \in [0,1]$.

The proof of the following proposition will be given in Subsection 12.4 assuming Proposition 12.2.

Proposition 12.5 *For $V = X$, Y or Z,*

- *The map π is regular from $\tilde{C}_2(M_{[0,5]}) \setminus \tilde{C}_2(M_{[0,3[})$ to S^2, it factors through $C^5_{[4,5]}$ on $(\tilde{C}_2(M_{[4,5]}) = S^1 \times C^5_{[4,5]})$, and $\pi^{-1}(V)$ is a 4–submanifold of $\tilde{C}_2(M_{[0,5]}) \setminus \tilde{C}_2(M_{[0,3[})$.*
- *Let $G(V)$ be the closure in $\tilde{C}_2(M)$ of*

$$P\left(\{(m;(p_V, \tilde{f}(m)+h+u)); m \in \tilde{f}^{-1}(]0,1]) \subset \tilde{M}, u \in [0,1]\setminus\{1-h\}\}\right).$$

- *Let $G_\iota(V)$ be the closure in $\tilde{C}_2(M)$ of*

$$P\left(\{((-p_V, \tilde{f}(m)-h+u); m); u \in [0,1]\setminus\{h\}, m \in \tilde{f}^{-1}(]0,1]) \subset \tilde{M}\}\right)$$

oriented as $]0,1[\times\tilde{M}$.

There exists a rational 3*–chain* $E(V)$ *of* $C^5_{[4,8]}$ *such that the boundary of*

$$\Phi_V = \frac{1}{t-1}(tG(V) + G_\iota(V)) + \pi^{-1}(V) + \frac{1}{\delta}C_V + S^1 \times E(V)$$

is

$$s_\tau(M;V) - J_\Delta ST(q_V \times S^1) - \frac{1}{1-t}ST(\{p_V\} \times S^1) - \frac{1}{1-t}ST(\{-p_V\} \times S^1) + ST(\{*_V\} \times S^1)$$

where the 4*–chain* $S^1 \times E(V)$ *is in* $\left(\tilde{C}_2(M_{[4,8]}) = S^1 \times C^5_{[4,8]}\right)$*, and* C_V *is the rational chain of Proposition 12.2.*

Proposition 12.6 *1. The chains* Φ_X*,* Φ_Y *and* Φ_Z *of Proposition 12.5 have no algebraic triple intersection outside* $\tilde{C}_2(M_{[0,3]})$*.*

2. Set $K = \{*_W\} \times S^1$*, and recall Notation* $A(K)$ *from Lemma 4.9 then for* $V = X$*,* Y *or* Z*,*

$$\langle \Phi_V, A(K)\rangle_{e,\tilde{C}_2(M)} = 0.$$

3.

$$\mathcal{Q}(\mathbb{K},\tau) = \langle \Phi_X, \Phi_Y, \Phi_Z\rangle_{e,\tilde{C}_2(M)} = \frac{1}{\delta(M)(x)\delta(M)(y)\delta(M)(z)}\langle C_X, C_Y, C_Z\rangle_{e,\tilde{C}_2(M)}.$$

Proposition 12.7 *For the chains* F_V *(for* $V = X$*,* Y *or* Z*) of Proposition 4.10 and Proposition 5.2,* $\delta(M)(t-1)F_V$ *can be assumed to be rational.*

Proof of Propositions 12.3, 12.6, 12.7 and 12.4 assuming Propositions 12.2 and 12.5: Proposition 12.3 is contained in Proposition 12.6. Observe that the boundaries of the Φ have no triple intersection on $\mathbb{Z} \times ST(M)$. The supports of the chains C_X, C_Y and C_Z on $\tilde{C}_2(M_{[0,3]}) \setminus \tilde{C}_2(M_{[0,3[})$ are in $\pi^{-1}(X)$, $\pi^{-1}(Y)$ and $\pi^{-1}(Z)$, respectively, so that there is no triple intersection there. Clearly, the C pieces cannot intersect the other ones, and two pieces of the same kind among π^{-1}, G, G_ι cannot intersect each other so that the only triple intersections that could occur outside $\tilde{C}_2(M_{[0,3]})$ should involve the three kinds or they should live in $\tilde{C}_2(M_{[4,8]})$. Finally, they should live in $\tilde{C}_2(M_{[4,8]})$ where all the chains factor through S^1 as codimension 2-chains in a 5–dimensional manifold so that the algebraic intersection between the chains vanishes there.

Similarly, since $A(K)$ factors through S^1, the algebraic intersections $\langle \Phi_V, A(K)\rangle_{e,\tilde{C}_2(M)}$ vanish, too, and the first two assertions of Proposition 12.6 are proved.

To construct chains F_X, F_Y and F_Z such that

$$\mathcal{Q}(\mathbb{K},\tau) = \langle F_X, F_Y, F_Z\rangle_{e,\tilde{C}_2(M)},$$

from the chains Φ_V, use a product neighborhood $[-2,0] \times \partial\tilde{C}_2(M)$ of $\partial\tilde{C}_2(M) = \{0\} \times \partial\tilde{C}_2(M)$, shrink the part of Φ inside $[-2,0] \times \partial\tilde{C}_2(M)$ into $[-2,-1] \times \partial\tilde{C}_2(M)$ using $(t,u) \mapsto ((t+2)/2-$

$2, u)$. Consider three disjoint connected graphs Γ_X, Γ_Y and Γ_Z embedded in $\left(D^2_{[2,8]} \setminus \{*_W\}\right)$ such that, for $V = X$, Y and Z, Γ_V contains p_V, $(-p_V)$, q_V and $*_V$. Then construct F_V by completing the shrinked Φ_V by a cobordism between its boundary $\{-1\} \times \partial\Phi_V$ and

$$\partial F_V = s_\tau(M; V) - \left(J_\Delta + \frac{1+t}{1-t}\right) ST(\{p_V\} \times S^1)$$

supported in $[-1, 0] \times (s_\tau(M; V) \cup ST(\Gamma_V \times S^1))$. It is clear that

$$\langle F_V, A(K)\rangle_{e,\tilde{C}_2(M)} = \langle \Phi_V, A(K)\rangle_{e,\tilde{C}_2(M)} = 0.$$

This proves Proposition 12.7. According to Proposition 2.12 and Theorem 4.8, J_Δ must be equal to $\frac{t\Delta'(t)}{\Delta(t)}$. This proves Proposition 12.4.

Now since

$$\langle \Phi_X, \Phi_Y, \Phi_Z\rangle_{e,\tilde{C}_2(M)} = \langle F_X, F_Y, F_Z\rangle_{e,\tilde{C}_2(M)},$$

the last assertion of Proposition 12.6 is also proved according to Proposition 5.2. $\diamond$

Thus, we are left with the proofs of Proposition 12.5 and Proposition 12.2.

