# Can there be a general nonlinear PDE theory for existence of solutions ?

Elemer Elad Rosinger

# Can there be a general nonlinear PDE theory for existence of solutions ?

*Elemér E Rosinger*
*Department of Mathematics*
*University of Pretoria*
*Pretoria, 0002 South Africa*
*e-mail : eerosinger@hotmail.com*

*Dedicated to Marie-Louise Nykamp*

**Abstract**

Contrary to widespread perception, there is ever since 1994 a unified, general type independent theory for the existence of solutions for very large classes of nonlinear systems of PDEs. This solution method is based on the Dedekind order completion of suitable spaces of piece-wise smooth functions on the Euclidean domains of definition of the respective PDEs. The method can also deal with associated initial and/or boundary value problems. The solutions obtained can be assimilated with usual measurable functions or even with Hausdorff continuous functions on the respective Euclidean domains.
It is important to note that the use of the order completion method does *not* require any monotonicity condition on the nonlinear systems of PDEs involved.
One of the major advantages of the order completion method is that it *eliminates* the algebra based dichotomy "linear versus nonlinear" PDEs, treating both cases with equal ease. Furthermore, the order completion method does *not* introduce the dichotomy "monotonous versus non-monotonous" PDEs.
None of the known functional analytic methods can exhibit such a performance, since in addition to topology, such methods are significantly based on algebra.

## 1. A Sample of customary perception

The 2004 edition of the Springer Universitext book "Lectures on PDEs" by V I Arnold, [4], starts on page 1 with the statement :

> "In contrast to ordinary differential equations, there is *no unified theory* of partial differential equations. Some equations have their own theories, while others have no theory at all. The reason for this complexity is a more complicated geometry ..." (italics added)

The 1998 edition of the book "Partial Differential Equations" by L C Evans, [8], starts his Examples on page 3 with the statement :

> "There is no general theory known concerning the solvability of all partial differential equations. Such a theory is *extremely unlikely* to exist, given the rich variety of physical, geometric, and probabilistic phenomena which can be modeled by PDE. Instead, research focuses on various particular partial differential equations ..." (italics added)

And yet, in 1994, [18], MR 95k:35002, precisely such a general theory of existence of solutions for very large classes of nonlinear PDEs was published. For latest developments, see [1-3,37,44-52,58-60,66].

In the sequel, we present the main ideas and motivations which underlie the order completion method. The detailed mathematical developments can be found in the references mentioned above.

It is on occasion worth recalling that we all do mathematics based on certain underlying ideas and motivations. What happens is that we may hold to them for longer, and do so long enough, so that many of them may become rather automatic. And once that happens, we do no longer - and in fact, can no longer - review them, and do so at least now and then. This is, then, how perceptions are established, and we end up being subjected to them.
Here an attempt is made to go beyond such perceptions in the realms of solving PDEs. And since perceptions are inevitably formulated in a

"meta-language" - in this case "meta" with respect to the usual formal mathematical texts - much of what follows has to go along with that.

The fact is that, recently, two rather different and general, that is, *type independent* solution methods have been developed for very large classes of linear and nonlinear systems of PDEs with possibly associated initial and/or boundary conditions. One of them, [16], is using standard functional analytic methods, while the other, [18,1-3,37,44-52,58-60,66], is based on a new idea in the realms of PDEs, namely, the *Dedekind order completion* of spaces of smooth functions.

Contrary to widespread perceptions, it thus proves to be possible to implement no less than two powerful solution methods for a very large variety of linear and nonlinear PDEs. These two methods are *type independent* in the sense that they are *no longer* dependent on specifics of one or another of the countless particular types of PDEs.
In fact, the essence of both methods is that, each in its own way is able to solve far more general equations than PDEs. And it is precisely in this lifting to a higher level of generality, one beyond PDEs, that the two methods attain their respective type independent power.

These two solution methods have rather complementary strong, respectively, weak points. The one in [16] does in fact deliver not only the existence of solutions, but also efficient numerical methods for approximating them. On the other hand, the method in [18,1-3,37,44-52,58-60,66] can deal with considerably more general equations, and among them linear and nonlinear systems of PDEs with possibly associated initial and/or boundary conditions.

In view of the above, it is clear that in proving the *existence* of solutions with the mentioned *regularity* properties, the order completion method can *dispense* to a good extent with all sorts of spaces of generalized functions, distributions, hyperfunctions, and so on, and instead, focus on various classes of usual measurable functions or Hausdorff continuous ones defined on usual Euclidean domains.
Certainly, the use of generalized solutions need not be avoided completely. However, solution methods based on them are clearly far less

powerful, especially with respect to their existence, than the order completion method. As for the regularity of solutions obtained by the order completion method, it gives from the beginning solutions which are more regular than by far most generalized functions, since they can be assimilated with usual measurable functions or Hausdorff continuous ones on Euclidean domains.

## 2. The class of nonlinear systems of PDEs solved

In [18] it was show how to obtain solutions $U$ for all systems of nonlinear PDEs with associated inital and/or boundary value problems, where the equations are of the form

$$(2.1) \quad F(x, U(x), \ \dots, D_x^p U(x), \ \dots) \ = \ f(x), \ x \in \Omega \subseteq \mathbb{R}^n, \ |p| \leq m$$

Here $F$ is any function jointly continuous in all its arguments, the right hand term $f$ can belong to a class of discontinuous functions, the order $m \in \mathbb{N}$ is given arbitrary, while the domain $\Omega$ can be any bounded or unbounded open set in $\mathbb{R}^n$.
In fact, even the functions $F$ defining the nonlinear partial differential operators in the left hand terms of (2.1) can have certain types of discontinuities.

Here one can note the unprecedented generality, type independence, or universality of the corresponding result both on the *existence* and the *regularity* of solutions given in [18] for systems of nonlinear PDEs of the form (2.1).

Indeed, regarding the *existence* of solution, the generality of the PDEs in (2.1) is self-evident.

As for the *regularity* of the solutions obtained, one can note the following.

The solutions $U$ obtained in [18] can be assimilated with *usual measurable functions* on the respective domains $\Omega$.
Not much later, in [2,3,58-60], it was shown that such solutions are in

fact always considerably more regular, being in fact Hausdorff continuous functions on the whole of the domains $\Omega$ of the respective PDEs.

Recently, in [44-51] this general, type independent, universal, or balnket regularity result was further improved as it was shown that the solutions $U$ can be associated with usual smooth functions on the domains $\Omega$ of the PDEs concerned, provided that similar smoothness conditions are assumed on the functions $F$ and $f$ in the PDEs (2.1).

Here it is important to note that *Hausdorff continuous* functions are not much unlike usual real valued continuous functions, [1-3]. Indeed, on suitable *dense* subsets of their domains of definition, Hausdorff continuous functions have as values real numbers, and are completely *determined* by such values. On the rest of their domains of definition, Hausdorff continuous functions can have values given by bounded or unbounded closed intervals of real numbers. Also, every real valued function which is continuous in the usual sense will be Hausdorff continuous as well.

One of the major advantages of the order completion method is that it *eliminates* the dichotomy between linear, and on the other hand, nonlinear PDEs, treating both cases with equal ease. Indeed, the dichotomy between linear and nonlinear follows from the vector space structure of the spaces of functions on which the partial differential operators act. In this way, this dichotomy is of an algebraic nature. On the other hand, partial orders are more basic mathematical structures, and as such, they do not, and simply cannot, differentiate between linear and nonlinear.
Clearly, functional analytic methods, which rely not only on topological but also algebraic structures cannot exhibit such a performance, since are bound to discriminate between linear and nonlinear, see details in section 9.