12.4 Proof of Proposition 12.5 assuming Proposition 12.2

For $\alpha \subset [0, 8]$, let $M_\alpha \times_{\tilde{f},h} (\{p_V\} \times \mathbb{R})$ be

$$P\left(\{(m; (p_V, \tilde{f}(m) + h)); m \in \tilde{f}^{-1}(]0, 1]) \cap \tilde{M}_\alpha\}\right)$$

oriented by $\tilde{M}_\alpha$. Similarly, let $(\{-p_V\} \times \mathbb{R}) \times_{\tilde{f},h} M_\alpha$ be

$$P\left(\{((-p_V, \tilde{f}(m) - h); m); m \in \tilde{f}^{-1}(]0, 1]) \cap \tilde{M}_\alpha\}\right)$$

oriented by $\tilde{M}_\alpha$.

$$\| (p_V, h) \| V = (p_V, h) = (5, z(V), h).$$

The boundary of $G(V)$ is

$$\partial G(V) = (1 - t^{-1}) \left(M \times_{\tilde{f},h} (\{p_V\} \times \mathbb{R})\right) \cup t^{-1} ST(\{p_V\} \times \mathbb{R}).$$

(About the sign in front of $t^{-1}ST(\{p_V\} \times \mathbb{R})$, M reads $(\tilde{f}^{-1}(s), s)$ where s is the parameter for $\mathbb{R}$ so that we have the orientation $(-(u, \tilde{f}^{-1}(s), s))$ for $G(V)$. When $u > 1 - h$, u is an inward normal, $\tilde{f}^{-1}(s)$ has the orientation opposite to that of a surrounding sphere, but it is first instead of second.)

The chain $G_\iota(V)$ is the closure in $\tilde{C}_2(M)$ of

$$P\left(\{((-p_V, \tilde{f}(m) - h + u); m); u \in [0,1] \setminus \{h\}, m \in \tilde{f}^{-1}(]0,1]) \subset \tilde{M}\}\right)$$

oriented as $[0,1] \times \tilde{M}$.

$$\partial G_\iota(V) = (t-1)(\{-p_V)\} \times \mathbb{R}) \times_{\tilde{f},h} M) \cup ST(\{-p_V\} \times S^1).$$

Assuming Proposition 12.2, the boundary of the 4-chain $\left(\pi^{-1}(V) \cup \frac{1}{\delta} C_V\right)$ is

$$\pi^{-1}_{|\partial\tilde{C}_2(M_{[0,5]})\setminus\partial\tilde{C}_2(M_{[0,5[})}(V) \cup s_\tau(M_{[0,5]}; V) \cup (-J_\Delta)K_V \times_\tau S^2.$$

Set

$$\partial_{[0,5]\leftrightarrow 5} = \pi^{-1}_{|\partial\tilde{C}_2(M_{[0,5]})\setminus\partial\tilde{C}_2(M_{[0,5[})}(V).$$

$$\partial_{[0,5]\leftrightarrow 5} = \partial_{[0,4],5} \cup \partial_{5,[0,4]} \cup \partial_{[4,5]\leftrightarrow 5}$$

where

$$\partial_{[4,5]\leftrightarrow 5} = \pi^{-1}_{|\partial C_2(M_{[4,5]})\setminus\partial C_2(M_{[4,5[})}(V)$$
$$\partial_{[0,4],5} = -(M_{[0,4]} \times_{\tilde{f},h} (\{p_V\} \times \mathbb{R})$$

(since M^2 is oriented as $(M, \text{outward normal to } M_{[0,5]}, \partial M_{[0,5]})$, and since π maps $\partial M_{[0,5]}$ to S^2 in an orientation-preserving way, we get the minus sign above) and

$$\partial_{5,[0,4]} = -((\{-p_V\} \times \mathbb{R}) \times_{\tilde{f},h} M_{[0,4]}.$$

Now, the boundary of the chain

$$\frac{1}{t-1}(tG(V) + G_\iota(V)) + \pi^{-1}(V) + \frac{1}{\delta} C_V$$

is $\partial_r + \partial_e$ where

$$\partial_r = s_\tau(M_{[0,5]}; V) \cup (-J_\Delta)ST(K_V) + \frac{1}{t-1} ST(\{p_V\} \times S^1) + \frac{1}{t-1} ST(\{-p_V\} \times S^1)$$

is part of $\partial\Phi_V$ and where the remaining part is

$$\partial_e = \left(M_{[4,8]} \times_{\tilde{f},h} (\{p_V\} \times \mathbb{R})\right) + \left(\{-p_V\} \times \mathbb{R}) \times_{\tilde{f},h} M_{[4,8]}\right) + \partial_{[4,5]\leftrightarrow 5}.$$

Recall that

$$\tilde{C}_2(M_{[4,8]}) = S^1 \times C^5_{[4,8]}$$

where the orientation of $C^5_{[4,8]}$ is induced by this product structure. Note that $\partial_e = S^1 \times \partial^2_e$, and that $s_\tau(M_{[5,8]}; V) = S^1 \times s_\tau(M_{[5,8]} \cap f^{-1}(1); V)$. Proposition 12.5 is now the consequence of the following lemma.

Lemma 12.8 *There exists a rational 3–dimensional chain $E(V)$ in $C^5_{[4,8]}$ such that*

$$\partial E(V) = \partial^2_e - s_\tau(M_{[5,8]} \cap f^{-1}(1); V) - ST(*_V).$$

PROOF: Let $s_+(M_{[5,8]} \cap f^{-1}(1))$ be a section that is in the hemisphere of the $\mathbb{R}$ direction and that coincides with $s_\tau(M_{[5,8]} \cap f^{-1}(1); V)$ on $\partial M_{[5,8]} \cap f^{-1}(1)$. Then $(\partial^2_e - s_+(M_{[5,8]} \cap f^{-1}(1)))$ is a 2–cycle in the closure of $P\left(\{((z \in D^2_{[4,8]}, 0); (z_2 \in D^2_{[4,8]}, t \in]0, 1/2[))\}\right)$ that is homotopy equivalent to $D^2 \times D^2 \times [1/2]$. Therefore this 2-cycle bounds a rational 3–chain in $C^5_{[4,8]}$. Now, $s_+(M_{[5,8]} \cap f^{-1}(1)) - s_\tau(M_{[5,8]} \cap f^{-1}(1); V) = ST(*_V)$ since the difference of the degrees of the Gauss maps of the sections of $T(S^2 \subset \mathbb{R}^3)$ given by the outward normal section, on one hand, and a trivialisation, on the other hand, is one. $\diamond$

12.5 Homology of $\tilde{C}_2(M_{[0,3]})$

Let $E = M_{[0,3]}$, E is the exterior of K, E is a rational homology torus, and we are going to compute the homology of $\tilde{C}_2(E)$ as in Section 2. First note that $H_2(E; \mathbb{Z}) = H_3(E; \mathbb{Z}) = 0$, $H_0(E; \mathbb{Z}) = \mathbb{Z}$ and $H_1(E; \mathbb{Z})/\text{Torsion} = \mathbb{Z}[K_X]$.