## 3. A short history of difficulties in solving linear and nonlinear PDEs

The first general, that is, *type independent* existence result for solu-

tions of rather arbitrary nonlinear systems of PDEs was obtained in 1874, when upon the suggestion of K Weierstrass, Sophia Kovalevskaia gave a rigorous proof for an earlier theorem of Cauchy, published in the 1821, in his Course d'Analyse. This result although completely general as far as the type independent nonlinearities involved ar concerned, assumes however, that in the systems of PDEs of the form (2.1) both $F$ and $f$ are analytic. In addition one also assumes initial value problems on non-characteristic analytic hypersurfaces, while boundary value problems are not treated by the respective Cauchy-Kovalevskaia theorem.
However, in such a highly particular situation concerning the regularity of the PDEs and the data involved, the solutions obtained are proved to exist always, and also to be unique and analytic.
The problem with that classical *existence, uniqueness* and *regularity* result is that, typically for nonlinear PDEs, such analytic solutions do not - and in general, cannot - exist globally on the whole of the domain of the respective PDEs, but only in certain neghbourhoods of the analytic hypersurfaces on which the initial values are given. This is, therefore, not due to the specific method of proof of Kovalevskaia.

Here however it is important to note that the failure of the existence of global analytic solutions is but a part of a far more general phenomenon, since even linear, let alone nonlinear PDEs may fail to have smooth, or even merely classical solutions, even in the case of solutions of major applicative interest. After all, even linear constant coefficient PDEs have nonclassical solutions of particular interest, such as those given by Green functions. In the nonlinear case, difficulties start with the simple ODE given by $U_t = U^2$ which does not have global classical solutions either, except for the trivial solution $U = 0$. As for the nonlinear shock wave equation $U_t + UU_x = 0$, its nonclassical solutions are precisely those which model shocks, thus have a considerable applicative interest.

What may be interesting, and also worthwhile to note with respect to the mentioned Cauchy-Kovalevskaia theorem, are the following three facts :

- The rigorous proof by Kovalevskaia of that theorem on solutions of general nonlinear systems of PDEs predates by about two

decades the corresponding general theorem on solving systems of nonlinear ODEs defined by continuous expressions. Indeed, the existence of solutions for such ODEs was given by Charles Emile Picard in his 1894 Comptes Rendu Acad. Sci. Paris paper, where the associated Cauchy problem was solved by the method of successive approximations.

- The only so called "hard" mathematics used in the proof of the Cauchy-Kovalevskaia theorem is the formula for the summation of a convergent geometric progression, the rest of the proof being but a succession of rather elementary, even if quite involved, estimates of terms in power series. In this way, the proof of the Cauchy-Kovalevskaia theorem does *not* involve methods of functional analysis, and certainly it could *not* involve such methods at the time in the 1880s when it was given. On the other hand, the proof of the corresponding general existence result for solutions of nonlinear systems of ODEs does involve a fixed point argument in suitable spaces of functions which are complete in their respective topologies.

- The result in the Cauchy-Kovalevskaia theorem - when considered on its own original terms of type independent nonlinear generality - could *not* so far be improved in those very terms, regardless of all the advances in functional analysis of the last more than a century. The only such improvement of the classical result in the Cauchy-Kovalevskaia theorem was obtained in 1985, without however using functional analytic methods, see section 4. Indeed, when it comes to *type independent nonlinear generality*, the functional analytic methods used in solving PDEs could bring about improvements - and often quite dramatic ones - only in a variety of far more particular cases than the type independent nonlinear generality dealt with in the Cauchy-Kovalevskaia theorem. In this way, in spite of more than one century of functional analysis, the classical Cauchy-Kovalevskaia theorem still remains a *maximal* result, except for its extension mentioned in section 4, which does not use functional analysis.

In the early 1950s, soon after the introduction of the linear theory of distributions by L Schwartz, it was proved independently by Mal-

grange and Ehrepreis that in case of a single PDE of the form (2.1), if the left hand term $F$ is linear and with constant coefficients, while $f$ is the Dirac delta distribution, then (2.1) always has a global, so called, fundamental solution given by a suitable Schwartz distribution.

This rather general linear result appeared to suggest that a similar result could be obtained in the more general case when $F$ in (2.1) is linear and with smooth coefficients. L Schwartz himself is known to have conjectured such a generalization, and furthermore, as it appears, he suggested it at the time to Francois Treves as a subject for his doctoral thesis.

However, in 1957, Hans Lewy showed that the rather simple linear first order PDE in three space variables and with first degree polynomial coefficients

$$(3.1) \quad (D_x + iD_y - 2(x+y)D_z)U(x,y,z) \;=\; f(x,y,z),$$

$$(x,y,z) \in \mathbb{R}^3$$

does not have any Schwartz distribution solutions in any neighbourhood of any point in $\mathbb{R}^3$, for a large class of smooth right terms $f$. In 1967, Shapiro gave a similar example of a smooth linear PDE which does not have solutions in Sato's hyperfunctions.

In the early 1960s, L Hörmander gave certain *necessary* conditions for the solvability in distributions of arbitrary linear smooth coefficient PDEs, see [27, pp. 37-39], [28, pp. 212-214].

Regarding the perception that nontrivial general, type independent results are just about impossible to obtain related to PDEs, it is worth noting that the Malgrange-Ehrenpreis result on fundamental solutions is precisely such a nontrivial general and type independent existence result within the range of all linear and constant coefficient PDEs.
The necessary condition for the existence of distributions solutions given by Hörmander is also a nontrivial general and type independent result, this time within the much larger class of all linear smooth coefficient PDEs.

In this regards, of course, the classical Cauchy-Kovalevskaia theorem is the most impressive general and type independent nonlinear result regarding existence, regularity and uniqueness of solutions, although it is obtained without any kind of functional analysis.

## 4. Nonlinear algebraic theory of generalized functions

This nonlinear theory - see 46F30 in the AMS Subject Classification 2010 - was started in the 1960s, [21-37,55-57,61,64,65], and it is based on the construction of all possible differential algebras of generalized functions which contain the Schwartz distributions. That theory has managed to come quite near to solving the Lewy impossibility. Yet it did not solve it completely, although it obtained generalized function solutions for large classes of linear and nonlinear PDEs. As an example, back in 1985, it obtained the first *global* existence result for the general nonlinear PDEs in the classical Cauchy-Kovalevskaia theorem. And the respective global solutions are analytic on the whole of the domain of the PDEs, except for certain closed and nowhere dense subsets, which can be chosen to have zero Lebesgue measure, see [27, pp. 259-266], [28, pp. 101-122], [29,36,37].

## 5. The order completion method

Surprisingly, the order completion method in solving general nonlinear systems of PDEs of the form (2.1) is based on certain very simple, even if less than usual, approximation properties, see [18, pp. 12-20]. To give here an idea about the ways the order completion method works, we mention some of these approximations here in the case of one single nonlinear PDE of the form (2.1).

Let us denote by $T(x, D)$ the left term in (2.1), then we have the basic approximation property :

**Lemma 5.1**

$\forall \;\; x_0 \in \Omega, \;\; \epsilon > 0 \;\; :$

$\exists \;\; \delta > 0, \;\; P \;\; \text{polynomial in} \;\; x \in \mathbb{R}^n \;\; :$

$$||x - x_0|| \;\leq\; \delta \;\;\Longrightarrow\;\; f(x) - \epsilon \;\leq\; T(x,D)P(x) \;\leq\; f(x)$$

□

Consequently, we obtain :

**Proposition 5.1**

$\forall \;\; \epsilon > 0 \;\; :$

$\exists \;\; \Gamma_\epsilon \subset \Omega \;\; \text{closed, nowhere dense in} \;\; \Omega, \;\; U_\epsilon \in C^\infty(\Omega) \;\; :$

$$f - \epsilon \;\leq\; T(x,D)P \;\leq\; f \;\; \text{on} \;\; \Omega \setminus \Gamma_\epsilon$$

Furthermore, one can also assume that the Lebesgue measure of $\Gamma_\epsilon$ is zero, namely

$$mes\;(\Gamma_\epsilon) \;=\; 0.$$

□

Let us now note that, see [1-3]

$$C^0(\Omega) \subset \mathbf{H}(\Omega)$$

and the set $\mathbf{H}(\Omega)$ of *Hausdorff continuous* functions on $\Omega$ is Dedekind order complete.
Consequently, we obtain the following basic result on the *existence* and *regularity* of solutions for nonlinear PDEs of the form (2.1) :

**Theorem 5.1**

$$T(x,D)U(x) \;=\; f(x), \;\; x \in \Omega$$

has solutions $U$ which can be assimilated with Hausdorff continuous functions, for a class of discontinuous functions $f$ on $\Omega$, class which

contains the continuous functions on $\Omega$.

□

We give here some more details related to Theorem 5.1 above. In view of Proposition 5.1, we shall be interested in spaces of piecewise smooth functions given by

$$(5.1) \quad C^l_{nd}(\Omega) \;=\; \left\{ u \;\middle|\; \begin{array}{l} \exists\ \ \Gamma \subset \Omega \ \text{ closed, nowhere dense } \ : \\ \\ \quad *)\ u : \Omega \setminus \Gamma \to \mathbb{R} \\ \\ \quad **)\ u \in C^l(\Omega \setminus \Gamma) \end{array} \right\}$$

where $l \in \mathbb{N}$. It is easy to see that we have the inclusions

$$(5.2) \quad T(x,D)\ C^m_{nd}(\Omega) \;\subseteq\; C^0_{nd}(\Omega) \;\subset\; \mathbf{H}(\Omega)$$

In this way, we obtain the following more precise formulation of the result in Theorem 5.1 on the existence and regularity of solutions :

**Theorem 5.1***

$$(5.3) \quad T(x,D)^{\#}\ (C^m_{nd}(\Omega))^{\#}_T \;=\; (C^0_{nd}(\Omega))^{\#} \;\subset\; \mathbf{H}(\Omega)$$

□

Here $(C^m_{nd}(\Omega))^{\#}_T$ and $(C^0_{nd}(\Omega))^{\#}$ are Dedekind order completions of $C^m_{nd}(\Omega)$ and $C^0_{nd}(\Omega)$, respectively, when these latter two spaces are considered with suitable partial orders. The respective partial order on $C^m_{nd}(\Omega)$ may depend on the nonlinear partial differential operator $T(x,D)$ in (5.2), while the partial order on $C^0_{nd}(\Omega)$ is the natural point-wise one at the points where two functions compared are both continuous.
The operator $T(x,D)^{\#}$ is a natural extension of the nonlinear partial differential operator $T(x,D)$ in (5.2) to the mentioned Dedekind order completions.

The meaning of (5.3) is twofold :

- for every right hand term $f \in (C^0_{nd}(\Omega))^{\#}$ in (2.1), there exists a solution $U \in (C^m_{nd}(\Omega))^{\#}_T$, and the set $(C^0_{nd}(\Omega))^{\#}$ contains many discontinuous functions beyond those piecewise discontinuous ones, see [18],

- the solutions $U$ can be assimilated with Hausdorff continuous functions on $\Omega$, see [1-3].

## 6. Comparison with methods in Functional Analysis

The order completion method is a powerful *alternative* to the usual functional analytic ones, when solving linear or nonlinear PDEs. Details in this regard are presented in [18, chap. 12]. Certainly, the order completion method is *not* meant to replace the functional analytic ones, the latter being useful in obtaining stronger results in a large variety of particular PDEs.
Here, we shall only mention the following. Functional analytic methods in solving PDEs are based on the *topological completion* of uniform spaces, such a normed or locally convex vector spaces of suitably chosen functions. In this respect, the comparative advantages of the order completion method can shortly be formulated as follows :

- unlike the functional analytic methods, which are geared more naturally to the solution of linear PDEs, the order completion method performs equally well in the case of both linear and nonlinear PDEs, see section 9 below,

- unlike the functional analytic methods, which face considerable difficulties when dealing with initial, and especially, boundary value problems, the order completion method performs without significant additional troubles in such situations,

- the order completion method gives solutions which can be assimilated with usual measurable, or even Hausdorff continuous functions, and thus the solutions obtained are not merely distributions, generalized functions or hyperfunctions.

As an illustration of the comparative situation regarding these two methods let us consider on a bounded Euclidean domain $\Omega$, which has a smooth boundary $\partial\,\Omega$, the following well known linear boundary value problem

$$\begin{aligned} &\Delta\, U(x) \;=\; f(x), \quad x \in \Omega \\ &U \;=\; 0 \quad \text{on} \;\; \partial\,\Omega \end{aligned} \tag{6.1}$$

As is well known, for every given $f \in C^\infty(\widetilde{\Omega})$, this problem has a unique solution $U$ in the space

$$X \;=\; \left\{ \; v \in C^\infty(\widetilde{\Omega}) \;\mid\; v \;=\; 0 \;\; \text{on} \;\; \partial\,\Omega \; \right\} \tag{6.2}$$

It follows that the mapping

$$X \ni v \;\longmapsto\; ||\,\Delta v\,||_{L^2(\Omega)} \tag{6.3}$$

defines a norm on the vector space $X$. Now let

$$Y \;=\; C^\infty(\widetilde{\Omega}) \tag{6.4}$$

be endowed with the topology induced by $L^2(\Omega)$. Then in view of (6.1) - (6.4), it follows that the mapping

$$\Delta : X \;\to\; Y \tag{6.5}$$

is a uniform continuous linear bijection. Therefore, it can be extended in a unique manner to an isomorphism of Banach spaces

$$\Delta : \widetilde{X} \;\to\; \widetilde{Y} \;=\; L^2(\Omega) \tag{6.6}$$

In this way one has the classical existence and uniqueness result

$$\forall \quad f \in L^2(\Omega) \quad :$$

$$(6.7) \qquad \exists\, !\ U \in \widetilde{X} \quad :$$

$$\Delta U = f$$

The power and simplicity - based on linearity and topological completion of uniform spaces - of the above classical existence and uniqueness result is obvious. This power is illustrated by the fact that the set $\widetilde{Y} = L^2(\Omega)$ in which the right hand terms $f$ in (6.1) can now be chosen is much *larger* than the original $Y = C^\infty(\widetilde{\Omega})$. Furthermore, the existence and uniqueness result in (6.7) does not need the a priori knowledge of the structure of the elements $U \in \widetilde{X}$, that is, of the respective generalized solutions. This structure which gives the regularity properties of such solutions can be obtained by a further detailed study of the respective differential operators defining the PDEs under consideration, in this case, the Laplacian $\Delta$. And in the above specific instance we obtain

$$(6.8) \qquad \widetilde{X} \ = \ H^2(\Omega) \cap H_0^1(\Omega)$$

As seen above, typically for the functional analytic methods, the generalized solutions are obtained in topological completions of vector spaces of usual functions. And such completions, like for instance the various Sobolev spaces, are defined by certain linear partial differential operators which may happen to *depend* on the PDEs under consideration.