Let $\Sigma = S \cap E$, Σ is obtained from S by removing an open disk. The pair $(M \setminus S, E \setminus \Sigma)$ has the same homology as the pair $(D^2 \times \mathbb{R}, S^1 \times \mathbb{R})$ by excision. Therefore, Lemma 2.1 implies the following similar lemma:

Lemma 12.9 $H_i(E \setminus \Sigma; \mathbb{Z}) = \{0\}$ *for any* $i \geq 2$, $H_0(E \setminus \Sigma; \mathbb{Z}) = \mathbb{Z}[*^+]$.
Let $(z_i)_{i=1,\ldots 2g}$ *and* $(z_i^*)_{i=1,\ldots,2g}$ *be two dual bases of* $H_1(\Sigma; \mathbb{Z})$ *such that* $\langle z_i, z_j^* \rangle = \delta_{ij}$. *Then*

$$H_1(E \setminus \Sigma; \mathbb{Q}) = \bigoplus_{i=1}^{2g} \mathbb{Q}[z_i^+ - z_i^-]$$

and for any $v \in H_1(E \setminus \Sigma; \mathbb{Q})$, $v = \sum_{i=1}^{2g} lk(v, z_i^*)(z_i^+ - z_i^-)$.

The bases in the statements of Lemmas 2.1 and 12.9 can and will be assumed to be the same.

Let $\tilde{E}$ be the infinite cyclic covering of E.

Lemma 12.10

$$H_0(\tilde{E}) = \frac{\Lambda_M}{(t_M - 1)}, H_2(\tilde{E}) = \{0\}, H_3(\tilde{E}) = \{0\}$$

and

$$H_1(\tilde{E}) = \oplus_{i=1}^k \frac{\Lambda_M}{(\delta_i(M))}$$

where $\prod_{i=1}^k \delta_i(M) = \Delta(M)$ *is the Alexander polynomial of* M *and* E.

Proof: Compare the homology of $\tilde{E}$ with the homology of $\tilde{M}$ given by Lemma 2.2 using the long exact sequence associated with the pair $(\tilde{M}, \tilde{E})$ whose homology is the homology of $(D^2 \times \mathbb{R}, S^1 \times \mathbb{R})$. Again, $H_i(D^2 \times \mathbb{R}, S^1 \times \mathbb{R}) = 0$ when $i \neq 2$ and $H_2(D^2 \times \mathbb{R}, S^1 \times \mathbb{R}) = \mathbb{Q}[D^2]$ where $[D^2]$ is the image of the generator $[S]$ of $H_2(\tilde{M})$ under the composition of the natural map to $H_2(\tilde{M}, \tilde{E})$ with the excision map. $\diamond$

Now, the homology of $\widetilde{E^2}$ can be computed as in Subsection 2.4 to find the following proposition. Assume without loss that the Σ_i, the c_i and the $C(\Sigma_i \times \Sigma_j)$ of Proposition 2.6 live in $\tilde{E}$ or in $\widetilde{E^2}$.

Proposition 12.11 *The rational homology of $\widetilde{E^2}$ reads as follows*

$$\begin{array}{rcl} H_0(\widetilde{E^2}) & = & \frac{\Lambda}{(t-1)}[* \times *] \\ H_1(\widetilde{E^2}) & = & \frac{\Lambda}{(t-1)}[\mathrm{diag}(K^2)] \\ H_2(\widetilde{E^2}) & = & \oplus_{(i,j)\in\{1,2,\ldots,k\}^2} \frac{\Lambda}{(\delta_{\min(i,j)})}[c_i \times c_j] \\ H_3(\widetilde{E^2}) & = & \oplus_{(i,j)\in\{1,2,\ldots,k\}^2} \frac{\Lambda}{(\delta_{\min(i,j)})}[C(\Sigma_i \times \Sigma_j)] \\ H_4(\widetilde{E^2}) & = & H_5(\widetilde{E^2}) = 0 \end{array}$$

where the Σ_i, the c_i and the $C(\Sigma_i \times \Sigma_j)$ are the same as in Proposition 2.6.

12.6 Proof of Proposition 12.2

We must find a 4-dimensional rational chain C_V of $\tilde{C}_2(E)$ whose boundary is

$$\delta(M)\left(\pi^{-1}_{|\partial\tilde{C}_2(E)\setminus\partial\tilde{C}_2(M_{[0,3[})}(V) \cup s_\tau(E;V) \cup \lambda K_V \times_\tau S^2\right),$$

for some λ in $\frac{1}{\delta(M)}\mathbb{Q}[t, t^{-1}]$, and that is transverse to $\partial\tilde{C}_2(E)$.

Set

$$A = \delta(M)\left(\pi^{-1}_{|\partial\tilde{C}_2(M_{[0,3]})\setminus\partial\tilde{C}_2(M_{[0,3[})}(V) \cup s_\tau(E;V)\right).$$

A is a 3-cycle of $\partial\tilde{C}_2(E)$ that can be first pushed inside $\tilde{C}_2(E)$ using a product cobordism in the neighborhood of the boundary towards a 3-cycle B that is parallel to A but that does not meet the boundary anymore. According to Proposition 12.11, $\delta(M)B$ bounds a 4-chain in $\widetilde{E^2}$. This chain can be assumed to be transverse to the preimage of the diagonal of E^2 under the covering map. This chain can be modified by the addition of 4-cycles of the form $\partial(B^3 \times G)$ where G is a 2–cobordism in the preimage of the diagonal of E^2, and B^3 is a normal section of the diagonal of E^2. Thus, our chain can be furthermore assumed to intersect this preimage along the preimage of the diagonal of K_V^2. This shows that there exists a 4-dimensional rational chain C_V of $\tilde{C}_2(E)$ whose boundary is

$$\delta(M)\left(\pi^{-1}_{|\partial\tilde{C}_2(E)\setminus\partial\tilde{C}_2(M_{[0,3[})}(V) \cup s_\tau(E;V) \cup \lambda K_V \times_\tau S^2\right),$$

and that is transverse to $\partial\tilde{C}_2(E)$, for some λ of $\frac{1}{\delta(M)}\mathbb{Q}[t, t^{-1}]$. $\diamond$

13 On the augmentation of $\mathcal{Q}$.

13.1 The result and the sketch of the proof

This section is devoted to the proof of the following theorem.

Theorem 13.1 *Let $M_{\mathbb{K}}$ denote the rational homology sphere obtained from M by surgery along $\mathbb{K}$. Then*

$$\mathcal{Q}(\mathbb{K} \subset M)(1,1,1) = 6\lambda(M_{\mathbb{K}}).$$

The proof of this proposition heavily relies on Propositions 12.2 and 12.3. Using the same notation as in these propositions, and denoting the meridian disk of the torus reglued during the surgery on $(M, \mathbb{K})$ by $D^2_{[7,8]}(\hat{K})$, write

$$M_{\mathbb{K}} = M_{[0,7]} \cup_{\partial M_{[0,7]} \sim (-\partial(S^1 \times D^2_{[7,8]}(\hat{K})))} S^1 \times D^2_{[7,8]}(\hat{K})$$

where

$$p_M(7z, u) \in \partial M_{[0,7]} = \{p_M(7z, u); z \in S^1, u \in [0,1]\}$$

(with the notation of Subsection 12.2) is identified to $(z, \exp(2i\pi u) \in \partial D^2_{[7,8]}(\hat{K})) \in S^1 \times D^2_{[7,8]}(\hat{K})$.