In the above example, for instance, the topology on the space $X$ obviously *depends* on the specific PDE in (6.1). Thus the topological completion $\widetilde{X}$ in which the generalized solutions $U$ are found according to (6.7), does again *depend* on the respective PDE.

On the other hand, with the method of order completion we are *no longer* looking for generalized solutions, and instead, a *type independent, universal* or *blanket* regularity property is attained, since the solutions obtained can always be assimilated with usual measurable functions, or even with Hausdorff continuous functions. Similar to the

functional analytic methods, however, the order completion method obtains the solutions in spaces which may again be related to the specific nonlinear partial differential operators $T(x, D)$ in the equations of form (2.1).

## 7. Solving equations by extending their domains of definition : the three classical methods

The ancient case of *solving an equation*, which shocked Pythagoras two and a half millennia ago, is given by

$$x^2 = 2 \tag{7.1}$$

This is of the general form

$$E(x) = c \tag{7.2}$$

where we are given a mapping

$$E : X \to Y \tag{7.3}$$

together with a specific $c \in Y$, and then we want to find a solution $x \in X$ so that (7.2) holds.

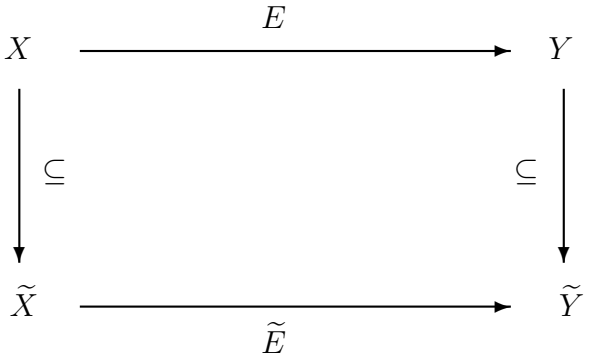
What shocked Pythagoras was that (7.1) could not be solved if one restricted oneself to $X = \mathbb{Q}$ in (7.3). And it took no less than about two millennia or more, till we could rigorously extend $X = \mathbb{Q}$ to $\widetilde{X} = \mathbb{R}$, and thus obtain a well defined solution $x = \pm\sqrt{2}$ of (7.1).

In this way, ever since, we have the following model lesson in solving equations :

- if one cannot solve (7.2) within the framework of (7.3), then one can try to solve it in the *extended* framework

$$\widetilde{E} : \widetilde{X} \to \widetilde{Y} \tag{7.4}$$

where $X \subset \widetilde{X}$, $Y \subseteq \widetilde{Y}$, and $\widetilde{E}$ is such that we have the commutative diagram

(7.5)



Here however, we face the following problems :

- how to choose or construct $\widetilde{X}$, and then how to interpret the new, or so called *generalized* solutions $x \in \widetilde{X} \setminus X$, which two questions altogether constitute but the celebrated regularity problem,
- how to do the same for $\widetilde{Y}$, which nevertheless need not always be done, since we can often stay with $Y$ in (7.4) and only have to extend $X$ to $\widetilde{X}$,
- how to define the extension $\widetilde{E}$, which often, and typically in the nonlinear case, is not a trivial problem.

Fur further detail, we can now recall that with the equation (7.1) we had to

(7.6) $\quad go\ from\quad \mathbb{Q}\quad to\quad \widetilde{\mathbb{Q}} = \mathbb{R}$

On the other hand, with the equation

(7.7) $\quad x^2 + 1 = 0$

we had to

(7.8) $\quad go\ from\quad \mathbb{R}\quad to\quad \mathbb{C}$

However, there is a vast difference between (7.6) and (7.8), respectively, between solving (7.1) and (7.7). Indeed, we solve (7.7) through the extension (7.8) which is a mere *algebraic adjoining* of an element, in this case, of $i = \sqrt{-1}$ to $\mathbb{R}$.

On the other hand, when solving (7.1), the extension (7.6) can be seen as at least *three different*, even if in this particular case equivalent, constructions, namely, through :

- topology
- algebra
- order.

And to be more precise, we have :

- The Cauchy-Bolzano method is *ring theoretic* plus *topological*, and it is applied to $\mathbb{Q}$, as it obtains $\mathbb{R}$ according to the quotient construction in algebras

  $$\mathbb{R} \;=\; \mathcal{A}/\mathcal{I} \tag{7.9}$$

  where $\mathcal{A} \subset \mathbb{Q}^{\mathbb{N}}$ is the algebra of Cauchy sequences of rational numbers, while $\mathcal{I}$ is its ideal of sequences convergent to zero.

- The method of Dedekind is based on the *order completion* of $\mathbb{Q}$.

However, the Cauchy-Bolzano method can be generalized in two directions :

- In the topological generalization the algebraic part can be omitted, and instead, one only uses the *topological completion* of uniform spaces, here of the usual metric space on $\mathbb{Q}$.
- In the algebraic generalization it is possible to extract the abstract essence of (7.9), and simply start with a suitable algebra $\mathcal{A}$, and an ideal $\mathcal{I}$ in it. Such a construction can indeed be rather abstract, since it need *not* involve any topology on $\mathbb{Q}$ or $\mathbb{R}$, as it happens for instance, when constructing the nonstandard reals

${}^*\mathbb{R}$, namely

$${}^*\mathbb{R} \;=\; \mathcal{A}/\mathcal{I}$$

Here one takes $\mathcal{A} = \mathbb{R}^{\mathbb{N}}$, that is, the algebra of all sequences of real numbers, while the ideal $\mathcal{I}$ is defined by any given free ultrafilter on $\mathbb{N}$.
A rather general version of such an abstract approach, which however makes a certain limited use of topology, has been introduced and extensively used in the nonlinear algebraic theory of generalized functions under the AMS classification index 46F30, as mentioned in section 4 above.

What is done in the method in 46F30 is to generalize the Cauchy-Bolzano method by retaining its ring theoretic algebraic aspect, while the topological one is weakened to the certain extent of being confined to the topologies of Euclidean spaces only.

What is done in the method introduced in [18], and further developed in [1-3,44-51,58-60], is the extension of the classical Dedekind order completion method, used in the construction of $\mathbb{R}$ from $\mathbb{Q}$, to suitable spaces of piece-wise smooth functions.

An important fact to note is that both the topological and order completion methods give us the property that

- $\mathbb{Q}$ is *dense* in $\mathbb{R}$

in the respective sense of topology or order. In this way, the elements of the extension of $\mathbb{Q}$, that is, the elements of $\mathbb{R}$, are in the corresponding sense *arbitrarily near* to the elements of the extended space $\mathbb{Q}$. Thus the elements in the extension can arbitrarily be *approximated* by elements of the extended space, be it in the sense of topology, or respectively, order.
Furthermore, both through the methods of topology and order, one obtains $\mathbb{R}$ in a *unique* manner, up to a respective isomorphism.

In this way both the topological and order completion methods have the *double* advantage that

- the elements of the extension are not too strange conceptually,

and furthermore

- the elements of the extension are near to elements of the extended space, within arbitrarily small error.

This *density* property remains also in the general Dedekind order completion method used in [18,1-3,44-51,58-60]. Indeed, in (5.3) we have that $C^m_{nd}(\Omega)$ and $C^0_{nd}(\Omega)$ are order dense in $(C^m_{nd}(\Omega))^{\#}_T$ and $(C^0_{nd}(\Omega))^{\#}$, respectively.

Connected with the general extension method in (7.3) - (7.5) one can note that, on occasion, the following *convenient* situation may occur : the extended mapping

$$(7.10) \quad \widetilde{E} : \widetilde{X} \longrightarrow \widetilde{Y}$$

may turn out to be an *isomorphism* of the respective algebraic, topological or order structures used on $X$ and $Y$, when constructing the corresponding extensions $\widetilde{X}$ and $\widetilde{Y}$. In such a case, and when one has a better understanding of the structure of the elements in $\widetilde{Y}$, one can obtain in addition a *regularity* type result concerning the so called generalized solutions $x \in \widetilde{X} \setminus X$ of the equations (7.2), since such generalized solutions can be *assimilated* - through the isomorphism $\widetilde{E}$ - with the corresponding elements $\widetilde{E}(x) \in \widetilde{Y}$.

A classical example of such an isomorphism (7.10) happens, for instance, in (6.5), (6.6), when the boundary value problem (6.1) is solved by using well known functional analytic methods.
In that specific instance, however, the suitable further use of functional analytic methods can lead to the *additional* regularity property of generalized solutions in $\widetilde{X}$, as given in (6.8). Nevertheless, the Banach space isomorphism (6.6) - which in that case is but the particular form taken by (7.10) - is in itself already a *first* regularity result about the structure of the elements of $\widetilde{X}$.

The above convenient situation of an isomorphism of type (7.10) can appear as well when using the order completion method in solving

nonlinear system of PDEs. This is the reason why the solutions obtained in [18] could be assimilated with usual measurable functions, while in [1-3], they can be assimilated with the much *more regular* Hausdorff continuous functions.
More specifically, in (5.3), the extended mappings $T(x,D)^{\#}$ prove to be *order isomorphisms* between the spaces $C^m_{nd}(\Omega)^{\#}_T$ and $C^0_{nd}(\Omega)^{\#}$.
This is then, in essence, the reason why the solutions of nonlinear systems of PDEs of the form (2.1) could earlier be assimilated with usual measurable functions, and can now be assimilated with Hausdorff continuous functions.

## 8. The need for extensions in the case of solving PDEs.

Let us now associate with each nonlinear PDE in (2.1) the corresponding nonlinear partial differential operator defined by the left hand side, namely

$$(8.1) \quad T(x,D)U(x) \;=\; F(x,U(x),\;\ldots,D^p_xU(x),\;\ldots),\quad x\in\Omega$$

*Two* facts about the nonlinear PDEs in (2.1) and the corresponding nonlinear partial differential operators $T(x,D)$ in (8.1) are important and immediate

- The operators $T(x,D)$ can *naturally* be seen as acting in the *classical* context, namely

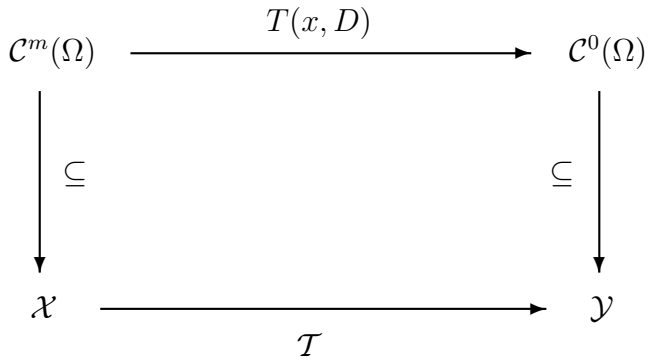
$$(8.2) \quad T(x,D)\;:\;\mathcal{C}^m(\Omega)\ni U\quad\longmapsto\quad T(x,D)U\in\mathcal{C}^0(\Omega)$$

while, unfortunately on the other hand

- The mappings in this natural classical context (8.2) are typically *not* surjective. In other words, linear or nonlinear PDEs in (2.1) typically *cannot* be expected to have *classical* solutions $U\in\mathcal{C}^m(\Omega)$, for arbitrary continuous right hand terms $f\in\mathcal{C}^0(\Omega)$.
  Furthermore, it can often happen that nonclassical solutions have a major applicative interest, thus they have to be sought out beyond the classical framework in (8.2).

This is, therefore, how we are led to the *necessity* to consider *generalized solutions* $U$ for PDEs of type (2.1), that is, solutions $U \notin \mathcal{C}^m(\Omega)$, which therefore are no longer classical. This means that the natural classical mappings (8.2) must in certain suitable ways be *extended* to *commutative diagrams*

(8.3)




which are expected to have certain kind of *surjectivity* type properties, such as for instance

(8.4) $\quad \mathcal{C}^0(\Omega) \subseteq T(\mathcal{X})$

We conclude with a few comments :

- Traditionally, ever since Hilbert and Sobolev, starting before WW II, *functional analysis* has been used in solving PDEs, and suitable uniform topologies are defined on the domains and ranges of the corresponding partial differential operators $T(x, D)$. These domains and ranges are given by various vector spaces of generalized functions, and in particular, distributions. Thus these operators obtain certain continuity properties. Then the extensions $\mathcal{X}$ and $\mathcal{Y}$ in (8.3) are defined as the *completions* in these uniform topologies of the respective domains and ranges of $T(x, D)$. Finally, the continuity properties of $T(x, D)$ may allow the construction of suitable extensions $\mathcal{T}$ which would give the commutative diagrams (8.3), and also satisfy some version of the surjectivity property (8.4), see for details [18, chap. 12, pp. 237-262].
- Since the 1960s, the *algebraic nonlinear* method in 46F30 can

alternatively be used especially in the case of *nonlinear* partial differential operators $T(x, D)$. In this respect, large, and in fact, infinitely many classes of *differential algebras of generalized functions* containing the Schwartz distributions were constructed as the sought after extensions $\mathcal{X}$ and $\mathcal{Y}$ in (8.3).
The most general classes of such algebras were introduced and used in [21-37], starting with the 1960s. Later, in the 1980s, a particular class of such algebras was introduced in [6], and it has known a certain popularity.
However, due to the specific polynomially limiting *growth conditions* required in the construction of Colombeau algebras, their use in the study, for instance, of Lie group symmetries of PDEs, or singularities in General Relativity is limited, since in both cases one may have to deal with transformation whose growth can be arbitrary. In this way, such transformations cannot be accommodated within the Colombeau algebras of generalized functions. In fact, in the Colombeau algebras, as well as in other ones which fail to be flabby sheaves, one cannot even formulate - let alone solve - the Global Cauchy-Kovalevskaia Theorem. Similarly, one cannot define globally arbitrary Lie group actions.
On the other hand, arbitrary smooth transformations and operations can easily be dealt with in some of the other classes of algebras of generalized functions introduced earlier in [21-37]. Also as mentioned, the Global Cauchy-Kovalevskaia Theorem can be formulated and solved, and arbitrary Lie group actions can be defined globally in such algebras. In fact, based on such a definition of arbitrary global Lie group actions on generalized functions, in [32] a first time complete solution of Hilbert's Fifth Problem was presented.
The severe limitations on dealing with singularities which various vector spaces of distributions or algebras of generalized functions suffer from is closely related to their failure to be *flabby sheaves*. Among such algebras are the Colombeau algebras which fail to be flabby sheaves due to the mentioned growth conditions which paly an essential role in their definition.
Details related to the ability of various vector spaces of distributions and algebras of generalized functions to deal with large classes of singularities are presented in [37]. And it turns out

that those vector spaces of distributions and algebras of generalized functions which fail to be flabby sheaves also fail in dealing with large enough classes of singularities.