Referring to the configuration space construction of the Walker invariant in Subsection 6.1, assume that $B(M_{\mathbb{K}}) = B_{M_{\mathbb{K}}}(3)$ is embedded in $M_{\mathbb{K}}$ and that the ball $(M_{\mathbb{K}} \setminus B(M_{\mathbb{K}}))$ is a small ball inside $M_{]6,7[}$. Recall $E = M_{[0,3]}$ and set $B(M_{\mathbb{K}})_{[3,8]} = B(M_{\mathbb{K}}) \setminus M_{[0,3[}$. Note that $B(M_{\mathbb{K}})_{[3,8]}$ is a solid torus minus a small open ball that is independent of $(M, \mathbb{K})$.

For $V = X, Y$ or Z that will be assumed to be almost horizontal, let $p(C_V)$ be the image of the chain C_V of Proposition 12.2 in $C_2(E)$ by the covering map. Since the covering map sends t to 1,

$$\partial p(C_V) = \pi^{-1}_{|\partial \tilde{C}_2(M_{[0,3]}) \setminus \partial \tilde{C}_2(M_{[0,3[})}(V) \cup s_\tau(M_{[0,3]}; V).$$

We are going to construct 4–chains D_V of $C_2(B(M_{\mathbb{K}}))$

- with support outside the interiors of $C_2(E = M_{[0,3]})$ and $C_2(B(M_{\mathbb{K}})_{[3,8]})$
- such that D_X, D_Y and D_Z have no triple intersection,
- and such that there exist transverse 4–chains E_V of $C_2(B(M_{\mathbb{K}})_{[3,8]})$ independent of $M_{\mathbb{K}}$ such that

$$6\lambda(M_{\mathbb{K}}) = \langle p(C_X) + D_X + E_X, p(C_Y) + D_Y + E_Y, p(C_Z) + D_Z + E_Z \rangle + C(p_1) - \frac{p_1(\tau)}{4}$$

  where $C(p_1)$ is a rational number that does not depend on $(M, \mathbb{K})$.

Therefore, we shall have

$$6\lambda(M_{\mathbb{K}}) = \langle p(C_X), p(C_Y), p(C_Z)\rangle_{C_2(E)} + \langle E_X, E_Y, E_Z\rangle + C(p_1) - \frac{p_1(\tau)}{4}$$

where

$$\mathcal{Q}(\mathbb{K} \subset M)(1,1,1) = \langle p(C_X), p(C_Y), p(C_Z)\rangle_{C_2(E)} - \frac{p_1(\tau)}{4}$$

so that

$$\mathcal{Q}(\mathbb{K} \subset M)(1,1,1) - 6\lambda(M_{\mathbb{K}})$$

will be independent of $(M, \mathbb{K})$. The theorem will follow since $\mathcal{Q}(S^1 \times S^2, S^1 \times u) = 0 = \lambda(S^3)$.

13.2 The construction of D_V

Set $B(M_{\mathbb{K}})_{[5,8]} = B(M_{\mathbb{K}}) \setminus M_{[0,5[}$. The chain D_V will read

$$p\left(\pi^{-1}_{|\tilde{C}_2(M_{[0,5]})\setminus\left(\tilde{C}_2(M_{[0,3[})\cup\tilde{C}_2(M_{]3,5]})\right)}(V)\right) + d_V + d'_V$$

where d_V will be supported in $M_{[0,3]} \times B(M_{\mathbb{K}})_{[5,8]}$ and d'_V that will be close to $\iota(d_V)$ will be supported in $B(M_{\mathbb{K}})_{[5,8]} \times M_{[0,3]}$. The chain d_V is made of the following 3 pieces:

- $-\bigcup_{u\in[0,1]} f^{-1}_{|E}(\exp(2i\pi(u-h))) \times p_M\left(\{p_V\} \times [0,u]\right)$,
- $M_{[0,3]} \times [p_M(p_V, 0), 3V]$
  where $[p_M(p_V, 0), 3V]$ is a path in $B(M_{\mathbb{K}})_{[5,8]}$ that goes from $(p_M(p_V, 0) \in \partial M_{[0,5]})$ to $(3V \in \partial B(M_{\mathbb{K}}))$, and the paths $[p_M(p_X, 0), 3X]$, $[p_M(p_Y, 0), 3Y]$ and $[p_M(p_Z, 0), 3Z]$ are pairwise disjoint,
- $f^{-1}_{|E}(\exp(2i\pi(-h))) \times \left(p_V \times D^2_{[5,8]}(\hat{K})\right)$
  where $\left(p_V \times D^2_{[5,8]}(\hat{K})\right)$ is a meridian disk of $\hat{K}$ in $B(M_{\mathbb{K}})_{[5,8]}$ with boundary $p_M(p_V \times [0,1])$.

Split

$$\partial p\left(\pi^{-1}_{|\tilde{C}_2(M_{[0,5]})\setminus\left(\tilde{C}_2(M_{[0,3[})\cup\tilde{C}_2(M_{]3,5]})\right)}(V)\right)$$

as the following sum

$$\partial_{[0,3],5} + \partial_{5,[0,3]} + \partial_{[3,5]\leftrightarrow 3} + \partial_{[0,3]\leftrightarrow 3}$$

where the subscripts reflect the values of the map r of the beginning of Subsection 12.2 of the points of an ordered pair of $C_2(M_{[0,5]})$.

$$\partial_{[0,3],5} = -\bigcup_{u\in[0,1]} f_{|E}^{-1}(\exp(2i\pi(u-h))) \times p_M\left(\{p_V\}\times\{u\}\right).$$

$$\partial_{5,[0,3]} = -\bigcup_{u\in[0,1]} p_M\left(\{-p_V\}\times\{u\}\right) \times f_{|E}^{-1}(\exp(2i\pi(u+h))).$$

Lemma 13.2

$$\begin{aligned}\partial d_V = \ & -M_{[0,3]}\times\{3V\} - \partial_{[0,3],5} \\ & +\textstyle\bigcup_{u\in[0,1]} \partial f_{|E}^{-1}(\exp(2i\pi(u-h))) \times p_M\left(\{p_V\}\times[0,u]\right) \\ & +\partial M_{[0,3]}\times[p_M(p_V,0),3V] \\ & +\partial f_{|E}^{-1}(\exp(2i\pi(-h)))\times\left(p_V\times D^2_{[5,8]}(\hat{K})\right).\end{aligned}$$

Proof: Note that the part $-f_{|E}^{-1}(\exp(2i\pi(1-h)))\times p_M\left(\{p_V\}\times[0,1]\right)$ cancels with

$$f_{|E}^{-1}(\exp(2i\pi(-h)))\times\left(p_V\times\partial D^2_{[5,8]}(\hat{K})\right).$$

◇

The chain d'_V is made of the following 3 pieces:

- $-\bigcup_{u\in[0,1]} p_M\left(\{-p_V\}\times[0,u]\right)\times f_{|E}^{-1}(\exp(2i\pi(u+h)))$,
- $[-3V,p_M(-p_V,0)]\times M_{[0,3]}$ where $[-3V,p_M(-p_V,0)]$ is a path in $B(M_{\mathbb{K}})_{[5,8]}$ that goes from $(-3V)$ in the boundary of $B(M_{\mathbb{K}})$ to $p_M(-p_V,0)$, and the paths $[-3X,p_M(-p_X,0)]$, $[-3Y,p_M(-p_Y,0)]$ and $[-3Z,p_M(-p_Z,0)]$ are pairwise disjoint,
- $\left(\{-p_V\}\times D^2_{[5,8]}(\hat{K})\right)\times f_{|E}^{-1}(\exp(2i\pi h))$ where $\left(\{-p_V\}\times D^2_{[5,8]}(\hat{K})\right)$ is a meridian disk of $\hat{K}$ in $B(M_{\mathbb{K}})_{[5,8]}$ with boundary $p_M(\{-p_V\}\times[0,1])$.