- The *order completion* method, introduced and developed in 1994 in [18], and further improved in [1-3,37,44-52,58-60], constructs the extensions $\mathcal{X}$ and $\mathcal{Y}$ in (8.3) as the *Dedekind order completion* of spaces naturally associated with the partial differential operators $T(x, D)$, and the spaces $\mathcal{C}^m(\Omega)$ and $\mathcal{C}^0(\Omega)$ in (8.2).

Related to the advantages of the order completion method in solving nonlinear PDEs let us mention here in short the following.

Neither the functional analytic, nor the algebraic methods can so far come anywhere near to solve nonlinear PDEs of the generality of those in (2.1), let alone systems of such nonlinear PDEs together with associated initial and/or boundary value problems.

In fact, the functional analytic methods are still subjected to the celebrated 1957 Hans Lewy impossibility which they are nowhere near to manage to overcome, even in the general smooth coefficient linear case.

The extent of the inevitability of this failure within the present functional analytic methods is clearly illustrated by the necessary condition for the solvability in distributions of linear smooth coefficient PDEs given by Hörmander in the 1960s, a condition which is also sufficient in the case of first order such PDEs, as shown not much later by Nirenberg & Treves, see [27, pp. 37-47] for details.

As far as the algebraic method is concerned, it has among others come quite near to the solution of the Lewy impossibility, see [6], and for a short respective account [27].
Further powerful results in solving various classes of nonlinear PDEs, not treated so far by the functional analytic method, can be found in [25-30], [6], or [17].
Among such results is the *global* solution of arbitrary analytic nonlinear systems of PDEs, when considered with analytic non-characteristic Cauchy initial values, mentioned in section 4 above. The respective generalized solutions obtained are *analytic* functions, except for *closed*

and *nowhere dense* subsets $\Gamma$ of the domains $\Omega$ of definition of the given PDEs. In addition, these subsets $\Gamma$ can also be chosen to have *zero* Lebesgue measure.

On the other hand, the order completion method introduced and developed in [18], and further improved in [1-3,37,44-52,58-60] as far as the *regularity* of solutions is concerned, can not only deliver global solutions for systems of nonlinear PDEs of the generality of those in (2.1), but it can also obtain a general, type independent, universal or blanket *regularity* result for such solution, namely, it can prove that the solutions obtained can be assimilated with usual measurable functions, or even with *Hausdorff continuous* functions, and in fact, with smooth functions, under appropriate smoothness conditions on the respective nonlinear PDEs.

Clearly, as one of the consequences of solving nonlinear systems of PDEs of the generality of those in (2.1), the order completion method in [18,1-3,37,44-52,58-60] is the only one so far which fully manages to overcome the Lewy impossibility. Furthermore, it does so with a *large* nonlinear margin.

## 9. Order Completion Abolishes the Dichotomy "Linear versus Nonlinear"

The *dichotomy linear versus nonlinear* relating to equations or operators is in its essence an issue of *algebra*, and more specifically, of *vector space* structures. In this way, it is present *both* in the functional analytic *and* algebraic methods for solving PDEs. And needless to say, dealing with the nonlinear case proves to be incomparably more difficult than it is with the linear one. Consequently, and unfortunately, the presence of this dichotomy is one of the *major disadvantages* of both the functional analytic and algebraic methods in solving PDEs, even if by now it is taken so much for granted that no attempt is made to overcome it, let alone abolish what appears to be its inevitable presence.

On the other hand, order structures are of a more *basic* type than the

algebraic ones.
Consequently, if instead of algebraic structures we consider order structures on the spaces of smooth functions on which the partial differential operators act naturally when considered in the classical context, see for instance (8.2), then these order structures - being more basic than algebra - *can no longer distinguish* between the linearity or nonlinearity of such partial differential operators. In this way, the traditional dichotomy between the linear and nonlinear is simply set aside, and then the only problem left is whether indeed one can solve PDEs in the completion of such order structures.
Fortunately, as shown in [18,1-3,37,44-52,58-60], such a solution is possible and useful.

Here of course, one may think that it may help if the respective partial differential operators are monotonous. And then one may be concerned that all we managed to do was simply to get rid of the dichotomy between linear and nonlinear, so that instead, now we have to face the dichotomy monotonous versus arbitrary partial differential operators, with the latter being quite likely again far more difficult to deal with, than the former.
Such a particular approach in which the dichotomy monotonous versus non-monotonous prevails has recently been pursued in [5], for instance, with the consequent and not surprising considerable limitations on the types of PDEs to which it can be applied.

This however is clearly *not* the way with the method in [18,1-3,37,44-52,58-60]. Indeed, although in this method order structures and order completions are essentially used in solving PDEs, one nevertheless need *not* require a priori any sort of monotonicity property related to the equations solved. Certainly, the generality of nonlinear PDEs in (2.1), or of systems of such nonlinear PDEs, clearly illustrates the fact that the respective equations are *not* supposed to satisfy any a priori monotonicity conditions whatsoever.

What happens is very simple in fact, and is *similar* with the way the operators $T(x, D)$ and $\mathcal{T}$ in (8.3) acquire continuity type properties when the functional analytic method is used in the construction of such commutative diagrams. Indeed, with the functional analytic

method, when one starts, say, with the natural mappings (8.2), the uniform topologies one considers on the classical domains and ranges of the operators in order to obtain commutative diagrams (8.3) are not arbitrary, but typically are related to the respective operators, see section 6 above, or [18, chap. 12].

The same happens when order completion is used in [18,1-3,37,44-52,58-60] for the construction of commutative diagrams (8.3). More precisely, the order structures on the spaces of smooth functions on which the partial differential operators naturally act, see for instance (8.2), will typically be defined *dependent* to a certain extent on these operators. In this way, the respective operators, no matter how arbitrary within the class of those in (2.1), will nevertheless *become* monotonous, thus so much easier to deal with.

Such a procedure obviously cannot be imitated within algebra, since a nonlinear operator cannot in general define a vector space structure in which it would become linear.
Furthermore, such a procedure obviously goes far beyond the approach in [5], for instance, where one starts with given natural order relations on *both* the domains and ranges of ODEs or PDEs, and then severely restricts oneself only to those rather small classes of equations whose associated operators, or rather, inverse operators, are monotonous in the a priori given orders.

## 10. The Hidden Power of Methods Based on Partial Orders

As it happens, there is a rather widespread perception in mathematics that order structures are too simple, and thus powerless, especially in analysis, therefore, they can deliver less than algebra, which on its turn, can deliver less than topology.
Accordingly, since the emergence of functional analytic methods in the solution of PDEs at the beginning of the 20th century, with the respective wealth of topologies on a variety of spaces of functions, the perception prevails that there simply cannot be any other more powerful methods in present day mathematics which could deal with such equations.

This perception has obviously further been strengthened by the fact that - lacking actually a sufficient ability to deal in a type independent manner with solving linear and nonlinear PDEs - the power of functional analytic methods got fragmented into dealing with a larger and larger variety more and more particular types of linear and nonlinear PDEs. And needless to say, such an increasing fragmentation has led to more and more powerful results, each of a less and less wider applicability.
Since that process has by now been ongoing ever since the 1930s, it has entrenched the perception that there is not, and there cannot be any other way forward ...

In view of such a perception it may therefore appear rather surprising, if not in fact hardly credible, to see results such as in [1-3,37,44-52,58-60], and the earlier ones in [18], results obtained through order structures, and which so far could not be approached anywhere near by functional analytic methods.
Indeed, this method - based on order completion - does solve systems of PDEs of the nonlinear generality of those in (2.1), together with associated initial and/or boundary value problems, and furthermore, delivers for them global solutions which can be assimilated with usual measurable, or even Hausdorff continuous functions, and yet more smooth ones, under corresponding smoothness conditions on the respective PDEs. It is in this way that the order completion method is not only unprecedented, but it may also look rather strange in view of the mentioned perception in mathematics related to order structures.

Therefore, one should indeed address the apparent *secret of the power* of the order completion method in solving such large classes of systems of nonlinear PDEs, together with their associated initial and/or boundary value problems. And one can do so by questioning the mentioned perception. This can perhaps best be done by the presentation of certain classical - even if less well known - examples which illustrate the power of order structures in yielding what usually are called *deep theorems*.

In this regard a rather impressive, yet less well known fact is given by the 1936 "Spectral Theorem" of Freudenthal, see [11, chap. 6].

Let us recall here in short some of its rather deep consequences.

The mentioned "Spectral Theorem" is a theorem about partially ordered structures, and it was proved by Freudenthal exclusively in terms of such structures. Yet, what is of special relevance is that by suitable particularizations, one can obtain from it the following three results which are in fields as diverse as Operator Theory, Measure Theory, and linear PDEs :

- the celebrated spectral representation theorem for normal operators in Hilbert spaces,
- the highly nontrivial Radon-Nikodym theorem in measure theory, and
- the Poisson formula for harmonic functions in an open circle.

## 11. Conclusions

The unprecedented power of the order completion method in solving very general systems of nonlinear PDEs and the associated initial and/or boundary value problems, and obtaining solutions given by usual functions, stems from two facts :

- Partial orders are more basic mathematical structures than algebra or topology. And being more basic than algebra, partial orders do *not* distinguish between linear and nonlinear equations, operators, and so on. Consequently, partial orders treat the linear and nonlinear cases in the same manner. Functional analytic method clearly cannot do the same.
- When using order completion for solving PDEs, one need *not* assume any monotonicity properties of the respective equations.

In the order completion method, the partial orders on the spaces of functions which are the domains of definition of the partial differential operators considered are defined in relation to these operators. This is similar to the way the topologies on such domains are defined, when

functional analytic methods are used. Further details in this regard can be found in [18, chap. 12, 13] and [1-3,37,44-52,58-60].

# References


[1] Anguelov R : Dedekind order completion of C(X) by Hausdorff continuous functions. Quaestiones Mathematicae, 2004, Vol. 27, No. 2, 1533-169

[2] Anguelov R, Rosinger E E : Hausdorff continuous solutions of nonlinear PDEs through the order completion method. Quaestiones Mathematicae, Vol. 28, 2005, 1-15, arXiv:math.AP/0406517

[3] Anguelov R, Rosinger E E : Solving large classes of nonlinear systems of PDEs. Computers and Mathematics with Applications. Vol. 53, No. 3-4, Feb. 2007, 491-507, arXiv:math/0505674

[4] Arnold V I : Lectures on PDEs. Springer Universitext, 2004

[5] Carl S, Heikkilä S : Nonlinear Differential Equations in Ordered Spces. Chapman & Hall/CRC, Monographs and Surveys in Pure and Applied Mathematics, VOl. 111, Boca Raton, 2000

[6] Colombeau J-F : Elementary Introduction to New Generalized Functions. North-Holland Mathematics Studies, Vol. 113, Amsterdam, 1985

[7] Ehrenpreis L : Solutions of some problems of division I. Amer. J. Math., Vol. 76, 1954, 883-903

[8] Evans L C : Partial Differential Equations. AMS Graduate Studies in Mathematics, Vol. 19, 1998

[9] Grosser M, Kunzinger M, Oberguggenberger M, Steinbauer R : Geometric Theory of Generalized Functions with Applications to General Relativity. Kluwer, Dordrecht, 2002

[10] Lewy, H : An example of smooth linear partial differential equation without solutions. Ann. Math., vol. 66, no. 2, 1957, 155-158

[11] Luxemburg W A J, Zaanen A C : Riesz Spaces, Vol. I. North-Holland, Amsterdam, 1971

[12] Malgrange B : Existence et approximation des solutions des equations aux derivees partielles et des equations de convolutions. Ann. Inst. Fourier¡ grenoble, Vol. 6, 1955-56, 271-355

[13] Mallios A, Rosinger E E [1] : Abstract differential geometry, differential algebras of generalized functions, and de Rham cohomology. Acta Appl. Math., vol. 55, no. 3, Feb. 1999, 231-250

[14] Mallios A, Rosinger E E [2] : Space-time foam dense singularities and de Rham cohomology. Acta Appl. Math., vol. 67, 2001, 59-89, arxiv:math/0406540

[15] Mallios A, Rosinger E E [3] : Dense singularities and de Rham cohomology. In (Eds. Strantzalos P, Fragoulopoulou M) Topological Algebras with Applications to Differential Geometry and Mathematical Physics. Proc. Fest-Colloq. in honour of Prof. Anastasios Mallios (16-18 September 1999), pp. 54-71, Dept. Math. Univ. Athens Publishers, Athens, Greece, 2002

[16] Neuberger J W : Prospects for a central theory of partial differential equations. The Mathematical Intelligencer, Vol. 27, No. 3, Summer 2005, 47-55

[17] Oberguggenberger M B : Multiplication of Distributions and Applications to PDEs. Pitman Research Notes in Mathematics, Vol. 259. Longman, Harlow, 1992

[18] Oberguggenberger M B, Rosinger E E : Solution of Continuous Nonlinear PDEs through Order Completion. North-Holland Mathematics Studies, Vol. 181. North-Holland, Amsterdam, 1994

[19] See review MR 95k:35002

[20] Oxtoby J C : Measure and Category. Springer, New York, 1971

[21] Rosinger E E [1] : Embedding of the $\mathcal{D}'$ distributions into pseudo-topological algebras. Stud. Cerc. Math., Vol. 18, No. 5, 1966, 687-729.