Lemma 13.3

$$\begin{aligned}\partial d'_V = \ & -\{-3V\}\times M_{[0,3]} - \partial_{5,[0,3]} \\ & -\textstyle\bigcup_{u\in[0,1]} p_M\left(\{-p_V\}\times[0,u]\right)\times\partial f_{|E}^{-1}(\exp(2i\pi(u+h))) \\ & -[-3V,p_M(-p_V,0)]\times\partial M_{[0,3]} \\ & +\left(\{-p_V\}\times D^2_{[5,8]}(\hat{K})\right)\times\partial f_{|E}^{-1}(\exp(2i\pi(h))).\end{aligned}$$

Proof: The proof is similar, of course. ◇

Lemma 13.4

$$\partial(p(C_V) + D_V) = \partial_r + \partial_e$$

where $\partial_r = -M_{[0,3]} \times \{3V\} - \{-3V\} \times M_{[0,3]} + s_\tau(M_{[0,3]}; V)$ *and*

$$\begin{aligned}\partial_e = \ & \partial_{[3,5]\leftrightarrow 3} \\ & - \bigcup_{u\in[0,1]} p_M\left(\{-p_V\} \times [0,u]\right) \times \partial f^{-1}_{|E}(\exp(2i\pi(u+h))) \\ & + \bigcup_{u\in[0,1]} \partial f^{-1}_{|E}(\exp(2i\pi(u-h))) \times p_M\left(\{p_V\} \times [0,u]\right) \\ & -[-3V, p_M(-p_V,0)] \times \partial M_{[0,3]} + \partial M_{[0,3]} \times [p_M(p_V,0), 3V] \\ & + \left(\{-p_V\} \times D^2_{[5,8]}(\hat{K})\right) \times \partial f^{-1}_{|E}(\exp(2i\pi(h))) + \partial f^{-1}_{|E}(\exp(2i\pi(-h))) \times \left(p_V \times D^2_{[5,8]}(\hat{K})\right).\end{aligned}$$

Note that the chains D_V have no triple intersection. Indeed D_V meets $M_{[0,3]} \times \partial M_{[0,5]}$ in $M_{[0,3]} \times p_M(p_V \times \mathbb{R})$ and the $p_M(p_V \times \mathbb{R})$ are pairwise disjoint so that two D_V cannot intersect in $M_{[0,3]} \times \partial M_{[0,5]}$. Similarly, two D_V cannot intersect in $\partial M_{[0,5]} \times M_{[0,3]}$. Then triple intersections of D_X, D_Y and D_Z must be triple intersections of d_X, d_Y and d_Z or triple intersections of d'_X, d'_Y and d'_Z that cannot occur under the further natural assumption that the meridian disks $((\pm p_V) \times D^2_{[5,8]}(\hat{K}))$ are disjoint.

13.3 The existence of E_V

The trivialisation τ of $M_{[0,7]}$ of Subsection 12.2 does not extend to $B(M_{\mathbb{K}})$ as a genuine trivialisation but it extends as a pseudo-trivialisation $\tilde{\tau}$ of Definition 10.11 where c is a parallel of $\hat{K}$, that is a meridian of K. Set $N = M_{\mathbb{K}}$. According to Proposition 10.12,

$$6\lambda(N) = \langle F_{N,X}(\tilde{\tau}), F_{N,Y}(\tilde{\tau}), F_{N,Z}(\tilde{\tau})\rangle_{C_2(B(N))} - \frac{p_1(\tilde{\tau})}{4}$$

where the boundary of $F_{N,V}(\tilde{\tau})$ is defined in Definition 10.11, for $V = X$, Y or Z.

Recall that $(B(M_{\mathbb{K}})_{[3,8]} = B(M_{\mathbb{K}}) \setminus M_{[0,3[})$ is a solid torus minus a small open ball that is independent of $(M, \mathbb{K})$.

Note that ∂_r is part of the wanted boundary $\partial F_{N,V}(\tilde{\tau})$, and that the cycle

$$e_V = (\partial F_{N,V}(\tilde{\tau}) - \partial_e - \partial_r)$$

is therefore supported in $\partial C_2(B(M_{\mathbb{K}})_{[3,8]})$.

Thus, e_V is a 3–cycle of $\partial C_2(B(M_{\mathbb{K}})_{[3,8]})$ that is independent of $(M, \mathbb{K})$. Let h_X, h_Y and h_Z be three distinct small positive numbers. Let $\hat{K}_V$ denote a horizontal parallel of $\hat{K}$ that reads $p_M(3S^1 \times \{h_V\})$ in $\partial M_{[0,3]}$. We are going to prove the following lemma.

Lemma 13.5 *There exists* $\mu \in \mathbb{Q}$ *such that*

$$e_V - \mu ST(\hat{K}_V)$$

bounds a 4*–chain* E_V *in* $C_2(B(M_{\mathbb{K}})_{[3,8]})$ *transverse to* $\partial C_2(B(M_{\mathbb{K}})_{[3,8]})$.

Let us show that Lemma 13.5 implies Theorem 13.1. Let Σ_X, Σ_Y and Σ_Z be three disjoint Seifert surfaces in $M_{[0,3]}$ for $\hat{K}_X$, $\hat{K}_Y$ and $\hat{K}_Z$, respectively. Let $N = M_{\mathbb{K}}$.

Then, we can use the chains

$$F_{N,V}(\tilde{\tau}) = p(C_V) + D_V + E_V + \mu ST(\Sigma_V)$$

to compute $(6\lambda(N) - \mathcal{Q}(\mathbb{K} \subset M)(1,1,1))$. Since the $ST(\Sigma_V)$ are pairwise disjoint, and since the other pieces of $F_{N,V}(\tilde{\tau})$ only meet $ST(M_{[0,3]})$ on $s_\tau(M_{[0,3]}; V)$, no triple intersection will involve the $ST(\Sigma_V)$ that can be forgotten. If $p(C_X)$ met D_Y or E_Y outside $ST(\Sigma_Y)$, it would be in $p(\pi^{-1}(X) \cap \pi^{-1}(Y))$, so they do not meet. If D_X met E_Y outside $ST(\Sigma_Y)$, it would be in ∂E_Y and actually in ∂D_Y. Therefore, since the D_V have no triple intersection,

$$6\lambda(N) = \langle p(C_X), p(C_Y), p(C_Z)\rangle_{C_2(E)} + \langle E_X, E_Y, E_Z\rangle_{C_2(B(M_{\mathbb{K}})_{[3,8]})} - \frac{p_1(\tilde{\tau})}{4}.$$

Lemma 13.6 *The difference* $(p_1(\tau) - p_1(\tilde{\tau}))$ *is independent of* $(M, \mathbb{K})$.