[22] Rosinger E E [2] : Pseudotopological spaces, the embedding of the $\mathcal{D}'$ distributions into algebras. Stud. Cerc. Math., Vol. 20, No. 4, 1968, 553-582.

[23] Rosinger E E [3] : Division of Distributions. Pacif.J. Math., Vol. 66, No. 1, 1976, 257-263

[24] Rosinger E E [4] : Nonsymmetric Dirac distributions in scattering theory. In Springer Lecture Notes in Mathematics, Vol. 564, 391-399, Springer, New York, 1976

[25] Rosinger E E [5] : Distributions and Nonlinear Partial Differential Equations. Springer Lectures Notes in Mathematics, Vol. 684, Springer, New York, 1978.

[26] Rosinger E E [6] : Nonlinear Partial Differential Equations, Sequential and Weak Solutions, North Holland Mathematics Studies, Vol. 44, Amsterdam, 1980.

[27] Rosinger E E [7] : Generalized Solutions of Nonlinear Partial Differential Equations. North Holland Mathematics Studies, Vol. 146, Amsterdam, 1987.

[28] Rosinger E E [8] : Nonlinear Partial Differential Equations, An Algebraic View of Generalized Solutions. North Holland Mathematics Studies, Vol. 164, Amsterdam, 1990.

[29] Rosinger E E [9] : Global version of the Cauchy-Kovalevskaia theorem for nonlinear PDEs. Acta Appl. Math., Vol. 21, 1990, 331–343.

[30] Rosinger E E [10] : Characterization for the solvability of nonlinear PDEs, Trans. AMS, Vol. 330, No. 1, March 1992, 203-225.

[31] Rosinger E E [11] : Nonprojectable Lie Symmetries of nonlinear PDEs and a negative solution to Hilbert's fifth problem. In (Eds. N.H. Ibragimov and F.M. Mahomed) Modern Group Analysis VI, Proceedings of the International Conference in the New South Africa, Developments and Applications, Johannesburg, January 1996, 21-30. New Age Inter. Publ., New Delhi, 1997

[32] Rosinger E E [12] : Parametric Lie Group Actions on Global Generalised Solutions of Nonlinear PDEs, Including a Solution to Hilbert's Fifth Problem. Kluwer Acad. Publishers, Amsterdam, Boston, 1998

[33] Rosinger E E [13] : Arbitrary Global Lie Group Actions on Generalized Solutions of Nonlinear PDEs and an Answer to Hilbert's Fifth Problem. In (Eds. Grosser M, Hörmann G, Kunzinger M, Oberguggenberger M B) Nonlinear Theory of Generalized Functions, 251-265, Research Notes in Mathematics, Chapman & Hall / CRC, London, New York, 1999

[34] Rosinger E E [14] : How to solve smooth nonlinear PDEs in algebras of generalized functions with dense singularities (invited paper) Applicable Analysis, vol. 78, 2001, 355-378, arXiv:math/0406594

[35] Rosinger E E [15] : Differential Algebras with Dense Singularities on Manifolds. Acta Applicandae Mathematicae. Vol. 95, No. 3, Feb. 2007, 233-256, arXiv:math/0606358

[36] Rosinger E E [16] : Space-Time Foam Differential Algebras of Generalized Functions and a Global Cauchy-Kovalevskaia Theorem. Acta Applicandae Mathematicae, DOI 10.1007/s10440-008-9326-z, Received: 5 February 2008 / Accepted: 23 September 2008, 05.10.2008, vol. 109, no. 2, pp. 439-462, arxiv:math/0607397

[37] Rosinger E E [17] : Survey on Singularities and Differential Algebras of Generalized Functions : A Basic Dichotomic Sheaf Theoretic Singularity Test. hal-00510751, version 1, 23 August, 2010

[38] Rosinger E E, Rudolph M : Group invariance of global generalised solutions of nonlinear PDEs : A Dedekind order completion method. Lie Groups and their Applications, Vol. 1, No. 1, July-August 1994, 203-215

[39] Rosinger E E, Walus E Y [1] : Group invariance of generalized solutions obtained through the algebraic method. Nonlinearity, Vol. 7, 1994, 837-859

[40] Rosinger E E, Walus E Y [2] : Group invariance of global generalised solutions of nonlinear PDEs in nowhere dense algebras. Lie Groups and their Applications, Vol. 1, No. 1, July-August 1994, 216-225

[41] See reviews MR 92d:46098, Zbl. Math. 717 35001, MR 92d:46097, Bull. AMS vol.20, no.1, Jan 1989, 96-101, MR 89g:35001

[42] Schwartz L : Sur l'impossibilite de la multiplication des distributions. C.R. Acad. Sci. Paris, vol. 239, 1954, 847-848

[43] Treves F : Applications of distributions to PDE theory. Amer. Math. Monthly, March 1970, 241-248

[44] Van der Walt J H : Generalized solutions to nonlinear first order Cauchy problems. UPWT2007/15, Universit of Pretoria.

[45] Van der Walt J H : The uniform order convergence structure on ML(X). Qaestiones Mathematicae, 31 (2008), 55-77

[46] Van der Walt J H : The order completion method for systems of nonlinear PDEs: Pseudo- topological perspectives. Acta Applicanda Mathematicae, 103 (2008), 1-17

[47] Van der Walt J H : The order completion method for systems of nonlinear PDEs revisited. Acta Applicanda Mathematicae 106 (2009), 149-176

[48] Van der Walt J H : On the completion of uniform convergence spaces and an application to nonlinear PDEs. Qaestiones Mathematicae, 32 (2009), 371-395

[49] Van der Walt J H : Solutions of smooth nonlinear PDEs. UPWT2009/02, University of Pretoria

[50] Van der Walt J H : The order completion method for systems of nonlinear PDEs: Solutions of initial value problems, UPWT2009/03, University of Pretoria

[51] Van der Walt J H : Singularities in Space-times Foam Algebras. To Appear in Applicable Analysis, Special Issue on Generalized

Functions and Applications.

Some of my web site papers :

[52] arXiv:math/0407026, Can there be a general nonlinear PDE theory for the existence of solutions ?

[53] arXiv:math/0604561, Genuine Lie semigroups and semi-symmetries of PDEs

[54] arXiv:math/0605109, New Symmetry Groups for Generalized Solutions of ODEs

[55] arXiv:math/0606358, Differential Algebras with Dense Singularities on Manifolds

[56] arXiv:math/0607082, Which are the Maximal Ideals ?

[57] arXiv:math/0607397, Space-Time Foam Differential Algebras of Generalized Functions and a Global Cauchy-Kovalevskaia Theorem

[58] arXiv:math/0608450, Solving General Equations by Order Completion

[59] arXiv:math/0609234, Extending Mappings between Posets

[60] arXiv:math/0610279, New Method for Solving Large Classes of Nonlinear Systems of

[61] arXiv:math/0611445, Junction Conditions, Resolution of Singularities and Nonlinear Equations of Physics

[62] arXiv:math/0611486, How to Define Global Lie Group Actions on Functions

[63] arXiv:math/0703515, Improvement on a Central Theory of PDEs

[64] arXiv:0912.4049, Local Functions : Algebras, Ideals, and Reduced Power Algebras

[65] arXiv:1002.0938, Inevitable Infinite Branching in the Multiplication of Singularities

[66] vixra:1008.0011, Reflections on an Asymmetry on the Occasion of Arnold's Paasing Away ...