Proof: Let W be 4–manifold with boundary $B_N(3) \cup (-[0,1] \times \partial B(3)) \cup (-B(3))$ with signature 0. $p_1(\tilde{\tau})$ is the obstruction to extend the trivialisation induced by $\tilde{\tau}_{\mathbb{C}}$ on ∂W to W. Glue a 3-ball B along $\partial B(3)$, so that $M_{\mathbb{K}} = B_N(3) \cup B$, extend $\tilde{\tau}$ as τ_B on B. Consider $W_B = W \cup_{[0,1]\times\partial B(3)} [0,1] \times B$. Then $p_1(\tilde{\tau})$ is the obstruction to extend the trivialisation induced by $\tilde{\tau}_{\mathbb{C}}$ and τ_B on ∂W_B to W_B, where the signature of W_B is zero. Glue a ball B^4 to the part $\{0\} \times (B(3) \cup B)$ of ∂W_B where $\tau_{B(3)}$ has been extended by τ_B. The obstruction to extend this trivialisation to B^4 will be denoted by $p_1(\tau_{B(3)} \cup \tau_B)$. Then $N = M_{\mathbb{K}} = \partial(W_B \cup B^4)$ and the obstruction to extend $\tilde{\tau}$ to $W_B \cup B^4$ is $p_1(\tilde{\tau}) + p_1(\tau_{B(3)} \cup \tau_B)$. Now, attaching a 2–handle along $\hat{K} \times D^2 \subset \partial(W_B \cup B^4)$ equipped with its canonical trivialisation produces a 4-manifold W_M with signature 0 and with boundary M and the obstruction to extend the trivialisation induced by τ to W_M is $p_1(\tau)$. Let $p_1(H)$ be the obstruction to extend the trivialisation induced by τ and $\tilde{\tau}_{\mathbb{C}}$ on the 2–handle, it is independent of M and $\mathbb{K}$. Then $p_1(\tau) = p_1(H) + p_1(\tilde{\tau}) + p_1(\tau_{B(3)} \cup \tau_B)$. $\diamond$

Since

$$\mathcal{Q}(\mathbb{K} \subset M)(1,1,1) = \langle p(C_X), p(C_Y), p(C_Z)\rangle_{C_2(E)} - \frac{p_1(\tau)}{4}$$

$$6\lambda(M_{\mathbb{K}}) - \mathcal{Q}(\mathbb{K} \subset M)(1,1,1) = \frac{p_1(H) + p_1(\tau_{B(3)} \cup \tau_B)}{4} + \langle E_X, E_Y, E_Z\rangle_{C_2(B(M_{\mathbb{K}})_{[3,8]})}$$

is independent of $(M, \mathbb{K})$. Thus, we are left with the proof of Lemma 13.5 to prove Theorem 13.1.

We shall use the following easy lemma.

Lemma 13.7 $H_3(C_2(B(M_{\mathbb{K}})_{[3,8]}); \mathbb{Q}) = \mathbb{Q}[\partial B(M_{\mathbb{K}}) \times \hat{K}] \oplus \mathbb{Q}[\hat{K} \times \partial B(M_{\mathbb{K}})] \oplus \mathbb{Q}[ST(\hat{K})]$. *A 3-cycle of* $C_2(B(M_{\mathbb{K}})_{[3,8]})$ *reads* $\mu ST(\hat{K})$ *if and only if its algebraic intersections with the two chains* $[P, 3W] \times D^2_{[3,8]}(\hat{K})_P$ *and* $D^2_{[3,8]}(\hat{K})_P \times [P, 3W]$ *vanish, where* $D^2_{[3,8]}(\hat{K})_P$ *is a meridian disk of* $\hat{K}$ *in* $B(M_{\mathbb{K}})_{[3,8]}$, $3W \in \partial B(M_{\mathbb{K}})$, $P \in \partial M_{[0,3]}$, *and* $[P, 3W]$ *is a path from* P *to* $3W$ *that does not meet* $D^2_{[3,8]}(\hat{K})_P$.

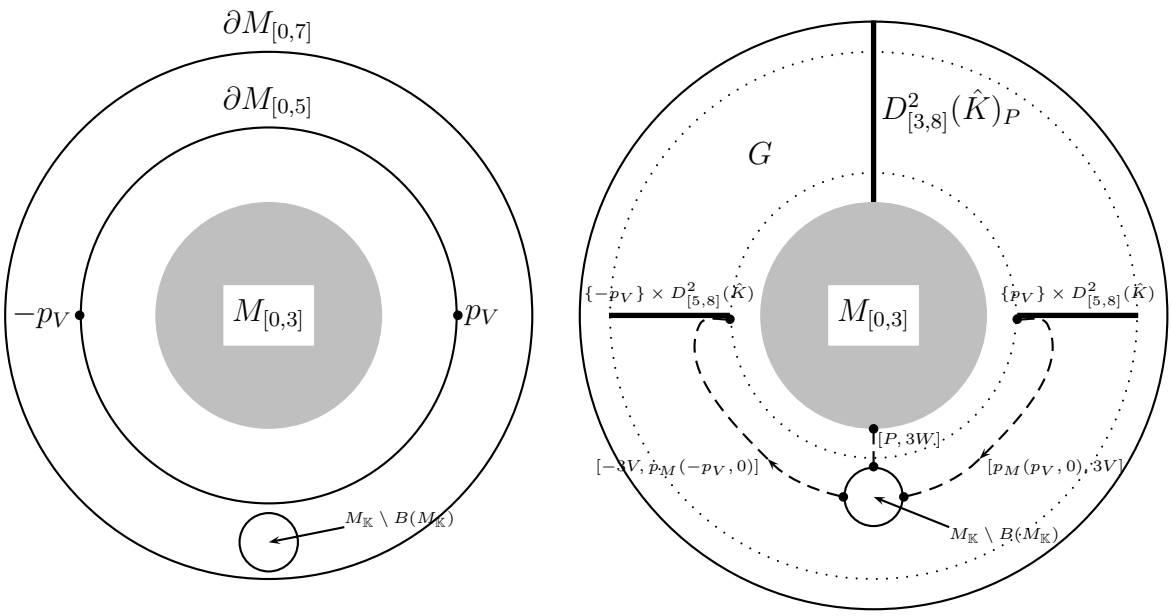
$\diamond$

Proof of Lemma 13.5:

It is enough to show that $D^2_{[3,8]}(\hat{K})_P$, $[P, 3W]$, $\{\pm p_V\} \times D^2_{[5,8]}(\hat{K})$, $[-3V, p_M(-p_V, 0)]$ and $[p_M(p_V, 0), 3V]$ can be fixed so that neither ∂_e nor $\partial F_{N,W}(\tilde{\tau})$ intersects

$$H = [P, 3W] \times D^2_{[3,8]}(\hat{K})$$

or $\iota(H) = D^2_{[3,8]}(\hat{K})_P \times [P, 3W]$. To do that, fix $\{\pm p_V\} \times D^2_{[5,8]}(\hat{K})$ so that it intersects $M_{[5,6]}$ as $\{(q(t, \pm z(V)), z); t \in [5, 6], z \in S^1\}$, with respect to the notation of Subsection 12.2, where $(p_V, h_V) = (5, z(V), h_V)$. Then complete these two vertical annuli with two parallel meridian disks of the tubular neighborhood $N_{[6,8]}(\hat{K})$ of $\hat{K}$ in $M_{\mathbb{K}}$. Here is a picture of the projection on $\mathbb{C}$ of $M_{[0,7]} \setminus M_{[0,3[}$ and the corresponding picture of $S^1 \times D^2_{[3,8]}(\hat{K})$. Note that, for $z \in S^1$, the preimage of $\mathbb{R}^+\{z\}$ in $(M_{[0,7]} \setminus M_{[0,3[})$ corresponds to $\{z\} \times D^2_{[3,8]}(\hat{K})$ in $(M_{\mathbb{K}} \setminus M_{[0,3[})$.



The complement of the two meridian disks $\{\pm p_V\} \times D^2_{[5,8]}(\hat{K})$ in the solid torus $N_{[5,8]}(\hat{K})$ has two connected components. Let G be the connected component that does not contain $M_{\mathbb{K}} \setminus B(M_{\mathbb{K}})$.

Let $\frac{5}{3}P$ be a point of $\partial D^2_{[0,5]}$ such that $(\frac{5}{3}P, 1) \notin \overline{G}$. Fix $D^2_{[3,8]}(\hat{K})_P$ so that it intersects $M_{[3,6]}$ as $\{(q(t, -\frac{P}{|P|}), z); t \in [3, 6], z \in S^1\}$, and complete it into a meridian disk in G. Fix the path $[P, 3W]$ so that it intersects $M_{[3,6]}$ as $\{(q(t, \frac{P}{|P|}), 1); t \in [3, 6]\}$ and so that it does not intersect $\overline{G}$. Thus, π maps $C_2(M_{[0,5]}) \cap (H \cup \iota(H))$ to $(\lambda P, h)$ for two real numbers λ and h, so that $\partial_{[3,5]\leftrightarrow 3}$ does not meet $(H \cup \iota(H))$, because $(\lambda P, h) \neq \pm V$. The path $[P, 3W]$ is furthermore assumed to be transverse to $\partial B(M_{\mathbb{K}})$ with $W \neq \pm V$ so that $\partial F_{N,V}(\tilde{\tau})$ does not meet $(H \cup \iota(H))$. It is

clear that $(H \cup \iota(H))$ does not meet

$$\begin{aligned}
&+\bigcup_{u\in[0,1]} p_M\left(\{-p_V\}\times[0,u]\right)\times\partial f_{|E}^{-1}(\exp(2i\pi(u+h)))\\
&+\bigcup_{u\in[0,1]} \partial f_{|E}^{-1}(\exp(2i\pi(u-h)))\times p_M\left(\{p_V\}\times[0,u]\right)\\
&+\Big(\{-p_V\}\times D^2_{[5,8]}(\hat{K})\Big)\times\partial f_{|E}^{-1}(\exp(2i\pi(h)))+\partial f_{|E}^{-1}(\exp(2i\pi(-h)))\times\Big(p_V\times D^2_{[5,8]}(\hat{K})\Big)
\end{aligned}$$

either, and since we can fix the paths $[-3V, p_M(-p_V,0)]$ and $[p_M(p_V,0),3V]$ so that they meet neither $D^2_{[3,8]}(\hat{K})_P$ nor $[P,3W]$, we conclude that ∂_e does not meet $(H\cup\iota(H))$ at all. Then the class of e_V reads $\mu ST(\hat{K}_V)$ in $H_3(C_2(B(M_{\mathbb{K}})_{[3,8]}))$ and Lemma 13.5 and therefore Theorem 13.1 are proved. $\diamond$

14 On the target vector space for $\overline{\mathcal{Q}}(M)$

14.1 Notation

Consider the polynomial ring

$$\Lambda_f = \mathbb{Q}[x^{\pm 1}, y^{\pm 1}] = \frac{\mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]}{xyz = 1}$$

and its subring

$$\mathbb{Q}_s[x^{\pm 1}, y^{\pm 1}] = \{P \in \Lambda_f; P(x,y) = P(x, z = (xy)^{-1}) = P(y,x) = P(x^{-1}, y^{-1})\}.$$

Then

$$\mathcal{Q}_2(\mathbb{K}) = \delta(x)\delta(y)\delta(z)\mathcal{Q}(\mathbb{K})$$

takes its values in $\mathbb{Q}_s[x^{\pm 1}, y^{\pm 1}]$, according to Proposition 12.1 and to Subsection 5.4. Define

$$P_k = P_k(M) = P_k(\Delta(M), \delta(M)) = P_k(\Delta, \delta) = \delta(x)\delta(y)\delta(z) \sum_{\mathfrak{S}_3(x,y,z)} \frac{(x^k - x^{-k})}{\delta(x)} I_\Delta(z).$$

Lemma 14.1 $P_k \in \mathbb{Q}_s[x^{\pm 1}, y^{\pm 1}]$.

$$P_k = \sum_{\mathfrak{S}_3} (x^k - x^{-k})\delta(y)\delta(z)\frac{z\Delta'(z)}{\Delta(z)} + \sum_{\circlearrowleft} \frac{\left((xz)^k - (xz)^{-k}\right)\delta(x) - \left(x^k - x^{-k}\right)\delta(xz)}{z-1}\delta(z)(z+1)$$

Proof: The symmetry properties of P_k are easy to check, and it is enough to check that P_k can be expressed as in the statement to prove that P_k is actually a polynomial.

$$\begin{aligned}
P_k &- \sum_{\mathfrak{S}_3} (x^k - x^{-k})\delta(y)\delta(z)\frac{z\Delta'(z)}{\Delta(z)} \\
&= -\sum_{\mathfrak{S}_3} (x^k - x^{-k})\delta(y)\delta(z)\frac{z+1}{z-1} \\
&= -\sum_{\mathfrak{S}_3} (x^k - x^{-k})\delta(xz)\delta(z)\frac{z+1}{z-1} \\
&= \sum_{\circlearrowleft} \frac{\left((xz)^k - (xz)^{-k}\right)\delta(x) - \left(x^k - x^{-k}\right)\delta(xz)}{z-1}\delta(z)(z+1).
\end{aligned}$$

$\diamond$

Let $\Lambda(M) = \Lambda(\Delta, \delta)$ be the quotient of $\mathbb{Q}_s[x^{\pm 1}, y^{\pm 1}]$ by the rational vector space generated by the $P_k(M)$ for $k \in \mathbb{N}$, $k \geq 1$. According to Theorem 1.5, the class $\overline{\mathcal{Q}}_2(M)$ of $\mathcal{Q}_2(\mathbb{K})$ in $\Lambda(M)$ is independent of $\mathbb{K}$. Therefore, it is worth studying this quotient.

I conjecture that $\overline{\mathcal{Q}}_2(M)$ is equivalent to an invariant previously defined by Ohtsuki in [O3] when $\delta = \Delta$, and that it is a refinement of this Ohtsuki invariant when $\delta \neq \Delta$. Ohtsuki has proved more results on $\Lambda(\Delta, \Delta)$ in [O3].

14.2 Detection of constants

In this subsection we prove the following proposition.

Proposition 14.2 *If $\Delta(M)$ has no multiple roots, then $\Delta(M) = \delta(M)$ and $\delta(x)\delta(y)\delta(z)$ does not vanish in $\Lambda(M)$. In particular, $\overline{\mathcal{Q}}_2(M\sharp N) \neq \overline{\mathcal{Q}}_2(M)$ for any rational homology sphere N with non trivial Walker invariant.*

We have the following tautological lemma.

Lemma 14.3 *In general, $\delta(x)\delta(y)\delta(z)$ vanishes in $\Lambda(M)$ if and only if the following equation is satisfied*

$$\delta(x)\delta(y)\delta(z) = \sum_{i=1}^{s} \beta_i P_i \tag{14.4}$$

where $s \in \mathbb{N} \setminus \{0\}$, $\beta_i \in \mathbb{Q}$, for $i = 1, \ldots, s$, and $\beta_s \neq 0$.

$\diamond$

Set

$$p_k = p_k(\Delta, \delta) = P_k(x, x^{-1}, 1).$$

Of course, if Equation 14.4 is satisfied, then

$$\delta^2(x) = \sum_{i=1}^{s} \beta_i p_i(x). \tag{14.5}$$

Lemma 14.6

$$p_k(\Delta, \delta) = 2(x^k - x^{-k})\delta(x)\left(-\frac{x\Delta'(x)}{\Delta(x)} - \frac{x\delta'(x)}{\delta(x)} + \frac{x+1}{x-1}\right) + 2k(x^k + x^{-k})\delta(x).$$

$$p_k(\Delta, \Delta) = -4x\Delta'(x)(x^k - x^{-k}) + 2\Delta(x)\left(k(x^k + x^{-k}) + (x+1)\frac{x^k - x^{-k}}{x-1}\right)$$
$$= -4x\Delta'(x)(x^k - x^{-k}) + 2\Delta(x)\left((k+1)(x^k + x^{-k}) + 2(x^{k-1} + x^{k-2} + \ldots + x^{2-k} + x^{1-k})\right).$$

Proof:

$$\begin{aligned} p_k \;=\; & -2(x^k - x^{-k})\delta(x)\tfrac{x\Delta'(x)}{\Delta(x)} \\ & +\lim_{z\to 1}\tfrac{\left((xz)^k - (xz)^{-k}\right)\delta(x) - \left(x^k - x^{-k}\right)\delta(xz)}{z-1}\delta(z)(z+1) \\ & +\tfrac{\left(x^k - x^{-k}\right)}{x-1}\delta(x)(x+1) - \tfrac{\left(x^k - x^{-k}\right)}{y-1}\delta(y)(y+1) \end{aligned}$$

where $y = x^{-1}$. Of course, the mentioned limit is nothing but the derivative at $z = 1$ that is

$$2\left(k\left(x^k + x^{-k}\right)\delta(x) - \left(x^k - x^{-k}\right)x\delta'(x)\right).$$

The second equality is an easy consequence of the first one. $\diamond$

If $\Delta = 1$, then $p_k = 2\left((k+1)(x^k + x^{-k}) + 2(x^{k-1} + x^{k-2} + \ldots + x^{2-k} + x^{1-k})\right)$. Therefore, the degree of p_k is k for any k and Equation 14.4 cannot hold with some $\beta_s \neq 0$. Therefore, Proposition 14.2 is true in this case.

Lemma 14.7 *Assume that*

$$\Delta = \delta = \sum_{i=1}^{r} \alpha_i (t^i + t^{-i}) + 1 - 2\sum_{i=1}^{r} \alpha_i,$$

with $\alpha_r \neq 0$ and $r > 0$. If Equation 14.4 is satisfied, then $s = r$, $r > 1$ and $\alpha_r = (2-2r)\beta_r$.

Proof: The leading term in

$$p_k = -4x\Delta'(x)(x^k - x^{-k}) + 2\Delta(x)\left((k+1)(x^k + x^{-k}) + 2(x^{k-1} + x^{k-2} + \ldots + x^{2-k} + x^{1-k})\right).$$

is

$$(2(k+1)\alpha_r - 4r\alpha_r)x^{k+r}$$

unless $k = 2r-1$, and in this latter case the degree of p_k is lower than $(k+r)$.

The degree of the left-hand side in Equation 14.5 is $2r$ while the degree of the right-hand side is $s+r$ if $s \neq 2r-1$ and is lower than $s+r$ if $s = 2r-1$. Therefore, if Equation 14.5 holds with $s \neq r$, then $s = 2r-1$ and $r < s$, and therefore $r > 1$.

In particular, if $r = 1$, Equation 14.5 could only hold with $r = s = 1$ but in this case $s = 2r-1$, and the degree of p_1 is smaller than 2. Thus, if Equation 14.5 is satisfied, then $r > 1$.

Then Lemma 14.7 will be proved as soon as the following lemma is proved.

Sublemma 14.8 *If $\Delta = \delta = \sum_{i=1}^{r} \alpha_i (t^i + t^{-i}) + 1 - 2\sum_{i=1}^{r} \alpha_i$, with $\alpha_r \neq 0$ then Equation 14.4 cannot hold if $s = 2r-1 > r > 1$.*

Proof of the sublemma: We compare the terms of lexicographically higher degrees in both sides of the equation:

$$\Delta(x)\Delta(y)\Delta(z)(x-1)(y-1)(z-1) = (x-1)(y-1)(z-1)\sum_{i=1}^{s} \beta_i P_i$$

in $\mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. On the left-hand side we have $\alpha_r^3 x^{2r} y^{2r}(-xy)$. On the right-hand side, since $s > r > 1$, the term of highest degree in

$$(x-1)(y-1)(z-1)P_s$$

$$= \sum_{\mathfrak{S}_3} (x-1)(x^s - x^{-s})(y-1)\delta(y)\left((z-1)z\delta'(z) - \delta(z)(z+1)\right)$$

is $2(xy)^{s+1}(r-1)\alpha_r^2 (xy)^r$. Then since $s > r$ the degree of the right-hand side is bigger than the degree of the left-hand side. ◇

The fact than $\alpha_r = (2-2r)\beta_r$ can be deduced by identifying the leading terms of both sides in Equation 14.5 or in Equation 14.4.

◇

End of proof of Proposition 14.2: If Δ has no multiple root then Δ is coprime with Δ' and therefore to $x\Delta'$.

Then if Equation 14.5 is satisfied, according to Lemma 14.6, Δ must divide $\sum_{i=1}^{s} \beta_i(x^i - x^{-i})$ and since Δ is symmetric, the ratio is of degree at least 1. Then $s \geq r+1$. This concludes the proof of Proposition 14.2. $\diamond$

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